CONFINEMENT IN RELATIVISTIC POTENTIAL MODELS

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Abstract

In relativistic potential models of quarkonia based on a Dirac-type of equation with a local potential there is a sharp distinction between a linear potential $V$ which is vector-like and one which is scalar-like: There are normalizable solutions for a scalar-like $V$ but not for a vector-like $V$. It is pointed out that if instead one uses an equation of the no-pair type, which is more natural from the viewpoint of field theory, this somewhat bizarre difference disappears.

Since the discovery of the narrow resonances in the GeV region, interest in a potential model description of these mesons and less charming ones as quark-antiquark bound states has continued unabated. A long-standing problem which arises in this connection is the following: If one tries to include relativistic effects with a Dirac-type of equation involving a purely local potential there is a dramatic difference between a linear potential which is vector-like, $V = k_v r$, and one which is scalar-like, $V = \beta_1 \beta_2 k_s r$. For the vector-like case there are no normalizable solutions. In view of the continued interest in such models\cite{1,2}, it may be useful to point out that if one uses equations of the no-pair type \cite{3}, which are much more natural in the context of field theory, this dichotomy is not present.

By way of review and for simplicity let us consider the case of an antiquark much heavier than the quark and take as a starting point a Dirac equation of the form

$$ (h_D + V)\psi = E\psi, \quad h_D = \alpha \cdot p_{\gamma} + \beta m, $$

with $V$ linear in $r$. A simple way to see the trouble which arises for $V = k_v r$ is to decompose the wave function $\psi$ into a sum

$$ \psi = \psi^+ + \psi^-, \quad \psi^\pm \equiv \beta^\pm \psi $$
where the $\beta^\pm$ are zero-momentum projection operators, defined by

$$\beta^\pm \equiv (1 \pm \beta)/2. \quad (3)$$

In the standard representation of the Dirac matrix $\beta$ this is essentially a decomposition into upper and lower components, but no use need be made of this fact. From (1) and (2) we have

$$\psi^- = (E + m - k_v r)^{-1} \alpha \cdot p_{op} \quad (4)$$

With $E > 0$, this implies that $\psi^-$ has a pole singularity at $r = (E + m)/k_v$ and is therefore not integrable. Since

$$\langle \psi^- | \psi^- \rangle = \langle \psi^+ | \psi^+ \rangle + \langle \psi^- | \psi^- \rangle, \quad (5)$$

the norm of $\psi^-$ will be infinite even if that of $\psi^+$ is finite. However, if the potential is scalar-like, $V = \beta k_s r$, the minus sign in the denominator in (4) changes to a plus sign,

$$\psi^- = (E + m + k_s r)^{-1} \alpha \cdot p_{op} \psi^+, \quad (6)$$

and there is no singularity. Thus, if both scalar and vector confining potentials are used it is necessary to have $k_s > k_v$. The same feature holds in the two-body equations of a similar type.

2. The corresponding no-pair equation does not suffer from this dichotomy. The counterpart of (1) is now

$$(h_D + \Lambda^\alpha_{op} U \Lambda^\alpha_{op}) \psi_+ = E \psi_+ \quad (7)$$

where $\Lambda^\alpha_{op}$ is the positive-energy Casimir projection operator, defined by

$$\Lambda^\alpha_{op} = (E_{op} + h_D)/2 E_{op}, \quad E_{op} \equiv (p_{op}^2 + m^2)^{1/2}. \quad (8)$$

The subscript "+" indicates that $\psi_+$ satisfies

$$\Lambda^\alpha_{op} \psi_+ = \psi_+ \quad (9)$$

and is thus a superposition of only positive-energy plane waves. From (9) it follows that

$$\psi_+^- = R_{op} \psi_+^+, \quad R_{op} \equiv (E_{op} + m)^{-1} \alpha \cdot p_{op} \quad (10)$$

Thus, regardless of the choice of $U$, there is now no singularity involved in the equation relating $\psi_+^+$ and $\psi_+^-$. Furthermore, we have

$$\langle \psi_+^- | \psi_+^- \rangle = \langle \psi_+^+ | R_{op}^\dagger R_{op} | \psi_+^+ \rangle$$

and in p-space the operator $R_{op}^\dagger R_{op}$ is just $p^2/[E(p) + m]^2$, which is bounded by unity. Thus

$$\langle \psi_+^- | \psi_+^- \rangle < \langle \psi_+^+ | \psi_+^+ \rangle \quad (11)$$

and if the norm of $\psi_+^+$ is finite, so is that of $\psi_+^-$ and hence that of $\psi_+^+$.  \hfill 2
3. To complete the argument let us compare the Schrödinger-Pauli form of the eigenvalue problem for the two cases of interest. As in earlier work it is convenient to introduce a new wave function $\psi_+^\pm$ which in $p$-space differs from $\psi_+^\pm$ by a slowly varying factor [3, 4],

$$\psi_+^\pm = A_{op}\phi, \quad A_{op} \equiv (E_{op} + m)/2E_{op}. \quad (12)$$

Then $\psi_+ = \psi_+^\pm + \psi_+^- = (1 + R_{op})\psi_+^\pm$, or

$$\psi_+ = S\phi, \quad S \equiv A_{op}(1 + R_{op})\beta^+. \quad (13)$$

It is easy to verify that because of the extra factor $\beta^+$, $S$ is pseudo-unitary, $S^\dagger S = \beta^+$. Since

$$\beta^+\phi = \phi \quad (14)$$

it follows that $\psi_+$ and $\phi$ have the same norm. On multiplying (9) on the left by $S^\dagger$ one finds that $\phi$ satisfies the equation

$$H_{red}\phi = E\phi \quad (15)$$

where

$$H_{red} = "S^\dagger HS". \quad (16)$$

The quotes indicate that $\beta$ is to be replaced by unity when acting directly on $\phi$. For a potential $U$ of the generic form

$$U = U_v + \beta U_s \quad (17)$$

computation yields

$$H_{red} = E_{op} + V_{red}, \quad (18)$$

where

$$V_{red} = A_{op}U_+A_{op} + (2E_{op}A_{op})^{-1}\sigma \cdot p_{op}U_-\sigma \cdot p_{op}(2E_{op}A_{op})^{-1} \quad (19)$$

with

$$U_\pm = U_v \pm U_s. \quad (20)$$

Since

$$\sigma \cdot p_{op}U_-\sigma \cdot p_{op} = p_{op} \cdot U_-p_{op} + \sigma \cdot (grad U_-) \times p_{op},$$

the main difference between a pure vector and pure scalar potential is a change in sign of part of the spin-independent relativistic correction and in the sign of the spin-orbit interaction. Since these corrections do not dominate the effective interaction, one expects that there are normalizable solutions both in the scalar case and in the vector case [5].

4. Of the making of potentials, as for books, there is no end. One criterion in a semi-phenomenological analysis of systems for which it is makes sense to attempt a description in terms of relativistic Schrödinger-like equations is simplicity. The use of purely local potentials lends itself to this because it limits the proliferation of parameters. Another criterion is to take note of the implications of field theory. For two spin-1/2 particles three-dimensional equations tied to field theory inevitably lead to effective interactions which involve projection operators. A reasonable compromise is therefore to consider equations of the no-pair type [3, 4]:

$$[(\alpha_1 \cdot p_{op} + \beta_1 m_1) + (-\alpha_2 \cdot p_{op} + \beta_2 m_2) + V_{++}]\psi = E\psi \quad (21)$$
where
\[ V_{++} = \Lambda_{++} U \Lambda_{++}, \]
\( \Lambda_{++} \) is the projection operator product \( \Lambda_+^D(1) \Lambda_+^D(2) \) and
\[ \Lambda_+^{op}(i) \psi = \psi \quad (i = 1, 2). \]

One may choose \( U \) to be purely local without running into difficulties. Note that the nonlocality of the projection operators does not introduce any new parameters, since it involves only the constituent masses, already present in the Dirac Hamiltonians.

As confirmation of the fact that no problems arise even if \( U \) is purely scalar-like, it should be noted that no difficulties are encountered with the numerical solution of (21) when \( U \) is chosen to have the scalar form [6]
\[ U = k \beta_1 \beta_2 r. \]

Acknowledgements

This work was supported in part by the National Science Foundation. I thank Marshall Baker and Martin Olsson for raising the issue addressed in this note.

References


[5] Statements to the contrary are made in Ref. 2. **Added note:** The authors of Ref. 2 now agree that there are normalizable solutions in either case; what they want to stress is the absence of Regge behavior in the no-pair vector case. [M. Olsson, private communication.]