Bosonization and Duality of Massive Thirring model

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Abstract

Starting from a formulation of the Thirring model as a gauge theory, we consider the bosonization of the $D$-dimensional massive Thirring model ($D \geq 2$) with four-fermion interaction of the current-current type. Especially we pay attention to the case of very massive fermion $m \gg 1$ in $(2+1)$ and $(1+1)$ dimensions. Up to the next-to-leading order of $1/m$, we show that the $(2+1)$-dimensional massive Thirring model is mapped to the Maxwell-Chern-Simons theory and that the $(1+1)$-dimensional massive Thirring model is equivalent to the massive free scalar field theory. In the process of the bosonization of the Thirring model, we point out the importance of the gauge-invariant formulation.

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1 Introduction and main results

Recently the Thirring model [1, 2] was reformulated as a gauge theory and was identified with a gauge-fixed version of the corresponding gauge theory by introducing the Stückelberg field $\theta$ in addition to the auxiliary vector field $A_{\mu}$ which is in that formulation identified with the massless gauge field [3, 4]. This is a consequence of the general formalism for the constrained system by Batalin and Fradkin [5]. This gives the general procedure by which the system with the second class constraint is converted to that with the first class one and the new field which is necessary to complete this procedure is called the Batalin-Fradkin field [6]. In the massive gauge theory the Batalin-Fradkin field is nothing but the well-known Stückelberg field as shown in [7]. By maintaining the manifest gauge symmetry, controversy among the papers [8, 9, 10, 11, 12] will be resolved as emphasized in [3].

We consider the mapping from quantum field theory of interacting fermions onto an equivalent theory of interacting bosons. In this paper such an equivalent bosonic theory to the original massive Thirring model is obtained starting from the formulation of the $D$-dimensional Thirring model ($D \geq 2$) as a gauge theory. This is a kind of bosonization. Especially, in this paper we consider the large bare fermion mass limit $m \gg 1$, in $(2+1)$ and $(1+1)$ dimensional cases.

In a special case of $(2+1)$-dimensions, we show that, up to the next-to-leading order in the inverse bare fermion mass, $1/m$, the $(2+1)$-dimensional massive Thirring model is equivalent to the Maxwell-Chern-Simons theory, the topologically massive $U(1)$ gauge theory. This equivalence in three dimensions has been shown, to the lowest order in $1/m$, by Fradkin and Schaposnik [14]. However we think that their treatment is unsatisfactory in the following points.

1. The original Thirring model has no gauge symmetry. Nevertheless the equivalent bosonic theory, the Maxwell-Chern-Simons theory, has $U(1)$ gauge symmetry. Where does this gauge symmetry come from?

2. The master Lagrangian (which is called the interpolating Lagrangian in [14]) of Deser and Jackiw [15] was suddenly introduced without notice as a device which is needed to show this equivalence. However the origin of the master Lagrangian was never shown.

3. The master Lagrangian is invariant under the independent gauge transformations for the auxiliary vector field $A_{\mu}$ and another vector field. In integrating out another vector field in the master Lagrangian, they had to introduce the gauge-fixing term ad hoc.

We resolve these problems by starting from the gauge-invariant or more precisely the BRS-invariant formulation of the Thirring model. Our approach is more direct than theirs and is able to derive the equivalent Lagrangian to the interpolating Lagrangian as a natural consequence (in an intermediate step).
of bosonization and determines definitely the procedure which is needed to fix the gauge invariance appearing in the master Lagrangian [15]. Our method may be easily extendable to the non-Abelian case [15, 16].

Finally we discuss the (1+1)-dimensional case. The Thirring model in (1+1)-dimensions is rewritten into the equivalent scalar field theory [1, 2]. It is well known that the (1+1)-dimensional massless Thirring model is exactly solvable in the sense that the model is equivalent to the massless free scalar theory [1]. In this paper we show, as a special case of the above formalism, the massive Thirring model in (1+1)-dimensions is equivalent to the massive free scalar field theory in (1+1)-dimensions, up to the next-to-lowest order in $1/m$.

In the previous paper [4] we have studied the spontaneous breakdown of the chiral symmetry in the massless Thirring model, $m = 0$. In this paper we consider another extreme limit $m \to \infty$. According to the universality hypothesis, the critical behavior of the model will be characterized by a small number of parameters appearing in the original Lagrangian of the model such as symmetry, range of interaction and dimensionality. Therefore, in the large bare fermion mass limit, the critical behavior of the Thirring model will be characterized by studying the equivalent bosonic theory according to the above bosonization.

## 2 Bosonization

The Lagrangian of the $D$-dimensional Thirring model ($D \geq 2$) is given by $^1$

$$L_{Th} = \bar{\psi}^j i\gamma^\mu \partial_\mu \psi^j - m \bar{\psi}^j \psi^j - \frac{G}{2N} (\bar{\psi}^j \gamma^\mu \psi^j)(\bar{\psi}^k \gamma^\mu \psi^k),$$

where $\psi^j$ is a Dirac spinor and the indices $j, k$ are summed over from 1 to $N$, and the gamma matrices $\gamma^\mu (\mu = 0, ..., D - 1)$ are defined so as to satisfy the Clifford algebra, \{$\gamma^\mu, \gamma^\nu$\} $= 2g_{\mu\nu}1 = 2diag(1,-1,...,-1)$.

By introducing an auxiliary vector field $A_\mu$, this Lagrangian is equivalently rewritten as

$$L_{Th'} = \bar{\psi}^j i\gamma^\mu (\partial_\mu - i\frac{g}{\sqrt{N}} A_\mu) \psi^j - m \bar{\psi}^j \psi^j + \frac{M^2}{2} A_\mu^2,$$

where we have introduced a parameter $M(=1)$ with the dimension of mass, $dim[m] = dim[M] = 1$ and put $g = \frac{g^2}{M}$. Thanks to the parameter $M$, all the fields have the corresponding canonical dimensions: $dim[\bar{\psi}] = dim[\psi] = (D - 1)/2, dim[A_\mu] = (D - 2)/2$ and then the coupling constant has the dimension: $dim[g] = (4 - D)/2, dim[G] = 2 - D$.

$^1$This Lagrangian has $O(N)$ symmetry. If we adopt the mass term $\sum^N_{j=1} m_j \bar{\psi}^j \psi$ and put $m_j = m$ for $j = 1, ..., N - k$ and $m_j = -m, ..., -m$ for $j = N - k + 1, ..., N$, the Lagrangian has $O(N - k) \times O(k)$ symmetry. See the Vafa-Witten argument [17].
The theory with this Lagrangian is identical with that of the massive vector field with which the fermion couples minimally, since a kinetic term for $A_{\mu}$ is generated through the radiative correction although it is absent originally. As is known from the study of massive vector boson theory \cite{7}, the Thirring model with the Lagrangian \cite{28/2/29} is cast into the form which is invariant under the Becchi-Rouet-Stora (BRS) transformation by introducing an additional field $\theta$. The field $\theta$ is called the St"{u}ckelberg field and identified with the Batalin-Fradkin (BF) field \cite{6} in the general formalism for the constrained system \cite{5}. Then we start from the Lagrangian with covariant gauge-fixing \cite{4}:

$$
\mathcal{L}_{T,\mu} = \bar{\psi}^j i\gamma^\mu (\partial_\mu - i\frac{g}{\sqrt{N}} A_\mu) \psi^j - m \bar{\psi}^j \psi^j + \frac{M^2}{2} (A_\mu - \sqrt{N} M^{-1} \partial_\mu \theta)^2
$$

$$
- A_\mu \partial^\mu B + \frac{\xi}{2} B^2 + i \partial^\mu \bar{C} \partial^\mu C.
$$

(3)

Actually this Lagrangian is invariant under the BRS transformation:

$$
\delta_B A_\mu(x) = \partial_\mu C(x),
$$

$$
\delta_B B(x) = 0,
$$

$$
\delta_B C(x) = 0,
$$

$$
\delta_B \bar{C}(x) = i B(x),
$$

$$
\delta_B \theta(x) = \frac{M}{\sqrt{N}} C(x),
$$

$$
\delta_B \psi^j(x) = \frac{i g}{\sqrt{N}} C(x) \psi^j(x),
$$

(4)

where $C(x)$ and $\bar{C}(x)$ are ghost fields, and $B(x)$ is the Nakanishi-Lautrup Lagrange multiplier field.

By introducing the scalar field

$$
\varphi = \sqrt{N} \frac{V}{\sqrt{2}} e^{i \theta/V},
$$

(5)

with $V = M/g = 1/\sqrt{G}$, this theory can also be regarded with the Higgs model (or the gauged non-linear sigma model with a local constraint: $\varphi(x) \varphi^*(x) = \frac{N}{2G}$) with fermions:

$$
\mathcal{L}_H = \bar{\psi}^j i\gamma^\mu D_\mu [A] \psi^j - m \bar{\psi}^j \psi^j + (D_\mu [A] \varphi)^{1} (D^\mu [A] \varphi) - A_\mu \partial^\mu B + \frac{\xi}{2} B^2,
$$

(6)

where $D_\mu [A]$ is the covariant derivative: $D_\mu [A] = \partial_\mu - i\frac{g}{\sqrt{N}} A_\mu$.

First we consider the case of $D \geq 3$. The two-dimensional case is discussed separately in the final part. By introducing an auxiliary vector field $f_\mu$, the "mass term" of the gauge field is linearized: For $K_\mu = \sqrt{N} M^{-1} \partial_\mu \theta$,

$$
\int D\theta \exp \left\{ i \int d^Dx \frac{1}{2} M^2 (A_\mu - K_\mu)^2 \right\}
$$

4
\[ \int \mathcal{D}\theta \int \mathcal{D}f_\mu \exp \left\{ i \int d^Dx \left[ -\frac{1}{2} f_\mu f^\mu + M f^\mu (A_\mu - K_\mu) \right] \right\} \]
\[ = \int \mathcal{D}f_\mu \delta(\partial^\mu f_\mu) \exp \left\{ i \int d^Dx \left[ -\frac{1}{2} f_\mu f^\mu + M f^\mu A_\mu \right] \right\} , \quad (7) \]
where in the last step we have integrated out the scalar mode \( \theta \).

Applying the Hodge decomposition \([18] \)\(^2\) to the 1-form \( f_\mu \), \( f_\mu \) is written as
\[ f_{\mu_1} = \partial_{\mu_1} \phi + \epsilon_{\mu_1 \cdots \mu_D} \partial^\mu_2 H^{\mu_3 \cdots \mu_D} , \quad (8) \]
where we have introduced the antisymmetric tensor field \( H^{\mu_1 \cdots \mu_{D-2}} \) of rank \( D-2 \), which satisfies the Bianchi identity. Since \( f_\mu \) is divergence free, \( \partial^\mu f_\mu = 0 \), we can put \( f_{\mu_1} = \epsilon_{\mu_1 \cdots \mu_D} \partial^\mu_2 H^{\mu_3 \cdots \mu_D} \). Then we have
\[ \int \mathcal{D}\theta \exp \left\{ i \int d^Dx \frac{1}{2} M^2 (A_\mu - K_\mu)^2 \right\} \]
\[ = \int \mathcal{D}H_{\mu_1 \cdots \mu_{D-2}} \exp \left\{ i \int d^Dx \left[ \frac{(-1)^D}{2(D-1)} H_{\mu_1 \cdots \mu_{D-1}} H^{\mu_1 \cdots \mu_{D-1}} + M \epsilon_{\mu_1 \mu_2 \cdots \mu_D} A_{\mu_1} \partial_{\mu_2} H^{\mu_3 \cdots \mu_D} \right] \right\} , \quad (9) \]
where we have defined
\[ H^{\mu_1 \cdots \mu_{D-1}} = \partial_{\mu_1} H^{\mu_2 \cdots \mu_{D-1}} - \partial_{\mu_2} H^{\mu_1 \mu_3 \cdots \mu_{D-1}} + \ldots \]
\[ + (-1)^D \partial_{\mu_{D-1}} H^{\mu_1 \cdots \mu_{D-2}} . \quad (10) \]

This result is a generalization of \([31] \) for \( D = 3 \) and coincides with that of Ito et al. \([3] \).

Then we obtain the equivalent Lagrangian with the mixed term between \( A_\mu \) and \( H^{\mu_1 \cdots \mu_{D-2}} \):
\[ \mathcal{L}_{T, \mu} = \bar{\psi}^j i \gamma^\mu D_\mu \psi^j - m \bar{\psi}^j \psi^j \]
\[ + \frac{(-1)^D}{2(D-1)} H_{\mu_1 \cdots \mu_{D-1}} H^{\mu_1 \cdots \mu_{D-1}} \]
\[ + M \epsilon_{\mu_1 \cdots \mu_D} A_{\mu_1} \partial_{\mu_2} H^{\mu_3 \cdots \mu_D} - A_\mu \partial^\mu B + \frac{\xi}{2} B^2 . \quad (11) \]

Integrating out the fermion field \( \bar{\psi}, \psi \), we thus obtain the bosonised action of the Thirring model:
\[ S_B = N \ln \det [i \gamma^\mu D_\mu [A] + m] \]
\[^{2}\text{Let } \omega \text{ be a } p\text{-form. Then there is a } (p+1)\text{-form } \alpha, \text{ a } (p-1)\text{-form } \beta \text{ and a harmonic } p\text{-form } h \text{ (i.e., obeying } \delta h = 0 = dh) \text{ such that} \]
\[ \omega = \delta \alpha + d\beta + h. \]

We can restrict ourselves to the topologically trivial space \( \Omega \) for which there are no harmonic forms. This is equivalent to saying that each \( p\)-form \( \omega \) obeying \( d\omega = 0 \) is of the form \( \omega = d\beta \) (Poincare’s lemma) and we say \( \Omega \) has trivial (co)homology. From now on we assume that the harmonic form is absent: \( h = 0 \).
\[
+ \int d^Dx \left[ \frac{(-1)^D}{2(D-1)} \bar{H}_{\mu_1 \ldots \mu_{D-1}} \partial^{\mu_1 \ldots \mu_{D-1}} H_{\mu_1 \ldots \mu_{D-1}} \right.
\]
\[
+ M \epsilon^{\mu_1 \ldots \mu_D} A_\mu \partial_{\mu_2} H_{\mu_3 \ldots \mu_D} - A_\mu \partial^\mu B + \frac{\xi}{2} B^2 \right].
\]  \tag{12}

To see the origin of the field \( H_{\mu_1 \ldots \mu_{D-2}} \), we integrate out the gauge field. Then, we obtain the partition function:
\[
Z = \int DB \int D\bar{\psi} \int D\psi \int DH_{\mu_1 \ldots \mu_{D-2}} \times \delta \left( \frac{g}{\sqrt{N}} \bar{\psi}^j \gamma^\mu \gamma_\nu \psi^j - M \epsilon^{\mu_1 \ldots \mu_{D-2}} \partial_{\mu_2} H_{\mu_3 \ldots \mu_{D-2}} - \partial^\mu B \right)
\times \exp \left\{ i \int d^Dx \left[ \bar{\psi}^j i \gamma^\mu (\partial_\mu - i \frac{g}{\sqrt{N}} A_\mu - i b_\mu) \psi^j - m \bar{\psi}^j \psi^j + \frac{M^2}{2} (A_\mu)^2 \right. \right.
\]
\[
\left. \left. + \frac{(-1)^D}{2(D-1)} \bar{H}_{\mu_1 \ldots \mu_{D-1}} H_{\mu_1 \ldots \mu_{D-1}} + \frac{\xi}{2} B^2 \right] \right\}. \tag{13}
\]

This implies that the dual field \( H_{\mu_1 \ldots \mu_{D-2}} \) is a composite of the fermion and antifermion.

The correspondence between the original Thirring model and the bosonized theory is generalized to the correlation function. We introduce the source \( b_\mu \) for the current
\[
\mathcal{J}_\mu = \bar{\psi}^j \gamma_\mu \psi^j. \tag{14}
\]

Adding the source term \( \mathcal{J}_\mu b^\mu \) to the original Lagrangian Eq. (1), \( \mathcal{L}_{Th} [b_\mu] \equiv \mathcal{L}_{Th} + \mathcal{J}_\mu b^\mu \), we obtain
\[
\mathcal{L}_{Th} [b_\mu] = \bar{\psi}^j i \gamma^\mu (\partial_\mu - i \frac{g}{\sqrt{N}} A_\mu - i b_\mu) \psi^j - m \bar{\psi}^j \psi^j + \frac{M^2}{2} (A_\mu)^2. \tag{15}
\]

After introducing the BF field \( \theta \) and shifting \( A_\mu \rightarrow A_\mu - \frac{\sqrt{N}}{g} b_\mu \), we obtain
\[
\mathcal{L}_{Th}[b_\mu] = -A_\mu \partial^\mu B + \frac{\xi}{2} B^2 + i \partial^\mu \bar{C} \partial^\mu C. \tag{16}
\]

Repeating the same steps as before, we arrive at
\[
\mathcal{L}_{Th}[b_\mu] = \mathcal{L}_{Th} + \frac{\sqrt{N}}{g} M \epsilon^{\mu_1 \ldots \mu_D} b_\mu \partial_{\mu_2} H_{\mu_3 \ldots \mu_D}. \tag{17}
\]

This leads to the equivalence of the partition function in the presence of the source \( b_\mu \): \( Z_{Th}[b_\mu] = Z_{Bosonised}[b_\mu] \). Therefore the connected correlation function has the following correspondence between the Thirring model and the bosonised theory with the action \( S_B \):
\[
\langle \mathcal{J}_{\mu_1} \cdots \mathcal{J}_{\mu_n} \rangle_{Th} = \langle \eta_{\mu_1} \cdots \eta_{\mu_n} \rangle_{Bosonised}, \tag{18}
\]

6
where

\[ \eta^{\mu_1} = \frac{1}{\sqrt{G/N}} \epsilon^{\mu_1 \cdots \mu_D} \partial_{\mu_2} H_{\mu_3 \cdots \mu_D}. \]  

(19)

In the following we discuss how to integrate out the auxiliary field \( A_\mu \) to obtain the bosonic theory which is written in terms of the field \( H_{\mu_1 \cdots \mu_{D-2}} \) only.

\section{(2+1)-dimensional case}

In the three-dimensional case, \( D = 3 \),

\[ S_B = N \ln \det [i\gamma^\mu D_\mu [A] + m] + \int d^3x \left\{ -\frac{1}{4} \tilde{H}_{\mu\nu} \tilde{H}^{\mu\nu} + Me^{\mu\nu\rho} A_\mu \partial_\nu H_\rho - A_\mu \partial^\mu B + \frac{\xi}{2} B^2 \right\}, \]

(20)

where

\[ \tilde{H}_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu. \]

(21)

In what follows we consider the large fermion mass limit, \( m \to \infty \). The Matthews-Salam determinant in Eq. (20) is calculated with the aid of various regularization methods [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Appropriate choice of regularization leads to the regulator independent result.

For the \( 2 \times 2 \) gamma matrices corresponding to the two-component fermion,

\[ N \ln \det [i\gamma^\mu D_\mu [A] + m] = N \ln \det [i\gamma^\mu \partial_\mu + m] = N Tr \left[ 1 + (i\gamma^\mu \partial_\mu + m)^{-1} \frac{g}{\sqrt{N}} \gamma^\mu A_\mu \right] = sgn(m) \frac{i g^2}{16\pi} \int d^3 x \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho - \frac{g^2}{24\pi |m|} \int d^3 x F_{\mu\nu} F^{\mu\nu} + O \left( \frac{\partial^2}{m^2} \right). \]

(22)

\[ ^3 \text{This should be compared with the massless limit, } m = 0. \text{ In this case, it is shown \cite{30} that up to one-loop} \]

\[ N \ln \det [i\gamma^\mu (\partial_\mu - i \frac{g}{\sqrt{N}} A_\mu) + m] = sgn(m) \frac{i g^2}{16\pi} \int d^3 x \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + NI_{PC}[A_\mu], \]

where the parity-conserving term is given by

\[ I_{PC}[A_\mu] = \frac{1}{8\pi} \zeta (\frac{3}{2}) \int d^3 x \left( \frac{g}{2\sqrt{N}} F_{\mu\nu}^2 \right)^{3/2}. \]

\[ ^4 \text{There are various regularization methods: 1) Pauli-Villars \cite{19, 20, 21, 22}, 2) lattice \cite{23, 24}, 3) analytic \cite{25, 4}, 4) dimensional \cite{26, 27}, 5) Zavialov \cite{28}, 6) parity-invariant Pauli-Villars (variant of chiral gauge invariant Pauli-Villars by Frolov and Slavnov) \cite{29}, 7) high covariant derivative \cite{33}. However it should be remarked that the methods 1) and 2) give regulator dependent result for the Chern-Simons part.} \]
where \( sgn(m) \) denotes signature of the bare fermion mass \( m \), \( sgn(m) = m/|m| \).

This is understood as follows. Note that

\[
NTr \ln \left[ 1 + (i\gamma^\mu \partial_\mu + m)^{-1} \frac{g}{\sqrt{N}} \gamma^\mu A_\mu \right] = \frac{1}{2} \int d^D x A_\mu(x) \Pi_{\mu\nu}(\partial) A_\nu(x) + \ldots (23)
\]

For \( D = 3 \), the vacuum polarization tensor is given by

\[
\Pi_{\mu\nu}(\partial) = \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \Pi_T(-\partial^2, m) + i\epsilon_{\mu\nu\rho} \partial_\rho \Pi_O(-\partial^2; m), \tag{24}
\]

where

\[
\Pi_T(k^2; m) = -\frac{g^2}{2\pi} k^2 \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{[m^2 - \alpha(1-\alpha)k^2]^{1/2}}, \tag{25}
\]

and

\[
\Pi_O(k^2; m) = -\frac{g^2m}{4\pi} \int_0^1 d\alpha \frac{1}{[m^2 - \alpha(1-\alpha)k^2]^{1/2}}. \tag{26}
\]

In the large \( m \) limit, we obtain Eq. (22).

Thus the Thirring model in the large \( m \) limit is equivalent to the bosonized theory with the interpolating Lagrangian:

\[
\mathcal{L}_I[A_\mu, H_\mu] = -\frac{1}{4} \tilde{H}_{\mu\nu} \tilde{H}^{\mu\nu} + \frac{M}{2} \epsilon^{\mu\nu\rho} \tilde{H}_{\mu\nu} A_\rho + \frac{i\theta_{CS}}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho
\]

\[
- \frac{g^2}{24\pi|m|} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + \mathcal{O}\left( \frac{\partial^2}{m^2} \right), \tag{27}
\]

where \(^5\)

\[
\theta_{CS} = sgn(m) \frac{g^2}{4\pi}. \tag{28}
\]

Integrating out the \( \tilde{H}_{\mu\nu} \) field in the interpolating Lagrangian, we obtain the self-dual Lagrangian:

\[
\mathcal{L}_{SD}[A_\mu] = \frac{M^2}{2} A_\mu A_\mu + \frac{i\theta_{CS}}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho, \tag{29}
\]

up to the lowest order of \( 1/m \). This implies that the Thirring model is equivalent to the self-dual model with the Lagrangian \( \mathcal{L}_{SD} \) in the lowest order of \( 1/m \).

\(^5\)For the mass term \( \sum_{j=1}^N m_j \bar{\psi}_j \psi_j \), \( \theta_{CS} = \sum_{j=1}^N sgn(m_j) \frac{g^2}{4\pi} = (N - 2k) \frac{g^2}{4\pi} \). Hence, if we take \( k = N/2 \) in the first footnote, the theory has \( \mathcal{O}(N/2) \times \mathcal{O}(N/2) \) symmetry and \( \theta_{CS} = 0 \), although the kinetic term for the field \( A_\mu \) is unchanged.
We notice that the interpolating Lagrangian we have just obtained is essentially equivalent to the master Lagrangian of Deser and Jackiw [15]. The master Lagrangian for $A_\mu$ and $H_{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} H_{\sigma}$ is given by

$$\mathcal{L}_{DJ} = \frac{1}{2} A^\mu A_\mu - A_\mu H^{\mu\ast} + \frac{1}{2} \tilde{m} H^{\mu\ast} H_\mu.$$  \hspace{3.5cm} \text{(30)}$$

Note that the role of the auxiliary field $A_\mu$ and the new field $H_\mu$ is interchanged in our interpolating Lagrangian compared with that in [14] based on the master Lagrangian of Deser and Jackiw. Hence the integration over the auxiliary field is non-trivial in our interpolating Lagrangian.

On the other hand, for the $4 \times 4$ gamma matrices corresponding to the four-component fermion, the Chern-Simons term in the determinant disappear, since $tr(\gamma_\mu \gamma_\nu \gamma_\rho) = 0$. In this case, the Thirring model is equivalent to the master Lagrangian of Deser and Jackiw [5b, 5d]. The deconstruction is non-trivial in our interpolating Lagrangian.

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where $\Delta^{(1)} = 1/\partial^2$ and $\Delta^{(2)} = 1/\partial^4$ in the sense $\partial^2 \Delta^{(1)}(x, y) = \delta(x - y)$ and $(\partial^2)^2 \Delta^{(2)}(x, y) = \delta(x - y)$. By using this formula, $A_\mu$ integration is performed:

$$J_\mu K^{-1}_{\mu \nu} J_\nu = \frac{M^2}{i\theta_{CS}} \epsilon^{\mu \rho \nu} H_\mu \partial_\nu H_\rho + \frac{g^2 M^2}{12 \pi \theta_{CS}^2 m} \bar{H}^2 + \mathcal{O}\left(\frac{1}{m^2}\right),$$

which is independent of the gauge-fixing parameter $\xi$ in the original theory.

Thus we arrive at an effective bosonized Lagrangian for the dual field $H_\mu$ alone:

$$\mathcal{L}_{MCS} = -\frac{1}{4} \left(1 + \frac{g^2 M^2}{6 \pi \theta_{CS}^2 m} \right) \bar{H}_\mu \bar{H}_\mu + \text{sgn}(m) \frac{2 \pi i}{g^2} M^2 \epsilon^{\mu \nu \rho} H_\mu \partial_\nu H_\rho$$

$$+ \mathcal{O}\left(\frac{\partial^2}{m^2}\right).$$

Note that the gauge-parameter dependence has dropped out in the bosonized theory. In the interpolating Lagrangian $\mathcal{L}_I$ the gauge degree of freedom for the $A_\mu$ field is fixed by the gauge-fixing term $\frac{1}{2\xi} (\partial_\mu A_\mu)^2$. However there is an additional gauge symmetry for the new field $H_\mu$: the Lagrangian $\mathcal{L}_I$ is invariant under the gauge trasformation $\delta H_\mu = \partial_\mu \omega$ independently of $A_\mu$, which leads to $\delta f_\mu = 0$ in the master Lagrangian. Therefore we must add a gauge-fixing term for the $H_\mu$ field to the bosonized Lagrangian $\mathcal{L}_{MCS}$.

Thus, to the leading order of $1/m$ expansion the Thirring model partition function coincides with that of the Maxwell-Chern-Simons theory. This result agrees with that in [14] obtained up to the leading of $1/m$ where the less direct procedure is adopted to show this equivalence using the self-dual action by way of the interpolating action. Up to the next-to-leading order of $1/m$, we have shown that the equivalence between the low energy sector of a theory of 3-dimensional fermions interacting via a current-current term and gauge bosons with Maxwell-Chern-Simons term is preserved. The Thirring spin-$1/2$ fermion with the Thirring coupling $g^2/N$ is equal to a spin-1 massive excitation with mass $\pi/g^2 \left(1 - \frac{g^2 M^2}{6 \pi \theta_{CS}^2 m}\right) + \mathcal{O}(1/m^2)$, in 2+1 dimensions. In 2+1 dimensions there is the following correspondence between the original Thirring model and the bosonized theory:

$$\bar{\psi}^j \gamma^\mu \psi^j \leftrightarrow \frac{1}{\sqrt{G/N}} \epsilon^{\mu \nu \rho} \bar{\psi}_\nu H_\rho.$$
This fact was already pointed out in [31]. The missing kinetic term for $A_\mu$ is generated through the radiative correction as shown above. The Meissner effect in superconductivity is nothing but the Higgs phenomenon, the photon (massless gauge field) becomes massive gauge boson by absorbing the massless Nambu-Goldstone boson (scalar mode). The mixed Chern-Simons action does not break the parity in sharp contrast to the ordinary Chern-Simons term. Therefore this model may be a candidate for the high-$T_c$ superconductivity without parity violation, as suggested in [31].

Finally we wish to point out that, in the large $m$ limit, the Thirring model is also equivalent to the Chern-Simons-Higgs model up to the leading order of $1/m$ and to the Maxwell-Chern-Simons-Higgs model up to the next-to-leading order of $1/m$, since

$$L_{CSH} = (D_\mu \varphi)^\dagger (D^\mu \varphi) + sgn(m) \frac{ig^2}{16\pi} \int d^3 x \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho$$

$$- \frac{g^2}{24\pi |m|} \int d^3 x F_{\mu\nu} F^{\mu\nu} + O \left( \frac{\partial^2}{m^2} \right),$$

apart from the gauge-fixing term. This result is consistent with the assertion of Deser and Yang [32].

4 (1+1)-dimensional case

For $D = 2$, it is easy to show that the bosonized action is given by

$$S_B = N \ln \det \left[ i\gamma_\mu D_\mu \left[ A \right] + m \right] + \int d^2 x \left[ \frac{1}{2} \left( \partial_\mu H \right)^2 + M \epsilon^{\mu\nu} A_\mu \partial_\nu H - A_\mu \partial^\mu B + \frac{\xi}{2} B^2 \right].$$

The determinant is calculated as

$$N \ln \frac{\det \left[ i\gamma_\mu D_\mu \left[ A \right] + m \right]}{\det \left[ i\gamma_\mu \partial_\mu + m \right]}$$

$$= \frac{1}{2} \int d^2 x A_\mu(x) \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \Pi_T (-\partial^2; m) A_\nu(x),$$

where

$$\Pi_T (k^2; m) = -\frac{g^2}{\pi} k^2 \int_0^1 d\alpha \frac{\alpha(1 - \alpha)}{m^2 - \alpha(1 - \alpha)k^2}.$$

In the massless case, $m = 0$,

$$\Pi_T (k^2; m = 0) = \frac{g^2}{\pi},$$

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and hence
\[ K_{\mu\nu} = \frac{g^2}{\pi} \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \frac{1}{\xi} \partial^\mu \partial^\nu, \]  
(47)
whose inverse is given by
\[ K^{-1}_{\mu\nu} = \frac{\pi}{g^2} \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \xi \frac{\partial_\mu \partial_\nu}{\partial^4}. \]  
(48)

Putting
\[ J_\mu = M e^{i\nu} \partial_\nu H, \]  
(49)
and integrating out the \( A_\mu \) field, we obtain the bosonized theory:
\[ S_B = \int d^2 x \left[ \frac{1}{2} \left( 1 + \frac{\pi}{G} \right) (\partial_\mu H)^2 \right]. \]  
(50)
The massless Thirring model in two dimensions is equivalent to the massless scalar field theory, as long as the action is bounded from below, i.e., \( G := g^2/M^2 > 0 \) or \( G < -\pi \). For more details on the massless case, see reference [3].

On the other hand, in the large \( m \) limit, we find
\[ \Pi_T(k^2; m) = -\frac{g^2}{\pi} \left[ \frac{1}{6} \frac{k^2}{m^2} + \frac{1}{30} \frac{k^4}{m^4} \right] + \mathcal{O} \left( \frac{k^4}{m^4} \right), \]  
(51)
and hence
\[ K_{\mu\nu} = \frac{g^2}{6\pi} \frac{\partial^2}{m^2} \left( 1 - \frac{1}{5} \frac{\partial^2}{m^2} \right) \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \frac{1}{\xi} \partial^\mu \partial^\nu + \mathcal{O} \left( \frac{\partial^2}{m^2} \right). \]  
(52)
The inverse is obtained as
\[ K^{-1}_{\mu\nu} = \frac{6\pi m^2}{g^2} \frac{\partial^2}{\partial^2} \left( 1 - \frac{1}{5} \frac{\partial^2}{m^2} \right) \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \xi \frac{\partial_\mu \partial_\nu}{\partial^4} + \mathcal{O} \left( \frac{\partial^2}{m^2} \right). \]  
(53)
Hence we obtain the bosonized theory after integrating out the \( A_\mu \) field:
\[ S_B = \int d^2 x \left[ \frac{1}{2} \left( 1 + \frac{6}{5G} \right) (\partial_\mu H)^2 - \frac{3\pi m^2}{G} H^2 \right] + \mathcal{O} \left( \frac{\partial^4}{m^2} \right). \]  
(54)
Thus in two dimensions the massive Thirring model with large mass \( m \gg 1 \) is equivalent to the scalar field theory with large mass:
\[ m_H = \sqrt{\frac{6\pi m^2}{G} \left( 1 + \frac{6}{5G} \right)}. \]  
(55)
In the massive limit, the Thirring model is physically sensible for \( G > 0 \).
In two dimensions there is a correspondence between the Thirring model and the bosonised theory:

$$\bar{\psi} j^\mu j^\nu \psi \leftrightarrow \frac{1}{\sqrt{G/N}} \epsilon^{\mu\nu} \partial_\nu H.$$  \hfill (56)

The above results are reasonable as shown in the following. The massive Thirring model is not exactly soluble even in $(1+1)$-dimensions. However the $(1+1)$-dimensional massive Thirring model is equivalent to the sine-Gordon model [2]:

$$L_{sG} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{\alpha}{\beta^2} [\cos(\beta \varphi) - 1],$$  \hfill (57)

if the following identifications are made between the two theories:

$$1 + \frac{G}{\pi} = \frac{4\pi}{\beta^2},$$  \hfill (58)

$$\bar{\psi} \gamma^\mu \psi = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\nu \varphi,$$  \hfill (59)

$$m \bar{\psi} \psi = -\frac{\alpha}{\beta^2} \cos(\beta \varphi),$$  \hfill (60)

where a constant in the Lagrangian $L_{sG}$ is adjusted so that the minimum of the energy density is zero.

The massless limit $m \to 0$ of the massive Thirring model corresponds to the limit $\alpha \to 0$ in the sine-Gordon model, i.e., a massless scalar field theory in agreement with the above result. On the other hand, the massive limit $m \to \infty$ corresponds to the limit $\beta \to 0$ (or $\alpha \to \infty$) which inevitably leads to the limit $G \to \infty$ as $\frac{\beta}{\sqrt{G}} \sim \frac{1}{\sqrt{\alpha}}$. Hence in this limit the Lagrangian reduces to

$$L_{sG} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{\alpha}{2} \varphi^2 + O(\alpha \beta^2 \varphi^4).$$  \hfill (61)

Moreover the Eq. (59) recovers the above correspondence relation Eq. (56) for $N = 1$. Therefore the very massive limit $m \gg 1$ of the 2-dimensional Thirring model is equivalent to the massive free scalar field theory with mass $\sqrt{\alpha}$ which is identified with $m_H$ given above.

Acknowledgments:

The author would like to thank Atsushi Nakamura, Tadahiko Kimura, Kenji Ikekami, Toru Ebihara and Koichi Yamawaki for valuable discussions.
References


