CONTRIBUTIONS TO THE THEORY
OF THE BEAM GUIDE
by
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| I. INTRODUCTION | 1 |
| II. THE DIFFERENTIAL EQUATIONS OF MOTION | 1 |
| III. ORBIT EQUATIONS | 3 |
| IV. PROPERTIES OF THE ORBITS | 8 |
| V. EXTREME VALUES AND HELICAL ORBITS | 10 |
| VI. GRAZING RAYS | 16 |
| VII. MOMENTUM ACCEPTANCE | 18 |
| VIII. HIGH-ENERGY BEHAVIOUR OF THE BEAM GUIDE | 19 |
| IX. ANGULAR ACCEPTANCE | 24 |
| X. PHASE ACCEPTANCE | 28 |
| ACKNOWLEDGEMENTS | 28 |
| REFERENCE | 29 |
I. INTRODUCTION

In the beam guide, a magnetic field of axial symmetry is used to transport particles over great distances. A coaxial transmission line, fed with d.c. or pulsed current, plays the role of the focusing channel and the particles follow screw-type orbits in the space available between the inner and outer conductor.

Although the image-forming properties of such a device are quite poor, continuous focusing results in a relatively high acceptance and this makes the beam guide an interesting tool for a number of practical applications. For instance, a beam guide may be considered as a means of channeling pions over distances of a few hundred meters, and collecting the muons from their decay in flight for use in experiments.

A few problems connected with the theory of the beam guide will be considered in this paper.

II. THE DIFFERENTIAL EQUATIONS OF MOTION

On account of the given geometry (Fig. 1), it is appropriate to write the equations of motion in cylindrical co-ordinates. For a particle of charge $q$ and mass $m$ moving with velocity $v$ in a magnetic field $B$, the basic equation

$$\dot{\mathbf{m}} = \mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

becomes upon projection

$$m(\ddot{r} - r\dot{\theta}^2) = q(r\dot{B}_z - z\dot{B}_\theta)$$
$$\frac{m}{r} \frac{d}{dt}(r^2\dot{\theta}) = q(z\dot{B}_r - r\dot{B}_z)$$
$$m\ddot{z} = q(r\dot{B}_\theta - r\dot{B}_r) .$$
Here, the dots indicate differentiation with respect to time; \( m \) stands for the relativistic mass

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} ;
\]

and

\[
v^2 = r^2 + r^2 \dot{\theta}^2 + z^2 .
\]

No acceleration takes place and, therefore, \( v \) and \( m \) are constants. On the other hand, \( B_r = 0, B_z = 0 \), and the magnetic field has only one component

\[
B_\theta = \frac{\mu_0 I}{2\pi r} ,
\]

where \( I \) is the current flowing in the conductors (MKSA units are used throughout).

Substitution of the magnetic field components in Eqs. (2) immediately leads to the first integral

\[
r^2 \dot{\theta} = r_0^2 \dot{\theta}_0 = C ,
\]

\( C \) being a constant of the motion.

The \( r \) and \( z \) equations can then be written

\[
\frac{m}{q} \left( \ddot{r} - \frac{C^2}{r^3} \right) = - \frac{\mu_0 I}{2\pi r} \dot{z}
\]

\[
\frac{m}{q} \ddot{z} = \frac{\mu_0 I}{2\pi r} \dot{r} .
\]

Putting

\[
V = \frac{\mu_0 I q}{2\pi m}
\]

these equations take the form
\[ \dddot{r} - \frac{C^2}{r} = -V \frac{\dot{r}}{r} \]
\[ \dddot{z} = V \frac{\dot{r}}{r}, \]

\( V \) having the dimensions of a velocity.

III. ORBIT EQUATIONS

Rather than express \( r, z, \) and \( \theta \) as functions of the time, we try to get the orbit equations in a form not involving the time as this parameter only plays an auxiliary role in the problem under consideration. Moreover, instead of determining the true orbit \( r = r(z, \theta) \), we attempt to calculate:

a) the orbit \( r = r(z) \) in a plane rotating around the axis with the speed of the particle;

b) the projection \( r = r(\theta) \) of the orbit on a plane perpendicular to the axis.

Considering first the orbit in a plane rotating with the particle, we can write the transformation relations

\[ \dot{r} = \frac{dr}{dt} = r' \dot{z} \]
\[ \ddot{r} = \frac{d^2r}{dt^2} = r'' \ddot{z} + r' \dot{z}' \]

where the dashes indicate differentiations with respect to \( z \).

Using now the first integrals (4) and (6), we have

\[ \dot{z} = \pm \frac{\sqrt{v^2 - \frac{C^2}{r^2}}}{\sqrt{1 + r'^2}}, \]

where we must allow for negative values of the square root in order to take into account a possible inversion of the direction of propagation of a ray.
Replacing Eqs. (10) and (11) in Eqs. (9), we obtain the differential equation of the particle orbit in the rotating plane

\[ r'' \left( v^2 - \frac{C^2}{r^2} \right) - \frac{C^2}{r^3} (1 + r'^2) \pm \frac{V}{r} (1 + r'^2)^{3/2} \sqrt{v^2 - \frac{C^2}{r^2}} = 0. \] (12)

Alternatively, this may be written

\[ r'' = \frac{C^2 (1 + r'^2)}{r^3 \left( v^2 - \frac{C^2}{r^2} \right)} + \frac{V}{r} \left( 1 + r'^2 \right)^{3/2} \sqrt{v^2 - \frac{C^2}{r^2}} \] (13)

or

\[ \frac{C^2}{r^3} \sqrt{\frac{1 + r'^2}{v^2 - \frac{C^2}{r^2}}} - \frac{v^2 - \frac{C^2}{r^2}}{\sqrt{1 + r'^2}} = \pm \frac{V}{r}. \] (14)

It is readily seen that the last equation can be written in the form

\[ d \left( \sqrt{\frac{v^2 - \frac{C^2}{r^2}}{1 + r'^2}} \right) = \pm \frac{V}{r} \frac{dr}{r}. \] (15)

from which the first integral

\[ \sqrt{\frac{v^2 - \frac{C^2}{r^2}}{1 + r'^2}} = \pm V \ln r + \text{Const.} \] (16)

follows. The constant on the r.h.s. of Eq. (16) can be expressed in terms of the initial conditions \( \theta_0, r_0, \dot{r}_0 \) so that the r,z equation in the rotating plane becomes

\[ \sqrt{\frac{v^2 - \frac{C^2}{r^2}}{1 + r'^2}} = \pm V \ln \frac{r}{r_0} + \sqrt{\frac{v^2 - \frac{C^2}{r^2}}{1 + r'^2}} \] (17)

or alternatively.

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\[
\frac{\mathrm{d}r}{\mathrm{d}z} = \pm \frac{\sqrt{v^2 - \frac{C^2}{r^2}} - \left( \pm V \ln \frac{r}{r_0} + \sqrt{\frac{v^2 - \frac{C^2}{r_0^2}}{1 + \frac{r^2}{r_0^2}}} \right)^2}{\left( \pm V \ln \frac{r}{r_0} + \frac{v^2 - \frac{C^2}{r_0^2}}{\sqrt{1 + \frac{r^2}{r_0^2}}} \right)^2}.
\]

To find the projection of the orbit on a plane perpendicular to the axis, we use the relation
\[
\dot{\vartheta} = \frac{\mathrm{d}\vartheta}{\mathrm{d}z} = \frac{\mathrm{d}\vartheta}{\mathrm{d}r} r'.
\]

Solving for \(\frac{\mathrm{d}r}{\mathrm{d}\vartheta}\), replacing \(\dot{\vartheta}\) and \(r'\) by their values, and taking account of Eq. (6), we find
\[
\sqrt{\frac{v^2 - \frac{C^2}{r^2}}{1 + \frac{1}{r^2} \left( \frac{\mathrm{d}r}{\mathrm{d}\vartheta} \right)^2}} = \pm V \ln \frac{r}{r_0} + \sqrt{\frac{v^2 - \frac{C^2}{r_0^2}}{1 + \frac{r^2}{r_0^2}}}
\]

or alternatively
\[
\frac{\mathrm{d}r}{\mathrm{d}\vartheta} = \pm \frac{r^2}{C} \sqrt{v^2 - \frac{C^2}{r^2}} - \left( \pm V \ln \frac{r}{r_0} + \sqrt{\frac{v^2 - \frac{C^2}{r_0^2}}{1 + \frac{r^2}{r_0^2}}} \right)^2.
\]

The direction of a ray at any moment can be specified by means of two angles. One may use the angles defined in the following way (Fig. 2): Consider a first plane \(P_1\) passing through the axis and through the instantaneous position of the particle - this is the plane rotating with the particle - and a second plane \(P_2\) perpendicular to the first, parallel to the axis and also passing through the particle. The angle made with the axis by the projection of the tangent to the ray on the plane \(P_1\) will be called \(\beta\), whereas the angle made with the axis by the projection of the tangent to the ray on the plane \(P_2\) will be called \(\alpha\). We shall restrict ourselves to the case where the initial values satisfy the relations
\[
0 < \alpha_0 < \frac{\pi}{2},
\]
\[
-\frac{\pi}{2} < \beta_0 < \frac{\pi}{2}.
\]
In terms of the angles thus defined, Eq. (4) can be written

\[ v^2 = (1 + \tan^2 \alpha + \tan^2 \beta) \beta^2 \]  

(23)

On the other hand, Eq. (11) shows that

\[ \beta^2 = \left( v^2 - \frac{C^2}{r^2} \right) \cos^2 \beta \]  

(24)

Combining the last two equations, we find

\[ \frac{C^2}{r^2} = v^2 \frac{\cos^2 \beta \tan^2 \alpha}{1 + \cos^2 \beta \tan^2 \alpha} \]  

(25)

Putting

\[ \cos \beta \tan \alpha = \tan \gamma \]  

(26)

Eq. (25) becomes

\[ \frac{C}{r} = v \sin \gamma \]  

(27)

and obviously one also has

\[ \frac{C}{r_0} = v \sin \gamma_0 \]  

(28)

The angle \( \gamma \) has a simple geometrical meaning (Fig. 3): it is the angle made by the tangent to the orbit with the rotating plane. In fact, instead of defining the direction of the orbit by means of \( \alpha \) and \( \beta \), one can also use the angles \( \gamma \) and \( \beta \).

At this point, it seems adequate to introduce the non-dimensional quantities

\[ x = \frac{r}{r_0} \]  

(29)

\[ \lambda = \frac{V}{v} = \frac{\mu_0 I}{2\pi r_0 p/q} \]  

(30)
where \( p/q \) is the magnetic rigidity "Bρ" of the particle (momentum per unit charge), measured in webers/m in MKSA units. (For transforming to other units, one may apply the well-known relation \( Bρ = (1/300 Z)\sqrt{T^2 + 2TE_0} \), with \( Bρ \) in webers/m, \( T \) and \( E_0 \) in MeV, and \( Z \) being the number of elementary charges carried by the particle.) \( λ \) is essentially a positive quantity.

With these notations, we have immediately from Eqs. (27) and (28)

\[
x = \frac{\sin γ_0}{\sin γ},
\]

and the basic equations (17) and (20) describing the orbit behaviour in the beam guide in terms of the initial conditions \( r_0, β_0, γ_0 \) become

\[
1 - \frac{\sin^2 γ_0}{x^2} = (\cos β_0 \cos γ_0 \pm λ \ln x)^2
\]

\[
1 + r_0^2 \left( \frac{dx}{dz} \right)^2
\]

in the plane rotating with the particle, and

\[
1 - \frac{\sin^2 γ_0}{x^2} \left[ 1 + \frac{1}{x^2} \left( \frac{dx}{dδ} \right)^2 \right] = (\cos β_0 \cos γ_0 \pm λ \ln x)^2
\]

in the plane perpendicular to the axis of the beam guide.

The last two equations may be put in the form

\[
\cos β \cos γ = \cos β_0 \cos γ_0 \pm λ \ln \frac{\sin γ_0}{\sin γ}.
\]

Equations (32) and (34) show that, under all circumstances, the inequality in \( x \) or \( γ \)

\[
\cos β_0 \cos γ_0 \pm λ \ln x \leq 1
\]

\[
\cos β_0 \cos γ_0 \pm λ \ln \frac{\sin γ_0}{\sin γ} \leq 1
\]

must be satisfied.
It is useful to remark that the second derivatives, giving the sign of the curvature of the orbit, are found to be

\[
\gamma \frac{d^2 x}{d\gamma^2} = \frac{\sin^2 \gamma_0 \left( \cos \beta_0 \cos \gamma_0 \pm \lambda \ln x \right)}{x^3 \left( \cos \beta_0 \cos \gamma_0 \pm \lambda \ln x \right)^3} \pm \lambda x^2 \left( 1 - \frac{\sin^2 \gamma_0}{x^2} \right)
\]

(36)

\[
\sin^2 \gamma_0 \frac{d^2 x}{d\gamma^2} = x \left[ 2x^2 - 2x^2 (\cos \beta_0 \cos \gamma_0 \pm \lambda \ln x) \right] \pm \lambda x^2 (\cos \beta_0 \cos \gamma_0 \pm \lambda \ln x) - \sin^2 \gamma_0
\]

(37)

Finally, the orbit itself in the \(r, \phi\) plane and in the \(r, \theta\) plane is given by two quadratures

\[
\frac{z - z_0}{\rho_0} = \int \frac{(\cos \beta_0 \cos \gamma_0 \pm \lambda \ln x) dx}{\sqrt{1 - \frac{\sin^2 \gamma_0}{x^2} - (\cos \beta_0 \cos \gamma_0 \pm \lambda \ln x)^2}}
\]

(38)

\[
\theta - \theta_0 = \sin \gamma_0 \int \frac{dx}{x^2 \left[ 1 - \frac{\sin^2 \gamma_0}{x^2} - (\cos \beta_0 \cos \gamma_0 \pm \lambda \ln x)^2 \right]}
\]

(39)

The latter equation gives the rotation of the particle around the axis relative to its initial position.

IV. PROPERTIES OF THE ORBITS

Equation (31) shows that

\[
x = \sin \gamma_0
\]

(40)

Consequently, \(\sin \gamma_0\) is an absolute minimum for \(x\) and for \(\sin \gamma_0 \neq 0\), i.e. \(\dot{\gamma}_0 \neq 0\), the beam spirals around the axis [which already follows from Eq. (5)].
The type of the orbit described by the particle depends on the relation between the parameter $\lambda$ (which is, except for a constant, the current in the guide conductors divided by particle momentum) and the initial conditions. Equation (32) shows that the orbit has cusps (Fig. 4) for
\[ x = \sin \gamma_0 \]
\[ \cos \beta_0 \cos \gamma_0 + \lambda \ln x = 0 . \]  

$\lambda$ being essentially a positive quantity, the only solution of this system is
\[ \lambda_0 = -\frac{\cos \beta_0 \cos \gamma_0}{\ln \sin \gamma_0} . \]  

For $\lambda > \lambda_0$ (low-energy particles), the beam is bent round by the magnetic field and the orbit shows loops (Fig. 5). In this case, $r$ is a multiple-valued function of $z$, the upper branches of the orbit (A and C in Fig. 5) being described by the differential equation
\[ \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2}} = \cos \beta_0 \cos \gamma_0 + \lambda \ln x \]  
\[ 1 + r_0^2 \frac{dx}{dz^2} \]  

and the lower branch (B in Fig. 4) being represented by
\[ \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2}} = \cos \beta_0 \cos \gamma_0 - \lambda \ln x . \]  

These two orbits are of a different type. In particular, the curve described by Eq. (44) cannot have a maximum, as can be readily seen by inspection of Eq. (36); however, it does have a minimum given by the solution of the equation
\[ \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2}} = \cos \beta_0 \cos \gamma_0 - \lambda \ln x . \]  

9559/p/smb
If $\lambda < \lambda_0$ (high-energy particles), no inversion is possible in the direction of the orbit, $r$ is a single-valued function of $z$, and the beam therefore progresses in one direction (Fig. 6).

It is interesting to note that the orbit shown in Fig. 7 - loops on the upper part of the orbit - is not possible. By taking the lower signs in Eq. (36), it can indeed be seen that the curvature cannot be negative in this case. Therefore, loops can exist only on the lower part of the orbit (Fig. 5). Physically, this can be interpreted by noting that a particle (whose momentum is sufficiently low) can be bent round by the magnetic field if it penetrates into a stronger field (Fig. 5), but it cannot do so if it gradually penetrates into a weaker field (Fig. 7). (We have excluded injection conditions where $\cos \beta_0$ or $\cos \gamma_0$ could be negative.) For the same reason, an orbit where the cusps would point away from the axis (Fig. 8) is not possible.

If $\dot{\phi}_0 = 0$, we have $\sin \gamma_0 = 0$ and the particle stays in a meridian plane. In this case, $\lambda_0$ as given by Eq. (42) is zero and, consequently, all planar orbits have loops. From Eq. (32), we find for the equation of the meridian orbits

$$\frac{1}{\sqrt{1 + r^2_0 \left(\frac{dx}{dz}\right)^2}} = \cos \beta_0 \pm \lambda \ln x. \quad (46)$$

V. EXTREME VALUES AND HELICAL ORBITS

In the case of the planar orbits just considered, there are two extreme values given by

$$\cos \beta_0 \pm \lambda \ln x = 1, \quad (47)$$

i.e.

$$x = \exp \pm \left(\frac{1 - \cos \beta_0}{\lambda}\right). \quad (48)$$
Except for meridian orbits, we henceforth limit ourselves to the case where \( \lambda < \lambda_0 \); on other words, we consider only orbits without loops corresponding to medium- or high-energy particles. From Eqs. (32) or (33), we then find that the maxima and the minima of the orbits are given by

\[
1 - \frac{\sin^2 \gamma_0}{x^2} = (\cos \beta_0 \cos \gamma_0 + \lambda \ln x)^2,
\]

(49)

with

\[
\sin \gamma_0 \leq x \leq \exp\left(\frac{1 - \cos \beta_0 \cos \gamma_0}{\lambda}\right),
\]

the second inequality following from inequality (35).

Putting

\[
F(x) = 1 - \frac{\sin^2 \gamma_0}{x^2} - (\cos \beta_0 \cos \gamma_0 + \lambda \ln x)^2,
\]

(51)

it is easily seen that

\[
F(\sin \gamma_0) < 0
\]

\[
F(1) > 0
\]

\[
F\left[\exp\left(\frac{1 - \cos \beta_0 \cos \gamma_0}{\lambda}\right)\right] < 0.
\]

(52)

Consequently, Eq. (49) generally has two roots which we shall call \( x_m \) and \( x_M \).

We have

\[
\sin \gamma_0 < x_m < 1 < x_M < \exp\left(\frac{1 - \cos \beta_0 \cos \gamma_0}{\lambda}\right),
\]

(53)

so that \( x_m \) corresponds to a minimum of the orbit whereas \( x_M \) gives its maximum. Therefore, the particle periodically oscillates between \( x_m \) and \( x_M \) with a period \( Z \) in the \( r, z \) plane and \( \Theta \) in the \( r, \Theta \) plane. From Eqs. (38) and (39), we have

9559/p/smb
\[ \frac{Z}{r_0} = 2 \int_{x_m}^{x_M} \frac{\cos \beta_0 \cos \gamma_0 + \lambda \ln x}{x^2 \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2} - (\cos \beta_0 \cos \gamma_0 + \lambda \ln x)^2}} \, dx \]  

\text{(54)}

and

\[ \Theta = 2 \sin \gamma_0 \int_{x_m}^{x_M} \frac{dx}{x^2 \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2} - (\cos \beta_0 \cos \gamma_0 + \lambda \ln x)^2}} \]  

\text{(55)}

Although the roots of Eq. (49) cannot be written out explicitly, there is a simple relation between them. We have, indeed, for given initial conditions

\[ \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2 \_M}} = \cos \beta_0 \cos \gamma_0 + \lambda \ln x_M \]  

\text{(56)}

\[ \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2 \_m}} = \cos \beta_0 \cos \gamma_0 + \lambda \ln x_m \]  

\text{(57)}

and therefore

\[ \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2 \_M}} - \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2 \_m}} = \lambda \ln \frac{x_M}{x_m} \]  

\text{(58)}

Putting

\[ u_M = \frac{x_M}{\sin \gamma_0} \quad u_m = \frac{x_m}{\sin \gamma_0} \]  

\text{(59)}

the relation between the normalized maximum and minimum excursions becomes

\[ \sqrt{1 - \frac{1}{u^2 \_M}} - \sqrt{1 - \frac{1}{u^2 \_m}} = \lambda \ln \frac{u_M}{u_m} \]  

\text{(60)}

The initial conditions have disappeared from this relation and the only parameter left is \( \lambda \), the ratio of the current to the magnetic rigidity of the particle.
Figure 9 shows the set of curves \( u_M = u_M(u_m, \lambda) \), \( \lambda \) being taken as a parameter.

Figure 10 gives the ratio \( u_M/u_m \) as a function of \( u_m \), \( \lambda \) again being taken as a parameter.

If \( u_m \gg 1 \), Eq. (60) takes the somewhat simpler form

\[
\frac{1}{u_m^2} - \frac{1}{u_M^2} = 2\lambda \ln \frac{u_M}{u_m}
\]  
(61)

Consideration of the second derivative allows the completion of the inequalities (53) by another sequence. It is seen from Eqs. (36) and (37) that the sign of the second derivative of a maximum or a minimum depends on the sign of the quantity \( 1 - \lambda (x^2/\sin^2 \gamma_0) \sqrt{1 - (\sin^2 \gamma_0/x^2)} \). Consequently, for a maximum we should have

\[
1 - \lambda \frac{x_M^2}{\sin^2 \gamma_0} \sqrt{1 - \sin^2 \gamma_0 / x_M^2} < 0
\]  
(62)

i.e.

\[
x_M > \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda^2} \sin \gamma_0}}
\]  
(63)

whereas for a minimum, the inequality

\[
1 - \lambda \frac{x_m^2}{\sin^2 \gamma_0} \sqrt{1 - \sin^2 \gamma_0 / x_m^2} > 0
\]  
(64)

i.e.

\[
x_m < \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda^2} \sin \gamma_0}}
\]  
(65)

must hold.

Therefore, if we put

\[
\lambda_1 = \tan^2 \gamma_0 \cos \gamma_0
\]  
(66)
we have the sequence

\[ x_m < \sqrt{\frac{1}{2} + \frac{1}{4} \frac{1}{\lambda^2} \sin \gamma_0} < 1 < x_M \]  \hspace{1cm} (67)

for \( \lambda > \lambda_1 \), whereas \( \lambda < \lambda_1 \) implies

\[ x_m < 1 < \sqrt{\frac{1}{2} + \frac{1}{4} \frac{1}{\lambda^2} \sin \gamma_0} < x_M \]  \hspace{1cm} (68)

From Eq. (56) we also find for the turning points of the orbit the equation

\[ \cos \beta_0 \cos \gamma_0 + \lambda \ln x = \lambda \frac{x^2}{\sin^2 \gamma_0} \left( 1 - \frac{\sin^2 \gamma_0}{x^2} \right) \]  \hspace{1cm} (69)

Putting

\[ \phi(x) = \cos \beta_0 \cos \gamma_0 + \lambda \ln x - \lambda \frac{x^2}{\sin^2 \gamma_0} \left( 1 - \frac{\sin^2 \gamma_0}{x^2} \right) \]  \hspace{1cm} (70)

it is seen that

\[ \phi(x_m) = \sqrt{1 - \frac{\sin^2 \gamma_0}{x_m^2}} \left[ 1 - \lambda \frac{x_m^2}{\sin^2 \gamma_0} \right] \left( 1 - \frac{\sin^2 \gamma_0}{x_m^2} \right) > 0 \]

\[ \phi(1) = \cos \gamma_0 \left( \cos \beta_0 - \lambda \frac{\cos \gamma_0}{\sin^2 \gamma_0} \right) \]  \hspace{1cm} (71)

\[ \phi(x_M) = \sqrt{1 - \frac{\sin^2 \gamma_0}{x_M^2}} \left[ 1 - \lambda \frac{x_M^2}{\sin^2 \gamma_0} \right] \left( 1 - \frac{\sin^2 \gamma_0}{x_M^2} \right) < 0 \]

Consequently, if we put

\[ \lambda_2 = \tan^2 \gamma_0 \cos \gamma_0 \cos \beta_0 , \]  \hspace{1cm} (72)

the turning points will be on the lower side of the orbit for \( \lambda > \lambda_2 \), whereas for \( \lambda < \lambda_2 \) (high-energy particles), the turning points will be on the upper side of the orbit (Fig. 6).
Thus far, we have only considered the case where $x_m \neq x_M$. If
\[
x_m = x_M,
\]
the second derivative is zero, and so are all higher-order derivatives; the orbit is then a helix. The conditions for a helical orbit are therefore
\[
1 - \frac{\sin^2 \gamma_0}{x^2} = (\cos \beta_0 \cos \gamma_0 + \lambda \ln x)^2 \quad (74)
\]
\[
\sin^2 \gamma_0 = \lambda \sqrt{1 - \frac{\sin^2 \gamma_0}{x^2}} \quad (75)
\]
with
\[
x = 1. \quad (76)
\]
Replacing Eq. (76) in Eqs. (74) and (75), we find the conditions
\[
\beta_0 = 0 \quad (77)
\]
\[
\lambda = \lambda_1 = \tan^2 \gamma_0 \cos \gamma_0. \quad (78)
\]
On account of Eqs. (26) and (77), the condition (78) can be written
\[
\lambda = \tan^2 a_0 \cos a_0. \quad (79)
\]
For a given particle momentum and supposing $\beta_0 = 0$, there is therefore one starting angle given by
\[
\cos a_0 = \sqrt{\frac{\lambda^2}{4} + 1 - \frac{\lambda}{2}} \quad (80)
\]
which leads to a helix; this angle is independent of the initial position $r_0$. The inclination of the helix is given by
\[
\cot a_0 = \sqrt{\frac{1}{4} + \frac{1}{\lambda^2} - \frac{1}{2}} \quad (81)
\]
and its pitch by
\[
\frac{Z}{r_0} = 2\pi \sqrt{\frac{1}{4} + \frac{1}{\lambda^2} - \frac{1}{2}}. \quad (82)
\]
VI. GRAZING RAYS

If we want a ray just to graze the outer conductor, the radius $R_0$ of that conductor must correspond to a maximum. Therefore, we must have

$$\sqrt{1 - \left(\frac{R_0}{R_0}\right)^2 \sin^2 \gamma_0} = \cos \beta_0 \cos \gamma_0 + \lambda \ln \frac{R_0}{r_0}$$  \hspace{1cm} (83)$$

with the condition [following from inequality (63)]

$$\frac{R_0}{R_0} < \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{\lambda^2} \sin \gamma_0}}$$  \hspace{1cm} (84)$$

Similarly, if we want the minimum to take place at the surface of the inner conductor of radius $R_1$, we must have

$$\sqrt{1 - \left(\frac{R_0}{R_1}\right)^2 \sin^2 \gamma_0} = \cos \beta_0 \cos \gamma_0 + \lambda \ln \frac{R_1}{r_0}$$  \hspace{1cm} (85)$$

with the condition [following from inequality (65)]

$$\frac{R_0}{R_1} > \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{\lambda^2} \sin \gamma_0}}$$  \hspace{1cm} (86)$$

Equations (83) and (85) allow the determination of the injection angles $\beta_0$ and $\gamma_0$ (or $\beta_0$ and $\alpha_0$) which lead, at the same time, to a maximum at the inner surface of the outer conductor and to a minimum at the outer surface of the inner conductor.

Putting

$$y = \frac{R_0}{R_1}$$  \hspace{1cm} (87)$$

$$\rho = \frac{R_0}{R_1}$$
we have, by subtracting Eq. (85) from Eq. (83),
\[
\sqrt{1 - \left( \frac{y}{\rho} \right)^2 \sin^2 \gamma_0} - \sqrt{1 - y^2 \sin^2 \gamma_0} = \lambda \ln \rho .
\] (88)

Using the abbreviation
\[
G(\lambda, \rho) = \sqrt{\lambda \left[ 2\sqrt{\lambda \rho \ln \rho} \right]^2 + (\rho^2 - 1)^2 - \lambda (\rho^2 + 1) \ln \rho} \ln \rho
\] (89)
the solution of Eq. (88) is
\[
\sin \gamma_0 = \frac{\rho}{y(\rho^2 - 1)} G(\lambda, \rho) ,
\] (90)
with the condition [following from inequalities (84) and (85)]
\[
\frac{\rho^2 - 1}{\rho \sqrt{\frac{1}{2} + \frac{1}{\sqrt{4 + \frac{1}{\lambda^2}}}}} < G(\lambda, \rho) < \frac{\rho^2 - 1}{\sqrt{\frac{1}{2} + \frac{1}{\sqrt{4 + \frac{1}{\lambda^2}}}}} .
\] (91)

Using next Eq. (85), we find for the two angles needed for simultaneous grazing
\[
\tan \alpha_0 = \frac{\rho}{y(\rho^2 - 1)} G
\] (92)
\[
\cos \beta_0 = \frac{1 - \frac{\rho^2}{(\rho^2 - 1)^2} G^2 + \lambda \ln y}{\sqrt{1 - \frac{\rho^2}{(\rho^2 - 1)^2} G^2 + \lambda \ln y}} .
\] (93)

Therefore, these two angles depend only on the ratios \( r_0/R_1, R_0/R_1 \), and on \( \lambda \).
VII. MOMENTUM ACCEPTANCE

Equation (38) shows that for any orbit one must have

$$\sqrt{1 - \frac{\sin^2 \gamma_0}{x^2}} \geq \cos \beta_0 \cos \gamma_0 + \lambda \ln x \quad (94)$$

Applying this condition to the outer and the inner conductor, we must have

$$\sqrt{1 - \left(\frac{\rho}{y}\right)^2 \sin^2 \gamma_0} \geq \cos \beta_0 \cos \gamma_0 + \lambda \ln \frac{\rho}{y} \quad (95)$$

$$\sqrt{1 - y^2 \sin^2 \gamma_0} \geq \cos \beta_0 \cos \gamma_0 + \lambda \ln \frac{1}{y}$$

and therefore

$$\frac{\sqrt{1 - y^2 \sin^2 \gamma_0 - \cos \beta_0 \cos \gamma_0}}{\ln y} \leq \lambda \leq \frac{\sqrt{1 - \left(\frac{\rho}{y}\right)^2 \sin^2 \gamma_0 - \cos \beta_0 \cos \gamma_0}}{\ln \frac{\rho}{y}} \quad (96)$$

Except for a constant, this gives the reciprocal value of the accepted momentum range for a given current in the conductors and for given initial conditions \(r_0, \beta_0, \gamma_0\).

It is easily seen that for

$$y \sin \gamma_0 = \sqrt{1 - \cos^2 \beta_0 \cos^2 \gamma_0} \quad (97)$$

there is no cut-off at all on the high-energy side, whereas on the low-energy side the cut-off becomes

$$\lambda \leq \frac{\sqrt{1 - \frac{1}{\rho^2} (1 - \cos^2 \beta_0 \cos^2 \gamma_0) - \cos \beta_0 \cos \gamma_0}}{\ln \frac{\rho \sin \gamma_0}{\sqrt{1 - \cos \beta_0 \cos \gamma_0}}} \quad (98)$$

For \(\cos \gamma_0 = 0\) and \(\rho >> 1\), the quantity on the r.h.s. of the inequality simply becomes \(1/\ln \rho\). On the other hand, the lowest cut-off one can get
on the low-energy side with a given current is obtained by making \( \gamma_0 = 0 \) (planar orbits) and \( \beta_0 = \pi/2 \). With these conditions, the cut-off on the low-energy side becomes

\[
\lambda \leq \frac{1}{\ln \frac{2}{y}} 
\]

or else

\[
\frac{p}{q} \geq \frac{\mu q I}{2\pi} \ln \frac{2}{y} 
\]

By varying the current in the guide conductors or the initial conditions one can, therefore, cover the whole momentum range extending from zero to infinity.

VIII. HIGH-ENERGY BEHAVIOUR OF THE BEAM GUIDE

For particles of a few GeV energy or higher, and with present-day currents, \( \lambda \) is of the order of \( 1/100 \) at the most. On the other hand, the emission angles \( \alpha_0 \) and \( \beta_0 \) are small fractions of a radian. This situation introduces considerable simplification in beam guide theory; it will be considered exclusively in what follows.

Neglecting third-order terms in Eq. (26), we have

\[
\gamma = \alpha \quad (101)
\]

and consequently

\[
\gamma_0 = \alpha_0 \quad (102)
\]

With this simplification, Eqs. (31) and (34) can be written

\[
x = \frac{\alpha_0}{\alpha} \quad (103)
\]

\[
a^2 + \beta^2 = a_0^2 + \beta_0^2 - 2\lambda \ln \frac{\alpha_0}{\alpha} \quad (104)
\]
whereas the orbit equations (38) and (39) take the form

\[
\frac{z - z_0}{r_0} = \int_1^x \frac{dx}{\sqrt{\alpha_o^2 \left(1 - \frac{1}{x^2}\right) + \beta_0^2 - 2\lambda \ln x}} 
\]

\[
\theta - \theta_0 = \alpha_0 \int_1^x \frac{dx}{x^2 \sqrt{\alpha_o^2 \left(1 - \frac{1}{x^2}\right) + \beta_0^2 - 2\lambda \ln x}}.
\]

For meridian orbits the last two equations become

\[
\frac{z - z_0}{r_0} = \int_1^x \frac{dx}{\sqrt{\beta_0^2 + 2\lambda \ln x}}
\]

\[
\theta = \theta_0.
\]

It should, however, be emphasized that Eq. (107) is only applicable over the restricted range of small angles. In the case of meridian orbits where the angles will generally be large (there are always loops in the case of meridian orbits), one must resort to Eq. (38) and there make \(\gamma_0 = 0\).

In high-energy work, the characteristic values of \(\lambda\) become

\[
\lambda_0 = \frac{1}{\ln \frac{1}{\alpha_o}}
\]

\[
\lambda_1 = \lambda_2 = \alpha_o^2.
\]

From Eq. (49), we now find for the maximum and minimum excursions of the orbit

\[
\alpha_o^2 \left(1 - \frac{1}{x^2}\right) + \beta_0^2 = 2\lambda \ln x
\]

with

\[
\alpha_0 < x_m < \frac{\alpha_0}{\sqrt{\lambda}} < 1 < x_M < \exp\left(\frac{\alpha_o^2 + \beta_0^2}{2\lambda}\right)
\]

or

\[
\alpha_0 < x_m < 1 < \frac{\alpha_0}{\sqrt{\lambda}} < x_M < \exp\left(\frac{\alpha_o^2 + \beta_0^2}{2\lambda}\right)
\]
according to whether

\[ \lambda \gtrsim \lambda_1 \]  

(113)

The last relation also determines the position of the turning points on
the lower or the upper part of the orbit.

If

\[ \beta_0 = 0 \]
\[ \lambda = \lambda_1 \]  

(114)

the orbit is a helix; in other words, for a particle of given momentum,
the starting angles \( \beta_0 = 0 \), \( \alpha_0 = \sqrt{\lambda} \) lead to a helix with an inclination
given by

\[ \frac{1}{\alpha_0} = \frac{1}{\sqrt{\lambda}} \]  

(115)

and a pitch given by

\[ z = \frac{2 \gamma r_0}{\sqrt{\lambda}} \]  

(116)

Instead of trying to solve Eq. (111) which gives the extreme
excursions of the orbit, it is more convenient to regard \( x \) as a parameter
in this equation. Writing then

\[ X = \frac{\alpha_0}{\sqrt{\lambda}} \quad Y = \frac{\beta_0}{\sqrt{\lambda}} \]  

(117)

Eq. (111) takes the form

\[ x^2 \left( 1 - \frac{1}{x^2} \right) + y^2 = 2 \ln x \]  

(118)

Consequently, for a maximum \( x = x_M > 1 \), the locus of the \( X, Y \) point is
represented by the ellipse

\[ \frac{x^2}{2 \ln x_M} + \frac{y^2}{2 \ln x_M} = 1 \]  

(119)
whose semi-axes are

\[ A = \sqrt{2 \ln x_M - 1} \left( \frac{x_M^2}{x_M^2 - 1} \right) \]

\[ B = \sqrt{2 \ln x_M} . \]

For \( x \gg 1 \), the ellipse goes over into the circle

\[ X^2 + Y^2 = 2 \ln x_M \]  \hspace{1cm} (121)

while for \( x \) approaching 1, the ellipse goes over into that part of the X axis (Fig. 11) which is situated to the left of the point \( X = 1 \) (see also Ref. 1).

Similarly, for a minimum \( (x = x_m < 1) \), the locus of the \( X, Y \) point is represented by the hyperbola

\[ \frac{X^2}{2 \ln \frac{1}{x_m}} - \frac{Y^2}{2 \ln \frac{1}{x_m}} = 1 \]  \hspace{1cm} (122)

the semi-axes of which are

\[ C = \sqrt{2 \ln \frac{1}{x_m}} \]

\[ D = \sqrt{2 \ln \frac{1}{x_m}} . \]

For \( x \) approaching zero, the hyperbola goes over into the Y axis while for \( x \) approaching 1, the hyperbola goes over into that part of the X axis which is situated to the right of the point \( X = 1 \) (Fig. 11).

The point \( X = 1 \) itself represents a helical orbit as is apparent from Eqs. (110), (114) and (117).
In high-energy work, Eqs. (92) and (93) - which determine the conditions for having the beam envelope just touching the guide - go over into

\begin{equation}
\alpha_0 = \frac{2}{y} \sqrt{\frac{2\lambda}{\rho^2 - 1}} \frac{\ln \rho}{\rho^2 - 1} \tag{124}
\end{equation}

\begin{equation}
\beta_0 = \sqrt{2\lambda \left( \frac{\rho^2}{y^2} \left( \frac{y^2 - 1}{\rho^2 - 1} \right) \right) \ln \rho - \ln y} \tag{125}
\end{equation}

whereas the condition (91) becomes

\begin{equation}
\frac{\rho^2 - 1}{\rho} < \sqrt{2(\rho^2 - 1)} \ln \rho < \rho^2 - 1
\end{equation}

(126)

which is independent of \( \lambda \).

If \( \rho \gg 1 \), these relations simplify to

\begin{equation}
\alpha_0 = \frac{1}{y} \sqrt{2\lambda \ln \rho} \tag{127}
\end{equation}

\begin{equation}
\beta_0 = \sqrt{2\lambda \left( \frac{y^2 - 1}{y^2} \ln \rho - \ln y \right)}
\end{equation}

\begin{equation}
1 < \sqrt{2 \ln \rho} < \rho
\end{equation}

Finally, from the inequality (96) we find for the accepted range of \( \lambda \) in terms of the given current and the given initial conditions

\begin{equation}
\frac{\beta_0^2 - \alpha_0^2(y^2 - 1)}{2 \ln y} \leq \lambda \leq \frac{\beta_0^2 + \alpha_0^2 \left[ 1 - \left( \frac{y}{\rho} \right)^2 \right]}{2 \ln \frac{\rho}{y}} \tag{128}
\end{equation}

or in terms of the accepted momentum range

\begin{equation}
\frac{\mu_0 I}{\pi} \cdot \frac{\ln \frac{\rho}{y}}{\beta_0^2 + \alpha_0^2 \left[ 1 - \left( \frac{y}{\rho} \right)^2 \right]} \leq \frac{p}{q} \leq \frac{\mu_0 I}{\pi} \cdot \frac{\ln y}{\beta_0^2 - \alpha_0^2(y^2 - 1)} \tag{129}
\end{equation}
It is seen that for
\[ y = \sqrt{1 + \frac{\beta^2}{\alpha_0^2}} \]  
(130)
there is no cut-off at all on the high-energy side, whereas on the low-energy side the cut-off becomes
\[ \lambda \leq \left(1 - \frac{1}{\rho^2}\right) \frac{a^2 + \beta^2}{2 \ln \frac{\rho a_0}{\sqrt{a_0^2 + \beta^2}}} \]  
(131)
The last relation assumes that the high-energy approximation is still valid on the low-energy side; if not, Eq. (98) should be applied.

IX. ANGULAR ACCEPTANCE

If the maximum excursion of the orbit takes place at the surface of the outer conductor, the semi-axes of the corresponding ellipse are
\[ a = \sqrt{\frac{2}{\rho^2 - y^2} \ln \frac{\rho}{y}} \]  
(132)
\[ b = \sqrt{2 \ln \frac{\rho}{y}} \] .

Similarly, if the minimum excursion takes place at the surface of the inner conductor, the semi-axes of the corresponding hyperbola are
\[ c = \sqrt{\frac{2}{y^2 - 1} \ln y} \]  
(133)
\[ d = \sqrt{2 \ln y} \] .
The area enclosed by the limiting ellipse (132) and the limiting hyperbola (133) defines the total angular acceptance of the beam guide (Fig. 12) for particles originating from a point source placed at \( r_0 \).
From Fig. 12, we have for the co-ordinates of the intersection point of the limiting ellipse and the limiting hyperbola

\[ X_i = \frac{ac}{\sqrt{b^2 c^2 + a^2 d^2}} \]  
\[ (134) \]

i.e.

\[ X_i = \sqrt{2 \frac{\rho^2 \ln \rho}{y^2 (\rho^2 - 1)}} \]  
\[ (135) \]

and

\[ Y_i = \frac{bd}{\sqrt{b^2 c^2 + a^2 d^2}} \]  
\[ (136) \]

i.e.

\[ Y_i = \sqrt{2 \left[ \frac{\rho^2 (y^2 - 1)}{y^2 (\rho^2 - 1)} \right] \ln \rho - \ln y} \]  
\[ (137) \]

Equations (135) and (136) check with the values found previously [Eqs. (124), and (125)] for \( a_0 \) and \( \beta_0 \) in order to achieve simultaneous grazing of the outer and the inner conductor.

Figure 12 also shows that for \( \beta_0 = 0 \), the accepted range for \( \alpha_0 \) is maximum with

\[ \sqrt{2 \lambda \frac{\ln y}{y^2 - 1}} < \alpha_0 < \sqrt{2 \lambda \frac{\rho^2}{\rho^2 - y^2} \ln \rho} \]  
\[ (138) \]

whereas for

\[ \alpha_0 = \frac{\rho}{y} \sqrt{2 \lambda \frac{\ln \rho}{\rho^2 - 1}} \]

the accepted range for \( \beta_0 \) is maximum with

\[ 0 < \beta_0 < \sqrt{2 \lambda \left[ \frac{\rho^2 (y^2 - 1)}{y^2 (\rho^2 - 1)} \right] \ln \rho - \ln y} \]  
\[ (139) \]

To determine the total angular acceptance of the beam guide, we calculate the area enclosed by the limiting curves (Fig. 12). We have
\[ A = 4\lambda \left[ d \int_{c}^{X_1} \sqrt{\left(\frac{X}{c}\right)^2 - 1} \, dx + b \int_{X_1}^{a} \sqrt{1 - \left(\frac{X}{a}\right)^2} \, dx \right], \quad (140) \]

and carrying out the integrations we find

\[ \frac{A}{4\lambda} = ab \left[ \frac{1}{2} \text{arc cos} \frac{X_1}{a} - \frac{1}{4} \sin 2\left(\text{arc cos} \frac{X_1}{a}\right) \right] \]

\[ + cd \left[ -\frac{1}{2} \text{arc ch} \frac{X_1}{c} + \frac{1}{4} \sinh 2\left(\text{arc ch} \frac{X_1}{c}\right) \right]. \quad (141) \]

It is easy to check that

\[ ab \sin 2\left(\text{arc cos} \frac{X_1}{a}\right) = cd \sinh 2\left(\text{arc ch} \frac{X_1}{c}\right) \]

so that the total angular acceptance becomes

\[ A = 2\lambda \left( ab \text{arc cos} \frac{X_1}{a} - cd \text{arc ch} \frac{X_1}{c} \right) \]

and upon substitution of the relevant quantities

\[ \frac{A}{4\lambda} = \sqrt{\frac{\rho^2}{\rho^2 - y^2}} \left[ \text{arc cos} \sqrt{\frac{(\rho^2 - y^2)\ln \rho}{y^2(\rho^2 - 1)\ln(\rho/y)}} \right] \ln \frac{\rho}{y} \]

\[ \quad - \sqrt{\frac{1}{y^2 - 1}} \left[ \text{arc ch} \sqrt{\frac{\rho^2(y^2 - 1)\ln \rho}{y^2(\rho^2 - 1)\ln y}} \right] \ln y, \quad (144) \]

\( A \) being expressed in steradians.

To find the optimum position of the source point inside the beam guide, we calculate \( y \) from the equation

\[ \frac{dA}{dy} = 0. \quad (145) \]

Carrying out the calculations, this gives

\[ \frac{\rho}{\sqrt{\rho^2 - y^2}} \left( 1 - \frac{y^2}{\rho^2 - y^2} \ln \frac{\rho}{y} \right) \text{arc cos} \sqrt{\frac{(\rho^2 - y^2)\ln \rho}{y^2(\rho^2 - 1)\ln(\rho/y)}} \]

\[ + \frac{1}{\sqrt{y^2 - 1}} \left( 1 - \frac{y^2}{y^2 - 1} \ln y \right) \text{arc ch} \sqrt{\frac{\rho^2(y^2 - 1)\ln \rho}{y^2(\rho^2 - 1)\ln y}} \quad (146) \]

\[ = \rho \sqrt{\frac{\rho^2(y^2 - 1)\ln \rho - y^2(\rho^2 - 1)\ln y}{\ln \rho}} \ln \frac{\rho}{(\rho^2 - y^2)(y^2 - 1)} \].

9559/p/smb
This equation has been solved numerically for $y$ and the values have been replaced in Eq. (144). Figure 13 shows the optimum position of the source inside the guide as calculated from Eq. (146).

For $\rho >> 1$, an approximate solution of Eq. (146) is given by

$$\rho = \exp\left(y_0 \, \text{arc cos} \frac{1}{y_0}\right). \quad (147)$$

Figure 13 also shows this approximate solution as compared to the exact solution taken from Eq. (146). It is seen that for $\rho > 3$, the error involved is less than 10%.

Figure 14 gives the value of the maximum angular acceptance as calculated from Eq. (144) by using the solution of Eq. (146). If one takes the solution of Eq. (147), the maximum angular acceptance may be represented approximately by the expression

$$A_M = 4\lambda \left[ \left( \text{arc cos} \frac{1}{y_0} \right)^2 \left( y - \text{arc cos} \frac{1}{y_0} \right) \right]. \quad (148)$$

Figure 14 also shows the curve obtained by means of this approximate expression. It is seen that for $\rho > 8$, the error involved is less than 10%. However, for small values of $\rho$ the discrepancy is important.

In practical devices, $\rho$ will hardly be larger than 20. In this case, it is more appropriate to use the approximate value

$$\frac{A_M}{4\lambda} = \frac{1}{2}(\ln \rho)^2. \quad (149)$$

Figure 15 gives the comparison between the exact value of $A_M$ and the approximate value resulting from Eq. (149). It is seen that for $\rho < 30$ the error is less than ± 10%.

Figure 16 shows the set of curves $A(y,\rho)/A_M(\rho)$ as a function of $y$, $\rho$ being taken as a parameter. It is seen that for small values of $\rho$, the maximum is rather sharp so that the position of the source is quite important.

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X. PHASE ACCEPTANCE

For a given beam guide and a given source position, the angular acceptance is determined by Eq. (144). Letting the source position vary from the inner to the outer conductor and summing up, we obtain the phase acceptance of the guide, i.e.

$$ P = \int_{R_1}^{R_0} A \cdot 2\pi r \, dr \quad . $$  \hspace{1cm} (150)

This can be written

$$ P = 2\pi R_1^2 \int_{1}^{\rho} A(y, \rho)y \, dy \quad . $$  \hspace{1cm} (151)

$P$ being expressed in m² x steradians.

The integration indicated in Eq. (151) has been performed numerically using Eq. (144), and the function

$$ \frac{P}{8\pi \lambda R_1^2} $$

has been plotted in Figs. 17 and 18.

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1) S. van der Meer: The beam guide, a device for the transport of charged particles, CERN 62-16.
Fig. 11
Fig. 15.