REDUCTION OF THE N-COMPONENT SCALAR MODEL AT TWO LOOP LEVEL

Antal Jakováč
Department of Atomic Physics
Eötvös University
Budapest, Hungary
(February 17, 1995)

Dimensional reduction of high temperature field theories improves IR features of their perturbative treatment. A crucial question is, what three-dimensional theory is representing the full system the most faithful way. Careful investigation of the induced 3-dimensional counterterm structure of the finite temperature 4D \( O(N) \) symmetric scalar theory at 2-loop level leads to proposing the presence of non-local operators in the effective theory. On scales beyond \( \mathcal{O}(T^{-1}) \), the scaling behavior of the couplings, consequently, deviates from the usual three-dimensional scaling characteristic for superrenormalizable theories.

PACS numbers: 11.10.Wx, 64.60.-i, 95.30.Cq

I. INTRODUCTION

Numerical investigation of the finite temperature electroweak phase transition is unavoidable because in the symmetric phase no improved perturbation method is known to work reliably. The most adequate approach is to concentrate on the degrees of freedom left "massless", e.g. the static Matsubara modes of the magnetic gauge and Higgs fluctuations [1].

The effective reduced theory usually is arrived at after simple 1-loop integration over the non-static and the screened static modes [2,3], with the restriction that only those terms are retained in the effective action which are renormalizable in 4 dimensions. These theories being superrenormalizable in 3 dimensions one can work out their exact divergence structure and relate the physical, temperature dependent mass to the bare parameters of the 3 dimensional theory [4,5]. The general strategy behind this procedure is the matching of some important perturbatively computable quantities calculated both in the full finite temperature theory and in the effective superrenormalisable three-dimensional theory [6]. The difficult and most interesting question is how accurate is this 3 dimensional superrenormalizable representation of the theory. On one hand one should control higher dimensional operators with couplings inversely proportional to some power of \( T \) [3]. On the other hand on very general grounds intrinsic non-local behaviour of range \( T^{-1} \) is also expected to appear.

The last point can be illustrated, for instance, on the example of the so-called "sunset" diagram contributing to the propagation of a static mode. The contributions are naturally divided into three groups:

\[
-\quad + \quad + \quad \quad + \quad -
\]

(1)

(the solid lines represent non-static, the dashed ones the static propagators). The first class is accounted for when one integrates over the non-static modes with two-loop accuracy. The second class is considered in the course of the solution of the static effective model. The third ("mixed") class, however would be left out from the two-step solution.

Our proposition is to incorporate the mixed part of the "sunset" contribution into the effective model by defining a new kind of vertex, namely
This vertex is clearly nonlocal, since the lines compressed into it have a typical range $O(T^{-1})$. Similar conclusion has been emphasized recently also by [7]. Explicit estimate of the strength of the non-local corrections, to our knowledge will appear for the first time in the present paper. In this first attempt we limit our calculations to the N-component scalar model (for $N = 4$ the Higgs-sector of the standard model) in the hope, that the main lesson remains true for the full electroweak theory [8].

We did our computations using momentum cut-off regularization, that is we regularized the propagator to be zero, when its momentum exceeds the cut-off. The divergent counterterm structure obtained for the effective model by the 2-loop reduction and that of the 3 dimensional superrenormalisable theory computed with cut-off regularization both display extra power divergences which would be absent in dimensional regularization [2,9,10]. The need for extra pieces in the effective action becomes obvious by the mismatch discovered just between those quantities. The correct choice of the new terms will be undisputable if with their contributions the balance between three different type of power- and logarithmic divergences of the effective theory and those induced by the reduction will be reestablished.

Our investigation starts with a detailed presentation of the counterterm structure of the 3 dimensional O(N) symmetric model of the static modes induced upon 2-loop integration over nonstatic degrees of freedom (Section 2). This is then compared to the divergences of the effective potential calculated from the 3 dimensional local superrenormalizable O(N) model at the same 2-loop level (Section 3). A discrepancy between the results will be discovered and the origin of the mismatch will be located in Section 4. The extra divergencies needed for the consistency between the effective and the original theory come from intrinsically non-local operators in 3 dimensions with a characteristic non-locality range $T^{-1}$, as qualitatively explained above. In the Conclusion (Section 5) we shall outline the matching strategy for replacing the effective non-local theory by a local theory with nonremovable cut-off $\alpha T^{-1}$ ($\alpha = 1 - 5$). The interpretation of the computer simulation of a corresponding lattice system will also be shortly discussed. Also we shall touch upon the extension of the present work to gauge theories.

II. REDUCTION OF THE 4 DIMENSIONAL O(N) MODEL AT TWO LOOP LEVEL

The model is described by the following Hamiltonian:

$$H = \sum_{i=1}^{N} \frac{1}{2} \left( (\partial \varphi_i)^2 + m^2 \varphi_i^2 \right) + \frac{\lambda}{4!} \left( \sum_{i=1}^{N} \varphi_i^2 \right)^2 + H_{ct},$$  \hspace{1cm} (3)

where $H_{ct}$ represents the counterterms. First, we integrate out the nonstatic Matsubara modes, and find the potential energy density for the reduced 3 dimensional theory. For this we shift the fields $\varphi$ by a constant $\Phi_0$, what stands for the static part. Because of the $O(N)$ symmetry there exists a coordinate system, where $\Phi_0$ points along the N-th axis. The Hamiltonian breaks up into $\varphi$-independent, and quadratic pieces, and a higher power $\varphi$-dependent term:

$$H = H_0 + H_2 + H_4,$$  \hspace{1cm} (4)

where

$$H_0 = \frac{1}{2} (m^2 + \delta m^2) \Phi_0^2 + \frac{1}{4!} (\lambda + \delta \lambda) \Phi_0^4,$$

$$H_2 = \sum_{i=1}^{N-1} \frac{1}{2} \varphi_i \left( -\partial^2 + m^2 + \frac{\lambda}{6} \Phi_0^2 \right) \varphi_i + \frac{1}{2} \varphi_N \left( -\partial^2 + m^2 + \frac{\lambda}{2} \Phi_0^2 \right) \varphi_N,$$
\[ \mathcal{H}_f = \sum_{i=1}^{N-1} \frac{1}{2} \varphi_i \left( -(Z-1)\partial^2 + \delta m^2 + \frac{\delta \lambda}{6} \Phi_0 \right) \varphi_i + \frac{1}{2} \varphi_N \left( -(Z-1)\partial^2 + \delta m^2 + \frac{\delta \lambda}{2} \Phi_0 \right) \varphi_N \]
\[ + \frac{\lambda + \delta \lambda}{4!} \left( \left( \sum_{i=1}^{N} \varphi_i^2 \right)^2 + 2 \varphi_N^2 \sum_{i=1}^{N-1} \varphi_i^2 + 4 \Phi_0 \varphi_N \sum_{i=1}^{N-1} \varphi_i^2 + \varphi_N^2 + 4 \Phi_0 \varphi_N^2 \right). \]  

We compute the potential energy density of the effective theory \( f \) in the following way:

\[ fV = H_0 + \frac{1}{2} \text{Tr} \log \Delta - \langle e^{-H_t} - 1 \rangle_c, \]

where the index \( c \) refers to connected graphs, \( V \) is the volume of the system.

For two loop results the interaction part of the exponential must be expanded to \( \mathcal{O}(\lambda^2) \), and only \( \Phi_0 \)-dependent terms should be retained:

\[ - \langle e^{-H_t} - 1 \rangle_c = \sum_{i=1}^{N} \frac{1}{2} \left( \delta m^2 + \frac{\delta \lambda}{6} \Phi_0 \right) \langle \varphi_i^2 \rangle_c + \frac{1}{2} \left( \delta m^2 + \frac{\delta \lambda}{2} \Phi_0 \right) \langle \varphi_N^2 \rangle_c \]
\[ + \frac{\lambda}{24} \left[ \left( \langle \sum_{i=1}^{N-1} \varphi_i^2 \rangle_c \right)^2 \right. \]
\[ + \left. 2 \langle \varphi_N^2 \sum_{i=1}^{N-1} \varphi_i^2 \rangle_c + \langle \varphi_N^2 \rangle_c \right] \]
\[ - \frac{\lambda^2}{72} \Phi_0^2 \langle \varphi_N(x) \sum_{i=1}^{N-1} \varphi_i(x) \varphi_N(y) \sum_{i=1}^{N} \varphi_i^2(y) \rangle_c \]  

This result can be rewritten in terms of Feynman-graphs. We introduce the following notations

\[ l(m) := \frac{1}{V} \text{Tr} \log \Delta, \]
\[ K(m_1, m_2, m_3) := \text{\begin{center} \begin{tikzpicture}[scale=0.5]
\draw[thick] (0,0) circle (1);
\end{tikzpicture} \end{center}} \]

where \( m_1, m_2, m_3 \) are the masses of the propagators on the three internal lines of the diagram. It is clear, that

\[ \frac{\partial}{\partial m^2} I =: I'(m). \]  

We introduce the notations

\[ m_{G}^2 := m^2 + \frac{\lambda}{6} \Phi_0^2, \]
\[ m_{H}^2 := m^2 + \frac{\lambda}{2} \Phi_0^2, \]  

and get the following result:

\[ f = \frac{1}{2} (m^2 + \delta m^2) \Phi_0^2 + \frac{\lambda + \delta \lambda}{24} \Phi_0^4 + \frac{1}{2} (N-1) l(m_G) + \frac{1}{2} l(m_H) + \frac{1}{2} (\delta m^2 + \frac{\delta \lambda}{6}) (N-1) l'(m_G) \]
\[ + \frac{1}{2} (\delta m^2 + \frac{\delta \lambda}{2}) l'(m_H) + \frac{\lambda}{24} \left[ (N^2 - 1) l'(m_G)^2 + 2 (N-1) l'(m_G) l'(m_H) + 3 l'(m_H)^2 \right] \]
\[ - \frac{\lambda^2}{36} \Phi_0^2 \left[ 3 K(m_H, m_H, m_H) + (N-1) K(m_H, m_G, m_G) \right] \]  

The functions \( l \) and \( K \) can be expanded with respect to the masses, if no IR divergencies arise. This happens in our case, where the IR sensitive part is subtracted because no static mode is
allowed to propagate on the internal lines. Therefore, one is allowed to expand $I$ and $K$ in powers of $m^2/T^2$. If we don’t want to keep the operators suppressed by some inverse power of $T$, we can truncate the expansion after the first few terms (high temperature expansion):

$$I(m) = I_0 + I_1m^2 + I_2m^4 + \ldots,$$

$$I'(m) = I_1 + 2I_2m^2 + 3I_3m^4 + \ldots,$$

$$K(m_1, m_2, m_3) = K_0 + K_1 \frac{m_1^2 + m_2^2 + m_3^2}{3} + \ldots$$ (12)

(the function $K$ is symmetric in the three $m$’s). Relying on the divergence structure of the zero temperature model, and on dimensional analysis, the expected form of the coefficients in (12) is:

$$I_1 = I_1^0 \Lambda^2 + I_1^1 T \Lambda + I_1^2 T^2,$$

$$I_2 = I_2^0 \log \left( \frac{\Lambda}{T} \right) + I_2^0,$$

$$I_3 = I_3^0 \frac{T}{T^2},$$

$$K_0 = K_0^0 \Lambda^2 + K_0^1 T \Lambda + K_0^2 T^2 \log \left( \frac{\Lambda}{T} \right) + K_0^3 T^2,$$

$$K_1 = K_1^{2 \log} \left( \log \frac{\Lambda}{T} \right)^2 + K_1^{3 \log} \log \left( \frac{\Lambda}{T} \right) + K_1^0.$$ (13)

For future convenience let us introduce $\tilde{N} = (N + 2)/3$. At one loop level, if we choose the counterterms (in general renormalization scheme) as

$$\delta m_1 = \Delta m_1 - \tilde{N} \lambda \left( \frac{1}{2} I_1 \Lambda^2 + m^2 I_2 \log \left( \frac{\Lambda}{\mu} \right) \right),$$

$$\delta \lambda_1 = \Delta \lambda_1 - (\tilde{N} + 2) \lambda^2 I_2 \log \left( \frac{\Lambda}{\mu} \right),$$ (14)

where the terms $\Delta$ are finite (and specify the renormalisation scheme), we get for the effective potential:

$$f = \frac{1}{2} \Phi_n \left( m^2 + \Delta m_1^2 + \tilde{N} \lambda \left( \frac{1}{2} I_1 \Lambda^2 + \tilde{N} \lambda^2 I_2^0 \log \left( \frac{\Lambda}{\mu} \right) + \frac{1}{2} \lambda I_1^1 T \Lambda \right) + \frac{1}{4 \lambda} \phi \left( \lambda + \Delta \lambda_1 + (\tilde{N} + 2) \lambda^2 I_2^0 + (\tilde{N} + 2) \lambda^2 I_2 \log \left( \frac{\Lambda}{\mu} \right) \right) \right) + \ldots$$ (15)

At two loop level, with the same procedure we choose the counterterms as

$$\delta m_2 = \Delta m_2^2 - \tilde{N} \Lambda^2 \left( \frac{1}{2} \Delta \lambda_1 I_1 - \frac{1}{6} K_0^2 \Lambda^2 - \tilde{N} + \frac{2}{2} \lambda^2 I_2 \log \left( \frac{\Lambda}{\mu} \right) \right)$$

$$+ \tilde{N} \left( (\tilde{N} + 2) \lambda^2 I_2^0 \log \left( \frac{\Lambda}{\mu} \right) \right)^2 + \frac{1}{3} \lambda^2 I_2^0 \Lambda^2 \log \left( \frac{\Lambda}{\mu} \right) \log \left( \frac{\Lambda}{\mu} \right)$$

$$+ \tilde{N} \left( \frac{1}{6} K_0^1 \right) \Lambda^2 \log \left( \frac{\Lambda}{\mu} \right) \log \left( \frac{\Lambda}{\mu} \right)$$

$$- \tilde{N} \left( \Delta m_1^2 \lambda^2 \lambda + \Delta \lambda_1 \lambda^4 \log \left( \frac{\Lambda}{\mu} \right) \right)$$

$$+ \left( \frac{20 + 22 \tilde{N} + 3 \tilde{N}^2}{3} \right) \lambda^2 \left( \log \left( \frac{\Lambda}{\mu} \right) \right)^2$$

$$\delta \lambda_2 = \Delta \lambda_2 + \left( \frac{20 + 22 \tilde{N} + 3 \tilde{N}^2}{3} \right) \lambda^2 \left( \log \left( \frac{\Lambda}{\mu} \right) \right)^2$$
\[ +2 \frac{5\hat{N} + 4}{9} \left( 6(I_1^{log})^2 + K_1^{2,log} \right) \lambda^2 \log \left( \frac{\Lambda}{\mu} \right) \log \left( \frac{\mu}{T} \right) \]
\[ - \left( 2(\hat{N} + 2)\Delta \lambda I_1 I_2^{log} - 4 \frac{5\hat{N} + 4}{3} I_2^{log} I_1^{log} \lambda^3 - \frac{5\hat{N} + 4}{9} K_1^{2,log} \lambda^3 \right) \log \left( \frac{\Lambda}{\mu} \right). \] (16)

Using also the expression of the one loop counterterms (14) we obtain the 2-loop result valid in any general renormalization scheme. Some important systematics can be discovered in this expression (see the comments below eq.(22)), if one introduces the following shorthand notation:

\[ \hat{\lambda} = \lambda + \Delta \lambda + \hat{N} I_1^{log} \lambda^2 + (\hat{N} + 2) I_2^{log} \lambda^3 \log \left( \frac{\mu}{T} \right). \] (17)

Then the two-loop effective potential, renormalised from the point of view of four-dimensional ultraviolet behavior can be written with \( \mathcal{O}(\lambda^3) \) accuracy as

\[ f = \frac{1}{2} \Phi_0^2 \left[ (m^2 + \Delta m_2) \left( 1 + \hat{\lambda} \left( I_1^{log} + I_2^{log} \log \left( \frac{\mu}{T} \right) \right) \right) + \Delta m_2^2 + m^2 \hat{N} \hat{\lambda}^2 \left( \frac{3N}{2} I_1 I_3 - \frac{1}{6} K_1^{log} \right) \]
\[ - \left( 2 I_2^{log} I_2^{log} + \frac{1}{6} K_1^{log} \right) \log \left( \frac{\mu}{T} \right) - \left( 2(I_2^{log})^2 + \frac{1}{6} K_1^{log} \right) \left( \log \frac{\mu}{T} \right)^2 \]
\[ + \hat{N} T^2 \hat{\lambda} \left( \frac{1}{2} I_1^{log} - \frac{1}{6} K_1^{log} \hat{\lambda} - \hat{N} \hat{\lambda} \left( I_1^{log} I_2^{log} + \frac{1}{6} K_1^{log} \right) \log \left( \frac{\Lambda}{T} \right) \right) \]
\[ + \hat{N} \hat{\lambda} T \left( \frac{1}{2} I_1^{log} - \frac{1}{6} K_1^{log} \hat{\lambda} + \hat{\lambda} I_1^{log} \log \left( \frac{\Lambda}{T} \right) \right) \]
\[ + \frac{1}{24} \Phi_0^4 \left[ \hat{\lambda}^2 + 2 \hat{\lambda} + \lambda^2 + 2 \lambda \Delta \lambda - 2 \lambda \Delta \lambda_1 \left( \hat{N} I_1^{log} + (\hat{N} + 2) I_2^{log} \log \left( \frac{\mu}{T} \right) \right) + \frac{3\hat{N}^2 - 10\hat{N} + 4}{3} (I_2^{log})^2 \lambda^3 \]
\[ + \frac{3}{2} \hat{N} (\hat{N} + 2) I_1 I_3 \lambda^2 - \frac{5\hat{N} + 4}{9} K_1^{log} \lambda^3 + \hat{\lambda}^3 \log \left( \frac{\mu}{T} \right) \left( \frac{2}{3} \hat{N}^2 - 4 \hat{N} - 8 I_2^{log} I_2^{log} - \frac{5\hat{N} + 4}{9} K_1^{2,log} \right) \]
\[ + \hat{\lambda}^3 \left( \log \frac{\mu}{T} \right)^2 \right. \]
\[ \left. \left( \frac{3\hat{N}^2 + 2\hat{N} + 4}{3} (I_2^{log})^2 - \frac{5\hat{N} + 4}{9} K_1^{2,log} \right) \right] \] (18)

If we choose \( \mu = T \), to avoid large logarithms, and a special scheme, eg. the following (Weinberg-type) one

\[ \frac{\partial^2 f}{\partial \Phi_0^2} \bigg|_{\mu = T, \Phi_0 = 0} := m^2, \]
\[ \frac{\partial^3 f}{\partial \Phi_0^4} \bigg|_{\mu = T, \Phi_0 = 0} := \lambda, \] (19)

we can obtain a considerably simpler result. At one loop level

\[ f = \frac{1}{2} \Phi_0^2 \left( m^2 + \frac{1}{2} \hat{N} \lambda I_1 T^2 + \frac{1}{2} \hat{N} \lambda I_1 T \Lambda \right) + \frac{1}{24} \lambda \Phi_0^4 + \ldots \] (20)

which corresponds to the following choice of the finite parts:

---

1 The correct way would be to choose \( \mu \) to be equal to the scale where the parameters \( \lambda \) and \( m \) are fixed (eg. the mass of the Higgs boson in the present case). The large logarithms can be summed with the help of RG. This would give the above result with logarithmically T-dependent \( \lambda \) and \( m \).
\[ \Delta m_i^2 = -\hat{N}\lambda_i^0 m_i^2, \]
\[ \Delta \lambda_i = -(\hat{N} + 2)\lambda^3 I_i^0. \]  

(21)

One might notice, that at 1-loop the coefficients of \( T^2 \) and of \( \Delta T \) do not depend on the details, how the regularising cut-off is imposed.

At two loop level (without explicitly giving the complicated expressions for \( \Delta m_i^2 \) and \( \Delta \lambda_i \))

\[
\begin{align*}
  f &= \frac{1}{2} \Phi_i^2 \left[ m_i^2 + \hat{N} T^2 \left( \frac{1}{2} \lambda_i^0 I_i^0 - \lambda_i^2 \left( \frac{1}{6} K_1^0 + \frac{1}{6} K_1^0 I_i^0 + \frac{1}{6} T_1^0 I_i^0 \log \left( \frac{A}{T} \right) \right) \right) \right. \\
  &\quad + \hat{N} \Delta T \left( \frac{1}{2} \lambda_i^0 I_i^0 - \lambda_i^2 \left( I_i^0 I_i^0 + \frac{1}{6} I_i^0 I_i^0 I_i^0 \log \left( \frac{A}{T} \right) \right) \right] + \frac{1}{24} \lambda_i^0 \Phi_i^0. 
\end{align*}
\]

(22)

It can be seen from formula (18), that the coefficients of the terms \( \Phi_i^0 T^2 \) and \( \Phi_i^0 \Delta T \) at \( \mathcal{O}(\lambda^3) \) depend on the specific implementation of the cut-off (regularisation dependence), through the coefficients \( K_1^0 \) and \( K_2^0 \). Therefore for computing physical quantities it is necessary to apply a common unique regularization procedure, otherwise the 3D linear divergence would not be cancelled and some \( \mathcal{O}(\lambda^3) \) finite contributions would be unreliable. The coefficients of other induced divergencies of the type \( T^2 \log(A/T) \), \( \Delta T \log(A/T) \) do not depend on the specific implementation of the cut-off in multi-loop integrals.

The only remaining task is the computation of the constants defined in (13). The \( I_i \) quantities come from the tadpole graph:

\[
I_1 + 2I_2 M^3 + 3I_3 M^4 + \cdots = T \sum_{n \neq 0} \int \frac{d^3 p}{(2\pi)^3 \omega_n^2 + p^2 + M^2}. 
\]

(23)

The \( K_i \) constants are determined by the setting sun diagram:

\[
K_0 + K_1 M^2 + K_2 M^4 + \cdots = T^3 \sum_{n \neq 0} \int \frac{d^3 p d^3 q d^3 k}{(2\pi)^9} \frac{\beta(2\pi)^3 \delta_{n+m+l,i} \delta^{(3)}(p + q + k)}{(\omega_n^2 + E_p^2) (\omega_m^2 + E_q^2) (\omega_l^2 + E_k^2)},
\]

(24)

where \( E_k^2 = p^2 + M^2 \). The result of the integrations (see Appendix A and B) are:

\[
I_1 = \frac{\lambda^2}{8\pi^2} \left( \frac{\Delta T}{2\pi^2} + \frac{T^2}{12} \right), \\
I_2 = -\frac{\lambda^2}{16\pi^2} \log \left( \frac{\Lambda}{T} \right) + \frac{\lambda^2}{16\pi^2} \log(2\pi) + 2 \gamma_E, \\
I_3 = -\frac{\zeta(3)}{192\pi^4 T^2}, \\
K_0 = 0.000104133 \lambda^2 - 0.0029850437 \lambda T + \frac{5}{32\pi^2} T^2 \log \left( \frac{\Lambda}{T} \right) - 0.0152887686 T^2, \\
K_1 = -\frac{3}{128\pi^4} \left( \log \left( \frac{\Lambda}{T} \right) \right)^2 + 0.00108787 \log \left( \frac{\Lambda}{T} \right) + \text{const.} 
\]

(25)

The coefficients written in decimal form are results of numerical integrations, other coefficients were calculated analytically. The former depend on the special implementation of the cut-off procedure the latter do not.

With these values we obtain the following result for the 3 dimensional effective potential in Weinberg-type renormalization scheme:

\[
\begin{align*}
  f &= \frac{1}{2} \Phi_i^2 \left[ m_i^2 + \hat{N} T^2 \left( \frac{\lambda^2}{24} - 0.0013551443 \lambda^2 - \frac{\lambda^2}{48\pi^2} \log \left( \frac{\Lambda}{T} \right) \right) \right. \\
  &\quad + \hat{N} \Delta T \left( -\frac{\lambda^2}{4\pi^2} + 0.0011227544 \lambda^2 - \frac{\lambda^2}{32\pi^2} \log \left( \frac{\Lambda}{T} \right) \right] + \frac{\lambda^2}{24} \Phi_i^0.
\end{align*}
\]

(26)
The one loop result is well known [2,3], the two loop result is new. From this expression one can read off the induced 3 dimensional mass counterterm:

$$\delta m^2_{\text{ind}} = -N \left( \frac{\lambda}{4\pi^2}(1 - 0.048277409 \lambda)\Delta T + \frac{\lambda^2}{2\pi^2} \Delta T \log \left( \frac{\Lambda}{T} \right) + \frac{\lambda^2}{48\pi^2} T^2 \log \left( \frac{\Lambda}{T} \right) \right) \quad (27)$$

No new divergence related to $\lambda$ appears, supporting the superrenormalizable nature of the 3 dimensional theory. We have to compare this induced counterterm to the counterterm required for the 3 dimensional theory of the same form as (3) expected on the basis of superrenormalizability. The eventual cancellation is a precondition for the consistent representation of the 4D finite $T$ theory by this simplest minded 3D form. But as we shall see this hope doesn’t come true.

III. 2-LOOP EFFECTIVE POTENTIAL OF THE 3D HIGGS-MODEL

The result (26) of the previous section suggests a 3D effective representation of the original theory which could be of the form (3). This is again a local scalar $O(N)$ model, just now with temperature dependent mass parameter. The renormalization of the theory, however, on the 4D level has been accomplished already. The counterterms of the “embedded” 3D theory were generated by the reduction. A crucial check of consistency on the suggested model is the cancellation of its divergencies against the induced counterterms.

In this section we investigate the effective potential of the three dimensional theory defined by the action (3) at two loop level, when the bare mass is temperature dependent, just as described by (26). The results are known for even more complex theories [2], so this section will be just a short summary of well-known facts.

In three dimensions the theory is superrenormalizable, we expect divergent counterterm only to the mass parameter, and only up to two loop level. Otherwise the graphs are the same as in the 4 dimensional case (see (8)), but the integrals are 3 dimensional. The result of the integration is (cf. (11)):

$$f = \frac{1}{2}(m^2 + \delta m^2)\Phi^2_\Phi + \frac{\lambda}{24} \Phi_\phi^4 + \frac{1}{2}(N - 1)I(m_G) + \frac{1}{2}I(m_H) + \frac{1}{2} \delta m^2(N - 1)I'(m_G)$$

$$+ \frac{1}{2} \delta m^2 I'(m_H) + \frac{\lambda}{24} \left[ (N^2 - 1)I'(m_G)^2 + 2(N - 1)I'(m_G)I'(m_H) + 3I'(m_H)^2 \right]$$

$$- \frac{\lambda^2}{36} \Phi_\phi^2 \left[ 3K(m_H, m_H, m_H) + (N - 1)K(m_H, m_G, m_G) \right]. \quad (28)$$

On dimensional reasons, and using the expected form of the divergencies we get for the relevant parts of the functions $I$ and $K$ (Don’t forget that we deal with three-dimensional theory!):

$$I(m) = 2J_1 \lambda m^2 + 2J_0 m^3,$$

$$K(m, m, M) = L_{\text{log}} \log \frac{\Lambda^2}{\mu^2} + L_{\text{log}} - 2L_{\text{log}} \log \frac{2m + M}{\mu}. \quad (29)$$

Let us choose the counterterm

$$\delta m^2 = \Delta m^2 - \frac{1}{3}(N + 2)J_1 \lambda \Lambda, \quad (30)$$

what leads for the one loop part of (28) to the result:

$$\psi_{\text{eff}} = \frac{1}{2}(m^2 + \Delta m^2)\Phi^2_\Phi + \frac{\lambda}{24} \Phi_\phi^4 + J_0 \left( m^2 + \frac{\lambda}{2} \Phi_\phi^2 \right)^{3/2} + (N - 1)J_0 \left( m^2 + \frac{\lambda}{6} \Phi_\phi^2 \right)^{3/2}. \quad (31)$$

Using (30) combined with the two loop counterterm
\[ \delta m_i^2 = \Delta m_i^2 + \frac{\lambda^2}{18} (N + 2) L_{\text{log}} \log \frac{\Lambda^2}{\mu^2}, \]  

the two-loop effective potential (28) becomes:

\[
eff_{\text{2-loop}} = \frac{1}{2} \left( m^2 + \Delta m_i^2 + \Delta m_\text{eff}^2 - \frac{\lambda^2}{18} (N + 2) L_0 + \frac{\lambda^2}{8} (N^2 + 8) J_0^2 \right) \Phi_0^2 + \frac{\lambda}{24} \Phi_0^4 \\
+ J_0 \left( m^2 + \frac{\lambda}{2} \Phi_0^2 \right)^{3/2} + (N - 1) J_0 \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{3/2} + \frac{3}{2} \Delta m_i^2 J_0 \left( m^2 + \frac{\lambda}{2} \Phi_0^2 \right)^{1/2} \\
+ \frac{3}{4} (N - 1) \Delta m_\text{eff}^2 J_0 \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{1/2} \\
+ \frac{3}{4} (N - 1) \Delta m_\text{eff}^2 J_0 \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{1/2} \\
+ \frac{\lambda^2}{18} \Phi_0^2 L_{\text{log}} \left[ 3 \log \frac{3m_H}{\mu} + (N - 1) \log \frac{2m_G + m_H}{\mu} \right]. \tag{33}\]

We can choose, in particular, the scheme

\[
\Delta m_i^2 = 0, \\
\Delta m_\text{eff}^2 = \lambda^2 \left( \frac{N + 2}{18} L_0 - \frac{N^2 + 8}{8} J_0^2 \right), \tag{34}\]

which gives for the finite potential

\[
eff_{\text{2-loop}} = \frac{1}{2} m^2 \Phi_0^2 + \frac{\lambda}{24} \Phi_0^4 + J_0 \left( m^2 + \frac{\lambda}{2} \Phi_0^2 \right)^{3/2} + (N - 1) J_0 \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{3/2} \\
+ \frac{3}{4} (N - 1) \lambda J_0^2 \left( m^2 + \frac{\lambda}{2} \Phi_0^2 \right)^{1/2} \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{1/2} \\
+ \frac{\lambda^2}{18} \Phi_0^2 L_{\text{log}} \left[ 3 \log \frac{3m_H}{\mu} + (N - 1) \log \frac{2m_G + m_H}{\mu} \right]. \tag{35}\]

Explicitly performing the 3D tadpole and setting sun calculations (see Appendix A), the values of the constants are:

\[
J_1 = \frac{1}{4 \pi^2}, \\
J_0 = -\frac{1}{12 \pi}, \\
L_{\text{log}} = \frac{1}{32 \pi^2}, \\
L_0 = 6.70322 \cdot 10^{-3}. \tag{36}\]

With these values

\[
eff_{\text{2-loop}} = \frac{1}{2} m^2 \Phi_0^2 + \frac{\lambda}{24} \Phi_0^4 - \frac{1}{12 \pi} \left[ \left( m^2 + \frac{\lambda}{2} \Phi_0^2 \right)^{3/2} + (N - 1) \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{3/2} \right] \\
+ \frac{N - 1}{192 \pi^2} \lambda \left( m^2 + \frac{\lambda}{2} \Phi_0^2 \right)^{1/2} \left( m^2 + \frac{\lambda}{6} \Phi_0^2 \right)^{1/2} \\
+ \frac{\lambda^2}{576 \pi^2} \Phi_0^2 \left[ 3 \log \frac{3m_H}{\mu} + (N - 1) \log \frac{2m_G + m_H}{\mu} \right]. \tag{37}\]

The divergence structure of the 3D theory can be read off from (30) and (32):
where the primed integral is an abbreviation for the cut-off integration with respect to $\nu$.

With Feynman-parametrization one rewrites it as

$$I_\Lambda(k) = \int_\nu^\Lambda dy \int_p^\Lambda \frac{1}{(p^2 + (2\pi nT)^2 + k^2 y(1-y))^2},$$

(41)

This coincides with the result of [2], if we change $\lambda \to \lambda/6$, which corresponds to the choice of the coefficient of the scalar selfinteraction to be $\lambda/4$.

The mismatch between the induced counterterms (27) and the three-dimensional divergencies (38) is by now quite obvious. There are some divergences present in (27), which don't appear in (38) (eg. $\sim \Delta T \log(\Lambda/T)$), but even the operators present in both cases have different coefficients (eg. $\sim T^2 \log(\Lambda/T)$). This means, that in a regularisation scheme, where power-like divergences are simply subtracted (see [6]), one still could recognize the need for supplementary terms in the action of the effective theory. The divergence structure of the reduced theory is richer anyhow than we could expect from the simplest (superrenormalisable) representation of the theory, eg. the scale dependence (beta-function) of the faithfully reduced theory is different. For the resolution of this puzzle, and for reconstructing the correct scale dependence we have to introduce some new operators into the action of the effective model. This will be proposed in the following section.

IV. FILLING THE GAP: THE ONE LOOP FOUR POINT FUNCTION

To cure the disease of the naively constructed 3D model detected in the previous section we have to add some new operators to the basic (3) action. The arguments in the Introduction suggest to investigate the four-point function (2) and substitute it by a new vertex. Since, already all momentum-independent operators with couplings proportional to non-negative powers of $T$ are taken into account, the new operators have to be momentum-dependent. From the point of view of the full theory, as we argued in the Introduction, the new term should represent the diagrams with mixed static and non-static internal lines. Such operators can be generated only radiatively, namely in the process of the integration over the nonstatic modes at 1-loop. In a subsequent 3D 1-loop computation of its contribution to the effective potential, 2-loop divergencies yet missing in the comparison of the induced counterterms with the divergencies of the 3D local effective theory should emerge.

We are going to evaluate the 1-loop correction to the 4-point function with static external lines of non-zero spatial momentum. When contracting two of its external lines as part of the solution of the effective model one recognizes the "mixed" static-nonstatic sunset contribution to the static 2-point function.

The relevant integral ($k = p_1 + p_2$) is seen to be:

$$I_\Lambda(k) = \int_p^\Lambda \frac{1}{p^2 + (2\pi nT)^2} \frac{1}{(p + k)^2 + (2\pi nT)^2},$$

(39)

Fig. 1

where the primed integral is an abbreviation for the cut-off integration with respect to $p$

$$\int_p^\Lambda = T \sum_{n \neq 0} \int \frac{d^3p}{(2\pi)^3}.$$ 

(40)

With Feynman-parametrization one rewrites it as:

$$I_\Lambda(k) = \int_{\frac{1}{y}}^1 dy \int_p^\Lambda \frac{1}{(p^2 + (2\pi nT)^2 + k^2 y(1-y))^2},$$

(41)
If the external lines are labelled by $i$, $j$, $k$, $l$ respectively, the tensorial structure of the graph in Fig. 1. is proportional to

$$(N + 4)\delta_{ij}\delta_{kl} + 2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Having this tensorial decomposition it is enough to work with homogeneous background as in the previous sections.

The result of the integration is known for $k^2 = 0$:

$$I_A(k = 0) = \frac{D_0 - 1}{8\pi^2} = \frac{1}{8\pi^2} \left( \log \frac{\Lambda}{T} - \log 2\pi + \gamma_E - 1 \right).$$

In the other limit, $k^2 \gg T^2$, one can look for an expansion in positive powers of $T$ (there is no other scale in the integral). Using the Euler-Maclaurin resummation formula:

$$\sum_{i=\pi}^{n} f(i) = \int_{\pi}^{n} dx f(x) + \frac{1}{2}[f(0) + f(n)] + \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} [f^{(2k-1)}(n) - f^{(2k-1)}(0)],$$

(where $B_{2k}$'s are the Bernoulli numbers) we get the following result:

$$I_A(k \gg T) = \frac{1}{8\pi^2} \log \frac{2\Lambda}{k} - \frac{1}{8\pi^2} F \left( \frac{k}{\Lambda} \right) - \frac{1}{16\pi^2} \sum_{j=1}^{\infty} \frac{(-1)^j (j-1)! B_{2j}}{(2j)!} \left( \frac{32\pi^2 T^2}{k^2} \right)^j,$$

where

$$F(x) = \left( \frac{1}{x} - 1 \right) \ln \left( \frac{1 - x}{2} \right) + 1.$$  

Terms proportional $1/k^n$ ($n \geq 2$) give constant contribution in three dimensions – these are the finite corrections to the four point function. The first three terms, however, might lead to divergencies when in a subsequent 1-loop calculation two of the external lines of Fig.1 are contracted. If we want to separate only the divergent contribution to the effective potential upon contraction of two legs of Fig.1, we can interpolate between the relevant pieces of (43) and (45) with help of the following expression:

$$I_A(k) \approx \frac{1}{8\pi^2} \left[ \log \frac{\Lambda}{CT} - \Omega(k) \right],$$

where $C = 2\pi e^{1-\gamma_E}$ and

$$\Omega(k) = \frac{\pi^2 T k}{T^2 + k^2} + \frac{1}{2} \log \left( 1 + \frac{k^2}{4\pi^2 T^2} \right) + F \left( \frac{k}{\Lambda} \right).$$

In section III, we have obtained for the one loop level momentum-independent radiative correction (at $k = 0$) to the potential energy density from Fig.1:

$$\Delta V_{3D} = -\frac{\tilde{N}}{3} + \frac{2}{3} \frac{\lambda^2}{128\pi^2} \left[ \log \frac{\Lambda}{T} - \log 2\pi + \gamma_E - 1 \right] \Phi^4_0.$$  

This will now be modified for x-dependent fields $\Phi_0$ by adding the operators:

$$\Delta L_{3D} = \frac{\tilde{N}}{3} + 2 \frac{\lambda^2}{128\pi^2} \Phi_0^4 \Omega(\partial) \Phi^2_0.$$  

The Fourier-transform of (50) promptly reproduces the k-dependent part of (47). For the modification of the original theory (3) we have to project back (50) the $O(N)$ tensor-structure arising from (42). The corresponding two possible $O(N)$ symmetric operators
The leading divergencies are provided by

\[ \begin{align*}
O_1 &= \left( \sum_{i=1}^{N} \varphi_i^2 \right) \Omega(i\partial) \left( \sum_{i=1}^{N} \varphi_i^2 \right), \\
O_2 &= \sum_{i,j=1}^{N} \varphi_i \varphi_j \Omega(i\partial) \varphi_i \varphi_j
\end{align*} \tag{51} \]

have the relative weight \( N + 4 : 4 \), so for a general \( \phi \)-configuration we have to add the following operators to the basic action (3):

\[ \Delta \mathcal{L}_{3D} = \frac{\lambda^2}{128 \pi^2} \left( \frac{N + 4}{9} O_1 + \frac{4}{9} O_2 \right). \tag{52} \]

This is the central proposition of this paper.

These operators are generated at one-loop level, so for two loop calculations of the effective potential they must be treated at one loop in the three dimensional theory. Whence the one loop contributions of different modes are independent we can calculate the extra divergences produced by these operators from the Lagrangian

\[ \mathcal{L}_{3D} = \sum_{i=1}^{N-1} \frac{1}{2} \left( (\partial \varphi_i)^2 + m_T^2 \varphi_i^2 \right) + \Delta \mathcal{L}_{3D}. \tag{53} \]

Shifting the fields \( \varphi_i \to \varphi_i (i \neq N) \) and \( \varphi_N \to \varphi_N + \Phi_0 \) respectively, and using the fact \( \Omega(k = 0) = 0 \), we get for the quadratic terms:

\[ \mathcal{L}_{3D}^{(2)} = \sum_{i=1}^{N-1} \frac{1}{2} \left( (\partial \varphi_i)^2 + m_T^2 \varphi_i^2 + \frac{\lambda^2 \Phi_0^2}{72 \pi^2} \varphi_i \Omega \right) + \frac{1}{2} \left( (\partial \varphi_N)^2 + m_T^2 \varphi_N^2 + \frac{(N + 8) \lambda^2 \Phi_0^2}{144 \pi^2} \varphi_N \Omega \right). \tag{54} \]

The effective potential using the above expression is:

\[ \begin{align*}
v_{eff} &= \frac{1}{2} m_T^2 \Phi_0^2 + (N - 1) \frac{1}{2} \int_{\Lambda} \frac{d^3k}{(2 \pi)^3} \ln \left[ k^2 + m_T^2 + \frac{\lambda^2 \Phi_0^2}{72 \pi^2} \Omega(k) \right] \\
&\quad + \frac{1}{2} \int_{\Lambda} \frac{d^3k}{(2 \pi)^3} \ln \left[ k^2 + m_T^2 + \frac{(N + 8) \lambda^2 \Phi_0^2}{144 \pi^2} \Omega(k) \right]. \tag{55} \end{align*} \]

After expanding it with respect to \( \Phi_0^2 \) the contribution to the quadratic term is:

\[ \begin{align*}
v_{eff}^{(2)} &= \frac{1}{2} m_T^2 \Phi_0^2 + \left( \frac{N + 2}{3} \right) \frac{\lambda^2 \Phi_0^2}{64 \pi^4} \int_{0}^{\Lambda} dk \frac{k^2}{k^2 + m_T^2} \Omega(k). \tag{56} \end{align*} \]

The leading divergencies are provided by

\[ \begin{align*}
v_{eff}^{(2) \text{div}} &= \left( \frac{N + 2}{3} \right) \frac{\lambda^2 \Phi_0^2}{64 \pi^4} \int_{0}^{\Lambda} dk \left[ \log \left( \frac{k}{T} \right) + \frac{\pi^2 T k^2}{T^2 + k^2} + F \left( \frac{k}{T} \right) \right], \tag{57} \end{align*} \]

what after performing the integrations simplifies to

\[ \begin{align*}
v_{eff}^{(2) \text{div}} &= \left( \frac{N + 2}{3} \right) \frac{\lambda^2 \Phi_0^2}{64 \pi^4} \left[ \Lambda \log \Lambda + \pi^2 T \log \Lambda - \Lambda \left( \ln 4 \pi + 1 - \gamma_E + \int_{0}^{\frac{2}{x}} dx \frac{2 - x}{x} \ln \frac{2 - x}{2} \right) \right]. \tag{58} \end{align*} \]
The divergencies of the three dimensional model up to two loop level have to be cancelled against the induced counterterm $\delta m^2_{\text{ind}}$ (see (27)). There are divergencies coming from the local terms (38) and others coming from the nonlocal terms (58). Indeed, summing these terms all divergencies, including the linear ones are cancelled exactly. This means, that for the proper treatment of the divergencies (or for the proper treatment of the scale dependence of the reduced theory) we must include the new operators (52) into the effective description. These are nonlocal operators, the nonlocality is of the order $O(T^{-1})$ in the coordinate space. On scales larger than $O(T^{-1})$ these can be omitted. So a coarse grained local action with grain size $O(T^{-1})$ may be a good choice to examine the whole theory [7] on this scale the theory forgets of its 4 dimensional origin. In the course of coming down from scale $\Lambda$ to scale $O(T)$ in the momentum space the nonlocal operators modify the coupling constants of the local ones. The actual magnitude of this modification can be estimated only after doing this calculation, for instance, by matching some n-point functions. This is, however, the task of future calculations.

V. CONCLUSIONS

In this paper we advocated a non-local effective 3D theory, equivalent to the finite temperature $O(N)$ symmetric scalar theory on the 2-loop level. The proposed theory has the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}[(\partial_i \phi_\alpha)^2 + m^2(T)\phi_\alpha^2] + \frac{\lambda_3}{24}\phi_\alpha^4 + \frac{\lambda_2}{2\cdot 128\pi^2}[N + 4 + \frac{4}{9}\phi_\beta^2\Omega(i\partial)\phi_\beta + \frac{4}{9}\phi_\beta\phi_\gamma\Omega(i\partial)\phi_\beta\phi_\gamma],$$

with $m^2(T)$ given by the coefficient of $\Phi^2/2$ in (26), and $\Omega(i\partial)$ defined in (48). Now with a further step one can define a local approximation to this theory, when all static degrees of freedom are integrated out with $k >> T$. A local cut-off theory with $\lambda_3 = \kappa T, \kappa \sim 1 - 5$ will be arrived at what is very close in spirit to the effective theory approach of [6]. The simplest version of this theory has approximately the form

$$\mathcal{L}_{\text{eff}, \text{eff}} = \frac{1}{2}[(\partial_i \phi_\alpha)^2 + (M^2(T) + \Sigma)\phi_\alpha^2] + \frac{\lambda_3 + \Delta\lambda_3}{24}\phi_\alpha^4,$$

where $Z, \Sigma, \Delta\lambda_3$ should be found by matching some important quantities calculated from the respective theories (59) and (60). We propose to calculate the couplings of (60) by matching the low-k behaviors of the 2-point functions and the self-coupling (4-point function). The matching is achieved the easiest way in the symmetric (high-T) regime, where $m^2(T) > 0$.

In the non-local theory our construction ensures the finiteness of the 2-point and 4-point functions. In the cut-off theory the dependence on the cut-off is not absorbed by anything, but is expected to be very weak if $\kappa$ is varied around unity. It will be very interesting to see the relation of the result of this approach to that of [6].

The practical interest of this second approximation is its easy discretisation into a lattice theory. However, since it is a cut-off theory, it should not be understood as a continuum theory, but it is better to study it with finite lattice constant of $O(T^{-1})$ (coarse grained lattice). This means that the thermodynamics of this model should be studied with a dimensionless temperature $\Theta = aT$ varying around unity. This circumstance seems to be very advantageous in view of the very weak phase transition signals, experienced in the small $\Theta$ region in some effective models [5].

The analysis of the present paper should be carried over to gauged Higgs theories. The vector nature of the gauge potential does not present serious difficulty. More important is the task of analysing another class of diagrams contributing to the non-local behaviour of the effective theory, namely the four-leg box diagrams with non-vanishing external spatial momenta.

ACKNOWLEDGEMENTS

The author would like to express his special gratitude to András Patkós for his continuous advice and help in writing this paper. He would also like to thank Z. Fodor, K. Kajantie and M. Laine for valuable discussions and comments.
APPENDIX A: COMPUTING THREEFOLD CUT-OFF INTEGRALS IN 3 DIMENSIONS

We shortly discuss the method of performing 3D integrals relevant to the evaluation of the setting sun diagram contribution both in 3 and 4 dimensions. Here we use the regularisation convention that momenta on any internal line should not exceed the value of the cut-off, separately. (Another natural procedure would be to cut-off the loop momenta.) The integral to be performed is:

\[ I = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \Theta(\Lambda - |p|)\Theta(\Lambda - |q|)\Theta(\Lambda - |k|)(2\pi)^3\delta^{(3)}(p + q + k) f(p^2, q^2, k^2) \quad (A1) \]

Let us perform first the \( k \)-integral with help of the delta-function – then instead of \( k^2 \) we can write \((p + q)^2\). The remaining integral depends on \( p^2, q^2 \) and \( x^2 = (pq)^2/(p^2q^2) \) \((-1 < x \leq 1); \)

\( k^2 = p^2 + q^2 + 2|pq|/x \). One can replace \( x \) by the variable \( k \) (its allowed range of variation is \(|p - q| < k < p + q\)), then the integral will be symmetric in \( p, q, k \), with the domain of integration

\[ \Delta := \{k < p + q \quad p < q + k \quad q < k + p\}. \quad (A2) \]

Our expression after the angular integrations is of the form

\[ I = \frac{1}{8\pi^4} \int_\Delta \int \int dpdqdk \Theta(\Lambda - p)\Theta(\Lambda - q)\Theta(\Lambda - k) pqk f(p^2, q^2, k^2). \quad (A3) \]

or writing out the limits of integration (cf. Fig.2)

\[ I = \frac{1}{8\pi^4} \left[ \int_0^\Lambda \int_0^\Lambda \int_0^{p+q} dpdqdk - \int_0^\Lambda \int_0^{p+q} dpdqdk \right] pqk f(p^2, q^2, k^2) \quad (A4) \]

Often happens, that the integrand \( f \) is symmetric to the interchange of \( p \) and \( q \). In this case we can write a simpler formula:

\[ I = \frac{1}{4\pi^4} \left[ \int_0^\Lambda \int_0^\Lambda \int_0^{p+q} dpdqdk - \int_0^\Lambda \int_0^{p+q} dpdqdk \right] pqk f(p^2, q^2, k^2) \quad (A5) \]
The integral we have to evaluate is (24)

\[
K(m_p, m_q, m_k) = \int \frac{d^3p d^3q d^3k}{(2\pi)^6} T^3 \sum_{n,m,l\neq 0} \beta \delta_{n+m+l}(2\pi)^3 \delta(p + q + k) \frac{\beta \delta_{n+m+l}(2\pi)^3 \delta(p + q + k)}{\omega_n^2 + \omega_m^2 + \omega_l^2 + E_n^2 + E_m^2 + E_l^2},
\]

(\text{B1})

where \(E_n^2 = p^2 + m_n^2\). The summations can be performed with help of the "Saclay-method" [14].

We get the expression:

\[
K(m_p, m_q, m_k) = \int \frac{d^3p d^3q d^3k}{(2\pi)^6} \delta(p + q + k) D(E_p, E_q, E_k),
\]

(\text{B2})

where

\[
D(E_p, E_q, E_k) = \frac{2T^2}{E_p^2 E_q^2 E_k^2} + \frac{1}{4 E_p E_q E_k (E_p + E_q + E_k)}
\]

\[
- \frac{T}{E_p E_q E_k} \left[ D_1(E_p, E_q, E_k) + D_1(E_q, E_p, E_k) + D_1(E_k, E_q, E_p) \right]
\]

\[
+ \frac{1}{4 E_p E_q E_k} \left[ D_2(E_p, E_q, E_k) + D_2(E_q, E_p, E_k) + D_2(E_k, E_q, E_p) \right]
\]

\[
+ \frac{1}{4 E_p E_q E_k} \left[ D_3(E_p, E_q, E_k) + D_3(E_q, E_p, E_k) + D_3(E_k, E_q, E_p) \right],
\]

(\text{B3})

and

\[
D_1(E_p, E_q, E_k) = \frac{n_{E_p}}{2E_p} \left( \frac{1}{E_p + E_k} - \frac{1}{E_p - E_k} \right) + (q \rightarrow k) + \frac{1}{2E_q (E_q + E_k)}
\]

\[
D_2(E_p, E_q, E_k) = n_{E_q} \left( \frac{1}{E_p + E_q + E_k} - \frac{1}{E_p - E_q - E_k} \right),
\]

\[
D_3(E_p, E_q, E_k) = n_{E_q} n_{E_k} \left( \frac{1}{E_p + E_q + E_k} + \frac{1}{E_p - E_q - E_k} + \frac{1}{E_p + E_q - E_k} + \frac{1}{E_p - E_q + E_k} \right)
\]

(\text{B4})

and \(n_{E_p}\) is the Bose-Einstein function \((n_{E_p} = (e^{\beta E_p} - 1)^{-1})\). The parts depending on \(T\) only in the form \(n_{E_p}\) agree with the expression of Parwani [15].

In the course of the integration of each single term in (B3) UV and various IR divergence problems arise. In order to control them the best is to work from the start with a regularized propagator in the mixed \((\tau, p)\) representation [14]:

\[
\Delta_{\tau = \mu}(\tau, E_p) = \Delta(\tau, E_p + \varepsilon) \Theta(\Lambda - |\mathcal{V}|),
\]

(\text{B5})

where

\[
\Delta(\tau, X) = \frac{n_{X}}{2X} \left( \epsilon^{(\beta - \tau)X} + \epsilon^{\tau X} \right).
\]

(\text{B6})

With this propagator, using the general scheme to perform threefold integrals in 3 dimensions (Appendix A) we get the following expression for (B2):

\[
K_{\text{reg}} = \frac{1}{8\pi^4} \int \delta d\delta d\delta \Theta(\Lambda - p) \Theta(\Lambda - q) \Theta(\Lambda - k) D(E_p + \varepsilon, E_q + \varepsilon, E_k + \varepsilon).
\]

(\text{B7})
The full expression is IR safe, since we omitted all static parts. Therefore at the end of the calculations we may perform the $\varepsilon \to 0$ limit.

If we rescale all the variables by $T$, the full integral $K \sim T^3$. At high temperatures it is possible and convenient to expand the integral with respect to the masses:

$$K_{\text{reg}}(m_p, m_q, m_k) = T^2 K_{\text{reg}}(0,0,0) + m_p^2 \frac{\partial}{\partial m_p} K_{\text{reg}}(m_p, m_q, m_k)\bigg|_{m_p=0} + (p \leftrightarrow q) + (p \leftrightarrow k) + \ldots$$

(B8)

The omitted parts are proportional to $m^4/T^2$.

The regularization of the propagator regularizes all the arising divergencies, which makes possible the correct derivation of the finite parts of the integrals. This is important in the case of $K_{\text{reg}}(0,0,0)$, where the finite part is multiplied by $T^3$, which means, that it cannot be absorbed by counterterms in any T-independent renormalization scheme.

For the calculation of $K_{\text{reg}}(0,0,0)$ I followed the following general ideas. First one has to separate the (UV/IR) divergent parts. They should have the simplest possible form allowing analytic evaluation of the coefficients of the divergencies. Substracting the divergent pieces we can perform the $\Lambda \to \infty$ and the $\varepsilon \to 0$ limits respectively. If there is some Bose-Einstein functions $(n_E)$ involved in the integral containing $\Lambda$ or $\varepsilon$, we can use the relations:

$$\lim_{\varepsilon \to 0} \int dx \varepsilon n(\varepsilon x) f(x) = \int dx \frac{f(x)}{x} \quad \text{if it exists},$$

$$\lim_{\Lambda \to \infty} \int dx \Lambda^3 n_{\Lambda x} f(x) = \frac{\pi^2}{6} f'(0) \quad \text{if } f(0) = 0. \quad (B9)$$

After this step exclusively integrals over rational functions remain to be performed. The $k$-integration can be done analytically. The $p,q$ integrations have been transformed to a form (in general by introducing new integrational variables), where at most 1 numerical integration was left.

Armed with these ideas all the integrals in $K_{\text{reg}}(0,0,0)$ can be performed. Let us denote the integrational measure in (B8) after shifting all the variables by $\varepsilon$ by

$$\int dM_\varepsilon = \frac{1}{8\pi^2} \left[ \int_{\varepsilon}^{\Lambda} dp \int_{\varepsilon}^{\Lambda} dq \int_{\varepsilon}^{\Lambda} dk \right]$$

(B10)

In the second part of the measure no IR divergencies occur, the $\varepsilon \to 0$ limit can be taken directly. The results of the relevant integrals are (ignoring terms, which vanish as $\Lambda \to \infty$ or $\varepsilon \to 0$):

$$I_A = \int dM_\varepsilon \frac{1}{p^2 q^2 k^2} = \frac{1}{16\pi^2} \ln \frac{\Lambda}{\varepsilon} - 0.0190158229,$$

$$I_B = \int dM_\varepsilon \frac{1}{2pqk^2(p+q)} = 0.000950146 \Lambda,$$

$$I_C = \int dM_\varepsilon \frac{1}{2pqk^2} \left( \frac{n_p+n_q}{p+q} + \frac{n_q-n_p}{p-q} \right) = \frac{1}{16\pi^2} \ln \frac{\Lambda}{\varepsilon} - 0.0115413434,$$

$$I_D = \int dM_\varepsilon \frac{1}{4pqk(p+q+k)} = 0.0001041333 \Lambda^2,$$

$$I_E = \int dM_\varepsilon \frac{n_p}{4pqk(p+q+k)} = \frac{1}{192\pi^2} \ln \Lambda + 0.0000776673,$$

$$I_F = \int dM_\varepsilon \frac{n_q}{4pqk(q+k-p)} = \frac{1}{192\pi^2} \ln \frac{\Lambda}{\varepsilon} + 0.0003657838,$$

$$I_G = \int dM_\varepsilon \frac{n_p n_q}{4pqk} \left( \frac{1}{k+p+q} + \frac{1}{k+p-q} + \frac{1}{k-p+q} + \frac{1}{k-p-q} \right) =$$
\[ \frac{1}{64\pi^2} \ln \frac{1}{\varepsilon} - 0.0044038354. \]  

(B11)

With these integrals we can express the first term in the expansion of (B8):  

\[ K_{\text{reg}}(0,0,0) = 2I_A - 3I_B - 3I_C + I_D + 3I_E + 3I_F + 3I_G \]  

(B12)

The IR divergencies cancel each other as we expected, the rest gives the result appearing in the expression (25):  

\[ K_{\text{reg}}(0,0,0) = 0.0001041333\Lambda^2 - 0.0029850437\Lambda T + \frac{5}{32\pi^2}T^2 \log \left( \frac{\Lambda}{T} \right) - 0.0152887686T^2. \]  

(B13)

Now, it is worthwhile to give a few comments on the second term of the mass-expansion. Its finite part can be absorbed into the finite renormalization, so only the divergent pieces are important. Therefore less terms are to be preserved. The differentiation with respect to the mass squared can be changed to differentiation with respect to the energy or (since we put afterwards all masses \( = 0 \)) to the momentum under the integral:

\[ \left. \frac{\partial}{\partial m_k} \left( \int \frac{D}{\Lambda,\varepsilon} \right) \right|_{m_i=0} = \int \frac{1}{2E_k} \frac{\partial D}{\partial E_k} \right|_{m_i=0}. \]  

(B14)

When the derivation acts on \( n(E) \), we can integrate partially, so at the end only rational functions and eventually \( n(E) \) factors remain, and with the previous ideas the integrations can be reduced to one-dimensional numerically computable integrals.