Center for Gravitational Physics and Geometry
The Pennsylvania State University
University Park, PA 16802-6300

μηδεις αγεωμετρικος εισιτω

CGPG-94/11-5

CP

Recent developments in canonical quantum gravity

Jorge Pullin

*Center for Gravitational Physics and Geometry
Department of Physics, 104 Davey Lab
The Pennsylvania State University
University Park, PA 16802, USA*

CERN LIBRARIES, GENEVA

SCAN-9502262

Abstract. We give a summary of the nonperturbative approach to quantum gravity based on Ashtekar's new variables and loop quantization, intended for a general audience of physicists. We discuss the possible connections of this approach with other areas of physics and the basic ideas of quantizing the gravitational field.

3w 9509

INTRODUCTION

This is a talk intended for a general audience, I apologize to those people who expected more technical detail. I would like to give some basic ideas and motivations behind a program to nonperturbatively quantize general relativity. This program is based on the introduction of new set of variables for general relativity in terms of which the theory looks very much like a particle physics theory. This allows the introduction of new ideas in the quantization process, in particular the use of loop variables. There have been up to present over 350 papers published that in some sense are connected with this topic since its introduction in 1986, so I cannot make a detailed list of people involved. Let me just mention that the major contributors to this program are Abhay Ashtekar, Rodolfo Gambini, Carlo Rovelli and Lee Smolin. Ashtekar and Gambini are presenting more detailed talks at this conference [1,2].

Possibly the questions that matter most to this audience are: what is quantum gravity and why are physicists spending time on it. Broadly speaking, quantum gravity is the application of the rules of quantum mechanics to the gravitational field. There are many versions of the rules of quantum mechanics and many theories for the gravitational field, and therefore several combinations are possible. One could apply the rules of quantum mechanics in the traditional canonical formulation (which we will do here), in the path integral formulation, perturbatively or nonperturbatively and one could take more radical approaches, where new formulations of the rules of quantum mechanics are sought and applied. As theory of gravity one could take traditional general relativity (which is the simplest theory experimentally viable) or more elaborate theories, like supersymmetric extensions, string theory, theories with higher order Lagrangians or more radical theories that reformulate gravity in

the time evolution. When promoted to wave equations, the first three tell us the wavefunctions do not depend on the metric but rather on the “geometry” (properties of the metric that are invariant under diffeomorphisms — “reparametrizations” — of the three dimensional spatial slice). Unfortunately, there is no simple way to understand the fourth equation purely in terms of three dimensional quantities and therefore it has to be enforced as a functional equation with no help of geometric intuition. To make matters worse, it is a nonpolynomial equation, which complicates considerably the factor ordering ambiguities present when one promotes nonlinear classical equations to operatorial wave equations. The resulting equation is called the Wheeler-DeWitt equation and its complexity ground to a halt the effort to canonically quantize general relativity for almost twenty years.

A breakthrough came with the introduction of a new set of canonical variables by Ashtekar (a good review is given in [3], some more recent results are given in [5]). These new variables simplify substantially the constraint equations, making them polynomial expressions. More importantly, they make the theory resemble a Yang-Mills theory. Yang-Mills theories are the models used by particle physicists over the last decades to account for the physics of elementary particles. In essence, they are generalizations of Maxwell theory in which one has several electric and magnetic fields E_i^a , B_i^a (where i label the different fields) that interact nonlinearly. The usual gauge symmetry of Maxwell theory gets replaced by a more complicated symmetry that mixes the different fields and is associated with a group structure, called the gauge group. These theories have been remarkably successful in explaining the electromagnetic, weak and strong interactions and many techniques have been developed for their quantization. By putting general relativity in the Yang-Mills language, the Ashtekar new variables allow the importation of many techniques from Yang-Mills theories into gravity, in particular one of them, the use of loops.

The technical details of the canonical formulation of general relativity in terms of these new variables exceeds the scope of these lectures, I will just give a brief idea about what it looks like. In any Riemannian space one can introduce a local basis of vectors in which the metric is flat. This is what we do in everyday physics and why we do not worry about curvature in local experiments. The mathematical expression of this fact is $q^{ab} = E_i^a E_j^b \delta^{ij}$ (summation from 1 to 3 over repeated indices), which tells us that in the basis of E_i^a vectors, the curved metric q^{ab} looks flat, δ^{ij} being the three dimensional Euclidean metric. As can be seen from the above expression, all information about the curved metric q^{ab} is contained in the frame fields E_i^a . That means one could use them as evariables for a canonical formulation of gravity. The use of the letter E to describe them is intentional: the frame fields transform as vectors in the a index and under Euclidean rotations ($SU(2)$) in the i index. That is, they behave exactly like the electric fields of an $SU(2)$ Yang-Mills theory. The insight of Ashtekar was to notice that their canonically conjugate momentum can be chosen in such a way that it transforms like a vector poten-

tial of a Yang-Mills theory A_a^i (the definition of A in detail exceeds the scope of these lectures, it involves notions of self-duality; the connection A_a^i was first introduced by Sen).

To give an idea of the simplification of the equations achieved by the use of these variables we can write the Hamiltonian constraint (which in terms of the traditional variables was a complicated nonpolynomial expression), which is remarkably simple,

$$H = \epsilon^{ijk} \vec{E}_i \cdot \vec{E}_j \times \vec{B}_k \quad (1)$$

where again repeated indices are summed from 1 to 3 and the mixed product is the usual one from three dimensional vector calculus.

The simplification introduced in the constraint equations allows us to promote them (formally, ignoring regularization problems) to quantum wave equations. It is remarkable that a solution to those formal wave equations can actually be found (in the case of general relativity with a cosmological constant Λ) [6,7], it is given by,

$$\Psi[A] = \exp \left(-\frac{6}{\Lambda} \int d^3x (\delta_{ij} \vec{A}^i \cdot \vec{\nabla} \times \vec{A}^j + \frac{2}{3} \epsilon^{ijk} \vec{A}^i \cdot \vec{A}^j \times \vec{A}^k) \right) \quad (2)$$

where $\vec{A}^i$ are the Sen vector potentials and ϵ_{ijk} is the alternating symbol in three dimensions. The expression in the exponent is known as the Chern-Simons form, and is a general expression one can build given a vector potential of a gauge theory. It is known to play important roles in topological field theories and differential geometry and it is remarkable that it appears in connection with the dynamics of general relativity.

LOOPS

As we mentioned before, having cast general relativity in a language that resembles that of Yang-Mills theories allows to import many techniques from them to the gravitational case. One of them is the use of loops and the loop representation. The role of loops in gauge theories is simple to understand by analogy with the Abelian case of the usual Maxwell theory. In that case one is usually interested in functionals of the vector potential $\Psi[A]$ that are invariant under gauge transformations. An example of such a functional is $W_\gamma[A] = \exp(\oint_\gamma dy^a A_a)$, the exponential of the circulation of the vector potential around a closed loop γ . Because of Stoke's theorem, one knows it is equal to the integral of the magnetic field on a surface that is bounded by the loop γ . The point is that if one knows the value of $W_\gamma[A]$ *for all* possible loops γ one can completely reconstruct the magnetic field of the theory (since one knows its integral for all possible domains). Now, the magnetic field contains all the gauge invariant information of the vector potential A . That means that *any* gauge invariant function $\Psi[A]$ can be expressed as a (possibly

infinite) superposition of W_γ 's for different loops. The point of view of the loop representation is to take this statement seriously and propose a quantum representation purely in terms of loops $\Psi(\gamma)$ where the wavefunctions are the "coefficients" of the infinite expansion of the wavefunctions in the connection representation $\Psi[A]$ in terms of $W_\gamma[A]$'s. Specifically,

$$\Psi(\gamma) = \int dA W_\gamma[A] \Psi[A]. \quad (3)$$

The above expression is more easily understood if one compares it with the transform from the position to the momentum representation in ordinary quantum mechanics. There one has functions $\Psi(x)$, one expands them in a basis of functions $\exp(ikx)$ parametrized by a continuous parameter k and obtains a function of the continuous parameter $\Psi(k)$,

$$\Psi(k) = \int dx \exp(ikx) \Psi(x) \quad (4)$$

Similarly in the case of loops, one expands the functions $\Psi[A]$ in terms of a basis parametrized by a continuous parameter $W_\gamma[A]$ and obtains a function of the continuous parameter γ .

The above discussion is for an Abelian theory with one vector potential. It turns out that it goes through with the addition of a small technicality for a non Abelian theory. The Wilson loop is defined in that case as,

$$W_\gamma = \text{Tr}[P \exp(\oint_\gamma dy^a A_a)] \quad (5)$$

where because of the non Abelian nature of the theory the exponential is a path-ordered exponential and in order to produce a gauge invariant object one takes the trace of the resulting exponential.

So general relativity can be formulated at a quantum mechanical level purely in terms of loops, the wavefunctions are functions of loops $\Psi(\gamma)$. The reader may feel the discussion is becoming abstract and vague. It turns out however, that loop representations are quite real. They have been successfully used in the past to study the quantization of the following systems,

- Maxwell theory.
- QED
- Chern-Simons theory
- Gravity in 2+1 dimensions
- QCD in the continuum
- QCD on the lattice

In all cases where the answers are known by other methods, the loop techniques yield the same results. Something that reassures the usefulness of these techniques is the possibility of using them in theories as complex as QCD on the lattice. The fact that the techniques can be implemented on a computer and used to compute concrete things like mass spectra should dispel

any doubt of their concreteness and usefulness. Loop representations for Yang-Mills theories were extensively considered by Gambini and Trias in the eighties [8] and were first considered for quantum gravity by Rovelli and Smolin [9].

There is a fundamental difference between general relativity and other theories when formulated in terms of loops. As we mentioned above, three of the four Einstein equations in the canonical language tell us that the theory is invariant under diffeomorphisms of the three dimensional surface. That means we can deform the surface at will and the theory should be invariant. In the language of loops this means that the wavefunctions should depend on the loops in such a way that they are invariant under (smooth) deformations of the loops. Functions of closed curves that are invariant under smooth deformations of the curves are known as knot invariants and are the subject of study of the branch of mathematics called knot theory. Knot theory has attracted considerable recent interest due to its applications in topological field theories, solid state physics, biology and many other areas. The connection between knot theory and quantum gravity was first observed by Rovelli and Smolin [9].

Therefore we see that the use of the new set of variables not only has allowed us to make a connection between general relativity and Yang-Mills theory but also an unexpected connection with knot theory.

The study of properties of knots invariant under smooth deformations is a classical branch of mathematics. It was originally considered by Gauss in the 1820's and was given a great impetus towards the end of the last century. At that time Rayleigh and others were interested in constructing a theory that could explain atomic phenomena. The theory they came up with was to represent atoms through knotted lines of aether. The main developments of this model were worked out by P. G. Tait [10] in Scotland. Many of the outstanding questions of knot theory that are still unanswered today were formulated at that time, with the failed atomic theory as motivation. Maxwell, Faraday and others noted connections between knot theory and electromagnetic fields that turn out to have related roles to ideas that appear in modern loop representations.

The simplest knot invariant is the Gauss linking number, which is an invariant of two curves, which is one if the curves are linked and zero otherwise. It is quite evident that it is an invariant, since one cannot unlink two closed curves without cutting them, which is not a smooth deformation. Such invariant seems quite primitive, but it can actually be given an analytic formulation, given two closed curves γ_1 and γ_2 their linking number is given by the integrals,

$$\text{Linking}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} dx^a dy^b \epsilon_{abc} \frac{(x-y)^c}{|x-y|^3}. \quad (6)$$

Although we will not include the details here, it turns out that this simple expression plays a relevant role as a state of quantum gravity [11].

The central problem of knot theory is the classification of knots, that is, how to find out if two knots are continuously deformable into each other or

not. This problem is at the moment unsolved. An important development in the effort to tackle this problem has been the introduction of a technology to generate knot invariants known as knot polynomials. A knot polynomial is a polynomial in an arbitrary variable that is uniquely associated with each knot and that is invariant under smooth deformations of the knots. These polynomials can be defined through rather simple recursion relations called skein relations and each coefficient of the polynomial is a knot invariant. There are several of these polynomials known, most of them have been introduced quite recently and they have proper names: the Jones polynomial, the Kauffman bracket, the HOMFLY polynomial and others.

The reader may at this point be puzzled: sure, it seems one can rewrite everything in terms of loops, it is nice that the diffeomorphism invariance gets coded in knot invariance, but can one do something more explicit? It turns out one can. The constraints of quantum gravity can be entirely rewritten in terms of expressions purely dependent on loops [12,13], including the “time evolution” constraint (the Wheeler-DeWitt equation). The expressions in terms of loops are not very complicated, the Hamiltonian constraint has a topological interpretation in terms of rearrangements of loops at intersection points. Broadly speaking the rewriting in terms of loops goes along the following lines: the frame fields E_i^a are determined by the tangents to the loops and the field tensor F_{ab}^i is defined in terms of a differential operator in loop space known as loop derivative, that was first introduced by Mandelstam.

Another point to be highlighted is that we can construct solutions to the Wheeler-DeWitt equation in loop space. If one takes the state (2) and inserts it in the loop transform (3) one obtains an expression that has been considered in the context of Chern-Simons theory by Witten and others [14]. It turns out that the transform of the Chern-Simons state into the loop representation is given by a well known knot invariant, the Kauffman bracket knot polynomial. Therefore this knot polynomial, being the transform of a state that solves all the constraints of general relativity with cosmological constant in the connection representation, should solve all the constraints in the loop representation [7]. The remarkable fact is that we can check this fact: because we know *explicitly* the form of the Wheeler-DeWitt equation in the loop representation we can actually check that the Kauffman bracket is a solution. In explicitly checking this result, one finds a new solution, unknown by other methods. One finds that the second coefficient of a certain expansion of the Jones polynomial solves the Wheeler-DeWitt equation with zero cosmological constant [15,16]. This is the first nontrivial quantum state of general relativity ever found, and it is only known in the loop representation, we do not know what its counterpart is in terms of vector potentials.

FINAL REMARKS AND CONNECTION WITH PHYSICS

The original intention of this program was to find a mathematically consistent theory of quantum gravity. We are not finished in this task. We have been able to promote the constraint equations to operators and find some solutions, we still have to introduce an inner product in order to be able to compute expectation values. In spite of this one can jump ahead and speculate what sort of picture of quantum gravity does this approach lead to [17]. The first question that comes to mind is what is the spacetime geometry predicted by this approach in terms of loops. A classical geometry, which is a macroscopically smooth object should only arise in a limit. It is not difficult to imagine what this limit would be. Consider a loop that has the structure of a “weave”. Viewed from large scales, one does not notice its discrete structure and one averages over many threads when one computes physical quantities. The perception therefore is that the geometry is smooth over large scales. If one probes the weave at short distances (high energies) the picture changes completely: it is inherently discrete and in fact quite detached from the idea of a classical geometry.

It is quite remarkable that starting from the most conservative approach to quantization coupled with the most conservative theory of gravity we end up with a picture of spacetime that at short distances departs so radically from the conventional point of view in terms of classical geometries.

There is a lot yet to be done in the issue of introducing an inner product in order to compute expectation values and end the construction of a mathematically consistent theory of quantum gravity. Even more work will be needed, especially in approximation techniques to connect such a theory with the perceived problems of quantum gravity. The main hope that this approach offers is the large number of connections with other branches of physics and mathematics which allows us to use a variety of new techniques and insights. If it is successful or not will be decided in a not so distant future.

I will end by mentioning some recent technical developments whose detailed treatment exceeds the scope of these lectures, but which illustrate the level of cross-breeding with other areas of physics that is taking place. Ashtekar and collaborators [18] are at present developing a theory of integration in the space of nonabelian vector potentials. They have succeeded in defining rigorous measures that give formal meaning to the integral in (3). This work could have implications for the nonperturbative quantization of QCD and other non Abelian theories and could shed lights on issues like QCD confinement. The other development is the definition of an extension of the idea of loop and the group of loops by Gambini and collaborators [19]. This work could help define in a rigorous way issues having to do with the regularization of gauge theories in terms of loops. These two examples of techniques developed for the program I have described but that may result useful in other areas of physics

make the masses smaller at will. However, when one computes the energy of a point charge, one finds it diverges. Therefore its mass diverges and one cannot neglect gravitational corrections. In fact, if one throws in gravitational corrections in a naive calculation one finds that the negative gravitational self-energy of the charge can cancel the divergence (for details of the calculation see [3]). If one does the calculation with Newton's theory one finds that fine-tuning the bare mass and charge one can get a finite result. If one considers the types of nonlinearities present in general relativity (all energy contributes to gravity, not just the bare mass) the result is finite without fine tuning. Remarkably, the result is non-perturbative: if one were to expand it in terms of Newton's constant, one would get a series of divergent terms whose sum is finite. This calculation, although naive, conveys the flavor of the kind of ideas one expects to hold in general and how the unavoidable introduction of gravitational effects could help with the problems of divergences in quantum field theory.

The second of the slogans says that because general relativity is non-renormalizable, we should seek a "better" classical theory to quantize. This implies that non-renormalizability is something devastatingly bad for a theory that renders hopeless all attempts to quantization. This is not the case. The precise content of the statement is "we applied (some precise version of) perturbation theory to general relativity and it didn't work". This sounds considerably less strong. Couldn't it be that the particular version of perturbation theory is inadequate? Or maybe the theory does not admit a perturbative version. Perturbation theory implies considering spacetimes that differ little from flat spacetime and quantizing the modes that measure the departures from flatness. What one ends up quantizing is a theory that even classically is radically different from general relativity: it has no black holes, no big bang, and would not pass experimental tests. It differs from general relativity exactly in the high range of energies one is interested in. The failure of its quantization therefore says very little from the failure of general relativity as a whole, it tells us more of the failure of the perturbative method in itself. If this were actually the case it would not even be the first time a theory that is perfectly fine quantum mechanically is non-renormalizable. Examples of this are gravity in 2+1 dimensions (it was only realized how to quantize it perturbatively after it was shown by Witten [4] that it was exactly solvable) and more recently and to the point, the Gross-Neveu model in three dimensions).

The last of the slogans claims that the problem of quantum gravity will be automatically tackled when gravity is unified with the other interactions. Such unification, it is claimed, will introduce corrections in Einstein's theory that would cure all difficulties of quantum gravity. A slightly different point of view is the one that says that general relativity is not the correct theory of gravity to be quantized, and that suitable modifications (e.g. strings, supersymmetric theories, higher order theories) will remove the obstacles in quantization. I find this very unlikely. It is not that we are failing to quantize general relativity due to technicalities. We are failing because of good reasons, basically due to

the lack of understanding of how to quantize a theory without a given space-time. Unless something very radical is modified, all alternative theories or unified models will face the same difficulties. Therefore it seems natural to study these difficulties in the context of the simplest theory available, general relativity.

In fact the last phrase captures almost all the intent of the program I am about to describe. The idea is to attempt a non-perturbative canonical quantization of general relativity, facing the difficulties as explicitly as possible and trying to learn something from the attempt. The ultimate goal would be to construct at least one mathematically viable theory of quantum gravity. Its connection to physics (or more precisely to the “perceived problems” of quantum gravity) should then be explored, very likely through some approximation scheme.

CANONICAL QUANTIZATION: NEW VARIABLES

The canonical quantization of a physical system is the most understood method of quantization. It can be summarized in a series of steps:

1. Cast the theory in Hamiltonian form. Pick an algebra under Poisson brackets of classical quantities “large enough” to describe the physics one is interested in.
2. Promote the algebra to operators acting on a space of functions.
3. Promote the “field equations” (constraints) to wave equations and solve them. The solutions are called “physical states”.
4. Introduce an inner product under which the observable quantities (those that commute with the constraints) are self-adjoint operators.
5. Make physical predictions through the computation of expectation values.

For the particular case of general relativity, the Hamiltonian formulation is well understood. In order to cast the theory in Hamiltonian form, one splits spacetime into space and time and views spacetime as a collection of spaces that evolve in time. This splitting is fiducial: at the end of the day the formalism tells us that it does not matter how we do the splitting, as it should be since in general relativity space and time are on the same footing. The basic variables of the Hamiltonian theory are the metric of three dimensional space q^{ab} and its “time derivative” given by the extrinsic curvature. The metric and extrinsic curvature are conjugate variables under Poisson brackets. The traditional approach to canonical quantization is based on considering them as basic variables and promoting them to operators on a space of wavefunctionals depending on the metric $\Psi[q]$. The Einstein equations in the canonical formalism become four equations. Their meaning is the following. Three of the equations tell us that it does not matter how we parametrize the spatial surface. The fourth one tells us that it does not matter how we parametrize

non-metric ways. Almost all possible combinations of these approaches and theories have been explored up to some extent over the last 60 years. The result is a great deal of confusion. The situation confronting the field is quite unusual in physics. On one hand, there is no experimental pressure to find a theory of quantum gravity, i.e. there are no precise outstanding experiments that a candidate theory should explain. There are certain general criteria for viability of a theory, but the constraints they impose are not too severe. An unsuspecting physicist would imagine that theorists are having a field day in this area, i.e. they can cook up tons of successful theories due to the lack of experimental constraints. Actually the situation is quite the opposite: in spite of the lack of experimental constraints, we do not at present have a *single* mathematically consistent, viable theory of quantum gravity. This is not (hopefully) due to the incompetence of theorists but rather to the fact that the problems confronted when trying to quantize gravity are genuinely hard, and fundamentally quite novel with respect to our experience in the rest of physics.

This state of affairs has given rise to a series of slogans about the field, some of which are quite widespread, and which in my opinion add to the confusion. To try to illustrate the point of view sought in the approach we are presenting, I would like to briefly discuss three of these slogans, which in themselves are of slightly different nature and bearing, but all having a negative and confusing connotations,

- "There will never be observable effects of quantum gravity, so why bother studying it?"
- "Something is deeply wrong with gravity: general relativity is non-renormalizable".
- "Einstein plus something is better than Einstein alone: unification will eventually take care of gravity"

The first statement is based on the fact that if one uses order of magnitude estimates of at which energy quantum gravitational effects come into play, one ends up with a ridiculously high number, the Planck energy of 10^{19} GeV. To attain such energies a particle accelerator should have the size of a galaxy. It is quite evident that we will hardly ever find any situation in nature where such energies are present. Therefore many people take the point of view that quantum gravity will ever be a purely speculative subject, not suitable for examination by professional physicists. The main drawback of this point of view is that in the present paradigm of theoretical physics, more precisely in quantum field theories, we actually consider every day arbitrarily high energies —and therefore arbitrarily high masses— for which gravitational effects cannot be ignored. A simplified picture of this is given by the example of the calculation of the energy of a charged point particle in electromagnetism. This is a calculation we all learn at a second year university course in physics. The initial hypothesis in such courses is that gravitational effects are negligible. In general that hypothesis is correct, since one can maintain constant charges and