Abstract

Institute of Physics, University of Tokyo, Komaba, Tokyo 153 Japan

Ikuo Ichinose and Masaru Odaka

Statistics and Chiral Spin Order
Heisenberg Anti-ferromagnets: Fractional
Strongly-Coupled Massive O(3) and Frustrated

LT-Konbun 94-17
1 Introduction

At present it is established that interactions between elementary particles are described by the gauge theory. In the last several years, it has been realized that gauge theory also plays an important role in some phenomena in condensed matter physics, which include fractional quantum Hall effect[1], high-Tc superconductivity[2], quantum spin models[3,4], etc. In (2 + 1) dimensions, a special type of gauge theory appears, i.e., the Chern-Simons(CS) gauge theory, and actually it describes some of the above mentioned phenomena.

It is naturally expected that existence of CS term drastically changes dynamics of the gauge theory, because it works as a mass term of gauge field[5]. However, only a few studies on non-perturbative properties of gauge theories including CS term have been given so far[6]. This study is very important, because pure Maxwell gauge theories (with continuous gauge group) in (2+1) dimensions are all in the confinement phase, regardless of the gauge group and strength of the gauge coupling.

In this paper, we shall study (2+1)-dimensional SU(Nc) and U(Nc) massive QCD on a lattice. As well-known, the naive formulation of lattice fermion gives fermions with mass term ±m in the continuum limit. In the (2 + 1) dimensions, this fermion doubling is related to the parity symmetry, because sign of the Dirac mass changes under parity transformation. In order to generate fermions with the same mass, e.g., ±m, we introduce diagonal hopping terms of fermions, which explicitly breaks parity invariance. In the naive continuum limit, this lattice fermion generates two species of fermions with the same Dirac mass. In the massive QCD3 in the continuum, if quarks are integrated out and derivative expansion is applied, the Maxwell-CS theory appears. Therefore, it is expected that phase structures of these two theories are closely related with each other. It is not easy to formulate the CS theory on a lattice. The massive QCD3, which is formulated as mentioned above, is much easier to study. We believe that phase structure and dynamical properties of the massive QCD3 gives much insight for those of the Maxwell-CS theory.

This paper is organized as follows. In Sect.2, we give a detailed description of the massive QCD3 on a lattice. We employ the Hamiltonian formalism. Unusual assignment of quark hopping term is used to obtain Dirac fermions with definite-sign mass term in the continuum limit. Then, we derive an effective model by integrating out gauge field using strong-coupling expansion. This method has been used in various ways and is proved to be useful for obtaining qualitative picture of gauge theories[7]. Here, strong coupling means that the coefficient of the Maxwell term or the plaquette term is small, and it is not assumed that the model is in the confinement phase. In fact, it is observed by numerical calculation that for large number of flavors of quarks, QCD3 is not in the confinement phase, even if there is no plaquette term[8].

Derived effective models are essentially Heisenberg models of antiferromagnets with frustration.

In Sect.3, the effective spin model is studied for large Nc. In large Nc, classical treatment of spin operators becomes reliable to obtain the ground state of the model. We observe that there is a phase transition from a Neel ordered state to a state with chiral-spin order. Parity-breaking term, i.e., the quark mass term, is essential for this result.

In Sect.4, the results obtained in Sect.3 are examined from the viewpoint of the original gauge theory. It is shown that color magnetic flux perpendicular to the 2-dimensional space is spontaneously generated, and it induces fractional statistics excitations due to the Aharonov-Bohm effect. It is also suggested that the Neel-chiral-spin phase transition in the effective spin model corresponds to a confinement-deconfinement phase transition in the massive QCD3 and the Maxwell-CS theory.

Section 5 is devoted for discussion. Some technical details are summarized in Appendices.

---

1 In the strong-coupling limit of QCD, chiral symmetry breaking does not occur for large number of quark flavors[7]. This is an indirect evidence for the deconfinement.
2 Effective Quantum Spin Models

In this section we shall study SU($N_C$) and U($N_C$) gauge theories with fermions on a
$D$-dimensional spatial lattice in the Hamiltonian formalism. We first present Hamiltonian of gauge systems, and then derive effective spin models of gauge theories by the strong-coupling expansion.

2.1 Hamiltonian and gauge constraint

We shall discretize space as a cubic (or square) lattice and time is left continuous.
We use units in which the lattice spacing, the speed of light and the Planck constant are all equal to unity. Lattice sites are parameterized as

$$x = (x_1, \cdots, x_D),$$

where $x_i$'s are integers.

The nearest-neighbor (NN) lattice sites are connected by unit vectors

$$\hat{1} = (1, 0, \cdots), \quad \hat{2} = (0, 1, 0, \cdots), \cdots \quad (2.1)$$

and the oriented link between the lattice site $x$ and $x + i$ is denoted as $[x, i]$. The link oriented in the opposite direction is denoted as $-[x, i]$. Then links obey the identity

$$[x, -i] = -[x, i]. \quad (2.2)$$

Also, a square plaquette with corners $x, x + i, x + i + j, x + j$ and with sides $[x, i], [x + i + j, i], [x + j, -j], [x + j, -j]$ is denoted as $[x, i, j]$. It also obey the identities

$$[x, j, i] = -[x, i, j], \quad [x, -i, j] = -[x, -i, j]. \quad (2.3)$$

Gauge field $U[x, i]$ living on the link $[x, i]$ is a group element in the fundamental representation $[N_C]$ of SU($N_C$) and, if the color group is U($N_C$), it also carries local U(1) charge. It has the property

$$U[x + i, -i] = U^\dagger[x, i]. \quad (2.4)$$

The color electric field operators $E^a[x, i]$ associated with the link $[x, i]$ satisfy the Lie algebra

$$[E^a[x, i], E^b[y, j]] = if^a_{bc}E^c[x, j][k(x, i) - [y, j]] \quad (2.5)$$

and

$$E[x + i, -i] = -U[x, i]E[x, i]U^\dagger[x, i]. \quad (2.6)$$

where $E[x, i] \equiv E^a[x, i]T^a_{N_C}$ and $T^a_{N_C}$'s, $a = 0, \cdots, N_C^2 - 1$, are the generators of the Lie algebra of SU($N_C$) or U($N_C$) obeying $[T^a_{N_C}, T^b_{N_C}] = if^a_{bc}T^c_{N_C}$ with the structure constants $f^a_{bc}$. In the U($N_C$) case, $T^a_{N_C} \propto I$, the Unit matrix, represents U(1) and $T^a_{N_C}$'s, $p \neq 0$ give the representation $[N_C]$ of SU($N_C$). It generates the action of the Lie algebra on $U[x, i]$, i.e.

$$[E^a[x, i], U[y, j]] = U[y, j]T^a_{N_C}\delta(x, i) - [y, j]) = T^a_{N_C}U[y, j]^a(x, i) + [y, j]). \quad (2.7)$$

Lattice fermion operators satisfy the canonical commutation relation

$$\{\psi^{\dagger a}(x), \epsilon^{\dagger a}(y)\} = \delta^a_b\delta(x - y). \quad (2.8)$$

where $A, B = 1, \cdots, N_C$ are color indices and $a, b = 1, \cdots, N_f$ are lattice flavor indices, respectively. We shall employ the staggered fermion method which assigns component of the Dirac spinor as well as flavors to adjacent lattice sites.

The naive lattice QCD with dynamical fermions is formulated in terms of the above field operators. In order to consider fermions with masses of a definite sign, however, we must introduce hopping terms of fermions along diagonal links connecting nearest-neighbor (NN) sites besides the usual NN ones, and then the gauge field and the electric field operators living on diagonal links are necessary for gauge invariance. These operators satisfy the same formulas as the operators living on links which connect NN lattice sites. The Hamiltonian is

$$H = T + V + H_f. \quad (2.9)$$

$$T = \sum_{[x,k]} \frac{g^2}{2} E^a[x, k] E^a[x, k]. \quad (2.10)$$
\[ V = - \sum_{[x,i,j]} \frac{1}{2g^2} T \left[ U^*[x,i+j] U^*[x,j] U^*[x,i] + h.c. \right] \]

\[ + U^*[x,j] U^*[x+i,j] U^*[x,i+j] \]

\[ + U^*[x,j] U^*[x+j,i] U^*[x,j] + h.c. \right]. \]

(2.11)

\[ H_f = \sum_{[x,k]} \left\{ t_{[x,k]} \Psi^a_{[x,k]}(x) \right\} \Psi^a_{[x,k]}(x) + h.c. \right]. \]

(2.12)

where \([x,i \pm j]\) denotes the oriented link between the lattice site \(x\) and \(x + i \pm j\), and \([x,k]\) denote \([x,i]\) or \([x,i+j]\) or \([x,i-j]\) \((i \neq j)\). The hopping parameters \(t_{[x,k]} = t_{[x,-k,j]}\) satisfy \(t_{[x,k]} = t\), \(t_{[x,x+j]} = t\) and have non-trivial phases.

A few remarks about the Hamiltonian \(H\) are in order.

1. Each term of the magnetic term \(V\) is given by the product of the gauge fields along fundamental triangle plaquette. This term favors configuration with vanishing color magnetic flux in each fundamental triangle plaquette, i.e., \(T \left[ U^*[x,i+j] U^*[x+i,j] \right] \sim 1\). If we take products of the gauge fields along the square plaquettes instead, there is no suppression factor for color magnetic flux in the triangle plaquettes and we cannot expect smooth configuration of the gauge field even for small \(g^2\).

2. For the 3-dimensional massive QCD, the hopping parameters \(t_{[x,k]}\) are explicitly given as

\[ t_{[x,k]} = t_{[x,-k,j]} = t e^{i \phi_k} \]

\[ t_{[x,i+2]} = -t_{[x,i-2]} = t(-1)^{\sum_{i=1}^3 i} \]

(2.13)

where \([x]\) is +1 if \(\sum_{i=1}^3 x_i\) is even, and is -1 if \(\sum_{i=1}^3 x_i\) is odd. It is easily seen that the diagonal hopping term with (2.13) explicitly breaks the parity symmetry.

Let us consider the naive continuum limit of \(H_f\) with the above choice of \(t_{[x,k]}\). To this end, we simply put \(U^a_{[x,k]} = 1\), and omit the color and flavor indices for simplicity. The hopping Hamiltonian

\[ H^0_f \equiv H_f \]

has the periodicity of twice the lattice spacing. Then we introduce 4-component "spinor" field \(\Psi\) as follows,

\[ \Psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \]

(2.14)

When \(\tilde{t} = 0\), i.e., there is no diagonal hopping, \(H^0_f\) is invariant under the following transformation

\[ \psi_1(x) \rightarrow a \psi_1(x) \]

\[ \psi_2(x) \rightarrow a \psi_2(x + 1) \]

\[ \psi_3(x) \rightarrow a \psi_3(x + 1) \]

\[ \psi_4(x) \rightarrow a \psi_4(x) \]

(2.15)

where \(a = e^{i \pi}\), and this symmetry corresponds to the discrete chiral symmetry in the continuum. It is easily verified that the diagonal hopping explicitly breaks the above symmetry, and therefore it works as a mass term (see later discussion).

We introduce the Fourier-transformed fields as

\[ \tilde{\psi}_\gamma(k) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} \psi_\gamma(x), \quad \gamma = 1, ..., 4. \]

(2.16)

where \(a\) is the lattice spacing which we have set \(a = 1\) so far. In terms of them, \(H^0_f\) is written as

\[ H^0_f = \int d^3k \tilde{\Psi}^\dagger(k) \left\{ i \Gamma_N(k) \right\} \tilde{\Psi}(k). \]

(2.17)

where \(\Gamma_N(k)\) and \(\Gamma_{NN}(k)\) are \((4 \times 4)\) matrices which depend on only \(k\). Straightforward calculation shows that the energy eigenvalue takes its lowest value at \((k_1, k_2) = \ldots \)
\((\pi/2a, \pi/2a)\). Therefore rescaling the momentum as \(k_i = \frac{k_i}{\sqrt{\alpha}} + \phi_i\), \((i = 1, 2)\), the continuum limit of \(H^0_i\) is obtained as
\[
H^0_i \sim \int d^2p \psi^\dagger(p) \left\{ \sum_j \eta_j \Gamma_j + i \Gamma_0 \right\} \psi(p),
\]
where \(\Gamma_\mu\) \((\mu = 0, 1, 2)\) are constant \((4 \times 4)\) matrices like
\[
\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

By the following linear transformaton of \(\psi_i\),
\[
\chi_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad \chi_2 = \frac{1}{\sqrt{2}}(\psi_3 - \psi_4),
\]
\[
\chi_3 = \frac{1}{\sqrt{2}}(\psi_2 + \psi_4), \quad \chi_4 = \frac{1}{\sqrt{2}}(\psi_1 - \psi_3).
\]
\(\Gamma_\mu\) matrices factorize into two \((2 \times 2)\) matrices,
\[
H^0_i \sim \int d^2p \tilde{X}(p) \left\{ \sum_j \eta_j \sigma_j + i \right\} \tilde{X}(p).
\]
where \(\tilde{X}(p) = (\chi_1(p) \cdots \chi_4(p))^\dagger\), \(\tilde{X}(p) = X(p)\Sigma_0\) and
\[
\Sigma_0 = \begin{pmatrix} \sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} \sigma_+ & 0 \\ 0 & -\sigma_- \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} -\sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix}.
\]
The above result means that the naive continuum limit of \(H^0_i\) is the two flavor Dirac fields with the same mass \(\tilde{m}\). The result that two Dirac fields have the same mass inspite of the doubling problem of lattice fermion is not so surprising, because the diagonal hopping term with (2.13) explicitly breaks the parity invariance.

(3) The generator of time-independent gauge transformations is
\[
\mathcal{G}(x) = \sum_{\eta} E^\eta[x,k] + \psi^\dagger(x) T^\eta \psi(x),
\]
with \(p = 1, \cdots, N^2_0 - 1\). If the gauge group is \(U(N_c)\) rather than \(SU(N_c)\) there is also the \(U(1)\) generator
\[
\mathcal{G}(x) = \sum_{\eta > 0} \left( E^\eta[x,k] - E^\eta[x, -k] \right) + \frac{1}{2} [\psi^\dagger, \psi^\dagger].
\]
where \(\Sigma_{\eta > 0}\) means summing over half links radiate from the lattice site \(x\). The \(U(1)\) charge operator defined as above changes sign under time reversal as \(E\). It is easily verified that the Hamiltonian is gauge invariant.
\[
[\mathcal{G}(x), H] = 0, \quad \forall p.
\]

In following subsection, strong-coupling expansion will be employed. We first seek eigenstates of \(T\) and treat \(V + H_f\) as a perturbation. Each term is gauge invariant,
\[
[\mathcal{G}(x), T] = [\mathcal{G}(x), V] = [\mathcal{G}(x), H_f] = 0, \quad \forall p.
\]
and therefore, if we find a gauge invariant eigenstate of \(T\), perturbation calculation preserves gauge invariance at each order.

### 2.2 Strong-coupling expansion

We shall now derive the effective Hamiltonian by integrating out the gauge fields by the strong-coupling expansion. As discussed in Appendix A, we consider \(H_0 \equiv T\) as the leading order term and \(H = V + H_f\) as a perturbation. Let \(P\) be the projection operator on the subspace \(E^\eta[R,k] = 0\) and \(Q = 1 - P\). We can show that both the first order and the third order terms of the perturbation expansion are equal to zero, because \(U(N_c)\) and \(SU(N_c)\) integrals with odd number of \(U\)'s as integrand are vanishing. Therefore, to the fourth order, the effective Hamiltonian is
\[
H_{eff} = PHG_0 \hat{G} P + PHG_0 \hat{G} H G_0 \hat{G} P.
\]
where \(G_0 = Q(E_0 - H_0)^{-1}Q\) and \(E_0\) is the lowest eigenvalue of \(H_0\) (for derivation see Appendix A). For later convenience, we introduce formulas for the evaluation of \(U(N_c)\) and \(SU(N_c)\) integrals
\[
PU^\eta \psi^\dagger(x, k)U^\dagger \psi^\dagger(y, l)P = \frac{1}{N_c} \delta^\eta \delta^\eta (x^\dagger - y, l) / P,
\]
\[
PU^\mu \psi^\dagger(x, k)U^\dagger \psi^\dagger(y, l)P = 0.
\]
and the formula

$$G \mathcal{U}[x_1, k_1] \cdots \mathcal{U}[x_n, k_n] P = -\frac{2}{n q^2 C^2} \mathcal{U}[x_1, k_1] \cdots \mathcal{U}[x_n, k_n] P. \quad (2.29)$$

where $\mathcal{U} = U$ or $U^1$, every $[x_i, k_i]$ ($i = 1, \ldots, n$) is different from each other, and $C^2 = \sum_p T^2_p$. If $\rho$ runs either from 0 or 1 to $N^2_q - 1$. Using these formulas and a shorthand symbol for a fundamental square plaquette $(1, 2, 3, 4) = (x, x + i, x + i + j, x + j)$ ($i \neq j$), we get

$$H_{\text{eff}} = H_{\text{eff} 2} + H_{\text{eff} 3} + H_{\text{eff} 4}. \quad (2.30)$$

$$H_{\text{eff} 2} = \frac{4}{g^2 N^2_q C^2} \sum_{[x, k]} \left| t_{[x]} \right|^2 \left\{ \frac{1}{N^2_q} \rho(x) \rho(x + k) + 2 J_{x} \cdot J_{x + k} \right\} P. \quad (2.31)$$

$$H_{\text{eff} 3} = \left\{ \mathcal{H}_3 + \mathcal{H}_3^1 \right\} P,$n

$$\mathcal{H}_3 = \left\{ \mathcal{H}_{123} + \mathcal{H}_{132} + \mathcal{H}_{213} + \mathcal{H}_{231} \right\}, \quad (2.32)$$

$$\mathcal{H}_{123} = -\frac{16}{g^2 (N^2_q C^2)^2} \sum_{[x, k]} \left| t_{[x]} \right|^2 \left[ \rho(x) \rho(x + k) \right] \left\{ \mathcal{K}^{221}_{a+b+1} + \mathcal{K}^{212}_{+1-a} + \mathcal{K}^{a+1}_{+1} \right\} \left( \mathcal{K}^{a}_* - \mathcal{K}^{a+1}_* \right),$$

where $|t_{[x]}| = t$, $|t_{[x, k]}| = |t_{[x]}| = t$,

$$\rho(x) = \psi_{\alpha}(x) \gamma^{0}(x) N^2_q / 2, \quad J_{x} = \psi_{\alpha}(x) \gamma^{0}(x) N^2_q B \psi_{\alpha}(x). \quad (2.33)$$

and $\mathcal{K}^{(a+b)}_{(a+b)}$'s ($k = 1, 2, 3, 4$ and $\alpha = +, -$) are of third-ordered with respect to $\rho$ or $J_{x}$ and their explicit forms are given in Appendix B. $H_{\text{eff} 4}$ is fourth-ordered with respect to $\rho$ or $J_{x}$ but contains neither $\mathcal{K}^{(a+b)}_{(a+b)}$ nor $\mathcal{K}^{(a+b)}_{(a+b)}$. In the most of the analysis below, we shall neglect $H_{\text{eff} 4}$ and higher-order terms. We believe that this approximation still gives important insight for the phase structure of the gauge theories, as in the cases studied so far[7].

The above result is very complicated. However, for the specific choice of $t_i$, (2.13), which corresponds to $(2 + 1)$-dimensional massive Dirac fields, the effective Hamiltonian $H_{\text{eff}}$ has a very simple form. Using these hopping parameters and the formula in Appendix B, we have

$$H_{\text{eff} 2} + H_{\text{eff} 3} = \frac{4}{g^2 N^2_q C^2} P \left\{ \rho(x) \rho(x + i) + 2 J_{x} \cdot J_{x + i} \right\}$$

$$+ \frac{1}{N^2_q} \rho(x) \rho(x + i \pm 2) + 2 J_{x} \cdot J_{x + i \pm 2} \right\}$$

$$\left[ \mathcal{J}_{x} \cdot \left( \mathcal{J}_{x + 1} \times \mathcal{J}_{x + 1 \pm 2} \right) \right]$$

$$- \frac{128}{g^2 (N^2_q C^2)^3} P \left\{ \mathcal{J}_{x} \cdot \left( \mathcal{J}_{x + 1} \times \mathcal{J}_{x + 1 \pm 2} \right) \right\}.$n

where $J_{x} \times J_{x + 1}$ is the generalized spin operators of SU($N^2_q$). Therefore the effective Hamiltonian (2.34) is Heisenberg model of SU($N^2_q$) magnets. As the coefficients of NN and NNN terms are both positive, this spin model has frustration.

In Sect.3, we shall consider the simplest case $N^2_q = 2$. As $N^2_q \gg 1$, we can treat the spin operators as the classical ones. We shall study the ground state of the Heisenberg spin model, and find a phase transition from the Neel-ordered state to a new phase with chiral-spin order.
3 Phase Structure of Frustrated Heisenberg Model

In Sect.2, we obtained the effective spin model of the massive QCD$_4$ by integrating out the gauge field. When $\delta = 0$, i.e., there is no NNN terms, the ground state is expected to have the Neel order. This expectation has been verified for the case $N_C \gg N_\perp$ [7]. The existence of the Néel order means the spontaneous breakdown of the chiral symmetry (2.15). As $\delta$ and $1/g^2$ increase in the Heisenberg Hamiltonian (2.34), we can expect that new phase with chiral-spin order $J(x) \cdot \{J(x+1) \times J(x+1+2)\} \neq 0$ will appear. We shall call this phase chiral-spin state.

In order to investigate the phase structure of the model (2.34), we divide the lattice into four sublattices and label sites as follows,

<table>
<thead>
<tr>
<th>$(x_1, x_2)$</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>(even, even)</td>
<td>1</td>
</tr>
<tr>
<td>(odd, even)</td>
<td>2</td>
</tr>
<tr>
<td>(odd, odd)</td>
<td>3</td>
</tr>
<tr>
<td>(even, odd)</td>
<td>4</td>
</tr>
</tbody>
</table>

We have calculated $\mathcal{H}_{\text{class}}$, for various configurations of $S$, $\{i = 1, \ldots, 4\}$ and found that the following configuration, which is parameterized by an angle $\theta$, has the lowest classical energy for most of interesting parameter region.

\[
\begin{align*}
S_1 &= \langle |S| \sin \theta, 0, |S| \cos \theta \rangle, \\
S_2 &= \langle 0, |S| \sin \theta, -|S| \cos \theta \rangle, \\
S_3 &= \langle -|S| \sin \theta, 0, |S| \cos \theta \rangle, \\
S_4 &= \langle 0, -|S| \sin \theta, -|S| \cos \theta \rangle.
\end{align*}
\]

We shall now rewrite the Hamiltonian by substituting the above configuration (3.5),

\[
\mathcal{H}_{\text{class}} = \mathcal{H}_{\text{class, 2}} + \mathcal{H}_{\text{class, 3}}.
\]
\[ \mathcal{H}_{\text{class} 2} = \frac{64|S|^2}{g^2 N_c C_2^3} \left\{ -t^2 \cos^2 \theta + i (\cos^2 \theta - \sin^2 \theta) \right\}. \] (3.6)

\[ \mathcal{H}_{\text{class} 3} = \frac{-496|S|^3}{g^3 (N_c C_2)^3} t^2 i \sin^2 \theta \cos \theta. \]

Here, it should be remarked that if we start with parity-symmetric hopping terms instead of (2.13), \( \mathcal{H}_{\text{class} 2} \) remains unchanged but \( \mathcal{H}_{\text{class} 3} \) vanishes. In the range \( 0 \leq \theta \leq \pi \), there is one-to-one correspondence between \( \cos \theta \) and \( \theta \). Therefore, we shall change the variables \( \theta \rightarrow y = \cos \theta \). The Hamiltonian is

\[ \mathcal{H}_{\text{class}} = C y^3 + (2T - 1)y^2 - Cy - T, \] (3.7)

where

\[ C = \frac{256i}{96 N_c C_2^3} (> 0), \quad T = \frac{t^2}{r^2}. \] (3.8)

The local minimum appears naively at

\[ y_{\text{min}} = \frac{1}{3C} \left\{ -(2T - 1) + \sqrt{(2T - 1)^2 + 3C^2} \right\}. \] (3.9)

Since the condition for the existence of the chiral-spin ordered phase is \( 0 < y_{\text{min}} < 1 \), the phase structure is summarized as follows,

1. \( T < \frac{1}{2} \)
   - \( C > (1 - 2T) \): Chiral-spin order
   - \( C \leq (1 - 2T) \): Neel order
2. \( T = \frac{1}{2} \)
   - \( \forall C \): Chiral-spin order
   - including the limit \( C \rightarrow 0 \)
3. \( T > \frac{1}{2} \)
   - \( \forall C \): Chiral-spin order
   - excepting the limit \( C \rightarrow 0, y_{\text{min}} \rightarrow 0 (\theta = \frac{\pi}{2}) \)

In the last case \( T > \frac{1}{2} \) in the limit \( C \rightarrow 0 \), the ground state is neither the chiral-spin nor Neel ordered phase, but a new phase.

Frustrated Heisenberg models with the NN and NNN couplings \( S(x) \cdot S(x + \hat{i}) \). \( S(x) \cdot S(x + i \cdot j) \) have been discussed in various contexts and by various methods. It may be expected that in some parameter region the chiral-spin order appears. For large \( |S| \), the classical configuration gives (almost) correct result for the ground state.

As is in the previous case, the result of the phase structure of the system \( \mathcal{H}_{\text{class} 2} \) without \( \mathcal{H}_{\text{class} 3} \) is obtained as

1. \( T < \frac{1}{2} \)
   - Neel order
2. \( T = \frac{1}{2} \)
   - there is no definite ground state because \( \mathcal{H}_{\text{class} 2} = \text{constant} \)
3. \( T > \frac{1}{2} \)
   - the phase with \( \theta = \frac{\pi}{2} \)

Therefore, there is no chiral-spin phase in the present case contrary to the naive expectation. This means that the parity-breaking hopping term is essential for the existence of the chiral-spin phase.

Now we shall show graphically the numerical calculations, which support the above discussion. In all calculations, we set \( N_c = 3, C_2 = \frac{N_c - 1}{2N_c} \) and \( t = 1 \).

In Fig. 1, we can observe that phase transition occurs and the chiral-spin order is favored rather than Neel order as \( T = \frac{1}{2} \) is getting large.

In Fig. 2-1, 2-2 and 2-3, the value of energy is given as a function of \( \theta \) in the cases (1), (2) and (3), respectively.

Fig. 3 shows that, in the system with \( \mathcal{H}_{\text{class} 2} \) only, there is no chiral phase, and that there are two kinds of phases, the Neel ordered phase and the \( \theta = \frac{\pi}{2} \) phase.
4 Chiral spin order and fractional statistics

In the preceding sections, we showed that dynamics of the strongly-coupled massive QCD$_3$ is described by the frustrated Heisenberg model (2.34). Then, the ground state of the effective model is studied by the classical approximation and it was shown that there exists a phase with non-vanishing chiral-spin order $\Delta \equiv \langle S_1 \cdot (S_2 \times S_3) \rangle \neq 0$.

In this section, we shall study how this chiral spin state looks like in the original QCD$_3$.

In order to see the gauge dynamics, it is quite useful to introduce a spectator field, $\phi(x)$, into the system. We add the following term $H_S$ of the spectator field to the Hamiltonian,

$$H_S = \sum_{r,k} \left\{ r_{[r,k]} \phi(x+k)U[x,k] \phi(x) + \text{h.c.} \right\}, \quad (4.1)$$

where $r_{[r,k]}$'s are given similarly as $t_{[r,k]}$. It is straightforward to obtain an effective spin Hamiltonian including the spectator field by integrating out the gauge field. The term which describes hopping of the spectator field is given by

$$H_{\text{hop}} = -\frac{2}{g^2 N_C C_2} \sum_{r,k} \left\{ t_{[r,k]} r_{[r,k]} \phi^b(x+k) \phi^a(x+k) \phi^b(x) + \text{h.c.} \right\}. \quad (4.2)$$

From eq.(4.2), it is obvious that the composite operator $\phi^a(x) \phi^b(x+k)$ plays a role of the gauge field on the link $[r,k]$. Let us consider a fundamental triangle plaquette, e.g., $(1,2,3) = (x,x+i,x+i+j) (i \neq j)$. As the spectator travels around the plaquette as $1 \to 2 \to 3$, the following "phase factor" appears, which corresponds to the Wilson loop in the usual gauge theory

$$P_{l_{123}} = -\frac{8}{g^6 (N_C C_2)^3} 2 \pi \delta r \frac{\phi^b_{[1]}(3) \phi^a_{[2]}(1) \phi^b_{[3]}(3) \phi_{[3]}^b(1) \phi_{[3]}^b(2)}{\langle C_{\text{tr}} \rangle}, \quad (4.3)$$

where we have used formulas in Appendix B. Similarly for $1 \to 3 \to 2$

$$P_{l_{132}} = -\frac{8}{g^6 (N_C C_2)^3} 2 \pi \delta r \frac{\phi^b_{[1]}(3) \phi^a_{[2]}(1) \phi^b_{[3]}(3) \phi_{[3]}^b(1) \phi_{[3]}^b(2)}{\langle C_{\text{tr}} \rangle}. \quad (4.4)$$

From eqs.(4.3) and (4.4), straightforward calculation shows

$$P_{l_{123}} - P_{l_{132}} = \frac{32}{g^6 (N_C C_2)^3} \delta r \frac{\phi_{[3]}^b(1) \phi_{[3]}^b(2)}{\langle C_{\text{tr}} \rangle}. \quad (4.5)$$

Similarly, for a fundamental square plaquette $(1,2,3,4) = (x,x+i,x+j,x+j)$

$$P_{l_{124}} - P_{l_{132}} = \frac{32}{g^6 (N_C C_2)^3} \delta r \frac{\phi_{[3]}^b(1) \phi_{[3]}^b(2)}{\langle C_{\text{tr}} \rangle}. \quad (4.6)$$

where $\rho_{[x]} = \rho \pm N_C$. Therefore in the chiral-spin ordered state, the spectator field feels a non-vanishing magnetic flux perpendicular to the 2-dimensional space.

The above magnetic field is actually a color magnetic flux of the gauge field $U[x,k]$. This can be shown by going to the (continuous time with $A_0 = 0$ gauge) Lagrangian formalism. By adding source term $tr J[x,k] U[x,k]$ to the Lagrangian, we can integrate out the gauge field in the partition function as in the Hamiltonian formalism. As a result, terms like $t_{[r,k]} r_{[r,k]} \phi^a(x) \phi^b(x+k) J_{[r,k]}^a$ appear. Differentiating the partition function $Z$ with respect to the source $J[x,k]$, we obtain

$$\frac{1}{Z} \delta \frac{\delta}{\delta J_{[r,k]}^a} \cdots Z = \langle tr J_{[r,k]}^a \cdots \rangle = \langle tr r_{[r,k]} t_{[r,k]} \phi^a(x) \phi^b(x+k) \cdots \rangle. \quad (4.7)$$

Therefore, the chiral spin order $\Delta$ is essentially the color magnetic flux penetrating the fundamental triangle plaquette

$$\delta \sim \langle tr \left( \prod_{[1]} U \right) \rangle - \langle tr \left( \prod_{[1]} U \right) \rangle \quad (4.8)$$

We have integrated out the gauge field assuming the strong coupling. However, the above result means that the massive QCD$_3$ has the ground state in which color
magnetic field is spontaneously generated. Similar phenomenon has been observed in the Abelian Maxwell-CS theory coupled with fermions by RPA-type calculation[9]. In the ordinary confinement phase, color electric flux is squeezed and its conjugate magnetic field fluctuates violently. Therefore, dynamical generation of the magnetic field suggests that the chiral-spin state is a deconfinement phase. It should be remarked here that in some cases even in the strong-coupling limit system is not in confinement phase, as explained in the Introduction[8].

We have not considered effect of the spectator on the ground state of the chiral-spin phase so far. As explained in Sect.2 and Appendix B, the ground state of the Hamiltonian (2.34) is a half-filled state of quarks, and states at each site are constructed by the baryon operator as

\[ |A_1, \ldots, A_{Nc} > = c^{|a_{1} \ldots a_{Nc}|, \psi_{1}^{a_{1}}, \ldots, \psi_{Nc}^{a_{Nc}}|0 >. \]

which form the SU(2) representation of spin \( \frac{Nc}{2} \). Let us consider the spectator sitting on site \( x \). Obviously states at the site \( x \) are given as

\[ |A_1, \ldots, A_{Nc-1} > = c^{|a_{1} \ldots a_{Nc-1}|, \psi_{1}^{a_{1}}, \ldots, \psi_{Nc-1}^{a_{Nc-1}}, \rho_{Nc}^{a_{Nc}}|0 >. \]

and they form the SU(2) representation of spin \( \frac{Nc}{2} \). Therefore magnitude of the spin operator \( S(x) \) changes from \( \frac{1}{2} Nc \) to \( \frac{1}{2} (Nc - 1) \) by the existence of the spectator.

From the above consideration, it is obvious that the spectator \( \phi(x) \) decreases the chiral-spin order \( \Delta \) and, consequently, the magnetic flux around it by relative amount \( 1/Nc \). This means that the spectator carries effectively the \( 1/Nc \) (color) magnetic flux as well as the (color) electric charge. Therefore, the field \( \phi(x) \) has \( 1/Nc \) fractional statistics by the Aharonov-Bohm effect, as it is explained in Ref.[10].

There exists only the confinement phase in the \((2+1)\)-dimensional SU(\( Nc \)) (U(\( Nc \)) non-Abelian gauge theory without CS term. However, it is naturally expected that CS term makes the gauge field massive and the perturbative phase exists at least for small gauge coupling. The above picture of the chiral-spin phase apparently corresponds to the perturbative phase of the non-Abelian Maxwell-CS theory. Therefore it seems that the Neel-chiral-spin phase transition in the effective spin model corresponds to the confinement-deconfinement phase transition in the original gauge theory.

5 Discussion

In this paper, we studied the massive QCD \( 3 \) on a lattice. By integrating out the gauge fields, we derived effective model, which is generalized Heisenberg model of antiferromagnets. We investigated the model for large \( Nc \) and found that there is a phase transition from the Neel state to the chiral-spin state, as the mass getting large.

In the massless limit of QCD, it is believed that chiral symmetry is spontaneously broken in the confinement phase, and confinement-deconfinement phase transition is the same with that of chiral-symmetry restoration. This indicates that the Neel state for small quark mass in the present model is confinement phase with spontaneous breaking of chiral symmetry. On the other hand, for large mass, chiral symmetry is not a good symmetry any more, and non-vanishing expectation value of the spin operator \( S \) does not mean that the system is in confinement phase. As discussed in Sect.4, a spontaneous magnetic field is generated in the chiral-spin state. This suggests that the chiral-spin state corresponds to deconfinement perturbative phase.

Of course, this result is obtained by the strong-coupling expansion, and more detailed investigations, including numerical study, are required to get conclusive result.

ACKNOWLEDGMENT

One of the authors (11) exhibits his gratitude to Prof. T. Matsui for valuable discussions.
Appendix

A Perturbation and Effective theory

In this appendix, we shall consider perturbation expansion in the Hamiltonian formalism and derive effective theory. Let us divide the Hamiltonian into two parts, a leading-order part $H_0$ and a next-to-leading order part $\hat{H}$, which we treat as a perturbation. We introduce the projection operator $Q$ and the operator $G_0$.

$$Q = \sum_{k \in k_0} |k\rangle\langle k|,$$

$$G_0 = Q(E_0 - H_0)^{-1}Q,$$

(A.1) (A.2)

where $|i\rangle$ is an eigenstate of $H_0$ with the eigenvalue $E_i$ and $E_0$ is the minimum eigenvalue of $H_0$. Expectation value of the total Hamiltonian $H = H_0 + \hat{H}$ in the ground state is approximated by the following formula,

$$\langle G | H | G \rangle = E_0 + \langle g | \hat{H} | g \rangle + \langle g | \hat{H} G_0 | g \rangle$$

$$+ \langle g | \hat{H} G_0 \hat{H} G_0 H | g \rangle - \langle g | \hat{H} | g \rangle \langle g | \hat{H} G_0^2 | g \rangle$$

$$+ \langle g | \hat{H} G_0 \hat{H} G_0 H | g \rangle$$

$$\langle g | \hat{H} | g \rangle \langle g | \hat{H} G_0 H | g \rangle + \langle g | \hat{H} G_0^3 H | g \rangle$$

$$+ \langle g | \hat{H} G_0^2 \hat{H} G_0 H | g \rangle + \cdots.$$  

(A.3)

where $|G\rangle$ is the ground state of $H$, and $|g\rangle$ is an eigenstate of $H_0$ with the eigenvalue $E_0$.

If $\langle g | \hat{H} | g \rangle = 0$, then

$$\langle G | H | G \rangle = E_0 + \langle g | \hat{H} G_0 H | g \rangle + \langle g | \hat{H} G_0 \hat{H} G_0 H | g \rangle$$

$$+ \langle g | \hat{H} G_0 \hat{H} G_0 \hat{H} G_0 H | g \rangle$$

$$- \langle g | \hat{H} G_0 H | g \rangle \langle g | \hat{H} G_0^2 \hat{H} | g \rangle + \cdots.$$  

(A.4)

In this case, to the fourth order of the perturbation expansion and supposing that the term $\langle g | \hat{H} G_0 H | g \rangle $ can be neglected, the eigenvalue problem of $H$ is equivalent to that with the following effective Hamiltonian, which operates on the subspace composed of all the eigenstates of $H_0$ with the eigenvalue $E_0$.

$$H_{\text{eff}} = PHG_0 \hat{H} P + PHG_0 \hat{H} G_0 \hat{H} P + PHG_0 \hat{H} G_0 H G_0 \hat{H} P.$$  

(A.5)

where $P$ is the projection operator on the subspace.

B Fermionic representation of $SU(N_L)$

We introduce a basis for the $SU(N_L)$ algebra which is given by fermion bilinear operators,

$$J^A_A(x) = \frac{1}{2} \langle \psi^A_{\alpha}(x), \psi^{B_\alpha}(x) \rangle,$$

(B.1)

where $A, B = 1, \cdots, N_L$ and $\alpha, \beta = 1, \cdots, N_c$. The representation of the algebra on each site $x$ is fixed by specifying the local fermion number of the states,

$$\rho(x) = J_A^A.$$  

(B.2)

Let us consider the empty state $|0\rangle$, which has no fermions,

$$\psi^{\alpha_0}(x)|0\rangle = 0, \quad \forall A, \alpha, x$$  

(B.3)

and is singlet under $SU(N_L)$. The $SU(N_c)$ invariant states with $N_c$ fermions on a site $x$ are constructed as,

$$\epsilon^{\alpha_1 \cdots \alpha_{N_c}} \psi^A_{\alpha_1}(x) \cdots \psi^A_{\alpha_{N_c}}(x) |0\rangle,$$

(B.4)

and they form an irreducible representation of $SU(N_L)$ with a Young tableau with $N_c$ columns and one row, because the color-singlet creation operator, i.e., baryon operator

$$B^A_{\alpha_1 \cdots \alpha_{N_c}}(x) = \epsilon^{\alpha_1 \cdots \alpha_{N_c}} \psi^A_{\alpha_1}(x) \cdots \psi^A_{\alpha_{N_c}}(x)$$  

(B.5)
is symmetric with respect to the lattice flavor index. Fermi statistics allows at most $N_L$ color-singlet operators on each site. Thus, the allowed representation of the flavor SU($N_L$) algebra by color-singlet states on each site are the empty singlet and those with Young tableaux in Fig. 4.

For the gauge group U($N_C$) instead, gauge invariance requires that we impose the additional constraint

$$ G^a(x)\psi = 0, \quad \forall x. \quad (B.6) $$

This can be satisfied with states without electric fields, i.e., $\rho(x) = 0, \forall x,$ and this can be realized only for even $N_L.$ Then the allowed SU($N_L$) representation at each site $x$ is the Young tableau with $N_C$ columns and $N_L/2$ rows.

It is convenient to introduce the following operators

$$ J^i(x) \equiv \psi^{iA}(x) (T^i_{N_L})^A_B \psi^{B\lambda}(x) = J^i_B(x) (T^i_{N_L})^A_B, \quad (B.7) $$

where $T^i_{N_L} = T^i_{N_L}, \quad i = 1, \cdots, N^2_L - 1$ is a basis of the Lie algebra of SU($N_L$) in the fundamental representation normalized as

$$ T_{i} T_{j} = \frac{\delta_{i j}}{2}, \quad (B.8) $$

where

$$ (T^i_{N_L})^A_B (T^j_{N_L})^C_D = \frac{1}{2} \delta^A_D \delta^B_C - \frac{1}{2 N_L} \delta^A_C \delta^B_D. \quad (B.9) $$

we get the relation

$$ J^i_B(x) = \frac{1}{N_L} \rho(x) \delta^i_B + 2 J^i(x) (T^i_{N_L})^B_A. \quad (B.10) $$

We shall now consider representing particular products of fermionic operators in terms of $J^i, \rho.$ For convenience, we define the new symbols,

$$ \rho_+ \equiv \rho + \frac{N_L N_C}{2}, \quad \rho_- \equiv \rho - \frac{N_L N_C}{2}. \quad (B.11) $$

$$ \psi^{1A}(x), \psi^{2A}(x + i), \psi^{3A}(x + i + j), \psi^{4A}(x + j), \psi^{5A}(x + i + j + k), \psi^{6A}(x + j + k), \psi^{7A}(x + j + k + l), \psi^{8A}(x + k), \psi^{9A}(x + k + l), \psi^{10A}(x + k + l + m). \quad (B.12) $$

$$ J^i \equiv J^i, \quad J^i (J^j) \equiv f^{ijk} J^j J^k, \quad (B.13) $$

$$ J^i (J^j) \equiv f^{ijk} J^j J^k. \quad (B.14) $$

where $k, l, m, n = 1, \cdots, 4$ and $f^{ijk}$ is the structure constant in the fundamental representation $[N_L]$ of SU($N_L$). The products of fermionic operators that we are interested in are

$$ K^{k\lambda}_{(\alpha \beta \gamma \delta)} = \text{Tr}[M_{\alpha} M_{\beta} M_{\gamma} M_{\delta}], \quad (B.15) $$

$$ K^{k\lambda}_{(\alpha \beta \gamma \delta)} = \text{Tr}[M_{\alpha} M_{\beta} M_{\gamma} M_{\delta}]. \quad (B.16) $$

These operators are written in terms of $J^i$ and $\rho.$

$$ K^{k\lambda}_{(\alpha \beta \gamma \delta)} = \text{sing}(\alpha \beta) \left\{ \frac{k}{N_L} / \frac{\rho_+ \rho_- + 2 J^i \cdot J^i} \right\}, \quad (B.17) $$

$$ K^{k\lambda}_{(\alpha \beta \gamma \delta)} = \text{sing}(\alpha \beta \gamma) \left\{ \frac{k}{N_L} / \frac{\rho_+ \rho_- + 2 J^i \cdot J^i} \right\}, \quad (B.18) $$

$$ K^{k\lambda}_{(\alpha \beta \gamma \delta)} = \text{sing}(\alpha \beta \gamma \delta) \left\{ \frac{k}{N_L} / \frac{\rho_+ \rho_- + 2 J^i \cdot J^i} \right\}, \quad (B.19) $$
where \( \text{sign}(aJ\cdots) \) is +1 if the number of \( -\) is even, and −1 if the number of \( -\) is odd. Using these formulas, we also get

\[
\begin{align*}
\mathcal{K}_{(\alpha \beta)}^{\mu \nu} &= \mathcal{K}_{(\alpha \beta \gamma)}^{\mu \nu} = \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu}, \\
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} &= \mathcal{K}_{(\gamma \delta \beta \alpha)}^{\mu \nu}, \\
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \downarrow &= \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu}, \\
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \uparrow &= \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu},
\end{align*}
\]

(B.20)

\[
\begin{align*}
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} - \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \downarrow &= \text{sign}(\alpha \beta \gamma) \{ \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \downarrow \}, \\
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \uparrow &= \text{sign}(\alpha \beta \gamma) \{ \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \uparrow \},
\end{align*}
\]

(B.21)

where \( \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \) is defined as

\[
\begin{align*}
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} &= \text{sign}(\alpha \beta \gamma \delta) \{ \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \downarrow \}, \\
\mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \uparrow &= \text{sign}(\alpha \beta \gamma \delta) \{ \mathcal{K}_{(\alpha \beta \gamma \delta)}^{\mu \nu} \uparrow \},
\end{align*}
\]

(B.22)

References


Figure Captions

Fig.1: Energy is shown as a function of $\theta$ and $T = \frac{\eta}{g^2}$. The gauge coupling is fixed as $g = 1.2$. State with $\theta = 0$ corresponds to the Neel-ordered phase. Phase transition occurs as $T$ is getting large to state with non-zero $\theta$.

Fig.2-1, 2-2 and 2-3: Energies are shown for various values of $T$ and the gauge coupling $g$. In most of the cases, the chiral-spin state has $\theta \sim 1$.

Fig.3: For the spin system with $\mathcal{H}_{\text{class}}$ only, energy is shown as a function of $\theta$ and $T$.

Chiral-spin state does not appear as the ground state.

Fig.4: Young tableaux of color-singlet states for $\text{SU}(N_c)$.
$T = 1/3, g = 1.2$

Energy

$T = 1/3, g = 8$

Energy

$T = 1/2, g = 1.2$

Energy

$T = 1/2, g = 8$

Energy

Fig. 2-1

Fig. 2-2
Energy

(a)

Energy

(b)

Fig. 2-3

Fig. 3

Fig. 4

$T = 0.6, g = 1.2$

gauge coupling $g = 1.2$