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Gauge Theory of Non-Relativistic Fermions and Its Low-Energy Behavior

Ikuro Ichinose^{*1} and Tetsuo Matsui^{†2}

^{*}Institute of Physics, University of Tokyo, Komaba, Tokyo 153 Japan

[†]Department of Physics, Kinki University, Higashi-Osaka 577 Japan

Abstract

System of $(2+1)$ dimensional non-relativistic fermions with gauge interaction is studied by the renormalization-group analysis. Fluctuations of long-wavelength excitations of the gauge field are controlled as k^{2-b} , where k is the magnitude of momentum of gauge boson and b is a parameter of the present system. This model is motivated by the work of Halperin, Lee and Read, which discussed a $(2+1)$ -dimensional electron system in an external magnetic field with filling factor of Landau level $\nu = \frac{1}{2}$. The β function of gauge coupling and the anomalous dimension of fermion are systematically calculated, in the sense of $b-1$ expansion, both in the loop and the RPA- $1/N$ expansions (where N is the number of species of fermions). Results are different in these two calculations. In the loop expansion, there appears a non-trivial IR fixed point of gauge coupling for $b < 1$, but the anomalous dimension of fermion is vanishing. On the other hand, in the RPA calculation, renormalization of the gauge-fermion vertex does not occur, while the fermion wave function is renormalized because of a dissipative term in the dressed gauge propagator. These results are systematically understood by the Ward-Takahashi identity reflecting gauge invariance. Our calculations are compared with the recent work of Nayak and Wilczek.

¹e-mail address: ichinose@tansei.cc.u-tokyo.ac.jp

²e-mail address: matsui@phys.kindai.ac.jp

S 509

1 Introduction

Landau's Fermi liquid theory works quite successfully for a large variety of electron systems. Stability of the Fermi liquid is now well understood in the language of renormalization group[1]. Instability appears only in the superconducting channel of electron pair with momenta $\vec{k}$ and $-\vec{k}$ near Fermi surface, if interactions between electrons are all short-range.

However, behavior of the metallic phase of high- T_c cuprate are close to but definitely different from those of Fermi liquid. Interesting possibility is the appearance of gauge bosons that are generated dynamically as a result of the strong correlation between electrons[2]. The relationship between the dynamics of these gauge bosons and the phenomenon of charge-spin separation was recently discussed by the present authors[3].

Another interesting example of non-Fermi liquid is a 2-dimensional electron system in an external magnetic field with Landau filling factor $\nu = \frac{1}{2}$. Halperin, Lee and Read [4] studied this system by introducing Chern-Simons (CS) gauge field, being motivated by the analyses of the fractional quantum Hall effect [5]. They studied this "electron"- (CS) gauge system by the random phase approximation (RPA), and showed that anomalous behavior appears in the "electron" propagator due to interactions with CS gauge field. It may be expected that the existence of gauge field in this system is observed experimentally.

Similar models of fermions and the conventional Maxwell gauge field have been studied by Polchinski[6]. By using $1/N$ expansion, where N is the number of species of fermions, he found anomalous properties of fermion propagator, which appear due to the interaction with a transverse gauge boson. However, he did not study radiative corrections of the fermion-gauge boson vertex.

Very recently, being motivated from the above studies, Nayak and Wilczek [7] investigated related models by the renormalization-group method, and they suggested that there is a non-trivial infrared (IR) fixed point of the gauge-coupling β function.

They concluded that this fixed point essentially corresponds to the marginal fermi liquid [8], i.e., the fermion propagator has a branch cut rather than a pole. To obtain this result, they used a straightforward loop expansion, and certain specific parameterization of momenta near Fermi surface.

We examined the calculations of Nayak and Wilczek and found some difficulties to obtain their results, especially on the wave function renormalization of fermion field. Low-energy behavior of gauge-fermion systems is certainly of current interest. It is very important to study it by various methods and from various points of view, and each obtained results should be carefully re-examined. In this paper, we shall study a system of non-relativistic fermions with gauge interaction, which is essentially the same with the model of Nayak and Wilczek[7], by renormalization-group method. Fluctuations of gauge field are controlled as k^{2-b} , where k is the magnitude of gauge boson momentum and b is a parameter of the model. We shall employ both the loop expansion and the RPA- $1/N$ expansion. Final results are *different* in these two calculations. In both cases, however, the gauge interaction is not shielded (i.e., not becomes short-range) and it influences drastically the low energy properties of the model. In the loop expansion, the gauge-fermion coupling is renormalized. However, anomalous dimension of fermion is vanishing. On the other hand, in the RPA calculation, anomalous behavior of fermion appears because of a dissipative term in the corrected gauge propagator. However, renormalization of the gauge-fermion vertex does not occur. These results are systematically understood by the Ward-Takahashi identity reflecting the gauge invariance.

This paper is organized as follows. In Sect.2, model is introduced and Feynman rules are given. In Sect.3, fermion propagator and gauge coupling are studied by the loop expansion. At one-loop level, renormalization of fermi velocity and the gauge coupling appears. However, the wave-function renormalization of fermion does not appear. The β -function of gauge coupling is calculated. A non trivial ultraviolet(UV) fixed point exists for $b > 1$, while a non-trivial IR fixed point appears for $b < 1$.

Comparisons with the results of Nayak and Wilczek[7] is presented at the end. In Sect.4, the model is studied by the $1/N$ -RPA method. Gauge-field propagator is calculated in the leading order of $1/N$ or RPA. By using this dressed gauge propagator, radiative corrections to fermion propagator and the gauge-fermion vertex are studied in the next-leading order of $1/N$. Landau's damping term (dissipative term) in the dressed gauge propagator plays an important role. Actually, there appears wave function renormalization of fermion. The gauge coupling remains *marginal* for $b < 1$. For $b > 1$, it is renormalized to zero at low energy, i.e., to the Fermi liquid theory. Section 5 is devoted to discussion.

2 Model and Feynman Rules

In this paper we study the (2+1)-dimensional gauge theory of non-relativistic fermions, the partition function Z of which is given by

$$\begin{aligned} Z &= \int [d\psi][dA_i] \prod \delta(\partial_i A_i) \exp\left(\int_0^\beta d\tau A\right), \\ A &= -\int d^2x \bar{\psi}(\partial_\tau - \mu_c - \frac{1}{2m} D_i D_i) \psi(x, \tau) \\ &\quad - \frac{1}{2} \int d^2x \int d^2y B(x, \tau) V_B(x-y) B(y, \tau), \\ D_i &= \partial_i + ig A_i, \\ B &= \partial_1 A_2 - \partial_2 A_1, \end{aligned} \quad (2.1)$$

where $\psi(x, \tau)$ is a Grassmann number describing non-relativistic fermi particles¹ $A_i(x, \tau)$ is the abelian gauge field where $i(= 1, 2)$ is the direction index. D_i is the covariant derivative and g is the gauge-coupling constant. $B(x, \tau)$ is the magnetic field in the plane. μ_c is the chemical potential to adjust the fermion density $\rho(x, \tau) \equiv \bar{\psi}\psi$ to a given number. m is the band mass. The magnetic field interacts via a long-range

¹One may introduce N species of fermions. The calculations in Sections 3,4 are applicable if one multiplies a factor N for each fermion loop as long as there are no asymmetry among species.

potential V_B ;

$$\begin{aligned} V_B(x-y) &= v_B \int \underline{dk} \frac{\exp ikx}{k^b}, \\ \underline{dk} &\equiv (2\pi)^{-2} dk_1 dk_2, \end{aligned} \quad (2.2)$$

where v_B is a parameter.² The exponent b is an important parameter of the theory. The action is invariant under the following time-independent local $U(1)$ gauge transformation;

$$\begin{aligned} A_i(x, \tau) &\rightarrow A_i(x, \tau) + \partial_i \Lambda(x), \\ \psi(x, \tau) &\rightarrow \exp(i\Lambda(x)) \psi(x, \tau). \end{aligned} \quad (2.3)$$

We fix this freedom by taking the transverse gauge $\partial_i A_i = 0$ as enforced by the delta functions in Z .

The model (2.1) is motivated by the work of Halperin, Lee and Read[4], which discussed (2+1)-dimensional electron system in an external magnetic field with filling factor of Landau level $\nu = 1/2$. In the present paper, however, we study the model from more general point of view, and see how gauge field influences low-energy behavior of electrons with Fermi surface. We believe that results obtained by the present study will give important insight also for investigation on metallic phase of high- T_c materials.

We shall employ renormalization-group analysis. As it will be shown, we take $(1-b)$ as a small "expansion" parameter, and we extract singular terms in perturbation expansion in a spirit similar to the ϵ expansion[7, 9] for $d = 4 - \epsilon$ systems. Renormalization-group study with a momentum cutoff will be reported in a forthcoming paper[10].

We set up the perturbation theory assuming that the gauge-coupling g is small. We separate the action as $A = A_0 + A_1$ where the free part A_0 is obtained from A by

²Parameter v_B can be absorbed to the gauge coupling g . However, we have introduced it in order to present a clear scaling argument as given later.

setting $g = 0$. The fermion propagator is obtained from A_0 as

$$\begin{aligned}
G(x-y, \tau-\tau') &\equiv \langle \psi(x, \tau) \bar{\psi}(y, \tau') \rangle_0 \\
&= \int \frac{d\mathbf{k}}{\beta} \frac{1}{\beta} \sum_{n=\text{odd}} G(k, \omega_n) \exp[i\mathbf{k}(x-y) + i\omega_n(\tau-\tau')], \\
\omega_n &\equiv \frac{\pi n}{\beta}, \\
G(k, \omega_n) &\equiv \frac{\exp(-i\omega_n \eta)}{i\omega_n + E_k}, \\
E_k &\equiv \frac{1}{2m} k^2 - \mu_c,
\end{aligned} \tag{2.4}$$

where $\langle \dots \rangle_0$ means expectation value w.r.t. A_0 . The factor $\exp(-i\omega_n \eta)$, with η being an infinitesimal positive constant, comes from the ordering of path-integral quantization itself. This factor works as a convergence factor for some diagrams and we take the limit $\eta \rightarrow 0$ after the calculation.

The gauge-field propagator is given by

$$\begin{aligned}
\Delta_{ij}(x-y, \tau-\tau') &\equiv \langle A_i(x, \tau) A_j(y, \tau') \rangle_0 \\
&= \int \frac{d\mathbf{k}}{\beta} \frac{1}{\beta} \sum_{n=\text{even}} \Delta_{ij}(k, \omega_n) \exp[i\mathbf{k}(x-y) + i\omega_n(\tau-\tau')], \\
\Delta_{ij}(k, \omega_n) &\equiv \frac{1}{v_B k^{2-b}} (\delta_{ij} - \frac{k_i k_j}{k^2}).
\end{aligned} \tag{2.5}$$

Note the difference of the possible values n of Matsubara frequency ω_n , reflecting the antiperiodic and periodic boundary conditions, respectively.

The perturbed action $A_1 (\equiv A - A_0)$ contains two interaction vertices,

$$A_1 = \frac{1}{2m} \int d^2x \bar{\psi} (2ig A_i \partial_i - g^2 A_i^2) \psi. \tag{2.6}$$

The graphical symbols of these quantities are collected in Fig.1.

3 One-Loop Calculation

In the following sections, we shall study low-energy properties of the model (2.1) by the renormalization-group analysis in the $1-b$ expansion. In this section, we employ the straightforward loop expansion to obtain β -function of the gauge-coupling

constant and anomalous dimension of the fermion. We first show the one-loop calculation, and then discuss physical results.

3.1 Fermion propagator

The fermion correlation function,

$$\begin{aligned}
\hat{G}(x-y, \tau-\tau') &\equiv \langle \psi(x, \tau) \bar{\psi}(y, \tau') \rangle \\
&= \int \frac{d\mathbf{k}}{\beta} \frac{1}{\beta} \sum_{n=\text{odd}} \hat{G}(k, \omega_n) \exp[i\mathbf{k}(x-y) + i\omega_n(\tau-\tau')],
\end{aligned} \tag{3.1}$$

is evaluated up to the one-loop order at zero temperature ($\beta \rightarrow \infty$) as

$$\begin{aligned}
\hat{G}(k, \omega_n) &= G(k, \omega_n) - \frac{g^2}{2m} \Delta_{ii}(0, 0) G(k, \omega_n)^2 + \frac{g^2}{m^2} G(k, \omega_n)^2 \Sigma(k, \omega_n), \\
\Sigma(k, \omega_n) &\equiv \int \frac{d\mathbf{p}}{\beta} \frac{1}{\beta} \sum_{m=\text{even}} G(k-p, \omega_n - \epsilon_m) k_j (k-p)_i \Delta_{ij}(p, \epsilon_m),
\end{aligned} \tag{3.2}$$

where $\epsilon_m = \pi m / \beta$.

By using the formula (See Appendix for derivation)

$$\begin{aligned}
\frac{1}{\beta} \sum_{n=\text{even}} \frac{\exp(-i\omega_n \eta)}{i\omega_n + E} &= \frac{1}{\exp(\beta E) - 1} \rightarrow -\theta(-E) \quad (\beta \rightarrow \infty), \\
\frac{1}{\beta} \sum_{n=\text{odd}} \frac{\exp(-i\omega_n \eta)}{i\omega_n + E} &= -\frac{1}{\exp(\beta E) + 1} \rightarrow -\theta(-E) \quad (\beta \rightarrow \infty),
\end{aligned} \tag{3.3}$$

we get

$$\Sigma(k, \omega_n) = - \int \frac{d\mathbf{p}}{\beta} \theta(-E_{k-p}) \frac{k^2 \sin^2 \phi}{v_B p^{2-b}}, \tag{3.4}$$

where ϕ is the angle between the momenta k and p . (By the symbol k , we denote both the vector itself and its length. When some confusions may arise we specify them.) From (3.4), it is obvious that $\Sigma(k, \omega_n)$ does not depend on ω_n , because the integrand in (3.2) depends on it through the linear combination $\omega_n - \epsilon_n$. We are interested in the case where the momentum k is close to some fermi momentum k_0 , $k_0^2 = k_F^2$ where $k_F^2 / (2m) \equiv \mu_c$. Thus we expand $\Sigma(k, \omega_n)$ around $k = k_0$,

$$\Sigma(k, \omega_n) = \Gamma(k_0, \omega_n) + (k - k_0)_i \frac{\partial}{\partial k_i} \Gamma(k_0, \omega_n) + O((k - k_0)^2). \tag{3.5}$$

The first term is evaluated as

$$\begin{aligned}
\Sigma(k_0, \omega_n) &= -\frac{k_F^2}{(2\pi)^2 v_B} \int_{-\pi}^{\pi} d\phi \sin^2 \phi \int_0^{\infty} dp p^{b-1} \theta(2k_F \cos \phi - p) \\
&= -\frac{k_F^2}{(2\pi)^2 v_B} \int_{-\pi/2}^{\pi/2} d\phi \sin^2 \phi \int_0^{2k_F \cos \phi} dp p^{b-1} \\
&= -\frac{k_F^2}{(2\pi)^2 v_B} \int_{-\pi/2}^{\pi/2} d\phi \sin^2 \phi \frac{(2k_F \cos \phi)^b}{b} \\
&= -\frac{k_F^2 (2k_F)^b}{(2\pi)^2 v_B b} [C(b) - C(b+2)], \tag{3.6}
\end{aligned}$$

where we have used the formula [$\Gamma(x)$ is the usual gamma function];

$$\begin{aligned}
C(a) &\equiv \int_{-\pi/2}^{\pi/2} d\phi \cos^a \phi \\
&= \sqrt{\pi} \frac{\Gamma(\frac{a+1}{2})}{\Gamma(\frac{a+2}{2})}, \tag{3.7}
\end{aligned}$$

which is valid for $a > -1$. We note this first term is *not* singular at $b = 1$. In later calculation, we shall encounter $C(a)$ with $a \leq -1$. To prepare for such case, here we break the stream for a while and examine what the correct treatment of $C(a); a \leq -1$ is. Let us start by recalling that the well-defined case $a = -1 + \epsilon$ and small $\epsilon > 0$,

$$\begin{aligned}
C(-1 + \epsilon) &= \int_{-\pi/2}^{\pi/2} d\phi (\cos \phi)^{-1+\epsilon} \\
&\sim 2 \int_0^{x_0} dx x^{-1+\epsilon} \\
&= \frac{2}{\epsilon} x_0^\epsilon = \frac{2}{\epsilon} e^{\epsilon \ln x_0},
\end{aligned}$$

where x_0 is some finite constant. In the leading order of ϵ , $C(-1 + \epsilon)$ is approximately given as

$$C(-1 + \epsilon) \sim \frac{2}{\epsilon}. \tag{3.8}$$

This is common in the usual ϵ -expansion, i.e., logarithmic divergence is replaced with $\sim 1/\epsilon$.

On the other hand for $a = -1 - \epsilon$ ($\epsilon > 0$), the integral $C(-1 - \epsilon)$ has IR divergence, and an IR cutoff must be introduced. With the cutoff δ ,

$$C(-1 - \epsilon) = \int_{-\pi/2}^{\pi/2} d\phi (\cos \phi)^{-1-\epsilon}$$

$$\begin{aligned}
&\sim 2 \int_{\delta}^{x_0} x^{-1-\epsilon} \\
&= \frac{2}{\epsilon} (\delta^{-\epsilon} - x_0^{-\epsilon}) \\
&\sim \frac{2}{\epsilon} e^{-\epsilon \ln \delta}.
\end{aligned}$$

In the leading order of ϵ , one may conclude

$$C(-1 - \epsilon) \sim \frac{2}{\epsilon}. \tag{3.9}$$

Here we assumed that the limit $\delta \rightarrow 0$ is taken *after* the limit $\epsilon \rightarrow 0$ is taken. This assumption is reasonable because we confirmed that the prescription (3.9) gives rise to the results that agree with those obtained by another method, the decimation method [9] using momentum cutoff [10]. Therefore we shall use the formula

$$C(-1 + \epsilon) \sim \frac{2}{|\epsilon|} \tag{3.10}$$

for general ϵ in later calculations.³

Let us return to the expression (3.5). Its second term is evaluated as

$$\begin{aligned}
\frac{\partial}{\partial k_i} \Sigma(k_0, \omega_n) &= - \int dp \frac{1}{v_B p^{2-b}} [\delta(-E_{k_0-p}) \frac{(p-k_0)_i}{m} k_F^2 \sin^2 \phi \\
&\quad + \theta(-E_{k_0-p}) (2k_{0i} - \frac{2k_F \cos \phi}{p} p_i)], \\
&= - \int dp \frac{1}{v_B p^{2-b}} [\delta(p - 2k_F \cos \phi) \frac{k_F k_{0i}}{|\cos \phi|} (2 \cos^2 \phi - 1) \sin^2 \phi \\
&\quad + \theta(2k_F \cos \phi - p) \cdot 2k_{0i} \sin^2 \phi] \\
&= -\frac{k_i}{(2\pi)^2 v_B} \int_{-\pi/2}^{\pi/2} d\phi [2k_F^2 (2k_F \cos \phi)^{b-2} (2 \cos^2 \phi - 1) \sin^2 \phi \\
&\quad + 2 \sin^2 \phi \frac{(2k_F \cos \phi)^b}{b}] \\
&= -\frac{k_{0i} (2k_F)^b}{2(2\pi)^2 v_B} \left[-C(b-2) + (3 + \frac{2}{b})C(b) - (2 + \frac{2}{b})C(b+2) \right], \\
&\sim \frac{k_{0i} k_F}{(2\pi)^2 v_B} \frac{2}{|b-1|}, \quad (b \sim 1) \tag{3.11}
\end{aligned}$$

³This result is in strong contrast with the analytic continuation w.r.t. ϵ , which would read $C(-1 + \epsilon) \sim 2/\epsilon$.

where we decomposed p as $p_i = (p \cos \phi) k_0/k_F + (p \sin \phi) l_i$ (l_i is the unit vector perpendicular to k_0) and omitted the second term since the rest of integral is an even function of ϕ .

3.2 Gauge-Field Propagator

The gauge-field propagator

$$\begin{aligned}\hat{\Delta}_{ij}(x-y, \tau-\tau') &\equiv \langle A_i(x, \tau) A_j(y, \tau') \rangle \\ &= \int \frac{dk}{\beta} \frac{1}{\beta} \sum_{n=even} \hat{\Delta}_{ij}(k, \omega_n) \\ &\times \exp[ik(x-y) + i\omega_n(\tau-\tau')],\end{aligned}\quad (3.12)$$

is calculated up to the one-loop level as

$$\begin{aligned}\hat{\Delta}_{ij}(k, \omega_n) &= \Delta_{ij}(k, \omega_n) + \Delta_{il}(k, \omega_n) \Pi_{lm}(k, \omega_n) \Delta_{mj}(k, \omega_n), \\ \Pi_{lm}(k, \omega_n) &= \frac{g^2}{2m} G(x=0, \tau=0) \delta_{lm} - \frac{g^2}{m^2} \Omega_{lm}(k, \omega_n), \\ \Omega_{lm}(k, \omega_n) &\equiv \int \frac{dp}{\beta} \frac{1}{\beta} \sum_{m=odd} (k+p)_l p_m G(k+p, \omega_n + \epsilon_m) G(p, \epsilon_m) \\ &= \int \frac{dp}{\beta} \frac{1}{\beta} \sum_{m=odd} (k+p)_l p_m \left[\frac{1}{i(\omega_n + \epsilon_m) + E_{k+p}} - \frac{1}{i\epsilon_m + E_p} \right] \\ &\times \frac{\exp(-i(\omega_n + \epsilon_m)\eta)}{-i\omega_n + E_p - E_{k+p}} \\ &= - \int \frac{dp}{\beta} (k+p)_l p_m \frac{\theta(-E_{k+p}) - \theta(-E_p)}{-i\omega_n + E_p - E_{k+p}}.\end{aligned}\quad (3.13)$$

We are interested in the limit where the gauge-field momentum $k=0$ and $\omega_n=0$. Thus one might expand $\hat{\Delta}_{ij}(k, \omega_n)$ around these values;

$$\Omega_{lm}(k, \omega_n) = \Omega_{lm}(0, 0) + k_i \frac{\partial}{\partial k_i} \Omega_{lm}(0, 0) + \omega_n \frac{\partial}{\partial \omega_n} \Omega_{lm}(0, 0) + O(k^2, \omega_n^2). \quad (3.14)$$

However, a careful estimation of $\Omega_{lm}(k, \omega_n)$ that will be presented in the next section exhibits that it contains a nonanalytic term proportional to $|\omega_n|/k$. Therefore a naive expansion like (3.14) works only in the limit taken in the order $\lim(k \rightarrow 0) \lim(\omega_n \rightarrow 0)$ and *not* in the reversed ordering $\lim(\omega_n \rightarrow 0) \lim(k \rightarrow 0)$. We will encounter such

noncommutativity of ordering of limits also in calculating other quantities. Below we consider $\lim(k \rightarrow 0) \lim(\omega_n \rightarrow 0) \Omega_{lm}(k, \omega_n)$. By using the formula

$$\frac{d}{dE} \theta(-E) = -\delta(-E), \quad (3.15)$$

we get

$$\begin{aligned}\lim_{k \rightarrow 0} \lim_{\omega_n \rightarrow 0} \Omega_{lm}(k, \omega_n) &= - \int \frac{dp}{\beta} (k+p)_l p_m \frac{\theta(-E_{k+p}) - \theta(-E_p)}{E_p - E_{k+p}} \\ &= - \int \frac{dp}{\beta} p_l p_m \delta(\mu - \frac{1}{2m} p^2) \\ &= \frac{1}{2\pi} \int_0^\infty dp p \frac{p^2}{2} \delta_{lm} \frac{m}{p} \delta(p - k_F) \\ &= - \frac{m k_F^2}{4\pi} \delta_{lm}.\end{aligned}\quad (3.16)$$

This term cancels the first term in (3.13) since

$$\begin{aligned}G(0, 0) &= - \int \frac{dp}{\beta} \theta(-E_p) \\ &= - \frac{1}{2\pi} \int_0^{k_F} p dp = - \frac{k_F^2}{4\pi}.\end{aligned}\quad (3.17)$$

This implies that the mass of the transverse gauge field is vanishing as expected. The higher order terms in (3.14) are calculated in the next section.

3.3 Gauge coupling

The gauge vertex is defined as

$$\begin{aligned}\langle \psi(x, \tau) \bar{\psi}(y, \tau') A_i(z, \tau'') \rangle &\equiv \frac{1}{\beta^3} \sum_{\omega} \sum_{\omega'} \sum_{\omega''} \int \frac{dk}{\beta} \int \frac{dq}{\beta} \int \frac{du}{\beta} \Gamma_i(k, -q, u, \omega, -\omega', \omega'') \\ &\times \exp[i(kx - qy + uz) + i(\omega\tau - \omega'\tau' + \omega''\tau'')], \\ \Gamma_i(k, -q, u, \omega, -\omega', \omega'') &\equiv (2\pi)^2 \delta(k - q + u) \beta \delta_{\omega - \omega' + \omega'', 0} \tilde{\Gamma}_i(k, q, \omega, \omega') \hat{G}(k, \omega) \hat{G}(q, \omega').\end{aligned}\quad (3.18)$$

In the one-loop level, $\tilde{\Gamma}$ is given as

$$\tilde{\Gamma}_i(k, q, \omega, \omega') = \frac{ig}{m} \Delta_{ij}(q-k) i q_j + \Delta_{il}(q-k) [(\frac{-g}{m})^3 S_l - \frac{ig^3}{2m^2} T_l],$$

$$\begin{aligned}
S_l &\equiv \int d\underline{p} \frac{1}{\beta} \sum_{\epsilon} (q-p)_l G(k-p, \omega-\epsilon) G(q-p, \omega'-\epsilon) (k-p)_m q_n \Delta_{mn}(p, \epsilon). \\
T_l &\equiv \int d\underline{p} \frac{1}{\beta} \sum_{\epsilon} \Delta_{lm}(p, \epsilon) [G(q-p, \omega'-\epsilon) i q_m + i(k-p)_m G(k-p, \omega-\epsilon)],
\end{aligned} \tag{3.19}$$

with the sum $\epsilon = \pi m/\beta$ is over even integers m .

Let us consider S_l first. The summation over ϵ in S_l can be done as in (3.13);

$$S_l = \int d\underline{p} (q-p)_l (k-p)_m q_n \Delta_{mn}(p, \epsilon) \frac{-\theta(-E_{k-p}) + \theta(-E_{q-p})}{i(\omega' - \omega) + E_{q-p} - E_{k-p}} \tag{3.20}$$

We are interested in the case $\lim(q \rightarrow k)$ and $\lim(\omega \rightarrow \omega')$ with $k^2 = k_F^2$. Let us first consider the ordering $\lim(q \rightarrow k) \lim(\omega' \rightarrow \omega)$, which replaces the last factor of (3.20) to $-\delta(-E_{k-p})$;

$$\begin{aligned}
\lim_{q \rightarrow k} \lim_{\omega' \rightarrow \omega} S_l &= - \int d\underline{p} (q-p)_l (k-p)_m q_n \frac{1}{v_B p^{2-b}} \delta(-E_{k-p}) [\delta_{mn} - \frac{p_m p_n}{p^2}] \\
&= - \frac{1}{v_B (2\pi)^2} \int_{-\pi}^{\pi} d\phi \int_0^{\infty} dp p^{b-1} (k-p)_l \frac{m \delta(p - 2k_F \cos \phi)}{|k_F \cos \phi - p|} k_F^2 \sin^2 \phi \\
&= - \frac{k_l}{v_B (2\pi)^2} \int_{-\pi/2}^{\pi/2} d\phi (2k_F \cos \phi)^{b-1} (1 - 2 \cos^2 \phi) \frac{m}{k_F \cos \phi} k_F^2 \sin^2 \phi \\
&= - \frac{m k_l}{v_B (2\pi)^2} k_F^b 2^{b-1} [C(b-2) - 3C(b) + 2C(b+2)] \\
&\simeq - \frac{m k_l}{v_B (2\pi)^2} k_F \frac{2}{|b-1|}, \quad (b \simeq 1), \tag{3.21}
\end{aligned}$$

where the factor $(k-p)_l$ has been rewritten as $k_l(1 - p \cos \phi/k_F)$ before ϕ -integral, since the orthogonal component does not contribute as before. The last line shows that S_l has a singularity near $b = 1$.

Next let us consider the reversed ordering $\lim(\omega' \rightarrow \omega) \lim(q \rightarrow k)$. Here the last factor in (3.20) vanishes after $\lim(q \rightarrow k)$ is taken, so we conclude

$$\lim_{\omega' \rightarrow \omega} \lim_{q \rightarrow k} S_l = 0. \tag{3.22}$$

Another quantity, T_l , does not possess such noncommutativity and is calculated as

$$T_l = - \frac{2i k_l}{v_B} \int d\underline{p} \theta(-E_{k-p}) p^{b-2} \sin^2 \phi$$

$$\begin{aligned}
&= - \frac{2i k_l}{v_B (2\pi)^2} \int_{-\pi/2}^{\pi/2} d\phi \frac{(2k_F \cos \phi)^b}{b} \sin^2 \phi \\
&= - \frac{2i k_l}{v_B (2\pi)^2} \frac{(2k_F)^b}{b} [C(b) - C(b+2)], \tag{3.23}
\end{aligned}$$

where we have used $r_l \Delta_{lm} = 0$ and replaced the resulting tensor factor $k_l - p_l (p k_F \cos \phi)/p^2$ to $k_l(1 - \cos^2 \phi)$ since $p_l \rightarrow p \cos \phi (k_l/k_F)$. Thus T_l does not contain any singularities at $b = 1$.

3.4 Renormalization and Ward-Takahashi identity

Explicit calculation in the preceding subsections reveals that the dominant contributions of loop integrals come from the region such that (momentum of gauge boson) ~ 0 and $\cos \phi \sim 0$, where ϕ is the angle between fermion and gauge boson momenta. This fact is easily understood by the following consideration. Let us take the fermion momentum as $k = (k_1 \sim 0, k_2 \sim k_F)$. From the fermion-gauge coupling, the second component of gauge boson, $A_2(x, \tau)$, dominates in this fermion momentum region; i.e., A_i and k_i are parallel. This and the transversality condition of the gauge boson lead that the momentum of gauge boson should be perpendicular to the fermion momentum, which means $\cos \phi \sim 0$.

The above consideration justifies the (tree-level) scaling argument by Nayak and Wilczek[7]. In the region close to $k_0 = (0, k_F)$,

$$E_k = v_F (\bar{k}_2 + a k_1^2)|_{k_2=k_2-k_F}, \tag{3.24}$$

where $v_F = k_F/m$ and $a = (2k_F)^{-1}$. Renormalization-group transformation should be performed towards this point. By introducing the scaling parameter $s < 1$, the scaling dimensions of energy and momentum of fermion are given as

$$\omega \rightarrow s\omega$$

$$k_1 \rightarrow s^{1/2} k_1$$

$$\bar{k}_2 \rightarrow s \bar{k}_2$$

$$v_F \rightarrow v_F \quad (3.25)$$

$$E_k \rightarrow E_k. \quad (3.26)$$

By requiring that the quadratic terms of the action [see eq.(3.30) below] should be invariant under this scaling, the scaling dimensions of Fourier-transformed fields $\psi(k, \omega)$ and $A_2(p, \omega)$ are easily calculated as

$$[\psi] = -\frac{7}{4}, \quad [A_2] = \frac{b-7}{4}. \quad (3.27)$$

As explained above, dominant part of the fermion-gauge coupling is $gv_F\psi\psi A_2$, so

$$[g] = \frac{1-b}{4}. \quad (3.28)$$

In order to renormalize quantities calculated in the preceding subsections that have singularities at $b = 1$, we perform the renormalization-group transformation and introduce counter terms. Explicitly we introduce the renormalized fermion field ψ , the renormalization constants, Z , Z_{v_F} and Z_g , and the renormalized gauge coupling g through;⁴

$$\begin{aligned} \psi_0 &= Z^{1/2}\psi, \\ v_{F0} &= Z_{v_F}v_F \quad \text{or} \quad \frac{1}{m_0} = \frac{Z_{v_F}}{m}, \\ g_0 &= \mu^{\frac{1-b}{4}}g \frac{Z_g}{Z Z_{v_F}}, \end{aligned} \quad (3.29)$$

where we have introduced an energy scale μ to make g dimensionless. (We attach the suffix 0 to the bare quantities here and below, while quantities without suffix 0 indicate renormalized ones.) The action integral of (2.1) at $\beta = \infty$ in terms of Fourier

⁴Since the vacuum polarization diagrams of gauge-field propagator do not contain any singular terms at $b = 1$, we need not introduce the renormalization of gauge field. Also we shall not write down the formula for renormalization of the chemical potential $\mu_c = k_F^2/(2m)$ (or of the fermi momentum k_F) for simplicity, although the renormalization exists, since it does not affect at all the physical results we shall discuss below.

components then reads as

$$\begin{aligned} \int_0^\infty d\tau A_{\text{total}} &= - \int_{-\infty}^\infty d\omega d^2k \left\{ \psi^\dagger (iZ\omega + Z Z_{v_F} E_k - \mu_c) \psi \right\} \\ &\quad + \mu^{\frac{1-b}{4}} g Z_g \int d\omega d\omega' d^2k d^2q \psi^\dagger(k+q, \omega+\omega') \psi(k, \omega) \frac{2k_i + q_i}{2m} A_i(q, \omega') \\ &\quad - \frac{v_B}{2} \int d\omega d^2k \epsilon_{ij} \epsilon_{mn} k_i k_m k^{-b} A_j(k, \omega) A_n(-k, -\omega) + \dots \end{aligned} \quad (3.30)$$

where the part ... denotes the remaining interaction $g^2 A A \psi^\dagger \psi$. This total action is decomposed as $A_{\text{total}} = A_{\text{renormalized}} + A_{\text{counter}}$, where the renormalized action $A_{\text{renormalized}}$ is obtained by setting $Z = Z_{v_F} = Z_g = 1$ in A_{total} , while the counter term action A_{counter} is the rest. A_{counter} contains the so called counter terms associated with factors like $Z - 1$, which are to be treated as extra vertices in Feynman rules. The renormalization constants are determined from the previous calculations of fermion self-energy and gauge vertex function so that the counter terms cancel out the singular parts at $b = 1$ of these quantities. From (3.6) and (3.11), it is straightforward to obtain

$$\begin{aligned} Z &= 1, \\ Z Z_{v_F} &= 1 + \frac{g^2 v_F}{2\pi^2 v_B} \frac{1}{|1-b|}. \end{aligned} \quad (3.31)$$

As for the gauge-fermion vertex, the calculation in the previous subsection shows that the correction is singular or regular depending on the order of the limit $\omega' \rightarrow \omega, q \rightarrow k$. As will be explained just below, the renormalization constant Z_g is defined via the vertex function evaluated in the double limit $\lim(q \rightarrow k) \lim(\omega' \rightarrow \omega)$. Therefore, from (3.19) and (3.21), we get

$$Z_g = 1 + \frac{g^2 v_F}{2\pi^2 v_B} \frac{1}{|1-b|}. \quad (3.32)$$

Thus we have

$$Z Z_{v_F} = Z_g. \quad (3.33)$$

in the one-loop order. This expression of Z_g gives rise to important properties of the renormalized gauge coupling (see the discussion in the following paragraphs).

The equality (3.33) is guaranteed at any finite order by the gauge invariance. To see this we start with the following Ward-Takahashi(W-T) identity,

$$\begin{aligned} \partial_\mu < T(j_\mu(z)\psi_0^\dagger(x)\psi_0(y)) > \\ = \delta(z-x) < T(\psi_0^\dagger(x)\psi_0(y)) > + \delta(z-y) < T(\psi_0^\dagger(x)\psi_0(y)) >, \end{aligned} \quad (3.34)$$

where $j_\mu(z)$ is the current composed of the *bare* fields,

$$\begin{aligned} j_0 &= \psi_0^\dagger \psi_0, \\ j_i &= \frac{1}{2m_0} (\psi_0^\dagger \partial_i \psi_0 - \partial_i \psi_0^\dagger \cdot \psi_0), \quad (i = 1, 2). \end{aligned} \quad (3.35)$$

The derivation is straightforward by using the standard technique of path integrals. After Fourier transforming (3.34), we get

$$\begin{aligned} ip_\mu < j_\mu(p)\psi_0^\dagger(p+k)\psi_0(k) > \\ = < \psi_0^\dagger(k)\psi_0(k) > + < \psi_0^\dagger(k+p)\psi_0(k+p) >, \end{aligned} \quad (3.36)$$

where $p_\mu = (\omega, p_1, p_2)$. The action of the bare fields alone has the same form with (2.1). Therefore, the calculations given so far can be applied to (3.36) directly. Comparing singular parts in the both hands sides gives a relationship between the renormalization constants. From the spatial-component part of (3.36), we have

$$\begin{aligned} \text{LHS} &= \frac{1}{m_0} (2k \cdot p + p^2) Z_g, \\ \text{RHS} &= \frac{1}{m_0} (2k \cdot p + p^2) Z Z_{v_F}, \end{aligned}$$

which implies

$$Z_g = Z Z_{v_F}. \quad (3.37)$$

Here we compared the coefficients of $p_{1,2}$, so that one should have taken the limit $\lim(\omega' \rightarrow \omega)$ and set $\omega = 0$ first in order to throw away the irrelevant terms proportional to ω, ω' . Similarly, from the time-component part of (3.36) and the fact $v_F j_0 = j_2$ for $k = (0, k_F)$, we get

$$\tilde{Z}_g = Z, \quad (3.38)$$

where $\tilde{Z}_g$ is the "renormalization constant" corresponding to the gauge-fermion vertex evaluated in the double limits $\lim(\omega' \rightarrow \omega) \lim(q \rightarrow k)$. Actually we compared the coefficients of ω here, so one needs to set $\lim(q \rightarrow k_0)$ and $\lim(k \rightarrow k_0)$ first to throw away the ω -independent terms in the Taylor expansion. From (3.22), $\tilde{Z}_g = 1$, as required by the W-T identity.⁵

From (3.29), the W-T identity (3.37) means that the gauge coupling constant itself is not renormalized. Physical meaning of this result is easily understood. To this end, let us consider a multi-band extension of the model (2.1). If eq.(3.37) did not hold, the renormalized gauge coupling would have different values for different v_F 's, for renormalization constants explicitly depend on the value of the fermi velocity v_F . That is, even if the value of the bare gauge coupling of each band is the same, fermions with different fermi velocity would have different gauge coupling constants by the renormalization effect. This is unexpected. For example in QED, cancellation between the vertex and the wave function renormalizations occurs by the gauge invariance, and the renormalized gauge coupling does not depend on the fermion mass. This guarantees that the charges of electron and muon are equal. In the same way, the W-T identity (3.37) guarantees that the gauge coupling constant is independent of fermi velocity.

From the above calculation, e.g., (3.31) and (3.32), it is observed that the real expansion parameter is $\alpha \equiv g^2 v_F$. Thus we calculate below the β function of α ; $\beta(\alpha) \equiv (\mu d/d\mu)\alpha$. By differentiating the bare quantity,

$$\alpha_0 = g_0^2 v_{F0} = \mu^{\frac{1-b}{2}} \frac{Z_g^2}{Z^2 Z_{v_F}} \alpha, \quad (3.39)$$

⁵The renormalization constants Z_g and $\tilde{Z}_g$ are, of course, related with the renormalization of the current and density operators, respectively. As there is no Lorentz invariance in the present model, the renormalization constants of these operators does not necessarily coincide. Discussion on the renormalization of composite operators will be given in a forthcoming paper[11].

with respect to μ , and using $d\alpha_0/d\mu = 0$, we get

$$(1-b)\alpha_0/2 + \mu^{\frac{1-b}{2}} \mu \frac{d\alpha}{d\mu} \frac{d}{d\alpha} \left(\frac{\alpha Z_g^2}{Z^2 Z_{v_F}} \right) = 0, \quad (3.40)$$

where we converted the μ -derivative into the α -derivative, noting that $Z_g^2 Z^{-2} Z_{v_F}^{-1}$ is a function of α . From this we calculate $\beta(\alpha)$ as

$$\begin{aligned} \beta(\alpha) &\equiv \mu \frac{d\alpha}{d\mu} \\ &= -\frac{1}{2}(1-b) \left[\frac{d}{d\alpha} \ln \left(\frac{\alpha Z_g^2}{Z^2 Z_{v_F}} \right) \right]^{-1} \\ &= -\frac{1}{2}(1-b)\alpha + \text{sgn}(1-b) \cdot \frac{1}{(2\pi)^2 v_B} \alpha^2. \end{aligned} \quad (3.41)$$

Similarly, the anomalous dimension of fermion is calculated as

$$\begin{aligned} \eta(\alpha) &\equiv \mu \frac{d}{d\mu} \ln Z \\ &= 0, \end{aligned} \quad (3.42)$$

and renormalization of the fermi velocity is

$$\begin{aligned} \eta_{v_F}(\alpha) &\equiv \mu \frac{d}{d\mu} \ln Z_{v_F} \\ &= \frac{1}{2\pi^2 v_B} \left[\text{sgn}(b-1) \frac{\alpha}{2} + \frac{\alpha^2}{2\pi^2 v_B} \frac{1}{1-b} \right]. \end{aligned} \quad (3.43)$$

(anomalous but not of the type of marginal-fermi-liquid) From the β function (3.41), we can observe the low-energy behavior of the model. For $b < 1$, there is a non-trivial IR fixed point $\alpha_{IR}^* = 2\pi^2 v_B(1-b)$, and the low-energy behavior of the system is controlled by this fixed point. At this fixed point, $\eta_{v_F} = (1-b)/2$. On the other hand for $b > 1$, there is a non-trivial UV fixed point $\alpha_{UV}^* = 2\pi^2 v_B(b-1)$, and the trivial IR fixed point $\alpha_{IR}^* = 0$. Therefore, if the coupling $\alpha < \alpha_{UV}^*$, the system flows into $\alpha_{IR}^* = 0$ and its low-energy behavior is described by the Fermi-liquid theory. For $\alpha > \alpha_{UV}^*$, the system flows to strong-coupling region and we need separate analysis beyond perturbation theory.

In the paper[7], Nayak and Wilczek argued that renormalization of the wave function appears, i.e., $Z \neq 1$ and also $ZZ_{v_F} = 1$, which are in contrast with our (3.31), and concluded that a marginal-fermi-liquid-like behavior appears. Their $\beta(\alpha)$ function is different from ours (3.41); their coefficient of the second term is 1 instead of our $\text{sgn}(1-b)$. The treatment of the WT identity is also different from ours; they obtained $Z_g = Z$ instead of our (3.38). It seems that they do not notice the importance of orderings of the two limits $\lim(q \rightarrow k)$ and $\lim(\omega' \rightarrow \omega)$. Also a point in contrast is the treatment of singularities at $b = 1$, like the quantity $C(a)$ of (3.7); we employed the prescription (3.10), while they seem to employ the analytic continuation, as mentioned in the footnote to (3.10).

4 Random-Phase Approximation

In the preceding section, we studied the gauge model of non-relativistic fermion (2.1) by the renormalization-group analysis. Straightforward loop expansion was employed there to calculate the β function and the anomalous dimension of the fermion. In this section, we shall apply RPA, instead of the loop expansion, to study low-energy behavior of the model. By introducing N species of fermions and applying the $1/N$ expansion, a systematic calculation of the renormalization constants, etc., is possible, which justifies RPA calculation. Here, however, we shall consider the case of single species of fermion for simplicity. Retaining the factor N is straightforward, and it does not change any results which we derive in this section. We shall start with calculation of dressed gauge propagator, which appears in the leading order in $1/N$ and plays an important role in RPA. Then, we shall study dressed fermion propagator and vertex correction by using the dressed gauge propagator.

4.1 Gauge-Field Propagator

In this subsection, we shall calculate the dressed gauge propagator. Final result is given by (4.22) and (4.23). Reader who is not interested in the detailed calculation of the dressed gauge propagator can skip this subsection.

Below we calculate $\Pi_{lm}(k, \omega)$ of (3.13) up to $O(\omega)$ and $O(k^2)$. First of all, by the gauge invariance, its tensorial structure is given as

$$\Pi_{lm}(k, \omega) = (\delta_{lm} - \frac{k_l k_m}{k^2}) \Pi(k, \omega), \quad (4.1)$$

where $\Pi(k, \omega) \equiv \sum_{ll} \Pi_{ll}(k, \omega)$. By defining

$$\begin{aligned} E &\equiv E_p \equiv \frac{p^2}{2m} - \mu_c, \\ \Delta E &\equiv E_p - E_{k+p}, \\ &= -\frac{k^2 + 2kp \cos \phi}{2m}, \\ D &\equiv \omega^2 + (\Delta E)^2, \end{aligned} \quad (4.2)$$

and noting $(-i\omega + \Delta E)^{-1} = (\Delta E + i\omega)/D$, the scalar part $\Pi(k, \omega)$ is expressed as

$$\begin{aligned} \Pi(k, \omega) &= \frac{2g^2}{m} G(0, 0) + \Pi_1(k, \omega) + \Pi_2(k, \omega), \\ \Pi_1(k, \omega) &= - \int \frac{d\underline{p}}{(kp \cos \phi + p^2)} [\theta(-E_{k+p}) - \theta(-E_p)] \frac{\Delta E}{D}, \\ \Pi_2(k, \omega) &= - \int \frac{d\underline{p}}{(kp \cos \phi + p^2)} [\theta(-E_{k+p}) - \theta(-E_p)] \frac{i\omega}{D}. \end{aligned} \quad (4.3)$$

Corresponding to the expansion

$$\theta(-E_{k+p}) - \theta(-E_p) = \Delta E \delta(-E) + \frac{1}{2} (\Delta E)^2 \delta'(-E) + \frac{1}{3!} (\Delta E)^3 \delta''(-E) + \dots \quad (4.4)$$

$\Pi_{1,2}$ are written as

$$\begin{aligned} \Pi_1 &= \Pi_{11} + \Pi_{12} + \Pi_{13} + \dots \\ \Pi_2 &= \Pi_{21} + \Pi_{22} + \Pi_{23} + \dots \end{aligned} \quad (4.5)$$

Let us calculate Π_{11} first. By using $dE/dp = p/m$, we write

$$\int \frac{d\underline{p}}{(kp \cos \phi + p^2)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_0^{\infty} dp p = \frac{m}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_{-\mu_c}^{\infty} dE \quad (4.6)$$

Then, by recalling $p(E=0) = k_F \equiv (2m\mu_c)^{1/2}$,

$$\begin{aligned} \Pi_{11} &= \frac{g^2}{4\pi^2 m} \int_{-\pi}^{\pi} d\phi \int_{-\mu_c}^{\infty} dE (kp \cos \phi + p^2) \frac{(\Delta E)^2}{D} \delta(-E) \\ &= \frac{g^2}{4\pi^2 m} \left[\int_{-\pi}^{\pi} d\phi (kp \cos \phi + p^2) \left(1 - \frac{\omega^2}{D}\right) \right]_{p=k_F} \\ &= \frac{g^2}{4\pi^2 m} [2\pi p^2 - \omega^2 (kp \cos \phi + p^2)]_{p=k_F}, \end{aligned} \quad (4.7)$$

where

$$\begin{pmatrix} I \\ J \end{pmatrix} \equiv \int_{-\pi}^{\pi} d\phi \frac{1}{D} \begin{pmatrix} 1 \\ \cos \phi \end{pmatrix}. \quad (4.8)$$

I can be expressed as

$$\begin{aligned} I &= \int_{-\pi}^{\pi} d\phi \frac{1}{(a + b \cos \phi)^2 + \omega^2} \\ &= \frac{i}{2\omega} \int_{-\pi}^{\pi} d\phi \left[\frac{1}{a + b \cos \phi + i\omega} - \frac{1}{a + b \cos \phi - i\omega} \right], \\ a &= \frac{k^2}{2m}, \quad b = \frac{kp}{m}. \end{aligned} \quad (4.9)$$

Its first integral is evaluated by the contour integral over the unit circle $z \equiv \exp(i\phi)$;

$$\begin{aligned} &\int_{-\pi}^{\pi} d\phi \frac{1}{a + b \cos \phi + i\omega} \\ &= \oint \frac{dz}{iz} \frac{1}{a + \frac{b}{2}(z + z^{-1}) + i\omega} \\ &\equiv \oint dz \frac{2}{ib(z - z_0)(z - z_1)} \\ &= \frac{1}{2\pi i} \oint dz \frac{4\pi}{b} \left(\frac{1}{z - z_0} - \frac{1}{z - z_1} \right) \frac{1}{z_0 - z_1} \\ &= \frac{4\pi}{b} \frac{1}{2i\sqrt{b^2 - (a + i\omega)^2}} [\theta(\omega) - \theta(-\omega)]. \end{aligned} \quad (4.10)$$

The last line is obtained by considering

$$z_{0,1} = \frac{1}{b} [-(a + i\omega) \pm \sqrt{(a + i\omega)^2 - b^2}]$$

$$\begin{aligned}
&= \frac{1}{b}[-a \pm \frac{a\omega}{\sqrt{b^2 - a^2}} \pm i(\sqrt{b^2 - a^2} \mp \omega)], \\
|z_{0,1}|^2 &= 1 \mp \frac{2\omega}{\sqrt{b^2 - a^2}} + O(\omega^2).
\end{aligned} \tag{4.11}$$

Therefore we finally obtain I for $k \ll k_F$ and small ω ,

$$\begin{aligned}
I &= \frac{-2\pi i}{\sqrt{b^2 - a^2} - 2i\omega} \frac{1}{\omega} [\text{sgn}(\omega) - \text{sgn}(-\omega)] \\
&= \frac{\pi m}{pk} \frac{2\text{sgn}(\omega)}{\omega} + O(k, \omega) \\
&= \frac{2\pi m}{pk} \frac{1}{|\omega|} + O(k, \omega).
\end{aligned} \tag{4.12}$$

Exactly in the same manner, J is calculated as

$$\begin{aligned}
J &= \int_{-\pi}^{\pi} d\phi \frac{\cos \phi}{(a + b \cos \phi)^2 + \omega^2} \\
&= \frac{-i}{2\omega b} \int_{-\pi}^{\pi} d\phi \left[\frac{a + i\omega}{a + b \cos \phi + i\omega} - \frac{a - i\omega}{a + b \cos \phi - i\omega} \right], \\
&= \frac{-i}{2\omega b} \frac{-2\pi i}{\sqrt{b^2 - a^2}} \text{sgn}(\omega) [(a + i\omega) - (a - i\omega)] \\
&= -\frac{\pi m}{p^2 |\omega|} + O(k, \omega).
\end{aligned} \tag{4.13}$$

Thus we get

$$\Pi_{11} = \frac{g^2 k_F}{2\pi m} \left[k_F - m \left(1 - \frac{k^2}{2k_F^2} \right) \frac{|\omega|}{k} \right] + O(\omega^2). \tag{4.14}$$

Π_{12} is calculated as follows;

$$\begin{aligned}
\Pi_{12} &= \frac{g^2}{4\pi^2 m} \int_{-\pi}^{\pi} d\phi \int_{-\mu}^{\infty} dE (kp \cos \phi + p^2) \frac{(\Delta E)^3}{2D} \delta'(-E) \\
&= \frac{g^2}{4\pi^2 m} \int_{-\pi}^{\pi} d\phi \frac{d}{dE} [(kp \cos \phi + p^2) \frac{(\Delta E)^3}{2D}]_{E=0} \\
&= \frac{g^2}{4\pi^2 m} \frac{m}{k_F} \frac{d}{dp} \left[\int_{-\pi}^{\pi} d\phi (kp \cos \phi + p^2) \frac{(\Delta E)^3}{2D} \right]_{p=k_F} \\
&= \frac{g^2}{8\pi^2 m} \frac{m}{k_F} \frac{d}{dp} \left[\int_{-\pi}^{\pi} d\phi (p^2 - \frac{k^2}{2} - m\Delta E) \Delta E (1 - \frac{\omega^2}{D}) \right]_{p=k_F}.
\end{aligned} \tag{4.15}$$

By using

$$\int_{-\pi}^{\pi} d\phi \frac{1}{D} \left(\frac{\Delta E}{(\Delta E)^2} \right) = \left(-\frac{k^2}{2m} I - \frac{kp}{m} J \right), \tag{4.16}$$

one gets $(d/dp)[\dots] = -4\pi k_F k^2/m + O(\omega k, \omega^2)$. So finally we have

$$\Pi_{12} = -\frac{g^2}{2\pi m} k^2 + O(\omega k, \omega^2). \tag{4.17}$$

The third term Π_{13} can be calculated in the similar manner. We give here only the result,

$$\Pi_{13} = \frac{g^2}{3\pi m} k^2 + O(\omega^2). \tag{4.18}$$

By a simple power-counting, one can convince himself that Π_{1n} ($n > 3$) contain only the higher-order terms like $O(\omega^2, \omega k^3, k^4)$.

Next let us calculate Π_{21} ;

$$\begin{aligned}
\Pi_{21} &= \frac{g^2}{4\pi^2 m} i\omega \int_{-\pi}^{\pi} d\phi \int_{-\mu}^{\infty} dE (kp \cos \phi + p^2) \frac{\Delta E}{D} \delta(-E) \\
&= \frac{g^2}{4\pi^2 m} i\omega \left[(p^2 - \frac{k^2}{2}) \left(-\frac{k^2}{2m} I - \frac{kp}{m} J \right) - m(2\pi - \omega^2 I) \right]_{p=k_F} \\
&= -i \frac{g^2}{2\pi} \omega + O(\omega^2/k),
\end{aligned} \tag{4.19}$$

where we have used $kp \cos \phi + p^2 = p^2 - k^2/2 - m\Delta E$ and eq.(4.16).

Π_{22} is calculated as

$$\begin{aligned}
\Pi_{22} &= \frac{g^2}{4\pi^2 m} i\omega \int_{-\pi}^{\pi} d\phi \int_{-\mu}^{\infty} dE (p^2 - k^2/2 - m\Delta E) \frac{(\Delta E)^2}{2D} \delta'(-E) \\
&= \frac{g^2}{4\pi^2 m} i\omega \frac{m}{2k_F} \frac{d}{dp} \left[\int_{-\pi}^{\pi} d\phi (p^2 - k^2/2 - m\Delta E) \frac{(\Delta E)^2}{D} \right]_{p=k_F} \\
&= \frac{g^2}{4\pi^2 m} i\omega \frac{m}{2k_F} \frac{d}{dp} \left[(p^2 - k^2/2)(2\pi - \omega^2 I) - m(1 - \frac{\omega^2}{D}) \left(-\frac{k^2}{2m} I - \frac{kp}{m} J \right) \right]_{p=k_F} \\
&= i \frac{g^2}{2\pi} \omega + O(\omega^2, k^2).
\end{aligned} \tag{4.20}$$

In a similar manner Π_{23} and higher terms are calculated. One sees that these terms gives only higher-order terms. Therefore, by collecting everything, we get the final expression,

$$\Pi(k, \omega) = -\frac{g^2}{6\pi m} k^2 - \frac{g^2 k_F}{2\pi} \frac{|\omega|}{k} + O(\omega^2, \omega k, k^4). \tag{4.21}$$

In the RPA, one sums up the bubble diagrams in a geometric series to obtain the gauge field propagator,

$$\begin{aligned}
\hat{\Delta}_{ij}(k, \omega) &= \Delta_{ij} + \Delta_{il}\Pi_{lm}\Delta_{mj} + \dots \\
&= \Delta_{il}[1 - \Pi\Delta]_{lj}^{-1} \\
&= \frac{1}{F(k, \omega)}(\delta_{ij} - \frac{k_i k_j}{k^2}), \\
F(k, \omega) &= v_B k^{2-b} - \Pi(k, \omega) \\
&= v_B k^{2-b} + \frac{g^2}{6\pi m} k^2 + \frac{g^2 k_F}{2\pi} \frac{|\omega|}{k} \\
&\equiv \gamma(k) [|\omega| + B(k)], \\
\gamma(k) &\equiv \frac{g^2 k_F}{2\pi k}, \\
B(k) &\equiv \gamma(k)^{-1} [v_B k^{2-b} + \frac{g^2}{6\pi m} k^2],
\end{aligned} \tag{4.22}$$

where we introduced $\gamma(k)$ and $B(k)$ for later convenience. If one considers N species of fermions and applies the $1/N$ expansion, the gauge propagator in the leading order of $1/N$ is given by (4.22).

4.2 Fermion Propagator

Let us calculate the dressed fermion propagator, which contains the same diagram as in the one-loop calculation but the gauge-field propagator appearing in it is replaced by the above one. Then as (3.2), $\Sigma(k, \omega)$ is expressed as

$$\begin{aligned}
\Sigma(k, \omega) &= \int \frac{dp}{\beta} \sum_{\epsilon} G(k-p, \omega-\epsilon) k_j (k-p)_i \hat{\Delta}_{ij}(p, \epsilon) \\
&= \int \frac{dp}{\beta} \frac{k^2 \sin^2 \phi}{\gamma(p)} J(E_{k-p}, B(p), \omega), \\
J(E, B, \omega) &\equiv \frac{1}{\beta} \sum_{\epsilon} \frac{e^{-i(\omega-\epsilon)\eta}}{i(\omega-\epsilon) + E} \frac{1}{|\epsilon| + B} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{e^{-i(x+\omega)\eta}}{i(x+\omega) + E} \frac{1}{|x| + B} \\
&= \frac{1}{2\pi i} \int_0^{\infty} dx \left\{ \frac{1}{x + \omega - iE} + \frac{1}{-x + \omega - iE} \right\} \frac{1}{x + B}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \int_0^{\infty} dx \frac{-2(\omega - iE)}{x^2 - (\omega - iE)^2} \frac{1}{x + B} \\
&= \frac{1}{2\pi i} [2iE - 2\omega + 2iE \cdot i\omega \frac{\partial}{\partial E}] K(E, B) + O(\omega^2),
\end{aligned} \tag{4.24}$$

where we introduced

$$\begin{aligned}
K(E, B) &\equiv \int_0^{\infty} dx \frac{1}{x^2 + E^2} \frac{1}{x + B} \\
&= \frac{1}{2(E^2 + B^2)} \left[\ell n \frac{(x+B)^2}{x^2 + E^2} + \frac{2B}{|E|} \arctan \frac{x}{|E|} \right]_0^{\infty} \\
&= \frac{1}{2(E^2 + B^2)} \left[-\ell n \frac{B^2}{E^2} + \frac{\pi B}{|E|} \right].
\end{aligned} \tag{4.25}$$

As before we are interested in the region $k \simeq k_0$; $k_0^2 = k_F^2$ and small ω , and thus make an expansion w.r.t. the vector Δ :

$$\begin{aligned}
k &= k_0 + \Delta, \\
\Sigma(k, \omega) &= \Sigma_m + \Sigma_{\Delta} \cdot (k_F \cdot \Delta) + \Sigma_{\omega} \cdot \omega + O(\Delta^2, \omega^2), \\
\Sigma_m &= \frac{1}{2\pi i} \int \frac{dp}{\beta} \frac{k_F^2 \sin^2 \phi}{\gamma(p)} 2iEK(E, B), \\
\Sigma_{\Delta} &= \frac{1}{2\pi i} \int \frac{dp}{\beta} \frac{k_F^2 \sin^2 \phi}{\gamma(p)} \left[\frac{d}{dE} (2iEK(E, B)) \frac{1}{m} (1 - \frac{p \cos \phi}{k_F}) + \frac{2}{k_F^2} 2iEK(E, B) \right], \\
\Sigma_{\omega} &= \frac{1}{2\pi i} \int \frac{dp}{\beta} \frac{k_F^2 \sin^2 \phi}{\gamma(p)} (-2 - 2E \frac{d}{dE}) K(E, B),
\end{aligned} \tag{4.26}$$

where we denote E_{k_0-p} and $B(p)$ by E and B respectively, suppressing their arguments. We use this simple notation for the rest of this subsection. This should not bring any confusion, although we used to write E for E_p in the previous subsection. To obtain this equation we used the fact that $\Delta_i dJ/dk_i = \Delta_i (dE/dk_i) dJ/dE$, together with $dE/dk_i = (k_0 + \Delta - p)_i/m$, $p_i = (p \cos \phi/k_F)k_{0i} + (p \sin \phi)l_i$ and its second term does not contribute as before since the other part of the integrand is an even function of ϕ .

Let us calculate Σ_m first. By changing variable from (p, ϕ) to (p, t) , $t \equiv E/B$, we have an expression;

$$\Sigma_m = \frac{k_F m}{(2\pi)^2} \int_0^{\infty} dp \frac{1}{\pi \gamma(p)} \int_{-T_1}^{T_2} dt |\sin \phi(t)| \left[\frac{t \ell n t^2}{1+t^2} + \pi sgn(t) \right],$$

$$T_1 = \frac{2pk_F - p^2}{2mB}, T_2 = \frac{2pk_F + p^2}{2mB}. \quad (4.27)$$

Due to the structure of the integrand, the integral is dominated by the region $E/B \simeq 0$, or equivalently by $p \simeq 0, \cos \phi \simeq 0$. Thus, to obtain the leading term, the p^2 -terms are neglected compared with the p -terms in numerators of $T_{1,2}$. Also the p^2 -term of $|\sin \phi(t)|$ that starts with $1 + O(p^2)$ for $b = 1$ is neglected, so we set $|\sin \phi| = 1$. Then Σ_m vanishes since $T_1, T_2 = T \equiv pk_F/mB$ and the integrand is an odd function of t . The remaining terms, coming from the interval (T_1, T_2) and from $\sin \phi(t) = 1$ are calculable. In particular, one sees that they have no singularity at $b = 1$ (see the discussion around eq(4.35) below);

$$\Sigma_m = 0 + \text{regular term at } b = 1. \quad (4.28)$$

Next, we calculate Σ_ω , again neglecting the $O(p^2)$ terms as for Σ_m :

$$\Sigma_\omega = i \frac{4mk_F}{(2\pi)^3} \int_0^\infty dp \frac{1}{\gamma(p)B} \int_0^T dt \left[\frac{1-t^2}{(1+t^2)^2} \ell n t^2 + \frac{2}{1+t^2} - \frac{2\pi t}{(1+t^2)^2} \right]. \quad (4.29)$$

Let us call the first term of the t -integral Q , which is calculated as

$$\begin{aligned} Q &= -2 \int dt \frac{\ell n t}{1+t^2} + 4 \int dt \frac{\ell n t}{(1+t^2)^2} \\ &= -2 \left[\frac{t \ell n t - t}{1+t^2} \right]_0^T - 4 \int dt \frac{t^2}{(1+t^2)^2} (\ell n t - 1) + 4 \int dt \frac{\ell n t}{(1+t^2)^2} \\ &= -2[\dots] + 2Q + \int dt \frac{4t^2}{(1+t^2)^2} \\ &= - \int dt \frac{4t^2}{(1+t^2)^2} \\ &= -\pi, \end{aligned} \quad (4.30)$$

where we put the definite integral to be zero by sending $T \rightarrow \infty$ to obtain the leading contribution. The next leading contribution is seen to be of $O(1/T^2) = O(p^{4-2b})$. The second and the third terms in the t -integral are elementary giving rise to π and $-\pi$ respectively. Therefore we get

$$\Sigma_\omega = i \frac{4mk_F}{(2\pi)^3} \int_0^{k_F} dp \frac{1}{v_B p^{2-b}} (-\pi)$$

$$= -i \frac{k_F^b m}{2\pi^2 v_B} \frac{1}{|b-1|}, \quad (4.31)$$

where we neglected the second term of $\gamma(p)B = v_B p^{2-b} + g^2(6\pi m)^{-1} p^2$, which is higher order for small p at $b \simeq 1$. Also we cut off the p -integral up to k_F since we assumed small p . (The original p -integral is of course convergent at large p and the contribution from this region is regular.) To obtain the exact value of the coefficient in front of $(1-b)^{-1}$, one needs more elaborated calculations, since it depends on the value of cut off. We do not present such calculations here, because the most important thing, as we shall see in the later sections, is the very existence of the singularity at $b = 1$ itself and not its coefficient.

Finally, we calculate the leading contribution to Σ_Δ . First we note the relation,

$$\frac{d}{dE} [EK(E, B)] = K(E, B) + E \frac{d}{dE} K(E, B) + \frac{\pi}{B} \delta(E). \quad (4.32)$$

the last term of which breaks the Leibnitz' law of differentiation due to the nonanalytic term $(d/dE) \text{sgn}(E) = 2\delta(E)$. Therefore the first term of Σ_Δ is given as

$$\begin{aligned} & -\frac{i}{m} \Sigma_\omega + \frac{1}{\pi m} \frac{1}{(2\pi)^2} \int_0^\infty dp p \int d\phi \frac{k_F^2 \sin^2 \phi}{\gamma(p)} \frac{\pi}{B} \delta(E) \\ &= -\frac{i}{m} \Sigma_\omega + \frac{k_F^2}{m} \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{2m}{k_F} \frac{1}{v_B p^{2-b}} \\ &= -\frac{m}{2\pi^2 v_B} \frac{k_F^b}{|b-1|} + \frac{m}{2\pi^2 v_B} \frac{k_F^b}{|b-1|} \\ &= 0. \end{aligned} \quad (4.33)$$

This cancellation can be understood in a more direct manner. Actually, if one neglects the t -dependence in $\sin^2 \phi$ as we did, the integrand of t -integral becomes a total derivative (note $d/dE = B^{-1} d/dt$). So the t -integral reduces to a surface term that vanishes as $p \rightarrow 0$.

The second term of Σ_Δ is nothing but $2k_F^{-2}$ times Σ_m itself. So we reached the simple result that

$$\Sigma_\Delta = 0. \quad (4.34)$$

Before ending this subsection, let us estimate the higher-order terms in p that are generated by the full t -integration. They are accompanied with p -integrals like

$$\int_0^{k_F} dp \frac{p^n}{p^{2-b}} = \frac{k_F^{b+n-2}}{b+n-1}, \quad n \geq 2. \quad (4.35)$$

Therefore our conclusion that Σ_m and Σ_Δ do not contain any singular terms at $b = 1$ does not change. We stress that the contrasting results for Σ_ω and Σ_Δ regarding to the $b = 1$ singularities come from the nonanalytic term $\text{sgn}(E)$. This term, in turn, comes from the nonanalytic behavior $|\omega|$ of the dissipative term in the dressed gauge propagator in RPA. In contrast, in the one-loop calculation, the integral for the self energy of fermion has an integrand that is analytic both in E and ω .

4.3 Vertex Correction by Dressed Gauge Propagator

The vertex kernel is given by using the dressed propagator $\hat{\Delta}_{ij}(p, \omega)$ as

$$\begin{aligned} \tilde{\Gamma}_i(k, q, \omega, \omega') &= \frac{ig}{m} \hat{\Delta}_{ij}(q - k, \omega' - \omega) i q_j + \hat{\Delta}_{il}(q - k, \omega' - \omega) \left[\left(\frac{-g}{m} \right)^3 \hat{S}_l - \frac{ig^3}{2im^2} \hat{T}_l \right], \\ \hat{S}_l &\equiv \int \frac{d\underline{p}}{\beta} \sum_{\epsilon} (q - p)_l G(k - p, \omega - \epsilon) G(q - p, \omega' - \epsilon) (k - p)_m q_n \hat{\Delta}_{mn}(p, \epsilon), \\ \hat{T}_l &\equiv \int \frac{d\underline{p}}{\beta} \sum_{\epsilon} \hat{\Delta}_{lm}(p, \epsilon) [i q_m G(q - p, \omega' - \epsilon) + i(k - p)_m G(k - p, \omega - \epsilon)]. \end{aligned} \quad (4.36)$$

Let us calculate $\hat{S}_l$ in the limit (i) $q_i \rightarrow k_i$ with $k_i, q_i = (k_F)_i$ and in the limit (ii) $\omega' \rightarrow \omega$ and then $\omega \rightarrow 0$. As we shall see below, the result depends on the order of taking two limits (i) and (ii).

Let us begin with the following expression for the product of fermion propagators in $\hat{S}_l$;

$$G(k - p, \omega - \epsilon) G(q - p, \omega' - \epsilon) = \left[\frac{1}{i(\omega - \epsilon) + E_{k-p}} - \frac{1}{i(\omega' - \epsilon) + E_{q-p}} \right] \frac{1}{i(\omega' - \omega) + E_{q-p} - E_{k-p}}. \quad (4.37)$$

Thus $\hat{S}_l$ is expressed by using $J(E, B, \omega)$ of (4.24) as follows;

$$\begin{aligned} \hat{S}_l &= k_l \int \frac{d\underline{p}}{\beta} \frac{k_F^2 \sin^2 \phi}{\gamma(p)} \frac{1}{i(\omega' - \omega) + E_{q-p} - E_{k-p}} \\ &\times [J(E_{k-p}, B, \omega) - J(E_{q-p}, B, \omega')], \end{aligned} \quad (4.38)$$

where ϕ is the angle between p and k as before. Therefore, when we take $\lim(q \rightarrow k)$ first, then $\hat{S}_l$ becomes

$$\begin{aligned} \lim_{\omega' \rightarrow \omega} \lim_{q \rightarrow k} \hat{S}_l &= k_l \int \frac{d\underline{p}}{\beta} \frac{k_F^2 \sin^2 \phi}{\gamma(p)} \frac{-1}{\pi} (1 + E \frac{d}{dE}) K(E, B) \\ &= i k_l \Sigma_\omega \\ &= k_l \frac{m k_F^b}{2\pi^2 v_B} \frac{1}{|b - 1|}. \end{aligned} \quad (4.39)$$

On the other hand, when we first take $\lim(\omega' \rightarrow \omega)$, then we get

$$\begin{aligned} \lim_{q \rightarrow k} \lim_{\omega' \rightarrow \omega} \hat{S}_l &= k_l \int \frac{d\underline{p}}{\beta} \frac{k_F^2 \sin^2 \phi}{\gamma(p)} \frac{d}{dE} [EK(E, B)] \\ &= k_l m \Sigma_\Delta \\ &= 0. \end{aligned} \quad (4.40)$$

Note that the above results are just the opposite to those obtained by the loop expansion (3.21, 3.22).

The term $\hat{T}_l$ is easy to calculate and it does not depend on the ordering of the limits. Actually we have

$$\begin{aligned} \hat{T}_l &= 2i k_l \int \frac{d\underline{p}}{\beta} \sum_{\epsilon} \frac{1}{\gamma(p)[|\epsilon| + B(p)]} \frac{k_F^2 \sin^2 \phi}{i(\omega - \epsilon) + E(k - p)} \\ &= 2i k_l \Sigma_m \\ &= 0. \end{aligned} \quad (4.41)$$

This completes our calculations in the RPA.

4.4 Scaling and renormalization

In the RPA, the gauge propagator has the following form;

$$\hat{\Delta}_{ij} = \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) \frac{1}{F(p, \omega)},$$

$$F(p, \omega) = v_B p^{2-b} + \frac{g^2}{6\pi m} p^2 + \frac{g^2 k_F |\omega|}{2\pi p}.$$

For $b > 0$, the dominant momentum region of loop integrals is

$$p^{3-b} \sim |\omega|. \quad (4.42)$$

Let us take fermion momentum as $k = (k_1 \sim 0, k_2)$, $k_2 \sim k_F$ as before, and the dominantly coupled gauge field is A_2 . We consider renormalization group transformation towards $k_0 = (0, k_F)$. Then the scaling of energy and momentum in the RPA (i.e., the leading order of $1/N$) is given as

$$\begin{aligned} \omega &\rightarrow s\omega, \\ p_x &\rightarrow s^{\frac{1}{3-b}} p_x, \\ p_y &\rightarrow s^{\frac{2}{3-b}} p_y, \\ E_k &\rightarrow s^{\frac{2}{3-b}} E_k. \end{aligned} \quad (4.43)$$

For the case $b = 0$, these scalings reduce to those given by Polchinski [6]. The scaling dimensions of fermion field, gauge field, and gauge coupling are read off from the quadratic terms and three-point vertex. Since the kernel of fermion field, $i\omega + E_p$ scales asymmetrically, $i s\omega + s^{2/(3-b)} E$, one should distinguish two cases; (i) $1 > 2/(3-b)$ ($b < 1$) where E -term dominates as $s \rightarrow 0$, and (ii) $1 < 2/(3-b)$ ($b > 1$) where ω -term dominates. For $b < 1$, we have

$$[\psi] = -\frac{8-b}{2(3-b)}, \quad [A_2] = -\frac{4-b}{3-b}, \quad [g] = 0. \quad (4.44)$$

Therefore, the gauge coupling is *marginal* up to the next-to-leading of the RPA (or $1/N$). For $b > 1$, we have

$$[\psi] = -\frac{9-2b}{2(3-b)}, \quad [A_2] = -\frac{4-b}{3-b}, \quad [g] = \frac{1-b}{3-b}. \quad (4.45)$$

Renormalization constants Z and ZZ_{v_F} are easily obtained from the calculation in this section. From (4.31) and (4.34),

$$\begin{aligned} Z &= 1 - \frac{g^2 v_F}{2\pi^2 v_B} \frac{1}{|1-b|}, \\ ZZ_{v_F} &= 1. \end{aligned} \quad (4.46)$$

The gauge-coupling renormalization constant Z_g is defined as before and determined from the vertex correction (4.40) as

$$Z_g = 1. \quad (4.47)$$

Thus the spatial-component WT identity $Z_g = ZZ_{v_F}$ holds. The renormalization constant $\tilde{Z}_g$ is calculated from (4.39) as

$$\tilde{Z}_g = 1 - \frac{g^2 v_F}{2\pi^2 v_B} \frac{1}{|1-b|}, \quad (4.48)$$

so the time-component WT identity $\tilde{Z}_g = Z$ holds. From Z of (4.46), $\eta = \mu d(\ell n Z)/d\mu \neq 0$, at low energy if there is a non-trivial IR fixed point of the renormalization group flow of the gauge coupling. Thus the fermion propagator has a branch cut rather than a pole due to and a marginal-fermi-liquid like behavior appears. However, as we shall see in the next paragraph, existence of such an IR fixed point is disproved for $b > 1$, and higher-order calculation is necessary to get conclusion for $b < 1$.

Let us consider the $\beta(\alpha)$ function. For $b < 1$, the gauge coupling constant remains marginal⁶, i.e., $\beta(\alpha) = 0$, in the RPA (in the next-to-leading order of $1/N$). One needs higher-order calculations to argue the flow of α . For $b > 1$, the straightforward calculation starting with $\alpha_0 = \mu^{2(1-b)/(3-b)} (Z_g^2 Z^{-2} Z_{v_F}^{-1} \alpha)$ gives rise to

$$\beta(\alpha) = \frac{2}{3-b} [(b-1)\alpha + \frac{sgn(1-b)}{2\pi^2 v_B} \alpha^2], \quad (4.49)$$

which has an UV fixed point $\alpha_{UV}^* = 2\pi^2 v_B (b-1)$. For $\alpha < \alpha_{UV}^*$, there is the IR fixed point $\alpha_{IR}^* = 0$ and the low-energy behavior of the model is described by the Fermi-liquid theory. For $\alpha > \alpha_{UV}^*$, we need further (non-perturbative) analysis to investigate the IR structure. We note that the value of α_{UV}^* and the global behavior of $\beta(\alpha)$ is the same as in the one-loop case.

⁶The word "marginal" here implies that there appear no relevant corrections to the gauge-fermion vertex with finite momentum transfer.

5 Discussion

In this paper, we studied low-energy behavior of the system of non-relativistic fermions with gauge interaction by renormalization-group analysis. We employed the $(1 - b)$ expansion in a spirit similar to the usual $\epsilon(\equiv 4 - d, \text{ or } d - 2)$ expansion. Calculation using the momentum cutoff is under study and results will be reported in a forthcoming paper[10].

It was shown that the loop expansion and the $1/N$ -RPA give different results for the β function, anomalous dimension of fermion and renormalization of the fermi velocity. In any case, however, marginal-fermi-liquid like behavior does not appear, contrary to the result of Nayak and Wilczek[7]. Result by the loop expansion is that for $b < 1$ the expansion parameter α has a non-trivial IR fixed point. On the other hand for $b > 1$, there is a non-trivial UV fixed point α_{UV}^* and the coupling flows to the trivial IR fixed point $\alpha_{IR}^* = 0$, i.e., the Fermi liquid, if $\alpha < \alpha_{UV}^*$. For $\alpha > \alpha_{UV}^*$, the system flows to the strong-coupling region. In both cases, the fermi velocity is renormalized but wave function renormalization does not occur.

By the $1/N$ -RPA calculation, the following results are obtained. For $b < 1$, anomalous behavior of the fermion propagator is observed. However, the gauge coupling is marginal up to the next-leading order of $1/N$. This includes the result of Polchinski, which corresponds to $b = 0$ of the present model. For $b > 1$, the $\beta(\alpha)$ function behaves similarly as in the one-loop case. For $\alpha < \alpha_{UV}^*$ the gauge coupling flows to the trivial IR fixed point, i.e., the Fermi-liquid theory.

Different results are obtained by the loop and by the $1/N$ -RPA calculation. A natural question arises; i.e., which result is reliable. It is not so easy to answer this question. Similar problem appears for the Coulomb interaction between electrons. In the RPA theory, shielding of the Coulomb interaction is naturally observed. However, it is not obvious whether the shielding is observed by the straightforward loop expansion. Origin of this difference is clear. In the RPA theory, the bubble diagrams are summed up in geometric series (see Fig.3), and this changes long-range proper-

ties of the Coulomb interaction to short-range one. Somewhat detailed discussion on renormalization-group analysis of the Coulomb interaction using loop expansion is given in Ref.[1]. As explained there, the final answer in that case is obtained by bosonization technique in $(2 + 1)$ dimensions; low-energy behavior of electron system with the Coulomb interaction is described by the Fermi liquid theory, as predicted by the RPA theory.

Most interesting result is that obtained by the $1/N$ -RPA calculation for $b < 1$. In this case, renormalization constants Z and Z_{v_F} are given by (4.46). This result indicates possible anomalous behavior of the fermion propagator at low-energy and close to fermi surface (although one needs higher-order or renormalization-group-improved calculations to confirm this possibility). This fact has been first observed by Halperin, Lee and Read[4] and discussed by Polchinski[6] for the case $b = 0$ from scaling point of view. Our study in the present paper shows that the essential feature of the model with $b = 0$ remains unchanged for general $b < 1$.

In the forthcoming paper[10], we shall study a system of fermions interacting with dissipative gauge field by the Wilson-type renormalization group scheme with momentum cutoff. Gauge field has a dissipative term just like that obtained by the RPA in this paper. However, coefficient of the dissipative term is a parameter there, and it scales non-trivially under a renormalization-group transformation. Possibility of marginal-fermi-liquid behavior will be discussed.

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Appendix

In this Appendix we present a derivation of eq.(3.3). By making use of the Poisson's summation formula and the contour integral, we rewrite the first one as follows;

$$\begin{aligned}
& \frac{1}{\beta} \sum_{n \in \mathbf{Z}} \frac{\exp(-2\pi i \beta^{-1} n \eta)}{2\pi i \beta^{-1} n + E} \\
&= \frac{1}{\beta} \sum_{t \in \mathbf{Z}} \int_{-\infty}^{\infty} d\chi e^{2\pi i \chi t} \frac{e^{-2\pi i \beta^{-1} \chi \eta}}{2\pi i \beta^{-1} \chi + E} \\
&= \theta(E) \sum_{t > 0} e^{-\beta E t} - \theta(-E) \sum_{t \leq 0} e^{-\beta E t} \\
&= \theta(E) \frac{e^{-\beta E}}{1 - e^{-\beta E}} - \theta(-E) \frac{1}{1 - e^{\beta E}},
\end{aligned}$$

which is just the middle expression of (3.3). The second formula can be derived in the exactly same manner.

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Figure Captions

Fig.1: The graphical symbols of (a) fermion propagator $G(k, \omega)$ of Eq.(2.4), (b) gauge propagator $\Delta_{lm}(k, \omega)$ of Eq.(2.5), (c) fermion-gauge boson vertex of order g in Eq.(2.6), (d) fermion-gauge boson vertex of order g^2 in Eq.(2.6).

Fig.2: Feynman diagrams in one-loop calculations. (a) Self energy $\Sigma(k, \omega)$ of Eq.(3.2). (b,c) Vacuum polarization $\Pi_{lm}(k, \omega)$ of Eq.(3.13). (d) Vertex correction S_l of Eq.(3.19). (e,f) Vertex correction T_l of Eq.(3.19).

Fig.3: Sum over bubble diagrams for gauge-field propagator in the RPA approximation (4.22).

Fig. 1

Fig. 2

Fig. 2

$$\begin{aligned}
 & \text{thick wavy line} = \text{thin wavy line} + \\
 & \text{thin wavy line} \circlearrowleft \text{thin wavy line} + \text{thin wavy line} \circlearrowleft \text{thin wavy line} \circlearrowleft \text{thin wavy line} \\
 & + \text{thin wavy line} \circlearrowleft \text{thin wavy line} \circlearrowleft \text{thin wavy line} \circlearrowleft \text{thin wavy line} \\
 & + \dots
 \end{aligned}$$

Fig. 3