JA.

IC/95/1

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CONFORMALLY COMPACTIFIED HOMOGENEOUS
SPACES

(Possible Observable Consequences)

84 9510

P. Budinich

SCAN-9503015

CERN LIBRARIES, GENEVA

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

MIRAMARE-TRIESTE

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CONFORMALLY COMPACTIFIED HOMOGENEOUS SPACES
(Possible Observable Consequences)

P. Budinich
International Centre for Theoretical Physics, Trieste, Italy
and
Laboratorio Interdisciplinare per le Scienze Naturali ed Umanistiche della SISSA,
Trieste, Italy.

ABSTRACT

Some arguments based on the possible spontaneous violation of the Cosmological Principle (represented by the observed large-scale structures of galaxies), the Cartan-geometry of simple spinors and on the Fock-formulation of hydrogen-atom wave-equation in momentum-space, are presented in favour of the hypothesis that space-time and momentum-space should be both conformally compactified and represented by the two four-dimensional homogeneous spaces of the conformal group, both isomorphic to $(S^3 \times S^1)/Z_2$ and correlated by conformal inversion. Within this framework, the possible common origin for the $SO(4)$ symmetry underlying the geometrical structure of the Universe, of Kepler orbits and of the H-atom is discussed. One of the consequences of the proposed hypothesis could be that any quantum field theory should be naturally free from both infrared and ultraviolet divergences. But then physical spaces defined as those where physical phenomena may be best described, could be different from those homogeneous spaces. A simple, exactly soluble, toy model, valid for a two-dimensional space-time is presented where the conjectured conformally compactified space-time and momentum-space are both isomorphic to $(S^1 \times S^1)/Z_2$, while the physical spaces are two finite lattices which are dual since Fourier transforms, represented by finite, discrete, sums may be well defined on them. Furthermore, a q -deformed $SU_q(1,1)$ may be represented on them if q is a root of unity.

MIRAMARE – TRIESTE

January 1995

1. Introduction

The Maxwell's equations conformal covariance discovered long ago [1], is the main support for the often proposed [2] conjecture that space-time may be conformally compactified and represented by the homogeneous space M_G :

$$M_G = \frac{G}{H_1} \simeq \frac{S^3 \times S^1}{Z_2} \quad (1)$$

where G stands for the conformal group and $H_1 = L \otimes D \times K$ for its eleven parameter subgroup, where L , D , K represent Lorentz-Dilatation- and Special conformal-transformations, respectively.

Observe that in such space-time any field theory should be naturally free from infrared divergences since in it the mere concept of infinite space-time intervals is absent, but will not be free from ultraviolet ones.

The conformal group G has another eleven parameter subgroup: $H_2 = L \otimes D \times P$, where P indicates Poincaré translations, and the homogeneous space

$$P_G = \frac{G}{H_2} \simeq \frac{S^3 \times S^1}{Z_2} \quad (2)$$

could be a natural candidate for the conformal compactification of $P = \mathbb{R}^{3,1}$ momentum-space: the Fourier dual of Minkowski space $M = \mathbb{R}^{3,1}$. The main support for this conjecture may be found in spinor-geometry (chapter 4) and in the fact that while the conformal inversion I transforms M_G to P_G :

$$I M_G I^{-1} = P_G$$

it also transforms a length in the inverse of a length (chapter 4.1).

Observe that in such a compactified momentum space any field theory should be naturally free from ultraviolet divergences, since in it the mere concept of infinite momentum (or infinitesimal distance for a quantum field theory) is absent, but will not be free from infrared ones.

Should conformal covariance be a fundamental symmetry of nature then it is tempting to conjecture that both space-time and momentum-space be simultaneously compactified. In this paper we will give some observational and theoretical arguments on why M_G and P_G might indeed both be realized in nature (chapters 2, 3, 4). In such a world, because of the above observations, the mere concepts of infinity and infinitesimal should be absent ab inizio and in it any quantum field theory would have the attractive feature to be naturally free from both infrared and ultraviolet

divergences, provided we may find the appropriate algorithm to deal with field theories in such manifolds. In fact, it is plausible to conjecture that this algorithm may not be represented optimally by standard infinitesimal analysis and differential geometry which are precisely based on the concept of infinitesimal and systematically employ the concept of infinity. In order to find an alternative way to deal with fields in M_G and P_G we will first try to extend to these manifolds the concept of Fourier transforms. It is well known that the Fourier transform of a function taking values on a compact space M_G , say, is a function taking values on a discrete, infinite lattice, which, in turn, may be considered as the Fourier dual of M_G . If we now impose that momentum space, conceived as the dual of M_G , be compact (P_G , say), then that discrete lattice should be finite (loosely speaking, contained in P_G). The opposite happens if we start with P_G . The result is that to deal with field theories in M_G and P_G we should end up with fields, or functions, taking values in two discrete, finite and dual lattices, possibly in M_G and P_G , where all divergences should naturally be eliminated, as anticipated above. In so far we succeed with such a programme we will call these discrete, finite lattices "physical spaces" to distinguish them from the smooth homogeneous spaces M_G and P_G . In fact we will show that this programme works, but only in the case where S^3 in eqs.(1) and (2) is restricted to S^1 where the isometry group $U(1)$ is abelian and a simple but instructive toy model may be constructed in which the Fourier transform may be well defined for finite lattices (chapter 5.1). For S^3 instead, the difficulties arising from the loss of commutativity of the isometry group seems to indicate that the recently developed instruments of non commutative geometry and of q -deformed Lie groups, might be more appropriate (chapter 6).

2. Compactified space-time. Arguments from cosmology

Should space-time be represented by the homogeneous space M_G above, then, as is often assumed, the space-time of cosmology could be:

$$M_{RW} = \frac{S^3 \times \mathbb{R}}{\mathbb{Z}_2} \quad (3)$$

where $\mathbb{R}$ represents the infinite covering of S^1 in eq.(1). In fact eq.(3) also represents Robertson-Walker space; the familiar space-time of most cosmological models (for a closed universe) whose metric may be unambiguously deduced [4] from the universally accepted Cosmological Principle (stating the isotropy and uniformity of the universe).

Both the space M_G and M_{RW} are, strictly speaking, homogeneous group-spaces determined by symmetry principles (conformal or cosmological respectively). Physical phenomena are always described by fields (Maxwell, Dirac, Einstein) on those manifolds, and on account of the postulated symmetry, those fields should be solutions of well defined field equations of motion on those homogeneous spaces [5]. (This is rigorously true for the free Maxwell field, for other fields one has to impose conditions compatible with other field equations like Einstein ones for the gravitational field).

For cosmology, then, any physical quantity, in particular, energy- (and subsequently matter-) density should be described in terms of some field, solution of well defined equations on M_G or M_{RW} . In particular, in the now generally adopted hot Big Bang model of the universe, physical space-time itself should arise as a solution of some field equation on M_G or M_{RW} (an example could be the scalar "inflaton field" in the inflationary model).

But then it is necessary to separate the concept of homogeneous spaces generated by symmetry principles and by the corresponding groups, from that of physical spaces represented by functions solutions of well defined differential field-equations on those homogeneous spaces. This distinction might be of great importance since, as we will see, it might suggest the mathematical and geometrical instruments to adopt in dealing with physical phenomena in physical spaces. Furthermore within the framework of Big Bang cosmology, the postulated symmetry and the corresponding homogeneous space should predetermine the possible geometrical structure of the physical space-time.

Since notoriously particular solutions of equations of motion may "spontaneously" violate the symmetry principles implied by the covariance group of those equations (the planetary orbits break "spontaneously" the $SO(3)$ symmetry of the two-body Newton equations of motion), if physical space-time results from the solution of field equations on M_{RW} , one could observe in the physical universe "spontaneous" violations of the Cosmological Principle and these violations could then constitute a confirmation of the hypothesis of the assumed compactifications of the homogeneous spaces M_G or M_{RW} .

Recent astronomical observation seem to indicate that the homogeneity of the universe postulated by the Cosmological Principle might indeed be violated.

3. Possible violations of the Cosmological Principle

Recent observations of galaxies distribution in the universe have revealed large-scale structures (of sizes larger than 50 Mpc/h where $h = 0.5-1$ stands for the Hubble's constant in units of $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$) which, as generally agreed, could not be due to gravity (at the origin of galaxies-clusters or-superclusters of the size up to 10-20 Mpc/h). They could then represent an unexpected violation of the uniformity and isotropy of the universe postulated by the Cosmological Principle.

The most famous of such structures is the Great Wall, a ~ 50 by ~ 100 Mpc/h-structure discovered in 1989 [6] at a distance of ~ 80 Mpc/h in the northern hemisphere. The most recent is the South Wall discovered in the southern hemisphere slightly nearer to us [7]. The most impressive are those discovered at large distances (up to 1000 Mpc/h) through "pencil beam surveys" which show striking regularity in galaxies distributions with a periodicity of 128 Mpc/h

(for more than 11 peaks) [8] (see Fig. 1).

Following the ideas exposed in the previous chapter, these large-scale structures could represent a spontaneous violation of the cosmological principle. In order to test this hypothesis one first has to write a field equation compatible with the cosmological principle in M_{RW} . We will start from the simplest case of a scalar field ψ (although it would perhaps be more appropriate to deal with a spinor field) and then the most natural equation is [5]:

$$\{R^{-3} \partial_t (R^3 \partial_t) - R^{-2} [\Delta(S^3) - 1]\} \psi = 0 \quad (4)$$

where R is the (expanding) radius of S^3 and $\Delta(S^3)$ is the Laplace-Beltrami operator on S^3 .

This equation may be solved by separation of variables with elementary solutions of the form:

$$\psi_{n\ell m} = f(t) Y_{n\ell m}$$

where the S^3 -spherical harmonics $Y_{n\ell m}$ satisfy:

$$\Delta(S^3) Y_{n\ell m} = -n(n+2) Y_{n\ell m} . \quad (5)$$

As in any field theory we may now define in terms of ψ the energy momentum tensor $T_{\mu\nu}$ and the energy density $T_{00} = H$ which, for elementary solutions has the form:

$$H_{n\ell m} = K(t) |Y_{n\ell m}|^2 \quad (6)$$

where $K(t)$ is an appropriate time dependent factor (which may be eliminated if we are merely interested in the relative density at a given time t [9]).

If the universe at a given time t finds itself in the eigenmode $Y_{n\ell m}$, then $H_{n\ell m}$ should represent its energy-density and then its matter density (both visible and invisible) after the energy has been converted to matter, by a mechanism to be determined. $H_{n\ell m}$ "spontaneously" breaks the $SO(4)$ symmetry of M_{RW} implied by the Cosmological Principle.

It is interesting to observe that the possibility of the Robertson-Walker universe represented by M_{RW} being subject to an eigenvibration of the form $Y_{n\ell m}$ was first proposed by E. Schrödinger, who adopted a suggestion by A.S. Eddington [10].

In order to try to explain with this model the mentioned observations on large-scale structures of galaxies in the universe we have adopted, in previous papers [9], [11], for $Y_{n\ell m}$ the most symmetric form $Y_{n,0,0}$ (also because the model should be applicable to Cosmic Background Radiation (CBR) whose main characteristic is its high isotropy), having the simple form:

$$Y_{n,0,0} = C_n \frac{\sin(n+1)\rho}{\sin\rho}, \text{ with } n = 46 \quad (7)$$

where C_n is a constant, ρ is the geodesic distance from the center P_c of the eigenmode and $n = 46$ was determined from the periodicity of 128 Mpc/h (in the hypothesis that $\rho_{\max} = R \pi/2 = 3008$ Mpc/h).

The free parameters of the model are the distance ρ_c from us of P_c and its equatorial coordinates δ_c, α_c . Inserting (7) in (6) one arrives, after appropriate coordinates transformations [11] at:

$$H_{46,0,0}(\rho, \delta, \alpha, \rho_c, \delta_c, \alpha_c) = \quad (8)$$

$$|C_{46}^1 (\cos\rho \cos\rho_c + \sin\rho \sin\rho_c [\sin\delta \sin\delta_c + \cos\delta \cos\delta_c \cos(\alpha - \alpha_c)])|^2$$

where C_{46}^1 is a Gegenbauer polynomial. $H_{46,0,0}$ was assumed to be proportional to the number of galaxies for unit volume element to be observed at a distance ρ from us in the direction with equatorial coordinates δ, α . To compare $H_{46,0,0}$ with the astronomical observations we have multiplied it by a factor $\sin^a\rho$, to mimic both the geometry of the particular observation ($\sin^2\rho$ for a cone) and its luminosity function, to obtain:

$$N_{46,0,0} = H_{46,0,0} \sin^a\rho \quad (8')$$

It is shown in [11] that most of the observed large-scale structures of galaxies (Great Wall [6], South Wall [7], pencil beam surveys [8], ESO Strip [12]) may be fairly well represented by (8) assuming

$$\rho_c = 40 \text{ Mpc/h}, \quad \delta_c = -35^\circ, \quad \alpha_c = 195^\circ \quad (9)$$

We reproduce as an example in Fig. 1 the comparison of $N_{46,0,0}$ (where we set $a = 1.6$) with the result of pencil beam survey [8] in the direction of N-S galactic poles, as reported in ref. [11].

At the present stage we may affirm that astronomical observations seem to confirm the hypothesis that the homogeneous space of our universe may be represented by M_{RW} given by eq. (3) compatible with the Cosmological Principle, which instead is "spontaneously" violated by large-scale distribution of galaxies (or physical space-time) according to the eigenmode $Y_{46,0,0}$.

Observations on large-scale structures of galaxies are going on and then, with time, it will be possible to know with certainty if the presented model is apt to describe the structure of the universe.

In fact the model might be easily falsified. This is mainly due to the fact that the present observations seem to suggest that we are relatively near^(*) to P_c (~ 40 Mpc/h). Therefore, if we compute $H_{46,0,0}$ for ρ large enough we might predict the distribution of galaxies which will be, to a large extent, independent from the variation of the free parameters ρ_c , δ_c , α_c . An example is given in Fig. 2 where we represent the relative number of galaxies as counted by an observer at a distance $\rho_c' = 600$ Mpc/h from us in a spherical sheet of 10 Mpc/h, centered on him, of radius $\rho_0 = 1000$ Mpc/h, if the density of galaxies is proportional to $H_{46,0,0}$ given by eq. (8). In Fig. 3 the same distribution is represented, after an appropriate coordinate transformation^(**) $\rho(\delta, \alpha)$, as it could be seen from where we are. Such predictions, if observed (or not) could then represent a proof (or a disproof) of the model.

3.1 The cosmic background radiation (CBR)

In this model $H_{46,0,0}$ given by eq.(8) could also be applied to CBR. It is well known that the most striking aspect of CBR is its remarkable isotropy .

In order to compute the anisotropy predicted by $H_{46,0,0}$ at a given distance ρ we have only to compute $H_{46,0,0}$ in a given direction (δ, α) and in the opposite direction $(-\delta, \alpha + \pi)$. If we indicate these two values with $H_{46,0,0}^+$ and $H_{46,0,0}^-$ then the measure α of the anisotropy, in these two antipodal directions of the celestial sphere may be defined by

$$\alpha = \frac{H_{46,0,0}^+ - H_{46,0,0}^-}{H_{46,0,0}^+} .$$

Considering that at the time of the last scattering, when CBR was emitted, the age of the universe was of about 300,000 years, with an approximate radius $\rho = \delta \approx 0.4$ Mpc/h, we easily find from eq.(8) that with the values of the parameters given in eq.(9) the maximal anisotropy α_m is

(*) The position of P_c , where the matter-density should be maximal, also seems to be confirmed from independent observations on the motions of nearby galaxies which are moving towards a location named the Great Attractor at a distance of about 43 Mpc/h at about 10° from P_c [13].

(**) The function $\rho(\delta, \alpha)$ is [11]:

$$\rho(\delta, \alpha) = \arctan \frac{-f \tan(\rho_c' - \rho_c) \cos \rho_0 + r}{\cos \rho_c' + r f \tan(\rho_c' - \rho_c)}$$

where

$$r = [\cos^2(\rho_c' - \rho_c) + \cos^2 \rho_0 + \sin^2(\rho_c' - \rho_c) f^2]^{1/2}$$

and

$$f = \sin \rho \sin \rho_c + \cos \delta \cos \delta_c \cos(\alpha - \alpha_c) .$$

$$\alpha_m \sim 10^{-1}$$

about 4 orders of magnitude larger than the observed one: $\alpha_{\text{obs}} \simeq 10^{-5}$.

The anisotropy α will depend on n of $H_{n,0,0}$, and for $n < 5$, it may be set in the approximate form:

$$\alpha \simeq 4 \delta \rho_c n (n+2)$$

Therefore for $n \leq 1$

$$\alpha \leq 6.10^{-5}$$

in accordance with the observed value.

This means that, within the framework of the hot big bang and of this model, if one assumes that the early universe, at the time of the CBR emission, was in an eigenstate near to the fundamental state ($n = 0$) this could explain the very high isotropy of the observed CBR. (Obviously in this case it remains to be explained by which dynamical mechanisms the universe, during its evolution, arrived at the harmonic $n = 46$).

4. Momentum-space compactification. Arguments from spinor geometry

The geometry of simple spinors was discovered by E. Cartan. He formulated the basic axioms and theorems which, for what concerns us, may be summarized as follows [14]:

Given a $2n$ -dimensional pseudoeuclidean space $W = \mathbb{R}^{n+1, n-1}$ and the corresponding Clifford algebra $Cl(n+1, n-1)$ with generators γ_a , $a = 1, 2 \dots 2n$, a spinor ϕ is a vector of the representation space S of $Cl(n+1, n-1) = \text{End } S$, defined by

$$p_a \gamma^a \phi = 0 \tag{10}$$

where p_a are the orthonormal components of a vector $p \in W$. This vector is null since, for $\phi \neq 0$, eq.(10) implies:

$$Q : p_a p^a = 0. \tag{11}$$

For fixed ϕ all $p \in W$ satisfying (10) and (11) define a totally null plane $M(\phi) \subset W$, whose vectors are both null and mutually orthogonal.

Let $\gamma_{n+1} = \gamma_1 \gamma_2 \dots \gamma_{2n}$ represent the volume element of $Cl(n+1, n-1)$, then the spinors ϕ_+, ϕ_- obeying:

$$\gamma_{2n+1} \phi_{\pm} = \pm \phi_{\pm}$$

are named Weyl spinors and for them the defining eq. (10) becomes:

$$p_a \gamma^a (1 \pm \gamma_{2n+1}) \phi_{\pm} = 0 \quad (12)$$

Any spinor ϕ associated with W may be considered as a sum of Weyl spinors:

$$\phi = \phi_+ + \phi_-.$$

A Weyl spinor ϕ_+ or ϕ_- is named simple if $M(\phi_{\pm}) \subset W$ is n -dimensional, that is maximal. For $n \leq 3$ all Weyl spinors ϕ_+, ϕ_- are simple. For $n = 3$, Weyl spinors $\phi_{\pm}$ were named twistors by R. Penrose [2].

The light cone (11) considered as a projective quadric PQ defines the $2n-2$ compact space V_c :

$$PQ: V_c = \frac{S^n \times S^{n-2}}{Z_2} \quad (13)$$

The pseudo-euclidean space $V = \mathbb{R}^{n,n-2}$ may be densely imbedded in V_c on which the conformal group in V , represented linearly by $O(n+1, n-1)$ acting on W , acts globally and transitively.

As shown by W. Pauli and P.A.M. Dirac, spinors play an important role in physics, more precisely in quantum mechanics of fermions. In particular, the basic equations of spinor geometry, summarized above, may be given physical meaning. We have only to consider p_a fixed and then ϕ as a field, function of p_a (common intersection of all $M(\phi)$). For $n = 2$, $W = \mathbb{R}^{3,1}$ and eq. (12) becomes

$$p_{\mu} \gamma^{\mu} (1 \pm \gamma_5) \phi_{\pm} = 0 \quad ; \quad \mu = 1, 2, 3, 0 \quad (14)$$

which is the Weyl equation for massless neutrinos, in momentum space. Therefore $W = \mathbb{R}^{3,1}$ may be conceived as the Fourier dual to Minkowski space-time^(*).

^(*) This correlation between spinor-geometry and physics in momentum space may be further extended. First, representing spinor as minimal left ideals of Clifford algebras, Maxwell's equations in momentum space may be obtained. Then one may represent the Dirac equation for a massive spinor, the one for the nucleon doublet coupled to a pseudoscalar pion triplet and the spinor skeleton of the electroweak standard model, all of them in momentum space [15].

The corresponding equations for twistors, or Weyl spinors associated with $\mathbb{R}^{4,2}$, is

$$p_a \gamma^a (1 \pm \Gamma_7) \phi_{\pm} = 0 \quad a = 1, 2, 3, 0, 5, 6 \quad (15)$$

and the projective quadric $p_a p^a = 0$ defines the conformally compactified momentum space

$$P_G = \frac{S^3 \times S^1}{Z_2} ,$$

identical to eq. (2).

These realizations of spinor geometry in (compactified) momentum-space of physics suggest that if conformal covariance is a fundamental symmetry of physics then momentum space should also be conformally compactified and, as a homogeneous space, it could be represented by $P_G = G/H_2$ as suggested in chapter 1 [3].

Another important feature of spinor-geometry, known to physicists and developed by E. Cartan, is that vectors of pseudo-euclidean spaces may be considered as bilinear functions of spinors. In particular, in our case we have that, for $W = \mathbb{R}^{n+1, n-1}$ its null vectors may be constructed from simple spinors associated with W . In fact, indicating with ψ an arbitrary spinor associated with W , the following proposition holds [15].

Proposition: The vectors $p^{\pm} \in \mathbb{R}^{n+1, n-1}$ with components

$$p_{\pm}^a = \langle \psi^c B, \gamma^a \phi_{\pm} \rangle$$

are null if and only if the spinors $\phi_{\pm} = 1/2 (1 \pm \gamma_{2n+1}) \phi$ are simple. $p_{\pm}^a$ are real (or imaginary) if n is even (or odd).

(We have indicated with ψ^c the charged conjugate of ψ defined by $\psi^c = C \bar{\psi}$ with $\bar{\psi} \equiv$ complex conjugate of ψ and the matrices B and C are defined by $\gamma_a B = \gamma_a^t B$, $C \gamma_a = \bar{\gamma}_a C$ where $\gamma_a^t \equiv \gamma_a$ -transposed).

We have, therefore, that the null vectors p_a , building up compactified momentum spaces, may be constructed, as bilinear spinor functions, from simple spinors.

The idea of considering the vectors of pseudoeuclidean spaces, and in particular of Minkowski space-time, as derived (bilinearly) from underlying spinor geometry is an old dream which has interested several distinguished physicists (Heisenberg, Penrose). From the above

considerations it would appear that simple spinors might naturally build up momentum space^(*), and, in particular, compactified momentum space [15] rather than Minkowski space-time. Furthermore, the null-vectors of such momentum spaces, generated by spinors, naturally appear in the basic equations of quantum mechanics (Weyl, Dirac, etc.). Therefore simple spinor-geometry could be the underlying geometry not only for momentum-space, but also for quantum mechanics which, in turn is underlying classical mechanics, naturally described in configuration space. This genesis of quantum mechanics in conformally compactified momentum space could offer the advantage of eliminating at the origin the difficulties of ultraviolet divergences which are the most severe ones of quantum field theory. One could also hope to avoid some of the unsolved paradoxes which are encountered when quantum mechanics is formulated in configuration space^(**).

4.1 The role of conformal inversion

The conformal inversion I transforms M_G to P_G :

$$I M_G I^{-1} = P_G$$

and viceversa. If we now interpret P_G as compactified momentum space then the result should be that in the corresponding densely imbedded flat spaces M and P both $= \mathbb{R}^{3,1}$ conformal inversion I should induce a map from $x \in M$ to $p \in P$. In fact if $x_\mu \in M$ we have:

$$I : x_\mu \rightarrow x'_\mu = \frac{x_\mu}{x^2} \quad \text{and} \quad I : x^2 \rightarrow x'^2 = \frac{1}{x^2}, \quad (16)$$

and if x_μ is a length, x'_μ is the inverse of a length.

The above transformations, if applied to x_μ space-like, transforms every point $P(|x|)$ inside a sphere S^2 of radius 1 in a point $P(1/|x|)$ outside that sphere and one could be tempted to call the set of points inside S^2 "small" and those outside "large". But these words are physically meaningless unless one introduces an arbitrary unit of length L by which (16) becomes:

(*) It also appears from the above Proposition that the minimal simple spinors to build up P_G seems to be the eight-dimensional ones associated with $Cl(5,3)$ (which also presents the triality symmetry discovered by E. Cartan). Each one of these simple spinors is the sum of two twistors.

(**) In fact they could be ascribed to the fact that both simple spinors (equivalent to totally null planes) and momentum space, which could appear to be the basic ingredients for the formulation of quantum mechanics, are concepts out of range of our everyday intuition and common sense which instead is well equipped to deal with macroscopic, classical physical phenomena which are evolving in configuration space.

$$I : x_{\mu} \rightarrow x'_{\mu} = \frac{x_{\mu} L^2}{x^2} \quad (16')$$

and the radius of S^2 becomes L (however conformal covariance is broken). But then the meaning "small" and "large" refer to our choice of L and are then arbitrary.

A solution to this dilemma may be obtained by considering the exponential $\text{Exp}(i x_{\mu} k^{\mu})$ representing plane waves (and appearing in Fourier integrals in $\mathbb{R}^{3,1}$). In fact $x_{\mu} k^{\mu}$ is dimensionless and conformal inversion I transforms $x^2 k^2$ as follows:

$$I : x^2 k^2 \rightarrow x'^2 k'^2 = \frac{1}{x^2 k^2}$$

and then the sphere S^2 of radius 1 is meaningful. If now we adopt the De Broglie equality $p_{\mu} = \hbar k_{\mu}$ we obtain:

$$I : x^2 p^2 \rightarrow x'^2 p'^2 = \frac{\hbar^2}{x^2 p^2}$$

and the radius of S^2 becomes $\hbar$. This suggests, then, that in order to give physical meaning to eq.(16) instead of using an arbitrary unit of length L one may better adopt the Planck's constant $\hbar$ (which might be considered invariant since $e^2/\hbar c = \alpha$) and to substitute (16) with

$$I : x_{\mu} \rightarrow x'_{\mu} = \hbar \frac{x_{\mu}}{x^2} = p_{\mu} \quad (16'')$$

and to indicate the I -transformed x'_{μ} with another letter p_{μ} since it belongs to another homogeneous space: $x'_{\mu} \in P_G$ than $x_{\mu} \in M_G$. Then the radius of S^2 equals $\hbar$ and the words "large" and "small" identify with those of quantum mechanics since the radius $\hbar$ of S^2 separates the world where the action measured by xp is larger than the Planck's constant from that in which it is comparable or smaller. These transformations may be explicitly performed in a toy model (see chapter 5.1).

In chapter 3 we have shown that in nature there is perhaps an observable phenomenon of the realization of M_G represented by the observed large scale structure of galaxies in the universe considered as a non relativistic system (we separated the space- from time- coordinates) and explicitly described by the eigenvibration of $Y_{n,0,0}$ of S^3 .

Then, supposing that also P_G might be realized in nature, one could guess that conformal

inversion could map that non-relativistic physical system to a corresponding non-relativistic system in momentum space; and if the universe is considered as a "large" (classical) system the conformally inverted system should be a "small" (quantum) system. Such system indeed exists and it is the most abundant atom in the universe: the hydrogen atom.

4.2 The hydrogen atom in momentum space

In 1935 V. Fock [17] has shown how the stationary Schrödinger equation for the H-atom, which, after a Fourier transformation, may be written as an integral equation in momentum space, may be formulated on S^3 : one point compactification of $\mathbb{R}^3$ -momentum space, after the inverse of a stereographic projection. On S^3 the Fock integral equation has elementary solutions of the form:

$$Y_{n\ell m}(p) = \frac{\lambda}{2\pi^2} \int_{S^3} \frac{Y_{n\ell m}(p')}{(p-p')^2} d^3 p' \quad (17)$$

with

$$\lambda = \frac{e^2}{\hbar c} \frac{mc}{p_0} = n + 1 \quad (18)$$

where p_0 is a unit of momentum, and assuming $p_0 = (-2mE)^{1/2}$ Fock obtains for E the energy levels of the H atom. $Y_{n\ell m}$ are the S^3 -spherical harmonics defined by eq.(5). In particular, he observed that the most symmetrical wave-function for the H-atom in S^3 -momentum space has the form^(*) (eq.(26) of ref. [17]):

$$Y_{n,0,0} = C_n \frac{\sin(n+1)\chi}{\sin\chi} \quad (7')$$

identical to eq.(7) in chapter 3 which seems to be apt to describe the observed large-scale structures of galaxies in the universe, though in ordinary space. In this case the result would be that the matter density of the universe, at large distances may be represented by (the modulus square of) an eigenfunction which is the same of that of H-atom in a stationary state (in momentum space). This correlation between "large" and "small" in our universe might recall the one advocated by Mach's principle long ago and further analyzed by D. Sciama [19]. Furthermore, the origin of $SO(4)$ symmetry of both systems could be global for both and deriving from configuration-space for the

(*) It is interesting to note that within the framework of dynamical groups one might represent the whole spectrum of eigenfunctions of the Hydrogen atom by imbedding $SO(4)$ in $SO(4,2)$. However, in order to do it, $so(4,2)$ -algebra has to be represented in momentum space [18].

Universe while from momentum-space for the H-atom^(*). This could then constitute a further indication that the hypothesis of conformal compactification of both M and P in M_G and P_G respectively might have some elements of truth. Let us then tentatively adopt it.

5. Simultaneous compactification of space-time and momentum space

We know that the flat Minkowski space M and its Fourier dual P both isomorphic to $\mathbb{R}^{3,1}$ may be densely imbedded in M_G and P_G respectively. However, as such they are not flat but conformally-flat and in this way the standard formalism of Fourier transforms may not be adopted. In order to guess how we could substitute it, let us first start from some heuristic considerations.

Let us suppose that only M is compactified in $M_G = (S^3 \times S^1)/Z_2$ and then, any single valued function on M_G may be expressed as an infinite, discrete Fourier sum on the spherical harmonics $Y_{n\ell m} e^{im'\phi}$ on M_G . Therefore the Fourier dual of that function will be represented on an infinite lattice where n will take all integer values from $n = 0$ up to infinity. This lattice then plays the role of momentum space. The spacing of this lattice will be defined by the common radius R of S^3 and S^1 ; precisely it will be of the order of $1/R$. Let us now suppose that also P is compactified in P_G , then this lattice can no longer be infinite but will have to be truncated, in some way, at some maximal $n = N$ depending on the radius M of S^3 and S^1 of P_G .

Repeating the same considerations, starting from P one could guess that we should end up with two finite, discrete lattices on M_G and P_G where the differential equations on the homogeneous spaces M_G and P_G will have to be substituted by difference-equations on finite lattices and the absence of infinities and infinitesimals should obviously follow. On such finite lattices field theories should result naturally free from both ultraviolet and infrared divergences as a consequence of conformal covariance (that this may result, in conformally covariant field theories, was already shown by G. Mack and I.T. Todorov [20]). This attractive programme is difficult to realize with standard methods in general. It may only be performed if $\mathbb{R}^3$ -space is restricted to $\mathbb{R}^1$ and, correspondingly, if, in the compactified version, S^3 is restricted to S^1 . In this case one may formulate an exactly soluble toy model from which it is possible to obtain interesting suggestions for the higher dimensional cases.

(*) It is well known that the $SO(4)$ symmetry of the H-atom may also be derived from the "long range" Coulomb potential: $Z e^2/r$ which in turn may be interpreted as the consequence of the photon zero mass. This interpretation is missing instead for the $SO(4)$ symmetry giving rise to the Universe structure at large distances, while it may be advocated for the $SO(4)$ symmetry of Kepler motions which in turn may be derived from the "long range" Newton potential: $K M m/r$ due to the zero mass of the graviton. The only common origin of the possible symmetry $SO(4)$ for the 3 systems Universe, Kepler motions and H-atom could be the global one. The local one ("long range") for the latter two could derive from it, as may be induced from the two equivalent equations of the H-atom stationary states: the Fock one in momentum-space and the Schrödinger one in configuration space.

5.1 The toy model

Let us consider the space $W = \mathbb{R}^{2,1}$ and its projective light cone

$$PQ: V_G = \frac{S^1}{Z_2}$$

which is the one point conformal compactification of $V = \mathbb{R}^1$, where for any regular function $f(x)$ with $x \in V$ the Fourier transforms:

$$f(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{ikx} F(k) dk; \quad F(k) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx \quad (19)$$

hold and $F(k)$, the Fourier transform of $f(x)$, takes values on $k \in P = \mathbb{R}^1$, Fourier dual of V .

The 3-parameter conformal group on V is:

$$G = D(x) T(x) K$$

isomorphic to $SO(2,1)$ in $W = \mathbb{R}^{2,1}$. V_G above is the homogeneous space

$$V_G = \frac{G}{H_1} = \frac{S^1}{Z_2} \quad (20)$$

where $H_1 = D(x) K$.

Any single valued function $f(\phi)$ on S^1 may be expressed, as usual, through the infinite Fourier series:

$$f(\phi) = (2\pi)^{-\frac{1}{2}} \sum_{m=-\infty}^{+\infty} e^{im\phi} F(m) \quad (21)$$

with

$$F(m) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{+\pi} e^{-im\phi} f(\phi) d\phi \quad (22)$$

Let us suppose that S^1 has a radius R and let us introduce in S^1 the linear coordinate

$$x = R \phi.$$

The periodicity condition $\phi + 2\pi = \phi$ becomes $x + 2\pi R = x$.

In terms of x eq.(21) becomes:

$$f(x) = (2\pi)^{-\frac{1}{2}} \sum_{m=-\infty}^{+\infty} e^{i k_m x} F(k_m) \quad (23)$$

where $k_m = m/R$. We see then that the conformal compactification V_G of $V = \mathbb{R}^1$ has restricted the dual momentum space $P = \mathbb{R}^1$ to the infinite lattice

$$k_m = \frac{m}{R} ; -\infty < m < +\infty \quad (24)$$

with spacing

$$\Delta k_m = \frac{1}{R}$$

Let us now suppose that the dual momentum space $P = \mathbb{R}^1$ is compactified and represented by the homogeneous space

$$P_G = \frac{G}{H_2} = \frac{\tilde{S}^1}{Z_2}$$

with $H_2 = D \times P$. Repeating the previous procedure and supposing that M is the radius of S^1 we end up with

$$F(k) = (2\pi)^{-\frac{1}{2}} \sum_{m=-\infty}^{+\infty} e^{-i x_m k} f(x_m) \quad (23')$$

with $x_m = m/M$ and $V = \mathbb{R}^1$ has been restricted to an infinite lattice:

$$x_\mu = \frac{m}{M} ; -\infty < m < +\infty \quad (24')$$

with spacing:

$$\Delta x_m = \frac{1}{M}$$

But now we may suppose the simultaneous compactifications of V and P in V_G and P_G respectively. We have only to impose that the number of points of the two dual lattices is equal to a number $2N$ of vertices of two regular polygons on S^1 and $\tilde{S}^1$:

$$2\pi M = \frac{2N}{R} \quad \text{and} \quad 2\pi R = \frac{2N}{M}$$

which is satisfied for

$$N = \pi M R \quad (25)$$

(remember that if R is a length, M is the inverse of a length and $\pi M R$ is a pure number).

Then we have that if $f(x_m)$ and $F(k_m)$ are the functions $f(x)$ and $F(k)$ restricted on the $2N$ point lattices on S^1 and $\tilde{S}^1$ the following equations hold:

$$f(x_n) = (2N)^{-\frac{1}{2}} \sum_{m=-N}^{N-1} e^{i k_m x_n} F(k_m) \quad (26)$$

$$F(k_m) = (2N)^{-\frac{1}{2}} \sum_{n=-N}^{N-1} e^{i x_n k_m} f(x_n)$$

where $x_n k_m = nm/RM$. Eqs.(26) are true for any $N = \pi M R$ since

$$\frac{1}{2N} \sum_{n=-N}^{N-1} e^{i n (m-m') \frac{\pi}{N}} = \delta_{mm'} \quad (27)$$

In fact, this equation is equivalent to the theorem stating that the sum of the $2N$ roots of 1 equals zero [21].

We have then that the two $2N$ -point lattices L_M and M_P , obtained by restricting M_G and P_G to the vertices of regular polygons inscribed in them, may be defined, because of (26), as Fourier dual.

It is easy to verify that for $\mathbb{R} \rightarrow \infty$ and $M \rightarrow \infty$ eq. (26) become the standard ones on $\mathbb{R}^1$ given in eq.(19), which then justifies the name of "dual" assigned to the lattices L_M and L_P , and then to the interpretation of $P_G = G/H_2$ as compactified momentum-space.

The above means that in the same way as the infinite set of circular harmonics (periodic on S^1):

$$\phi_m(\phi) = (2\pi)^{-\frac{1}{2}} e^{i m \phi} ; \quad -\infty < m < +\infty \quad (28)$$

build up a complete orthonormal set for the representation of any continuous, single-valued function $f(\phi)$ on S^1 , the finite set of harmonics (periodic on the lattices):

$$\phi_m(x_n) = (2N)^{-\frac{1}{2}} e^{i m n \frac{\pi}{N}} \quad -N \leq m, n \leq N-1 \quad (29)$$

build up a complete orthonormal set for the representation of any single valued function $f(x_n)$ on the $2N$ -point lattice L_M in S^1 .

Consequently restricting S^1 and $\tilde{S}^1$ to the two $2N$ -point lattices L_M and M_P respectively is equivalent to the truncation to N of the infinite Fourier series of eq. (21).

This formalism may be easily extended to the two dimensional Minkowski space $M = \mathbb{R}^{1,1}$, for which the conformal compactification will be:

$$M_G = \frac{S^1 \times S^1}{Z_2} \quad , \quad P_G = \frac{\tilde{S}^1 \times \tilde{S}^1}{Z_2} \quad (30)$$

and then one could find examples of conformally covariant dynamical problems on the conformally compactified straight line.

Here we have that while the homogeneous spaces determined by the postulated (conformal) symmetry on $M = \mathbb{R}^{1,1}$ are the smooth toruses M_G and P_G of eq.(30), the physical spaces are the $2N \times 2N$ dual lattices L_M and L_P on them.

Obviously in the physical spaces L_M, L_P any dynamical theory will be a priori free from divergences since the mere concepts of infinity and infinitesimal will be absent (like in non standard mathematics).

Another substantial difference between the homogeneous spaces M_G and P_G and the discrete

physical spaces L_M and L_P is that while on M_G and on P_G the group G acts transitively and also smoothly^(*) this cannot be true in L_M and L_P . In order to analyze this point, let us first go back to the toy model where $M_G = S^1/Z_2$ and the conformal group is the non compact $SU(1,1)$. What we now need is finite dimensional and unitary representations of this group and remarkably this is precisely a feature of quantum groups: q -deformations of Lie group, when q is a root of unity [21] (which we exploited in eq.(27)).

In order to explore this possibility let us start from $U(1)$ acting on S^1 . If we represent a point of S^1 by $\phi_m = (2\pi)^{-1/2} e^{i m \phi}$ then the generator T_δ of $U(1)$ may be represented by:

$$T_\delta = e^{i m \delta} \quad (31)$$

in fact

$$\phi_m(\phi) \rightarrow \phi_m(\phi + \delta) = T_\delta \phi_m(\phi) \quad (32)$$

and δ is a continuous parameter: $0 \leq \delta \leq 2\pi$ such that $T_0 = 1 = T_{2\pi}$.

On the $2N$ point lattice of S^1 instead, we represent any point with the harmonics on the lattice given in eq.(29):

$$\phi_m(x_n) = (2N)^{-\frac{1}{2}} e^{i \frac{m}{R} \frac{n}{M}} = (2N)^{-\frac{1}{2}} e^{i m n \frac{\pi}{N}} \quad (33)$$

Then defining

$$T_{\Delta x} = e^{i m \frac{\pi}{N}} \quad (34)$$

we have

$$\phi_m(x_n) \rightarrow \phi_m(x_{n+1}) = T_{\Delta x} \phi_m(x_n) \quad (35)$$

and

$$\phi_m(x_n) \rightarrow \phi_m(x_{n-1}) = T_{\Delta x}^* \phi_m(x_n) \quad (36)$$

where $\Delta x = \pi/N$ represents the minimal translation on the lattice, while $(T_{\Delta x})^{2N} = 1$ as it should. Because of (35) and (36) we may then consider $T_{\Delta \phi}$ and $T_{\Delta \phi}^*$ as creation and annihilation operators on the lattice:

(*) For this action, including conformal inversion I see ref. [21].

$$T_{\Delta\phi} = a^* \quad ; \quad T_{\Delta\phi}^* = a \quad (37)$$

This formalism may be extended, by adding two new creation and annihilation operators b^* and b , and a ladder representation of $SU_q(1,1)$ may be given which, for q root of unity, restricts the action of the conformal group to the lattice L_M on S^1 as shown in ref. [21].

6. Higher dimensional spaces

The above formalism may not be easily extended to S^n . Already in S^2 one finds difficulties due to the fact that while $U(1)$ is abelian, $SU(2)$ is not or, in other words, while we may inscribe in S^1 a regular polygon of an arbitrary number $2N$ of vertices which then covers it densely for $N \rightarrow \infty$, this is not possible for S^2 as is well known since the time of Plato. One may in this case use orthonormal (Hahn) polynomials $h_{N,\ell,m}$ which approximate the S^2 spherical functions $Y_{\ell m}(\phi, \phi)$ and identify with them in the limit $N \rightarrow \infty$ as shown in a previous paper [21]. In this way one may regularize both infrared and ultraviolet divergences but otherwise this procedure is not fully satisfactory insofar as the lattice we obtain is only an approximation of what we defined as physical spaces.

Rather, one should take advantage of the fact that quantum groups seem to be applicable to the toy model and try to systematically extend this algorithm to higher dimensional spaces. Furthermore, the separation of the concepts of homogeneous spaces from that of physical spaces defined by classes of functions on the former may induce the adoption of the techniques of non-commutative geometry for dealing with physical spaces. In fact this technique may also be adopted in the case of the toy model in order to define the finite lattices and to extend to them the concept of Fourier transform^(*) [22]. Then one may hope to be able to extend it also to S^n .

7. Conclusions and outlook

We have seen how some recent astronomical observations seem to support the hypothesis that the geometry of large-scale structures of galaxies might originate from an S^3 -spherical harmonic $Y_{n\ell m}$ (with $\ell = 0 = m$) and could therefore represent a spontaneous violation of the Cosmological Principle conceived, in this context, as a symmetry principle represented by $SO(4)$. Only further astronomical observations will tell us if this hypothesis may be true (thus confirming the old Eddington-Schrödinger [10] conjectures). For the moment let us adopt it and try to derive from it some further consequences of interest for physics.

(*) This observation is due to S. Majid, to whom I am grateful.

We have seen that such a model of matter distribution in the universe, deriving from the isometry group $SO(4)$ of the spatial section of M_G or M_{RW} , has a dual represented by the stationary states of the hydrogen atom which, as shown by Fock [16] may also be described by the same harmonics Y_{nlm} , though in compactified momentum space S^3 .

There would then exist two non relativistic physical systems: the universe and the H-atom in stationary states, both ruled by the isometry group $SO(4)$, the first classical, naturally described in (compactified) configuration space and the second, quantum mechanical, naturally described in (compactified) momentum space.

We know that conformal inversion I transforms the homogeneous space $M_G = G/H_1$, often interpreted as conformally compactified space-time, into $P_G = G/H_2$, which could represent conformally compactified momentum space.

We could then assume that the two mentioned physical phenomena both derive from conformally covariant systems described in M_G and P_G after reduction from conformal symmetry to Lorentz- (because of non zero masses) and subsequently to Galilei- (because of slow motions)-symmetry but maintaining the maximal compact symmetry $SO(4)$ (in a similar way as $SO(3)$ is maintained when restricting Lorentz- to Galilei-covariance).

The model of the universe presented above is obviously incomplete since it does not contemplate the role of gravitation. Therefore, one of the next steps of our study should be to consider a system of equations consisting of eq.(4) (or, better, their spinor counterpart) and Einstein's equations and to study the conditions for their compatibility, which should be found since galaxies and their clusters do exist and are certainly due to gravitation, and therefore they should be represented as solutions of Einstein's equation. From these equations one must also be able to obtain, as a particular case, non relativistic Newton mechanics and, in particular, the two-body equations of motions, which also notoriously manifest an $SO(4)$ symmetry (spontaneously broken by Kepler orbits). Therefore the result should be that also in Minkowski space-time M densely contained in M_G we should be able to break down the symmetry up to the galilean one, still maintaining the original symmetry $SO(4)$ of M_{RW} or of M_G (or the maximal compact subgroup $SO(4)$ of G) and it should be possible to obtain this not only from the local property of the gravitational field $g_{\mu\nu}$ and then interpretable as due to the zero mass of the graviton, but also globally from the maximal isometry group $SO(4)$ of M_G or of M_{RW} , representing the Cosmological Principle.

We have seen in this paper that if the two homogeneous spaces of the conformal group M_G and P_G are both assumed to have a role in the description of physical phenomena, then, as clearly exemplified in the toy model presented in chapter 5.1, one has to separate the concept of homogeneous spaces M_G and M_P whose structure and metric is determined by postulated symmetry principles and on which standard differential analysis and geometry are applicable, from that of physical spaces defined by some class of functions on those homogeneous spaces,

where this applicability might no longer be appropriate. In particular the often adopted hypothesis of conformal compactification leading to compact, smooth manifolds might be true for the homogeneous spaces (both space-time and momentum-space) but could not be true for the physical space where, among others, the symmetry principles might be spontaneously violated.

Furthermore such physical spaces could be the arena of the newly discovered non-commutative geometry and q -deformed Lie groups, which, if the hypothesis of conformal compactifications is true, would eliminate ab initio the uneasy concepts of infinite and infinitesimal when dealing with physical phenomena in physical spaces (as similarly proposed by nonstandard mathematics).

Acknowledgments

The author wishes to thank L. Dabrowski, G.F. Dell'Antonio, C. Fronsdal, S. Majid, J. Niederle, P. Nurowski, M. Ramella, R. Raczka, D. Sciama and A. Trautman for helpful discussions and suggestions.

REFERENCES

- [1] E. Cunningham, Proc. London Math. Soc. 8, 77 (1909).
H. Bateman, Proc. London Math. Soc. 8, 223 (1909).
- [2] O. Veblen, Proc. Nat. Acad. Sci., USA 90, 503 (1933).
R. Penrose, M.A.H. Mac Callum, Phys. Reports 6C, 242 (1973).
I.E. Segal, Mathematical Cosmology and Extragalactic Astronomy, Academ. Press, N.Y. (1976).
I.E. Segal, Nuovo Cimento 79B, 187 (1984).
- [3] P. Budinich and R. Raczka, Foundations of Physics, 23, 599 (1993).
- [4] S. Weinberg, Gravitation and Cosmology, J. Wiley and Sons, N.Y. (1972).
- [5] A.O. Barut, P. Budinich, J. Niederle and R. Raczka, Conformal Space-Times: The Arenas for Physics and Cosmology, SISSA Preprint, 1994 (to be published in Foundations of Physics).
- [6] M.I. Geller and J.P. Huchra, Science 246, 989 (1989).
- [7] L.N. Da Costa, M.I. Geller, P.S. Pellegrini, D.W. Latham, A.P. Fairall, N.A. Willmar, J.P. Huchra, J.H. Calderon, M. Ramella and R.O. Martzke, Ap. J. Letters, 424, L1 (1994).
- [8] T.I. Broadhurst, R.S. Ellis, N. Ellemann, D.C. Koo, J.A. Munn and A.S. Szalay, Proceedings of the VI Marcel Grossmann Meeting 1992, World Scientific (Singapore) (1992).
- [9] P. Budinich and R. Raczka, Europhys. Lett. 23, 295 (1993).
- [10] E. Schrödinger, Pontificia Academia Scienziarium, Commentationes II, 9, 227 (1938).
E. Schrödinger, Physica IV, 272 (1939).
- [11] P. Budinich, P. Nurowski, R. Raczka and M. Ramella: On Possible Wave Structure of the Universe (The Spontaneous Violation of the Cosmological Principle). SISSA Preprint 79/94/FM (1994).
- [12] G. Vettolani et al, A.S.P. Conference Series, Milano p. 74 (1992).
- [13] D. Lynden-Bell et al, Ap. J. 326, 19 (1988).

- [14] E. Cartan, *Lecons sur la Theorie des Spineurs*, Hermann, Paris (1937).
 C. Chevalley, *The Algebraic Theory of Spinors*, Columbia U.P., New York (1954).
 I.M. Benn and R.W. Tucker, *An Introduction to Spinors and Geometry*, Adam Hilger, Bristol (1987).
 P. Budinich and A. Trautman, *The Spinorial Chessboard (Trieste Notes in Physics)*, Springer, New York (1989).
- [15] P. Budinich: *Non-Compact Lie-Groups*, Kluwer Academic Publishers, 123 (1994).
- [16] P. Budinich, *Foundations of Physics*, 23, 949 (1993).
- [17] V. Fock, *Zeitschrift f. Physik*, 98, 145 (1935).
- [18] A.O. Barut and R. Wilson, *Phys. Rev. A*40, 1340 (1989).
- [19] D.W. Sciama, *The Unity of the Universe*, Anchor, New York (1961).
- [20] G. Mack and I.T. Todorov, *Phys. Rev. D*8, 1764 (1973).
- [21] P. Budinich, L. Dabrowski and W. Heidenreich, *On Dual Finite Lattices*, SISSA Preprint (1994).
- [22] A. Kempf and S. Majid: *Algebraic q-integration and Fourier theory on quantum and braided spaces*, 1994. *J. Math. Phys.*, to appear.

Figure Captions

Figure 1 The histograms of the pencil beam survey in the North-South galactic poles directions [8] compared with $N_{46,0,0}$ given by equations (8), (8') (solid line) with ρ_c , δ_c and α_c as in equation (9) and $a = 1.6$, as reported in ref. [11]. The maxima of the theoretical curves are only indicative. The amplitudes of the fifth peak in the north and of the fourth peak in the south have been normalized to the histogram.

Figure 2 The angular distribution of the density of galaxies at $\rho = 1000$ Mpc/h for an observer at a distance of 600 Mpc/h from the center of the vibrational system $Y_{46,0,0}$.

Figure 3 The same distributions as in Figure 2 plotted as they would appear on our sky.

Fig.1

Fig.2

Fig.3