Instanton Dynamics in the Broken Phase of the Topological Sigma Model

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February 1995

Abstract

The topological $\sigma$ model with the black hole metric of the target space is considered. It has been shown before that this model is in the phase with BRST-symmetry broken. In particular, vacuum energy is non-zero and correlation functions of observables show the coordinate dependence. However these quantities turned out to be infrared (IR) divergent. It is shown here that IR divergences disappear after the sum over an arbitrary number of additional instanton-anti-instanton pairs is performed. The model appears to be equivalent to Coulomb gas/Sine Gordon system.

SWAT 95/62
hep-th 9500000

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1 Introduction

Considerable progress has been made during last years in the study of topological field theories (TFT) [1, 2]. The main avenue of these studies is the relationship between 2D TFT’s and the low dimensional string theory. It was shown [3, 4] that string theory at $c < 1$ equivalent to topological minimal matter [5] coupled to topological gravity [3]. An important step forward was made in ref.[6] where the equivalence of $c = 1$ string and topological version of $SL(2R)/U(1)$ WZW coset model [7] at level $k = 3$ was shown.

Though much work has been done along these lines one of the most crucial problem about TFT still remains unsolved. In the field theory framework TFT has no physical degrees of freedom: all correlation functions of observables are just numbers. Therefore, as it was proposed already in the original Witten’s papers [1, 2], we need some mechanism of the spontaneous breakdown of BRST symmetry for TFT to have something to do with physics. The idea is that the physical theory may correspond to the broken phase of the TFT. Then we could have advantages from the existence of the underlying BRST-symmetry (say, good UV properties) in a theory with some physical degrees of freedom "liberated".

The above problem persists also in the framework of the topological string. The $c < 1$ non-critical string has no physical degrees of freedom, therefore it is not surprising that it is actually topological. The problem of relevance of the string theory beyond the $c = 1$ barrier to any TFT remains unsolved.

In our previous paper [8] the 2D topological $\sigma$ model [2] with black hole metric [9] of the target space in two dimensions was considered. Although the target space is not compact (it has the form of a semi-infinite cigar) the model is shown to possess world-sheet instantons. In fact, cigar-like metric appears to be on the “borderline” between compact and noncompact cases and needs careful regularization. The result in [8] is that the topological version of the model does have unsuppressed instantons. The noncompactness of the moduli space of these instantons produces new divergences. These give rise to the nonzero vacuum energy and to the coordinate dependence of correlation functions of observables. Hence, the BRST symmetry is broken [8].

Divergences of the integrals over the moduli space of instanton studied in [8] are both of UV and IR nature. The UV ones introduce the UV cutoff parameter $a$ (lattice spacing) dependence of observables. These are signals
of the presence of an extra conformal anomaly in the theory associated with
the noncompactness of the target space. However the IR divergences are
obviously artificial. They should disappear when all IR singular effects are
taken into account.

In this paper we continue to study the topological $\sigma$ model which has the
target space with the geometry of the two-dimensional black hole. On one
hand this model can be viewed as a toy example to study the mechanism
of BRST-symmetry breaking. On the other hand, the model has the same
geometry of the target space as $SL(2, R)/U(1)$ coset \cite{9} (although it is not
identical to the latter one). This means (in view of the results in ref.\cite{6}) that
the BRST symmetry breaking in the topological black hole could have some
parallel in the $c = 1$ string theory.

Our aim in this paper is to study the physics which emerges in the broken
phase of TFT. In particular, we consider the partition function and correlation
functions of observables in the instanton vacuum with arbitrary number
of instanton–anti-instanton ($I\bar{I}$) pairs added. We sum over all these IR trou-
blesome effects constructing the effective Lagrangian of the model. After
that IR divergences disappear and the mass scale is dynamically generated.
The model turns out to be equivalent to Sine Gordon (SG) theory. In fact,
the instanton physics in the topological black hole model appears to be very
similar to that in $O(3)$ $\sigma$ model. In the latter model instanton vacuum has
the analogous Coulomb gas description \cite{10}.

The organization of this paper is as follows. In Sec.2 we review the
properties of the topological $\sigma$ model with black hole metric and show how
the breakdown of the BRST-symmetry occurs. In Sec.3 we develop a certain
approximate scheme ($I\bar{I}$ approximation) and show that instanton vacuum of
the model is equivalent to Coulomb gas/Sine Gordon (CG/SG) system (or
to free massive fermions) in this approximation. In Sec.4 we calculate the
partition function in the background of two $I\bar{I}$ pairs and show that (with
the proper definition of the geometry of the modular space of instantons)
$I\bar{I}$ approximation becomes exact. In Sec.5 we present our final result for
the vacuum energy which turns out to be nonzero and IR-finite. Then in
Sec.6 we calculate the two-point correlation function of operators from the
cohomology of observables. It appears to be also IR finite and coordinate
dependent. In particular, it shows the power fall-off at large distances. We
interpret this behaviour as a propagation of the goldstino fermion associated
with the broken BRST symmetry. Sec.7 contains our final discussion.
§ 2 d = 2 topological $\sigma$-model with the black hole metric

First in this section we review some general properties of the topological $\sigma$ model [2, 3] (for a review see also [11]). The action of the model on the $d = 2$ Kahler manifold reads [2]

$$S = \frac{r^2}{\pi} \int d^2 x \left\{ \frac{1}{2} g_{i j}(w) \partial_\mu w^i \partial_\mu w^j - \frac{1}{2} J_{i j}(w) \epsilon_{\mu \nu} \partial_\mu w^i \partial_\nu w^j 
- i g_{i j} \lambda^\mu D_\mu \chi^j - \frac{1}{8} R_{i j k l} \chi^i \chi^j \lambda^k \lambda^\ell \right\}. \tag{2.1}$$

Here $\mu, \nu = 1, 2$ are world sheet indices, while $i, j = 1, 2$ are target space ones. The world sheet is considered to be flat for simplicity, while $g_{i j}$ and $J_{i j}$ denote metric and complex structure of the target space. In this paper we consider target space metric and complex structure of the form

$$g_{i j} = g(w) \delta_{i j}, \quad J_{i j} = g(w) \epsilon_{i j}. \tag{2.2}$$

$D_\mu$ is the covariant derivative, $D_\mu A^i = \partial_\mu A^i + \partial_\mu w^k \Gamma^i_{k \ell} A^j$, while $\Gamma^i_{k \ell}$ and $R_{i j k l}$ are connection and curvature tensor respectively. Fermion system $\chi^i, \lambda^\mu$ has spins 0, 1 and satisfies the constraint

$$\lambda^\mu + \epsilon^\mu_\nu J^j_\nu \chi^j = 0. \tag{2.3}$$

Fermions play the role of ghosts which cancel out boson degrees of freedom in correlation functions of observables.

The BRST operator acts as follows

$$\{ Q, w^i \} = \chi^i, \quad \{ Q, \lambda^\mu \} = 0, \quad \{ Q, \chi^i \} = 2i (\partial_\mu w^i - \epsilon^\mu_\nu J^j_\nu \partial_\nu w^j) + \lambda^\mu \Gamma^i_{j k} \lambda^k. \tag{2.4}$$

Observables $O$ of TFT are elements of the $Q$-cohomology

$$\{ Q, O \} = O, \quad \{ Q, \tilde{O} \} \neq 0. \tag{2.5}$$
This condition means that we consider only gauge invariant operators which are defined up to a gauge transformation.[12] Correlation functions of interest are of the form
\[ \langle O_1(x_1) \cdots O_n(x_n) \rangle, \] (2.6)
where \( O_i \) are from the \( Q \)-cohomology. They are independent of world-sheet and target space metric in the topological phase [2]. First of these properties means, in particular, that (2.6) is independent on \( x_1, \ldots, x_n \) and the second ensure its independence of the coupling constant \( 1/r^2 \). We will see later that both of these properties are broken in the \( \sigma \) model with the black hole metric.

The \( Q \)-cohomology of observables in \( d = 2 \) \( \sigma \) model is particularly simple. It consists of only two elements, one is the partition function and another one can be chosen in the form [2, 3]
\[ O = iJ_{ij}(w)\chi^i\chi^j(x). \] (2.7)

Let us now consider correlation function (2.6) with operator \( O \) from (2.7). As it cannot depend on \( r^2 \) we can take limit \( r^2 \Rightarrow \infty \). Thus, the semiclassical approach becomes exact. In particular, nonzero contributions come only from instantons (1) [2]. The latter are solutions of classical equations of motion in a given topological class with winding number \( k \). They are holomorphic functions
\[ w(z) = v \left( 1 + \sum_{\ell=1}^{k} \frac{\tilde{\rho}_\ell}{z - z_\ell} \right), \] (2.8)
which satisfy the equation
\[ \partial_\mu w^i - \epsilon^i_{\mu j} J^j_\phi \partial_\mu w^j = 0, \] (2.9)
or \( \partial w = 0 \). Here \( w = w^1 + iw^2, \tilde{w} = w^1 - iw^2 \). Instanton solution in (2.8) depends on \( 2k + 1 \) complex parameters: \( z_\ell \) are centers of multi-instanton, \( \tilde{\rho}_\ell \) characterize its sizes and orientations, while \( v \) is the overall boundary condition at infinity. Hence, \( I \) in (2.8) has \( 2k + 1 \) boson zero modes [2] \( \partial w^i/\partial v, \partial w^i/\partial z_\ell \) and \( \partial w^i/\partial \tilde{\rho}_\ell \), which correspond to variations with respect to these parameters. However the one associated with boundary condition \( v \) has a quadratically divergent norm on the world sheet taken to be a complex plane. In fact parameter \( v \) has a meaning of vacuum expectation value (VEV) for field \( w \). We are not going to include the integration over \( v \) in the instanton
measure [8]. The reason is that we usually do not integrate over VEV in QFT. The latter would mean summing up all the different vacuums of the theory. Instead, we minimize the vacuum energy with respect to VEV to find the true vacuum of the theory. Of course, if physics do not depend on \( v \) (like in topological \( \sigma \) models with the compact target space) than one could safety integrate over it [2]; it makes essentially no difference. However in the case of the \( \sigma \) model with black hole metric we are going to study here, the BRST symmetry is broken and physics depends on \( v \) as we will see later. Therefore, we keep \( v \) fixed.

Thus we are left with \( 2k \) boson zero modes to be included in the instanton measure. Fermion zero modes are given by the same expressions \( \partial w^i/\partial z_i \), \( \partial w^i/\partial \bar{\nu} \) as boson ones [2, 8], since they are solution of the equation

\[
\bar{D}\chi = 0, \tag{2.10}
\]

which is identical to the equation for the boson zero modes. Hence, we have \( 2k \) complex fermion zero modes. This means that the correlation function

\[
\langle O(x_1) \cdots O(x_n) \rangle, \tag{2.11}
\]

(here \( O \) is from eq.(2.7)) is nonzero in the instanton background only if \( n = 2k \). In order to calculate it we have to substitute (2.7) into (2.11), use expressions for fermion zero modes \( \chi^i \) and integrate over \( z_i, \bar{\nu} \) and over their fermion superpartners. The result can be written in an elegant form. Instead of integration over \( z_i, \nu \) let us proceed to new variables defined as follows. Fix points \( x_1, \ldots, x_n \) and consider \( 2k \) functions \( w(x_1) \ldots w(x_n) \) given by (2.8) as functions of \( z_i, \nu \). Then it is easy to see that fermion zero modes \( \chi^i \) (which are given by \( \partial w^i/\partial z_i, \partial w^i/\partial \bar{\nu} \)) represent the jacobian needed to pass from variables \( z_i, \nu \) to \( w(x) \). We get finally

\[
\langle O(x_1) \cdots O(x_n) \rangle = g_{I}^{k} \int g(w) d^2 w_1 \cdots g(w_n) d^2 w_n. \tag{2.12}
\]

Here \( w_p = w(x_p) \), \( p = 1 \ldots n \), while factors \( g(w_p) \) arise from factors \( J_{ij}(w) \) in (2.7) when (2.2) is taken into account. For the more detailed derivation of eq.(2.12) see [8]. The constant \( g_I \) in eq.(2.12) is

\[
g_I = e^{-S_I}, \tag{2.13}
\]
where $S_I$ is the instanton action. In σ models with compact target space $S_I = 0$, because the topological term in the instanton action (the second term in r.h.s. of (2.1)) exactly cancels the kinetic term (the first term in r.h.s. of (2.1)) for holomorphic function $w$. Thus $S_I = 0$ and the independence of the correlation function (2.11) on coordinates $x_1 \ldots x_n$ as well as on the coupling constant $1/r^2$ is manifest in (2.12), provided the integrals are convergent.

Let us now consider the case of the black hole metric of the target space. It has the form

$$g(w) = \frac{1}{1 + |w|^2}. \tag{2.14}$$

Its difference from, say, metric of the sphere for $O(3)$ σ model

$$g^{sphσ}(w) = \frac{1}{(1 + |w|^2)^2} \tag{2.15}$$

in its slow fall-off at large $|w|$. To see the relation of the σ model with metric (2.14) to the black hole let us perform the change of variables

$$w = \sinh r e^{-iθ}, \quad \bar{w} = \sinh r e^{iθ}. \tag{2.16}$$

The kinetic term in eq.(2.1) with the metric (2.14) becomes

$$S_{kin} = \frac{r^2}{2\pi} \int d^2 x \left\{ (\partial_\mu r)^2 + \tanh r^2 (\partial_\mu θ)^2 \right\}. \tag{2.17}$$

The latter is the familiar metric of the Euclidean black hole studied in [9] in the framework of gauged $SL(2)\times U(1)$ WZW model. Note that gauged WZW model of ref. [9] includes also the dilation term which makes it conformal.

Let us now address a question [13, 8]: do holomorphic instantons (2.8) still exist in the σ model with metric (2.14). The main problem is that the topological term becomes logarithmically divergent at large $w$. However the coefficient in front of the logarithm does not depend on the regularization scheme and is still proportional to the winding number $k$. To see this let us substitute (2.8) into the second term in r.h.s. of (2.1). Using the metric from (2.14) we have

$$S_{top} = \frac{r^2}{\pi} \int d^2 x \left( \bar{\partial} w \partial \bar{w} - \bar{\partial} \bar{w} \partial w \right) = -2r^2 k \left[ \log \frac{1}{a} + \text{const} \right], \tag{2.18}$$
where we introduced the UV cutoff on the world sheet $1/a$ (lattice spacing). Thus we still have the configuration space divided into topological classes and holomorphic instantons (2.8) are still minimum points of the kinetic term in a topological class with a given winding number $k$. The explicit check that instanton (2.8) is a solution to equation of motion is performed in ref.[8].

What about instanton action $S_I$? One may worry that instantons are suppressed in the path integral if they have infinite action. It is shown in [8] that instanton (2.8) has nonzero but finite action due to the cancellation between kinetic and topological terms in the topological version of the $\sigma$ model (cf. ref.[13] where instantons are studied in the non-topological version of the $\sigma$ model with the black hole metric and shown to have infinite action). The result for $S_I$ is

$$ S_I = k \frac{r^2}{3}. \quad (2.19) $$

Hence, the constant $g_I$ in (2.12) becomes nontrivial

$$ g_I = e^{-r^2/3}. \quad (2.20) $$

It involves the dependence on $r^2$ which is the first signal for BRST symmetry breaking.

Let us note that the black hole metric is the limiting case to have unsuppressed instantons [8]. If the divergence of the topological term (2.18) were power rather than logarithm, then $S_I$ would be infinite [8].

Now let us consider the correlation function (2.11) in the $I$ background for the simplest case of $I$ with winding number $k = 1$

$$ w = v \left( 1 + \frac{\rho}{z - z_0} \right). \quad (2.21) $$

(2.11) is nonzero only for two point correlation function ($n = 2k = 2$). Eq.(2.12) gives

$$ \langle O(x_1)O(x_2) \rangle = g_I \int \frac{d^2w_1}{1 + |w_1|^2} \frac{d^2w_2}{1 + |w_2|^2}. \quad (2.22) $$

The integrals over modular space of $I$ become logarithmically divergent in (2.22). Introducing UV and IR cutoff on the world sheet ($1/a$ and $1/L$) we get with the double logarithmic accuracy [8]

$$ \langle O(x_1)O(x_2) \rangle = 2(2\pi)^2 g_I \log \frac{|x_{12}|}{a} \log \frac{L}{|x_{12}|} + O(\log), \quad (2.23) $$

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where $x_{12} = x_1 - x_2$. The UV logarithm arises here when the instanton centre is close to either point $x_1$ or $x_2$ (either $w_1$ or $w_2$ becomes large). The IR logarithm then comes from the integration over $w_2$ or $w_1$ respectively and can be rewritten as the logarithmic integral over the instanton size.

The $x_{12}$-dependence in (2.23) means BRST symmetry breaking. To see this explicitly consider $\partial_\mu O(x)$. We have

$$\partial_\mu O = \{Q, iJ_{ij} \partial_\mu w^i \chi^j\},$$

(2.24)

which means that $\partial_\mu O$ is $Q$-exact. Hence, nonzero value for $\langle \partial_\mu O(x_1), O(x_2) \rangle$ means that

$$Q|0 \neq 0.$$

(2.25)

Another way to see BRST symmetry breaking is to calculate the vacuum energy and to show that it is nonzero. Instantons by themselves cannot produce nonzero $E_{\text{vac}}$ because of the anomalous selection rule $n = 2k$ ($I'$s have fermion zero modes). In ref. [8] instanton–anti-instanton pair $(I\bar{I})$ was considered and shown to produce nonzero $E_{\text{vac}}$.

$I$ is an anti-holomorphic map from the world sheet to the target

$$w = v \left( 1 + \sum_{\ell=1}^p \tilde{\rho}_{a\ell} \frac{z - \tilde{z}_{a\ell}}{z_{a\ell}} \right),$$

(2.26)

where $p$ is the winding number, $x_{a\ell}, \tilde{\rho}_{a\ell}$ are new complex parameters. In nontopological versions of $\sigma$ models $I'$s and $\bar{I}'s$ come on the same ground. Instead, in topological $\sigma$ models with compact target space $I'$s come with zero action, while

$$S_I = 2r^2 pA,$$

(2.27)

where $A$ is the area of the target space. The reason for the result in (2.27) is that the topological term doubles the kinetic one for the anti-holomorphic map.

On general grounds the dependence on $r^2$ cannot appear in correlation functions if BRST symmetry is not broken. Thus, for topological $\sigma$ models with compact target space $\bar{I}'s$ plays no role [2]. Instead, for the topological $\sigma$ model with black hole metric (2.14) $\bar{I}'s$ do produce nonzero effects [8].

Consider the $I$ with winding number $-1$. Its action is logarithmically divergent:

$$S_I = 4r^2 \log \frac{|\rho_a|}{a},$$

(2.28)
Eq.(2.28) means that large size $\tilde{I}$'s are suppressed in the path integral. However small size $\tilde{I}$'s (with size $|\rho_a| \sim a$) induce a new point-like interaction. This is calculated in [8] in terms of an effective Lagrangian. The following vertex should be added to the action (2.1)

$$V_{\tilde{I}} = -g_{\tilde{I}} \int d^2x g(w) \tilde{\chi} \frac{\partial^2}{\partial \mu^2} (g(w) \tilde{\chi})$$

(2.29)

in order to mimic the effect of $\tilde{I}$'s. Here $g_{\tilde{I}}$ to be treated together with $g_I$ as two new coupling constants of the model. Four fermion fields in (2.29) account for four $\lambda^\mu$ zero modes of $\tilde{I}$. From (2.29) it is clear that $V_{\tilde{I}}$ is $Q$-exact and does not contribute to correlation functions if BRST symmetry is not broken.

However for the case of the black hole metric the vertex in(2.29) produces nonzero $I\tilde{I}$ contribution to $E_{\text{vac}}$. Observe first that the calculation of $E_{\text{vac}}^{II}$ is essentially the same as the one for the correlation function (2.22) in the one $I$ background. Eq.(2.29) gives

$$E_{\text{vac}}^{II} = \langle V_{\tilde{I}} \rangle_I = -g_{\tilde{I}} \int d^2x_1 \partial^2_{(x_2)} \langle O(x_1)O(x_2) \rangle \bigg|_{x_2-x_1}.$$  

(2.30)

Since correlation function $\langle O(x_1)O(x_2) \rangle$ shows $x_1\tau$-dependence for the case of black hole metric the r.h.s. in (2.30) is nonzero. Substituting (2.23) into (2.30) we get with logarithmic accuracy

$$E_{\text{vac}}^{II} = -16\pi^2 g_I g_{\tilde{I}} V \frac{V}{a} \left\{ \log \frac{L}{a} + O(1) \right\},$$

(2.31)

where $V$ is the volume of the world sheet.

The nonzero result for $E_{\text{vac}}$ in (2.31) confirms our conclusion that BRST symmetry is broken for the $\sigma$ model with black hole metric. However both results for correlation function in (2.23) and for $E_{\text{vac}}$ in (2.31) contain IR logarithmic divergences which come from the integration over the instanton size $\rho$. Our aim in this paper is to sum up all the IR divergent effects to get the IR finite results both for correlation functions and for $E_{\text{vac}}$. This allows us to interpret what physical degrees of freedom are "liberated" in eq.(2.23).

### 3 Coulomb gas description

At the end of the previous section we calculated vacuum energy induced by $I\tilde{I}$ pair. We have seen that it is IR divergent. The divergence comes from
the integration over size of $I$. This means that instanton in a given $II$ pair becomes of infinitely large size. Thus the single instanton approximation becomes invalid because $I$’s start to overlap and interact. In this section we are going to consider the gas of $II$ pairs and show that it is actually a Coulomb gas. To do so we use the method of instanton-induced effective Lagrangian [14, 8] which can be applied to any theory with instantons.

Let us first present the effective vertex for the single instanton (2.21) with $k = 1$. It has the form

$$V_I = - \int d\mu_I \left( \frac{i r^2}{2} \right)^4 g^4(v) |v|^4 \alpha_1 \lambda^{-} \bar{\alpha}_1 \lambda^{-} \rho \alpha_2 \partial \lambda^{-} \bar{\rho} \bar{\alpha}_2 \partial \lambda^{-}$$

$$\times \exp \left( 2v^2 g(v) [v \rho \partial \bar{w} + \bar{v} \bar{\rho} \bar{w}] \right).$$

(3.1)

Here $\alpha_1, \alpha_2$ are Grassmann variables which parametrize fermion zero modes of $I$ (2.21)

$$\chi_1 = \frac{\alpha_1 v}{z - z_0}$$

$$\chi_2 = \frac{\alpha_2 v \rho}{(z - z_0)^2}$$

(3.2)

while $d\mu_I$ is the instanton measure

$$d\mu_I = d^2 x_0 d^2 \rho d^2 \alpha_1 d^2 \alpha_2.$$  

(3.3)

The effective vertex (3.1) should be added to the action (2.1) to mimic the effect of $I$’s at the perturbative level. To check it let us calculate the following correlation function (cf.ref.[8])

$$\langle w(x_1) \ldots w(x_n) \chi(x'_1) \chi(x'_2) \bar{\chi}(x'_3) \bar{\chi}(x'_4) \rangle_I$$

(3.4)

in the one $I$ background.

On one hand (3.4) can be calculated (in the leading order in $1/r^2$) substituting classical expressions (2.21) and (3.2) for fields $w$ and $\chi$ into (3.4). This leads to

$$\prod_{i=1}^n \left( v + \frac{\rho}{z_i - z_0} \right) \frac{\alpha_1 v}{(z'_1 - z_0)(z'_1 - z_0)^2} \left( \frac{\bar{\alpha}_1 \bar{v}}{(z_1 - z_0)^2} \frac{\bar{\alpha}_2 \bar{\rho}}{(z_2 - z_0)^2} \right)$$

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+ permutations \( \left( \frac{z_1'}{z_2'} \leftrightarrow \frac{z_1''}{z_2''} \right) \).

On the other hand, the same result can be reproduced in the purely perturbative manner, inserting (3.1) into the action (2.1). Taking in the expansion of \( \exp -V_I \) the only first power in \( V_I \) (this corresponds to the one \( I \) contribution) and taking into account propagation functions

\[
\langle \phi(x), \bar{\phi}(0) \rangle = \frac{1}{g(v)r^2} \log \left| \frac{L}{|x|} \right| + v^2
\]

\[
\langle \lambda(x), \lambda^{\pm}(0) \rangle = -\frac{2i}{g(v)r^2} \frac{1}{z},
\]

\[
\langle \chi(x), \chi^{\pm}(v) \rangle = -\frac{2i}{g(v)r^2} \frac{1}{z},
\]

one gets the same answer for correlation function (3.4) as in (3.5).

As the effective Lagrangian should depend on field \( w \) rather on its VEV we generalize (3.1) making the substitution \( v \rightarrow w \) in (3.1). This takes into account higher loop corrections to (3.1) (note that we actually derived \( V_I \) above in the one loop approximation). Making also obvious generalization \( \partial \lambda^{\pm} \rightarrow D\lambda^{\pm}, \overline{\partial} \lambda^{\pm} \rightarrow \overline{D}\lambda^{\pm} \) and integrating over Grassmann variables \( \alpha_1, \alpha_2 \) in (3.1) we get finally

\[
V_I = -\frac{gnr^s}{16} \int d^2x d^2\rho \rho^2 |w|^4 g^2(w) g(w) \lambda^{\pm} \lambda^{\pm} g(w) D\lambda^{\pm} \overline{D}\lambda^{\pm}
\]

\[
\times \exp \left\{ 2r^2 g(w) [\rho \partial \bar{\partial} \bar{\phi} + \bar{\rho} \partial \bar{\partial} w] \right\}.
\]

Now we have two effective vertices for \( I \) and for \( \overline{I} \) in eqs. (3.7) and (2.29) respectively. These vertices determine, in principle, the instanton physics in the model.

It is clear that nonzero contributions to vacuum energy can come only from topologically trivial configurations with equal number \( I \)'s and \( \overline{I} \)'s. To study the medium containing both \( I \)'s and \( \overline{I} \)'s we will use \( I\overline{I} \) molecular gas approximation in this Section. Expanding \( \exp -(V_I + V_{\overline{I}}) \) in powers of \( V_I \) and \( V_{\overline{I}} \) we are going to contract fermion fields only in pairs. This gives the \( I\overline{I} \) effective vertex:

\[
V_{I\overline{I}}(w) = -\langle V_I V_{\overline{I}} \rangle.
\]

To give an idea what effects are taken into account in this approximation and what are ignored consider, say, two \( I \) -- two \( \overline{I} \) contribution.
It is clear from (3.8) that graphs in which fermion lines connect $I$'s and $ar{I}$'s only inside pairs are taken into account, whereas those graphs in which each $I$ and $ar{I}$ is connected to other three by fermion lines are ignored. This $I\bar{I}$ approximation is valid provided the instanton density is small, thus we assume $g_I \ll 1$, $g_{\bar{I}} \ll 1$ in this section.\footnote{This condition could be insufficient to ensure the validity of $I\bar{I}$ approximation if graphs with connected fermion lines contain too strong IR divergences. We will show in the next section that these graphs are IR finite.}

Let us now calculate the $I\bar{I}$ effective Lagrangian (3.8). This calculation is essentially of the same type as the one for $E_{\text{vac}}$ we have already performed in the previous section. Let us expand $w$ as $w = w_{\text{ext}} + w_{\text{qu}}$ in eqs. (3.7) and (2.29), where $w_{\text{ext}}$ is the external field and $w_{\text{qu}}$ is the quantum fluctuation. Averaging in (3.8) over fermions as well as over $w_{\text{qu}}$ using propagation functions (3.6) we arrive at

$$V_{I\bar{I}} = -8\pi g_I g_{\bar{I}} \int \frac{d^2 x}{a^2} \frac{d^2 \rho}{|\rho|^2} \epsilon^{2\rho \partial_\rho + \bar{\epsilon} \bar{\partial}_{\bar{\epsilon}}} \left[ \rho \bar{\rho} \partial_\rho \bar{\epsilon} + \bar{\rho} \bar{\partial}_{\bar{\epsilon}} \right],$$

(3.9)

where we put $w_{\text{ext}} \to w$ in the final equation. At $w = 0$ (3.9) reproduces our result for $E_{\text{vac}}$ (2.31). It is particularly clear from (3.9) that the logarithmic divergence in (2.31) comes from the integration over the instanton size $\rho$. Note that the size of $I$ is small $|\rho_0| \sim a$. Moreover, the integral over $I\bar{I}$ separation is dominated at small $|z_0 - z_0| \sim a$, thus the typical $I\bar{I}$ configuration corresponds to the very small $I$ located closely to the center of large $\bar{I}$. As we will see below the appearance of the factor $d^2 \rho/|\rho|^2$ is a signal for the Coulomb nature of interactions in the gas of $I\bar{I}$ pairs.

Let us compare (3.9) with the instanton induced effective vertex of $O(3)$ $\sigma$ model (the nontopological $\sigma$ model without fermions with target space metric (2.15)). It reads (cf. for example [13])

$$V_{I}^{O(3)} = -\text{const} \epsilon^{-2} \int \frac{d^2 x d^2 \rho}{|\rho|^4} \left( \frac{|\rho|}{a} \right)^b \epsilon^{2\rho \partial_\rho \sigma + \bar{\epsilon} \bar{\partial}_{\bar{\epsilon}} \sigma} \left[ \rho \bar{\rho} \partial_\rho \bar{\epsilon} + \bar{\rho} \bar{\partial}_{\bar{\epsilon}} \right].$$

(3.10)

The first coefficient of $\beta$-function $b = 2$ for $O(3)$ $\sigma$ model. Thus we have the same IR-logarithmic behaviour in (3.10) as in (3.9). In $O(3)$ $\sigma$ model each $I$ can be represented as a dipole of some "charge" and "anticharge" [10] (instanton quarks). These charges form the Coulomb gas system at the
inverse temperature $\beta = 1$ [10], where $\beta$ is the coefficient in the charge–anti-charge ($q\bar{q}$) interaction potential

$$e^{-U_+ - (x_1 - x_2)} = e^{-2\beta \ln \frac{|x_1 - x_2|}{\pi}}.$$  \hspace{1cm} (3.11)

The size of $I$ plays the role of the separation between charge and anticharge. Thus, the factor $d^4 \rho / |\rho|^2$ in (3.10) exactly corresponds to $\beta = 1$ in (3.11).

This temperature is above the point of Kosterlitz-Thouless phase transition [15] ($\beta = 2$), hence, the Coulomb gas is in the plasma state [16]. This means that the dynamically generated mass scale appears due to the Debye screening mechanism and all the IR divergences disappear. This "instanton induced" Coulomb gas is in fact equivalent to the SG theory [16].

As the $\Pi\bar{\Pi}$ vertex in (3.9) is of the same form as $I$ vertex in (3.10), we conclude that the $\Pi\bar{\Pi}$ pair in the black hole model plays the same role as a single $I$ in $O(3)$ $\sigma$ model. Hence, the gas of $\Pi\bar{\Pi}$ pairs in the black hole model represents the Coulomb plasma at the inverse temperature $\beta = 1$.

This can be verified directly without reference to $O(3)$ $\sigma$ model using $II$ vertex (3.10). For example, the interaction potential for two $\Pi\bar{\Pi}$ pairs

$$U^{(II)} = \langle 2r^2 g(w) [\rho_1 w \partial \bar{w} + \bar{\rho}_1 \bar{w} \partial w](x_1), 2r^2 g(w) [\rho_2 w \partial \bar{w} + \bar{\rho}_2 \bar{w} \partial w](x_2) \rangle$$

(3.12)

can be compared with that for the Coulomb system of two charges and two anticharges at the locations $x_1$, $x_1 + \rho_1$, $x_2$, $x_2 + \rho_2$. Classically (in the leading order in $r^2$) (3.12) is zero. Next-to-leading corrections in (3.12) can be analyzed and shown to reproduce the desirable Coulomb potential independently of $g(w)$, provided $g(w) \to 0$ if $|w| \to \infty$. We are not going to do it here.

The arguments above lead us to the conclusion that $\Pi\bar{\Pi}$ gas in the black hole $\sigma$ model in $\Pi\bar{\Pi}$ approximation can be described by the SG effective action

$$S^{(b)}_{eff} = \frac{1}{\pi} \int d^2 x \left\{ \frac{1}{2} (\partial \phi)^2 - \frac{2\pi g_4}{a^2} \cos 2 \phi \right\},$$

(3.13)

where $\phi$ is a real scalar field, $g_4$ is a "fugacity" of charges to be determined below. To fix $g_4$ in terms of $g_1g_1$ from (3.9), let us calculate the $q\bar{q}$ contribution to the vacuum energy and compare the result with $E^{II}_{\text{vac}}$ in (2.31). We have

$$E^{\Phi}_{\text{vac}} = g_4^2 \int \frac{d^2 x_1}{a^2} \frac{d^2 x_2}{a^2} \langle e^{2i\phi(x_1)}, e^{-2i\phi(x_2)} \rangle \text{ free boson}$$
\[
\frac{1}{a^2} \int \frac{d^2 \rho}{|\rho|^2} = -2\pi \frac{g_s^2 V}{a^2} \log \frac{L}{a},
\]
(3.14)

where \(\rho = x_1 - x_2\).

Note that we have got an IR logarithm in (3.14) because we have considered the contribution of a single \(q\) — single \(\bar{q}\) to \(E_{\text{vac}}\). The vacuum energy of the CG/SG system (3.13) is IR-finite, because field \(\phi\) acquires a dynamically generated mass in (3.13), as we note above. We postpone the calculation of \(E_{\text{vac}}\) for the black hole model till section 5.

Comparing \(E_{\text{vac}}\) with \(I\) vacuum energy (2.31) we get

\[
g_s^2 = 8\pi g_1 g_1. \tag{3.15}
\]

Now let us relate field \(w\) to field \(\phi\) from the effective action (3.13). Using again the equivalence of \(I\) gas in \(O(3)\) \(\sigma\) model and \(I\) gas in the topological black hole model, we can learn from [10] that

\[
\begin{align*}
    w &= e^{-i(\phi - \phi^*)} \\
    \bar{w} &= e^{-i(\phi + \phi^*)},
\end{align*}
\]
(3.16)

where \(\phi^*\) is the dual field to \(\phi\):

\[
\partial_{\mu} \phi^* = i\epsilon_{\mu\nu} \partial_{\nu} \phi. \tag{3.17}
\]

To make sense of (3.16) we assume that the constant mode of \(\phi\) is analytically continued to the imaginary values.

Let us check eq.(3.16). To do so consider field \(w\) produced by the single \(q\bar{q}\) and compare the result with that for the field \(w\) of a single \(I\) pair. Eq. (3.16) gives

\[
\langle w(x) \rangle_{\pi} = \langle e^{-i(\phi - \phi^*) |x|} \rangle \langle e^{2i\phi |\pi_0|} e^{-2i\phi |\pi_0 - \pi|} \rangle \text{ free boson}, \tag{3.18}
\]

where we represented charge and anti-charge by exponentials using (3.13). It is easy to check with the help of the definition of the dual field (3.17) that the propagation function of a chiral part \((\phi - \phi^*)\) of field \(\phi\) is given by

\[
\langle (\phi - \phi^*)(x), \phi(0) \rangle = \frac{1}{2} \ln \frac{L}{z}. \tag{3.19}
\]
Substituting (3.19) into (3.18) we arrive at

$$\langle w \rangle_{\tilde{H}} = \left(1 + \frac{\rho}{z - z_0}\right) e^{-i(\phi - \phi^{*})},$$

(3.20)

where $$\langle \phi - \phi^{*} \rangle$$ is the classical VEV of the field $$\phi - \phi^{*}$$.

What about the field of a single $$I\bar{I}$$ pair? From (3.9) it is clear that $$\langle w \rangle_{\tilde{H}}$$ coincides with instanton solution (2.21) in the leading order in $$r^2$$. Comparing (3.20) and (2.21) we see that they do coincide with the natural identification

$$v = e^{-i(\phi - \phi^{*})},$$

(3.21)

which relates the classical VEV's of fields $$w$$ and $$\phi$$ in accordance with (3.16).

To sum up, the above results mean that in the leading approximation at $$g_1 \ll 1$$, $$g_\bar{1} \ll 1$$, and $$r^2 \gg 1$$ the topological $$\sigma$$ model with black hole metric is equivalent to the sine Gordon theory (3.13) with $$g_1$$ given by (3.15). Any correlation function of field $$w$$ can be expressed (in the same approximation) in terms of correlation functions of SG model. Namely,

$$\langle F_1[w(x_1)], \ldots, F_n[w(x_n)] \rangle_{BH} = \langle F_1[e^{-i(\phi - \phi^{*})(x_1)}], \ldots, F_n[e^{-i(\phi - \phi^{*})(x_n)}] \rangle_{SG},$$

(3.22)

where $$\langle \ldots \rangle_{BH}$$ and $$\langle \ldots \rangle_{SG}$$ mean correlation functions in black hole and SG models respectively. However in the topological $$\sigma$$ model we are interested in correlation functions of type (2.11). In section 6 we will express these in terms of correlation functions of the SG model.

To conclude this section let us rewrite the effective theory (3.13) in terms of fermions to make it more practical in calculations. At the point $$\beta = 1$$ SG model (3.13) is equivalent to free massive fermions [16, 17]

$$S^{(f)}_{eff} = \frac{1}{\pi} \int d^2 x \{ \bar{\psi} i \gamma_\mu \partial_\mu \psi + im \bar{\psi} \psi \},$$

(3.23)

Here $$\psi$$ is a two component spinor $$\psi = (\psi_1 \psi_2)$$, while $$\gamma_1 = \sigma_1$$, $$\gamma_2 = \sigma_2$$ ($$\sigma_1$$ and $$\sigma_2$$ being Pauli matrices). The relation of the fermion field $$\psi$$ to SG field reads [18]

$$\bar{\psi}_1 = c : e^{-i(\phi - \phi^{*})} ; \quad \bar{\psi}_2 = c : e^{i(\phi - \phi^{*})} ;$$

$$\bar{\psi}_1 = c : e^{-i(\phi + \phi^{*})} ; \quad \bar{\psi}_2 = c : e^{i(\phi + \phi^{*})} :$$

(3.24)

The normalization constant $$c$$ can be fixed comparing, say, propagation function $$\langle \bar{\psi}_1, \bar{\psi}_2 \rangle$$ in models (3.13) and (3.23):

$$c^2 = \frac{1}{2i a},$$

(3.25)
Rewriting the mass term in (3.23) in components
\[ \frac{1}{\pi} \int d^2x \{ im\bar{\psi}_1\psi_1 + im\bar{\psi}_2\psi_2 \} \] (3.26)
and comparing it with (3.13) we can fix the value of the fermion mass in terms of bare coupling constant \( g_q \)
\[ m = -\frac{2\pi g_q}{a}. \] (3.27)
Thus our black hole model is equivalent to free massive fermions (3.23) at \( g_1 \ll 1, g_I \ll 1 \) and \( r^2 \gg 1 \). In the next section we will relax first two of these conditions, showing that \( I\bar{I} \) approximation becomes exact provided a proper definition of the geometry of the instanton modular space is used. As for perturbative corrections in \( 1/r^2 \) to the partition function and to correlation functions of observables (2.11), they should be zero since the theory is topological at the perturbative level. In other words, the dependence of observables on \( r^2 \) could come only in the combinations \( g_I \) and \( g_{\tilde{I}} \), because BRST symmetry is broken only by instanton effects. However we have no rigorous proof of this assertion here.

4 Exact \((I\bar{I})^2\) calculation

In the previous section we showed that the black hole model has CG/SG description in the \( I\bar{I} \) approximation. Here we will calculate the contribution of a two \( I\bar{I} \) pairs to the partition function exactly and compare the result with that given by the \( I\bar{I} \)-approximation. Our aim is to study the possible corrections to the Coulomb gas picture.

Like \( I\bar{I} \) contribution to the partition function (2.30) the \((I\bar{I})^2\) one can also be expressed in terms of correlation function (2.11) in the instanton background. Eq. (2.29) for \( V_I \) gives
\[ Z(I\bar{I}I) = \frac{1}{2} g_I \int d^2 x_1 d^2 x_3 \tilde{\partial}^2_{x_2} \tilde{\partial}^2_{x_4} \langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_{x_2-x_3}. \] (4.1)
According to our selection rule, \( n = 2k \), thus the correlation function in r.h.s. of (4.1) is nonzero only for \( I \) with winding number \( k = 2 \) (see (2.8))
\[ w(z) = v \left[ 1 + \frac{\tilde{\rho}_1}{z - z_1} + \frac{\tilde{\rho}_2}{z - z_2} \right]. \] (4.2)
For this correlation function the general eq.(2.12) gives

\[ \langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = g_l^2 \int \frac{d^2 w_1}{1 + |w_1|^2} \frac{d^2 w_2}{1 + |w_2|^2} \frac{d^2 w_3}{1 + |w_3|^2} \frac{d^2 w_4}{1 + |w_4|^2}, \]

where \( w_1 \ldots w_4 \) are values of (4.2) in the points \( x_1 \ldots x_4 \).

Let us first analyze (4.1) in the leading logarithmic approximation. The calculation is quite similar to the one for \( E_{vac}^{II} \) in (2.31). Two of the four logarithmic factors in (4.3) gives \( \delta(0) \sim 1/a^2 \) after operators \( \partial^2 \) are applied in (4.1). An other two can be written down as logarithmic integrals over \( \tilde{\rho}_1, \tilde{\rho}_2 \) (cf. eq.(3.9)). We have

\[ Z^{(II\tilde{I})} = \frac{1}{2} (2\pi)^2 g_l^2 g_{\tilde{l}}^2 \frac{V}{a^4} V \int \frac{d^2 x_1 d^2 \rho_1 d^2 \rho_2}{|\tilde{\rho}_1|^2 |\tilde{\rho}_2|^2}, \]

where \( x_1 = x_1 - x_3 \). Performing the \( \tilde{\rho} \) integrals in (4.4) we arrive with double logarithmic accuracy at

\[ Z^{(II\tilde{I})} = \frac{1}{2} (16\pi^2)^2 g_l^2 g_{\tilde{l}}^2 \frac{V^2}{a^4} \log^2 \frac{L}{a}. \]

The result in (4.5) should be expected. It is nothing other than the second nontrivial term in the expansion of the partition function \( Z = \exp(-E_{vac}) \) with \( E_{vac} \) taken in the \( I \tilde{I} \)-approximation (2.31).

The next question we are going to address concerning \( Z^{(II\tilde{I})} \) in (4.1), (4.3) is the presence of the corrections to (4.5) of the type

\[ g_l^2 g_{\tilde{l}}^2 \frac{V}{a^2} \log \frac{L}{a}. \]

The appearance of such a term would provide the \( O(g_l^2 g_{\tilde{l}}^2) \) correction to \( E_{vac}^{II} \) in (2.31). This would mean the presence of an extra IR divergence, besides the ones taken into account in the \( I \tilde{I} \) approximation.

To study the possible corrections of type (4.6) one can subtract the double logarithmic term (4.5) from (4.1) and look for the logarithmic divergence in the rest integral. The answer is that there are no dangerous contributions of type (4.6) in \( Z^{II\tilde{I}} \). This means that there are no new IR divergences in our model. In other words, all the IR divergences are taken into account in the \( I \tilde{I} \) approximation.
Now let us turn to non-logarithmic contributions to (4.1). To calculate a constant correction to logarithm one has to write down a more accurate expression for $Z^{II}$ instead of (4.4). However this is not very useful because the region of integration in (4.4) is not defined with appropriate accuracy. To give an example of such uncertainty let us relate parameters $\tilde{\rho}_1, \tilde{\rho}_2$ in (4.4) with instanton sizes $\rho_1, \rho_2$ which appear in the effective action approach of the previous section.

Using effective vertex (3.9) we can calculate the field of two $I\bar{I}$ pairs as $\langle w(x), V_I V_{\bar{I}} \rangle$. We have

$$\langle w(x) \rangle_{II} = \left(1 + \frac{\rho_1}{z - z_1}\right)\left(1 + \frac{\rho_2}{z - z_2}\right). \tag{4.7}$$

Comparing this with (4.2) (note, that instanton field coincides with that of $I\bar{I}$ pair at $r_2 \gg 1$) we have

$$\tilde{\rho}_1 = \rho_1 \left(1 + \frac{\rho_2}{z}\right),$$
$$\tilde{\rho}_2 = \rho_2 \left(1 - \frac{\rho_1}{z}\right), \tag{4.8}$$

where $z = z_1 - z_2$ is the distance between instanton centres. Observe now that constant corrections to (4.4) depend on the definition of the region of integration in (4.4). Say, we can choose the region

$$a \leq |\tilde{\rho}_1| \leq L,$$
$$a \leq |\tilde{\rho}_2| \leq L, \tag{4.9}$$

or instead

$$a \leq |\rho_1| \leq L,$$
$$a \leq |\rho_2| \leq L, \tag{4.10}$$

or make some other choice. In the $I\bar{I}$ approximation of the last section the choice (4.10) arises naturally, since $(I\bar{I})^2$ effect appears as a $V_{II}^2$ term in the expansion of $\exp(-V_{II})$, thus $\rho_1$ and $\rho_2$ come as an independent parameters.

We see that our model needs a more accurate definition of the geometry of the modular space of instantons. One can think of eq.(4.4) as being exact, and define the region of integration in it. We choose the one in (4.10),
The motivation is as follows. Suppose, instead of (4.10) we take (4.9) as the region of integration in (4.4). Then the answer (4.5) for $Z^{\Pi\Pi}$ becomes exact. This means that the contribution $O(g_0^2 V L^2 / a^4)$ which is needed to solve the IR problem is absent in $Z^{\Pi\Pi}$. The common belief is that the field theory with IR divergences is not reasonable. One has to use the freedom in the definition of the geometry of modular space of instantons to get a IR-finite theory. We will show below that the choice (4.10) will solve the IR problem.

To make (4.4) more transparent let us proceed from the integration over $\tilde{\rho}_1, \tilde{\rho}_2$ to the one over $\rho_1, \rho_2$ and assume that the region of integration is (4.10). Using (4.8) we get

$$Z^{\Pi\Pi} = \frac{1}{4} (2\pi)^2 g_1^2 g_2^2 \frac{16}{a^4} V \int \frac{d^2 x_{13} d^2 \rho_1 d^2 \rho_2 |x_{13}|^2 |x_{13} + \rho_2 - \rho_1|^2}{|\rho_1|^2 |\rho_2|^2 |x_{13} + \rho_2|^2 |x_{13} - \rho_1|^2}, \tag{4.11}$$

where we replace $z$ by $x_{13}$ because one $I$ comes close to $x_1$ and another to $x_3$. From (4.11) it is clear that we have to add two extra conditions to (4.10) in order to cut the integral (4.11) in the ultra-violet region. Namely, we have to impose

$$a \leq |x_{13} + \rho_2| \leq L, \quad a \leq |x_{13} - \rho_1| \leq L. \tag{4.12}$$

We will see below that the symmetry $\rho_1 \leftrightarrow x_{13} + \rho_2; \rho_2 \leftrightarrow x_{13} - \rho_1$ reflects the possibility of interchanging positions for anti-charges in the $(q\bar{q})^2$ system in the Coulomb gas description.

Let us compare (4.11) with the result for $Z^{\Pi\Pi}$ in $\Pi\Pi$ approximation. In this approximation $(\Pi\Pi)^2$ contribution to $Z$ is equal to the contribution of two $q\bar{q}$ pairs to the partition function. Using (3.13) we have

$$Z^{\Pi\Pi} = \frac{1}{4} g_0^4 \int \frac{d^2 x_1 d^2 \rho_1 d^2 x_3 d^2 \rho_2}{a^2 \frac{a^2}{a^2}} \times \left\langle e^{2i\phi(x_1)} e^{-2i\phi(x_1 - \rho_1)} e^{2i\phi(x_3)} e^{-2i\phi(x_3 - \rho_2)} \right\rangle_{\text{free boson}}, \tag{4.13}$$

where charges and anti-charges are taken to have coordinates $x_1, x_3$ and $x_1 - \rho_1, x_3 - \rho_2$ respectively. Using the relation

$$\left\langle e^{2i\phi(x)} e^{\pm 2i\phi(y)} \right\rangle_{\text{free boson}} = \left( \frac{|x - y|}{a} \right)^{\pm 2}. \tag{4.14}$$
It is easy to see that (4.13) gives the same integral as in (4.11). We conclude therefore that with the definition (4.10), (4.12) the \( H \) approximation becomes exact for \( Z_{IM} \). Furthermore, it is not difficult to check that the definition of the geometry of the modular space of instanton for any winding number \( k \) in terms of sizes \( \rho_1, \ldots, \rho_k \) like the one in (4.10), (4.12) for \( k = 2 \) makes our topological black hole model equivalent to the CG/SG system (3.13) exactly at any values of \( g_1, g_f \). Of course, this makes it IR-finite.

5 Vacuum energy

The \( H \) vacuum energy (2.31) shows the IR divergence. In this section we re-examine the calculation of the vacuum energy in the black hole model using the sine Gordon/free fermions description (3.13),(3.23).

First, let us minimize the classical vacuum energy in (3.13). The vacuum state can be chosen at

\[ \phi_0 = 0. \quad (5.1) \]

This means in accordance with (3.16) or (3.21) that

\[ v = e^{i\alpha} \quad (5.2) \]

with \( \alpha \) real. The reason for this result is that the constant mode of \( \phi^* \) is not fixed, see (3.17). Thus, as we mentioned before, the physics in the model is \( |v| \) dependent and the minimization of the vacuum energy gives

\[ \langle |w| \rangle = |v| = 1. \quad (5.3) \]

What about the U(1) symmetry \( w \to e^{i\gamma} w \) of the model. Eq.(5.3) shows that U(1) symmetry is not broken. To check this we can calculate the propagation function of "would-be" Goldstone boson. If U(1) were broken the phase of field \( w \) would be massless. Phase of \( w \) is related to \( e^{2i\phi^*} \), see (3.16).

Consider

\[ \langle e^{2i\phi^*(x)} e^{-2i\phi^*(y)} \rangle = (2i\alpha)^2 \langle \bar{\psi}_1 \psi_2(x), \bar{\psi}_1 \psi_2(y) \rangle, \quad (5.4) \]

\footnote{Note that this result is different from that in ref.[8]. It comes here as a consequence of a CG/SG description of the instanton vacuum.}
where we use the correspondence (3.24). The propagation function of a free massive fermion reads
\[
\langle \bar{\psi}(x) \psi(0) \rangle = \frac{i m}{2} \left\{ \left( \begin{array}{cc} 0 & \frac{z}{|z|} \\ \frac{z}{|z|} & 0 \end{array} \right) K'_0(m|x|) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) K_0(m|x|) \right\},
\]
(5.5)
where \( K_0 \) is the zero order modified Bessel function and \( K'_0 \) is its derivative with respect to the argument. Using (5.5) we get at \(|x - y| \to \infty \)
\[
\langle e^{2i \phi(x)} e^{-2i \phi(y)} \rangle = (2\pi)^2 g_q^2 [K'_0(m|x - y|) - K^2_0(m|x - y|)] \sim \frac{e^{-2m|x - y|}}{(m|x - y|)^{3/2}}.
\]
(5.6)
Thus we conclude that there is no Goldstone boson in the model: the phase of \( w \) couples to the same massive fermion field as \( w \) does.

Let us now consider the vacuum energy of our model in quantum theory. It is known that the SG model is renormalizable at \( \beta \leq 2 \) [17]. All UV divergences can be associated with the renormalization of the coupling constant \( g_q \), except vacuum energy. The vacuum energy is associated with another divergence at \( 1 \leq \beta \leq 2 \). Let us calculate it. We have in the fermion description
\[
E_{vac} = \frac{1}{2} \frac{i m}{\pi} \int d^2 x \langle \bar{\psi}_2 \psi_2 + \bar{\psi}_1 \psi_1 \rangle.
\]
(5.7)
Plugging (5.5) into (5.7) we arrive at
\[
E_{vac} = -\frac{m^2}{2\pi} V \log \frac{2}{m a}.
\]
(5.8)
Let us compare (5.8) with (3.14) or (2.31). The UV logarithm \( \log 1/a \) comes in (5.8) as a single \( q^2 \) effect like in (3.14), while the IR cutoff at \( 1/m \) arises due to the Debye screening in the Coulomb plasma. Note, that \( E_{vac} \) in (5.8) is still non-zero and implies the BRST symmetry breakdown.

6 Correlation functions

Now let us turn to the calculation of the correlation functions of observables (2.11) which interest us in the TFT. In this section we consider the two point correlation function (2.22) and show that it is IR-finite. As our SG effective boson action (3.13) arises as a result of integration over fermions \( \chi \) and \( \lambda \) we
expect that the correlation function of fermion fields (2.22) could appear to
be a complicated object (non-local object, in fact) in terms of the field of the
SG model.

The two point correlation function (2.22) is non-zero in the background
of a single $I$ plus an arbitrary number of $I\bar{I}$ pairs. Using the effective vertex
(2.29) for $\bar{I}$ we get the general expression

$$\langle O(x_1)O(x_2) \rangle_{I+I\bar{I}^n} = \frac{g_I^n}{Z} \int \prod_{i=1}^{n} d^2y_i \partial^2_\phi \langle O(x_1)O(x_2) \prod_{i=1}^{n} O(y_i)O(y_i') \rangle_{y_i=y_i}$$

for (2.22) in the background $I+(I\bar{I})^n$.

To simplify life let us consider $I+I\bar{I}$ contribution in (6.1) as an example,
like in section 4. Then (6.1) takes the form

$$\langle O(x_1)O(x_2) \rangle_{I+I\bar{I}} = \frac{g_I}{Z} \int d^2y \partial^2_\phi \langle O(x_1)O(x_2)O(y)O(y') \rangle_{y'=y}. \quad (6.2)$$

Here the four point correlation function is given by (4.3). The calculation
is quite similar to that for $Z_1^1I\bar{I}$ in section 4. One of the four logarithms
in (4.3) gives $\delta(0) \sim 1/a^2$ under the action of $\partial^2_\phi$ in (6.2), while another is
ultraviolet and gives $\log |x_{12}|/a$, like in (2.23). The remaining two can be
written down as logarithmic integrations over $\tilde{\rho}_1$ and $\tilde{\rho}_2$, like in (4.4). We get

$$\langle O(x_1)O(x_2) \rangle_{I+I\bar{I}} = 8(2\pi)^2 g_I^2 g_{\bar{I}} \log \frac{|x_{12}|}{a} \int \frac{d^2y \partial^2_\phi \tilde{\rho}_1 \partial^2_\phi \tilde{\rho}_2}{a^2(\tilde{\rho}_1^2 + |x_{12}|^2) \tilde{\rho}_2^2}. \quad (6.3)$$

Introducing new variable $\tilde{\rho}'_1$ with $|\tilde{\rho}'_1|^2 = |\tilde{\rho}_1|^2 + |x_{12}|^2$ we can rewrite the integral over $\tilde{\rho}_1$ in (6.3) as

$$\int \frac{d^2\tilde{\rho}'_1}{|\tilde{\rho}'_1|^2} \quad (6.4)$$

over the region

$$|\tilde{\rho}'_1| \geq |x_{12}|. \quad (6.5)$$

Let us now proceed from integration over $\tilde{\rho}'_1, \tilde{\rho}_2$ to the one over $\rho_1, \rho_2$
deﬁned as in (4.8) with $z = y - x_1$ (in fact $1/2 \log |x_{12}|/a$” arises when the
centre of one of $I$’s close to $x_1$ and another “$1/2 \log |x_{12}|/a$” arises when it
closes to $x_2$. We take $z = y - x_1$ for simplicity: formulas below should be

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understood as a symmetrization with respect to \( x_1 = x_2 \). After changing variables we get

\[
\langle O(x_1)O(x_2) \rangle_{T+T} = Z^{-1} 8(2\pi)^2 g_{f}^3 g_{j} \log \left( \frac{|x_{12}|}{a} \right) \times \frac{1}{2} \int \frac{d^2 z d^2 \rho_1 d^2 \rho_2 |z|^2 |z - \rho_1 + \rho_2|^2}{|\rho_1|^2 |\rho_2|^2 |z - \rho_1|^2 |z + \rho_2|^2}.
\]

This is the representation for Coulomb system of \((q\bar{q})^2\) with charges at points \( x_1, x_1 + z = y \) and anti-charges at points \( x_1 - \rho_1, y - \rho_2 \) (see eqs. (4.11), (4.13)). We have

\[
\langle O(x_1)O(x_2) \rangle_{T+T} = 2(2\pi) g_{j} g_{3} g_{f} \log \left( \frac{|x_{12}|}{a} \right) \times \int \frac{d^2 y d^2 \rho_1 d^2 \rho_2}{a^2} \cdot \left( e^{2i\phi(x_1)} e^{-2i\phi(x_1 - \rho_1)} e^{2i\phi(y)} e^{-2i\phi(y - \rho_2)} \right)_{\text{free boson}},
\]

where we made a distinction between \( g_{j} \) and \( g_{f} \) for a moment. The result in (6.7) is nothing other but the \( O(g_{j} g_{3}^2) \) term in the expansion of the following correlation function of the SG model:

\[
\langle O(x_1)O(x_2) \rangle = 2(2\pi) g_{f} \log \left( \frac{|x_{12}|}{a} \right) \times \int \frac{d^2 y}{a^2} d\tilde{q}_j \int \frac{d^2 \rho_1}{a^2} \left( e^{2i\phi(x_1)} e^{-2i\phi(x_1 - \rho_1)} \right)_{SG}.
\]

Eq. (6.8) is our desired expression of the correlation function (2.22) in terms of a correlation function of the SG model.

What about the constrain (6.5)? Can it be rewritten down as any condition on correlation functions of SG model? Consider the correlation function in the r.h.s. of (6.8). It can be written down in the form

\[
\left\langle e^{2i\phi(x_1)} e^{-2i\phi(x_1 - \rho_1)} \right\rangle_{SG} = \sum_{n=1}^{\infty} \frac{g_{j}^{2n} (n!)^2}{Z_{SG}} \int \prod_{i=1}^{n} \frac{d^2 y_i d^2 \rho_1 a^2}{a^2} \cdot \left( e^{2i\phi(y_1)} e^{-2i\phi(y_1 - \rho_1)} \right)_{\text{free boson}},
\]

where we introduced

\[
\rho_{1+} = \rho_1 \prod_{i=1}^{n} \frac{x_1 - y_i + \rho_i}{x_1 - y_i},
\]

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while

\[ \tilde{\rho}_{1-} = \rho_1 \prod_{i=1}^{n} \frac{x_1 - y_i - \rho_1}{x_1 - y_i - \rho_1 + \rho_i} \tag{6.11} \]

Here (6.10) is the obvious generalization of the expression for \( \tilde{\rho}_1 \) in (6.5) (see (4.8)) for arbitrary \( n \), while (6.11) is the same quantity as in (6.10) with all charges and anti-charges interchanged. Generalizing the condition (6.5) to the arbitrary \( n \), we get

\[ |\rho_{1+}| \geq |x_{12}|, \]
\[ |\rho_{1-}| \leq |x_{12}| \tag{6.12} \]

Substituting (6.13) into (6.9) we arrive finally at the constrain

\[ \left\langle e^{2i\phi(x_1)} e^{-2i\phi(x_1 - \rho_1)} \right\rangle_{SG} \leq \frac{a^2 |\rho_1|^2}{|x_{12}|^4} \tag{6.13} \]

Eq. (6.13) gives us the lower bound of \( \rho_1 \) in the integral over \( \rho_1 \) in the r.h.s. of (6.8). Note, that we considered the \( I + I^c \) system above only as a simplifying example. All the steps of the calculation leading to eqs.(6.8), (6.13) can be repeated for arbitrary \( n \).

Let us now calculate the correlation function (6.8). Using the fermion representation we get

\[ \left\langle e^{2i\phi(x)} e^{-2i\phi(y)} \right\rangle = (2ia)^2 \left\langle \bar{\psi}_2 \psi_2(x), \bar{\psi}_1 \psi_1(y) \right\rangle \tag{6.14} \]
\[ = (2\pi)^2 g^2 K_0^2 (m|\rho_1|). \]

Consider first the limit of small distances \( m|x_{12}| \ll 1 \). Approximating \( K_0' \) in this limit as

\[ K_0'(x) = -\frac{1}{x} + O(x) \tag{6.15} \]

and using eq.(3.27) we get from the constrain (6.13)

\[ |\rho_1| \geq |x_{12}|. \tag{6.16} \]

Performing the integral over \( \rho_1 \) in (6.8) in these limits we finally arrive at

\[ \left\langle O(x_1), O(x_2) \right\rangle = 2(2\pi)^2 g_I \log \frac{|x_{12}|}{a} \log \frac{2}{m|x_{12}|}. \tag{6.17} \]
Comparing this with (2.23) we see that the only modification is the appearance of IR cutoff parameter $1/m$. (6.17) still shows the $x_12$-dependence of $\langle O(x_1)O(x_2)\rangle$ which is a signal for BRST symmetry breaking.

Now let us estimate (6.9) at large distances $m|x_{12}| \gg 1$. In this limit the correlation function in (6.14) shows the exponential fall-off:

$$\left\langle e^{2i\phi(x_1)} e^{-2i\phi(x_1-\rho_1)}\right\rangle_{SG} \sim e^{-2m|\rho_1|}.$$ (6.18)

We assumed that $m|\rho_1| \gg 1$ here. We will check it is true below. Plugging (6.18) into the constrain (6.13) we get

$$|\rho_1| \geq \frac{1}{m} \text{log}(m^2|x_{12}|^2).$$ (6.19)

In particular eq.(6.19) shows that $m|\rho_1| \gg 1$, indeed. Integrating over $\rho_1$ in (6.8) within the limits determined by (6.19) we get at $m|x_{12}| \gg 1$

$$\langle O(x_1), O(x_2) \rangle \sim \frac{1}{m^4|x_{12}|^4}$$ (6.20)

We see that (6.20) shows a power fall-off at $|x_{12}| \to \infty$. This means the presence of massless particles in our theory.

As we showed above our topological black hole $\sigma$ model is equivalent to free massive fermions. One may worry therefore, how the power behaviour in (6.20) can appear. Technically it comes because correlation functions of the fermion field $\chi$ of the original model is expressed in a non-local manner in terms of correlation functions of SG model. Let us note however, that from the physical point of view the result in (6.20) could be expected. We interpret this behaviour as a propagation of the goldstino fermion which appears as a consequence of spontaneous BRST-symmetry breaking.

To see this, observe that the condition (2.25) means the existence of a (composite) goldstino fermion $\psi_g \sim Q|0>$. It is easy to see making the $Q$-transformation in the initial action of the model that this fermion couples to the current

$$\bar{\psi}_{g\mu} \sim J_{ij} \partial_\mu w^i \chi^j.$$ (6.21)

Observe now that the r.h.s. of (6.15) is related to $\partial_\mu O(x)$ according to (2.24). Hence, we have

$$\partial_\mu O \sim \bar{\psi}_{g\mu} \psi_g.$$ (6.22)
We see that the operator $O$ \((2.7)\) couples to the goldstino mode. Hence, the correlation function \(\langle O(x_1)O(x_2)\rangle\) should show the power behaviour at \(|x_{12}| \to \infty\), provided the spontaneous BRST symmetry breakdown takes place.

7 Conclusions

In this paper we have studied the topological $\sigma$ model with the black hole metric. Our results for correlation function \(\langle O(x_1),O(x_2)\rangle\) in (6.17) and (6.20) show its coordinate dependence. This is consistent with the nonzero result (5.8) for the vacuum energy and ensures the BRST-symmetry breaking.

The instanton vacuum of the model is equivalent to CG/SG system. This ensures the IR-finiteness of the physical observables due to the Debye screening phenomenon in the Coulomb plasma. The temperature of CG corresponds to $\beta = 1$. This means that the physical content of our theory is very simple: we deal with free massive fermions. Actually, we have proved the equivalence of the black hole model to free massive fermions in the weak coupling limit $r^2 \gg 1$, because we have not studied possible perturbative corrections on top of instanton effects. However, the topological nature of the model suggests that this holds true to any order in $r^2$ (the dependence on $r^2$ come only in combinations $q_I$ and $g_I$).

Let us now address a question: is the breakdown of the BRST-symmetry we observed in the black hole model an explicit one or a spontaneous one? One possible answer is that it is explicit and related to some sort of a holomorphic anomaly, like the one discovered in the topological gravity [19]. The argument in favour of this assertion is that our results for observables depend on $r^2$. Coupling constant $r^2$ is the coefficient in front of the $Q$-exact operator. The dependence on such coupling constants is interpreted in [19] as a holomorphic anomaly.

However, it seems more plausible to interpret the breakdown of the BRST-symmetry here as a spontaneous one. One argument for this is that the effective $I$ and $\bar{I}$ vertices (3.7) and (2.29) are $Q$-closed. This means that the effective action is $Q$-invariant and this is the choice of the vacuum state that breaks down the $Q$-symmetry. Another argument is the power behaviour (6.20) for the correlation function (2.22) at large distances. We interpret it as a propagation of the goldstino fermion associated with the spontaneous...
BRST-symmetry breaking.

Let us stress however, that the appearance of UV divergences in our results for physical observables shows the presence of the new conformal anomaly. Note, that on the level of perturbation theory the model has conformal anomaly (the model is not a conformal invariant one) [20], but the $\beta$ function associated with this anomaly does not contribute to physical observables [1, 2]. The new anomaly is of a non-perturbative nature and related to the noncompactness of the modular space of instantons.

From the point of view of the SG description of our model (3.13) this anomaly is associated with the tachyon operator $\cos 2\phi$. The coupling constant $g_3$ in front of this operator (which is related to couplings $g_I$ and $g_{\tilde{I}}$ via (3.15)) is renormalized according to the RG flow of the SG model. In particular, the fermion mass as defined in (3.27) is the RG-invariant.

The author is grateful to N.Dorey and A.Johansen for stimulating discussions and to the Particle Physics group of the University of Wales Swansea where part of this work was done for hospitality. This work was supported by the Higher Education Funding Council for Wales, by the Russian Foundation for Fundamental Studies under Grant No.93-02-3148 and by Grant No. NOD000 from the Internation Science Foundation.
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