Charged Dilaton Black Holes 
with Unusual Asymptotics

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Abstract

We present a new class of black hole solutions in Einstein-Maxwell-dilaton gravity in \( n \geq 4 \) dimensions. These solutions have regular horizons and a singularity only at the origin. Their asymptotic behavior is neither asymptotically flat nor (anti-) de Sitter. Similar solutions exist for certain Liouville-type potentials for the dilaton.

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1. Introduction

There has been great interest in recent years in $n$-dimensional ($n \geq 4$) dilaton gravity. It is important to investigate how the properties of black holes are modified when a dilaton is present. $n$-dimensional static spherically symmetric (SSS) electrically or magnetically charged dilatonic black hole solutions were first analyzed in some generality by Gibbons and Maeda [1], and the four-dimensional versions were rediscovered and clarified in an elegant note by Garfinkle, Horowitz and Strominger [2]. A number of authors have considered variations, such as SSS dyonic black hole solutions [3], or black holes when the dilaton acquires a mass [4,5].

In this paper, we obtain and discuss a new class of electrically (or magnetically) charged SSS black hole solutions to $n$-dimensional dilaton theories of gravity. The metrics associated with these solutions are neither asymptotically flat nor asymptotically (anti-) de Sitter; however the only curvature singularities are at the origin.

We first consider the four-dimensional action in which gravity is coupled to a dilaton and Maxwell field with an action

$$S = \int d^4x \sqrt{-g} (\mathcal{R} - 2(\nabla \phi)^2 - V(\phi) - e^{-2a\phi} F^2),$$

where $\mathcal{R}$ is the scalar curvature, $F^2 = F_{\mu\nu} F^{\mu\nu}$ is the usual Maxwell contribution, and $V(\phi)$ is a potential for $\phi$. The constant $a$ governs the coupling of $\phi$ to $F_{\mu\nu}$. We will consider three special cases: (i) $V(\phi) = 0$, (ii) $V(\phi) = 2\Lambda e^{2b\phi}$ and (iii) $V(\phi) = 2\Lambda_1 e^{2b_1\phi} + 2\Lambda_2 e^{2b_2\phi}$. The first case corresponds to the action considered in [1,2]. When $a = 1$, it reduces to the four-dimensional low-energy action obtained from string theory in terms of Einstein metric. Case (ii) corresponds to a Liouville-type gravity. We will refer to $\Lambda$ as the cosmological constant, although in the presence of a non-trivial dilaton field, the spacetime does not behave as either de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$). It has recently been shown that with the exception of a pure cosmological constant potential (i.e., $b = 0$), no asymptotically flat ($\Lambda = 0$), asymptotically de Sitter or asymptotically anti-de Sitter SSS solutions to the field equations associated with case (ii) exist [6]. However, non-static generalizations of the Kastor-Traschen-type [7] cosmological multi-black hole solutions are known [8]. The potential in case (iii) was previously investigated by a number of authors [9,10] in the context of FRW scalar field cosmologies. This kind of potential function can be obtained when a higher-dimensional theory is compactified to four-dimensional spacetime, including various supergravity and string models. In case (i), the non-asymptotically flat black hole
solutions we derive in this paper were previously obtained by several authors but they failed to recognize that the solutions admit regular horizons (see section 3). In cases (ii) and (iii), we will construct the first examples of SSS black hole solutions.

For the $n$-dimensional cases, we generalize the action (1.1) to

$$S = \int d^n x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-2} (\nabla \phi)^2 - V(\phi) - e^{-\frac{4a \phi}{n-2}} F^2 \right).$$

We will first derive the four-dimensional solutions and then generalize them to $n$ dimensions using (1.2). For simplicity, we will only consider pure electrically (or magnetically) charged cases.

The solutions we derive in this paper are not vacuum solutions in a strict sense since the action (1.1) (or (1.2)) contains a static “dilaton fluid” whose energy momentum tensor is nowhere vanishing. Here we adopt special dilaton fluid models in which the metric approaches neither the Minkowski nor (anti-) de Sitter metrics at spatial infinity. Although the dilaton field diverges for large $r$ ($\phi \propto \log r$ where $r$ is the usual radial coordinate), the quasilocal mass is finite for all values of $r$ outside the event horizon in every black hole solution. For all the black hole solutions, the Ricci and Kretschmann scalars are everywhere finite except at $r = 0$ but those singularities are hidden by event horizons. It can also be checked that all terms in the action (1.1) (with the any of the potentials (i)-(iii)) are finite for $\infty > r \geq r_h$ (where $r_h$ denotes the event horizon), and vanish as $r \rightarrow \infty$ in all the black hole solutions we obtain. As a consequence we believe that our black hole solutions are of some physical interest.

The organization of this paper is as follows. In section 2, we adopt a SSS ansatz and then write down the equations of motion for the action (1.1). The formula for calculating the quasilocal mass of a SSS black hole metric is briefly reviewed. We first consider a vanishing potential $V$ in section 3. When $a = 1$, the solution corresponds to a non-asymptotically flat string black hole in terms of the Einstein metric. We will discuss the solution in terms of the string metric as well. Regardless of whether $a = 1$ or not, we will see that these black hole metrics have no inner horizon. Furthermore, there is no extremal limit for the electric charge. More precisely, the charge can take any finite value without causing the event horizon to either vanish or become singular. Such a property is not shared by the asymptotically flat black holes in [1,2] or the Reissner-Nordstrom solution in General Relativity. In section 4 we solve the field equations for a simple Liouville-type potential and obtain three families of electrically charged Liouville black hole solutions. In
section 5, we consider the double Liouville potential and construct a family of black holes with two cosmological couplings for general $b_1$ and $b_2$. Finally, we solve the field equations for the action (1.2) and generalize the above four-dimensional solutions to $n$ dimensions in section 6. We summarize our results and discuss possible future work in the concluding section.

Our conventions are as in Wald [11].

2. Field Equations

We first consider the four-dimensional action (1.1). Varying (1.1) with respect to the metric, Maxwell, and dilaton fields, respectively, yields (after some manipulation)

\[
\mathcal{R}_{ab} = 2a \partial_a \phi \partial_b \phi + \frac{1}{2} g_{ab} V + 2e^{-2a \phi} \left( F_{ac} F^{c}_b - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right),
\]

\[
0 = \partial_a \left[ \sqrt{-g} e^{-2a \phi} F^{ab} \right],
\]

\[
\nabla^2 \phi = \frac{1}{4} \frac{\partial V}{\partial \phi} - \frac{a}{2} e^{-2a \phi} F_{ab} F^{ab}.
\]

We wish to find SSS solutions to (2.1) which admit non-singular horizons. In $3 + 1$ dimensions, the most general such metric has two degrees of freedom, and can be written in the form

\[
ds^2 = -U(r) dt^2 + \frac{dr^2}{U(r)} + R^2(r) d\Omega^2.
\]

For most of this paper, we will specialize to Maxwell fields corresponding to an isolated electric charge. The Maxwell equation (2.1b) can be integrated to give

\[
F_{tr} = \frac{Q e^{2a \phi}}{R^2},
\]

where $Q$ is the electric charge, defined through the integral $-\frac{1}{8 \pi} \int_S e^{-2a \phi} \epsilon_{\mu \nu \alpha \gamma} F^{\alpha \gamma}$, where $S$ is any two-sphere defined at spatial infinity and $\epsilon$ is the volume element (see [11]). With the metric (2.2) and Maxwell field (2.3), the equations of motion (2.1) reduce to three independent equations:

\[
\frac{1}{R^2} \frac{d}{dr} \left[ R^2 U \frac{d \phi}{dr} \right] = \frac{1}{4} \frac{dV}{d\phi} + a e^{2a \phi} \frac{Q^2}{R^4},
\]

\[
\frac{1}{R} \frac{d^2 R}{d r^2} = -\left( \frac{d \phi}{dr} \right)^2,
\]

\[
\frac{1}{R^2} \frac{d}{dr} \left[ U \frac{d}{dr} (R^2) \right] = 2 \frac{1}{R^2} - V - 2e^{2a \phi} \frac{Q^2}{R^4}.
\]
If we had instead chosen the Maxwell field to be an isolated magnetic charge at the origin, \( F_{\theta\phi} = Q \sin \theta \), then the equations of motion (2.4) would be unchanged except for the replacement \( a \rightarrow -a \). Thus, for any electrically charged solution we obtain for arbitrary \( a \), the corresponding magnetically charged solution can be found by simply replacing \( a \rightarrow -a \) everywhere (this symmetry is not valid for \( n > 4 \)). Thus we will only specifically mention the magnetically charged solution in one instance.

It is easy to see in (2.4b) that if \( R(r) = r \), then \( \phi \) must be constant. Thus one generally cannot have both \( g_{tt} = -1/g_{rr} \) and the angular part of the metric of the form \( r^2 d\Omega^2 \) when there is a non-trivial dilaton present. However in [12], a variety of solutions to the 2+1 dimensional action analogous to (1.1) were obtained by making the ansatz

\[
R(r) = \gamma r^N ,
\]

where \( \gamma \) and \( N \) are constants. Using (2.5) with (2.4b) immediately forces \( \phi \) to have the form

\[
\phi(r) = \phi_0 + \phi_1 \log r .
\]

A dilaton of the form (2.6) is common in two and three dimensional solutions, but is generally regarded as a sign of pathology in four or higher dimensions. We present in this paper solutions in four and higher dimensions using (2.5) and (2.6) that have finite mass, curvature, and charge at large \( r \), and thus should be regarded as physically interesting.

Because the metrics presented in this paper will not be asymptotically flat, we must use the quasilocal formalism to define the mass of the solutions (for a detailed review of the concepts of quasilocal mass, see [13]). If a SSS metric is written in the form

\[
ds^2 = -W^2(r)dt^2 + \frac{dr^2}{V^2(r)} + r^2 d\Omega_{n-2}^2 ,
\]

and the matter action contains no derivatives of the metric, then the quasilocal mass is given by

\[
\mathcal{M} = \frac{n-2}{2} r^{n-3} W(r) (V_0(r) - V(r)) .
\]

Here \( V_0(r) \) is an arbitrary function which determines the zero of the energy for a background spacetime and \( r \) is the radius of the spacelike hypersurface boundary. When the spacetime is asymptotically flat, the ADM mass \( M \) is the \( \mathcal{M} \) determined in (2.8) in the limit \( r \rightarrow \infty \). If no cosmological horizon is present, the large \( r \) limit of (2.8) is used to determine the mass. If a cosmological horizon is present once cannot take the large \( r \) limit to identify the quasilocal mass. However, one can still identify the “small mass” parameter in the solution [13].
3. Solutions with $V(\phi) = 0$

We begin by looking for four-dimensional solutions with $V(\phi) = 0$. The system of equations (2.4) has been completely solved before [1,5,6], but always with the aim of finding asymptotically flat solutions. The solutions we will present in this section were implicitly present in those works, but have not before been written down explicitly.

3.1. The string case, $a = 1$

We first consider the string theoretic case, where $a = 1$ in (1.1). This case has a number of interesting special features. Using (2.5) in (2.4), we find the solution

$$R(r) = \gamma r \frac{1}{r},$$

$$U(r) = \frac{1}{\gamma} (r - 4M),$$

$$\phi(r) = -\frac{1}{2} \log \left( \frac{2Q^2}{\gamma^2} \right) + \frac{1}{2} \log r.$$  

By performing a transformation $\gamma r \frac{1}{r} \rightarrow r$, we can write the metric

$$ds^2 = \frac{1}{\gamma^4} (r^2 - 4\gamma^2 M) dt^2 + \frac{4r^2}{r^2 - 4\gamma^2 M} dr^2 + r^2 d\Omega^2,$$  

with a dilaton

$$\phi(r) = -\frac{1}{2} \log (2Q^2) + \log r.$$  

$M$ is the quasilocal mass calculated from (2.8) and (3.2). A magnetically charged version of (3.2) for a specific value of $M$ and $Q$ can be found in [14].

The solution (3.2) has several interesting properties. First, it is easy to show that $\mathcal{R} = \frac{r^2 - 4\gamma^2 M}{2r^2}$. Thus, (3.2) is well behaved at the event horizon $r_h = 2\gamma \sqrt{M}$, and has a singularity at $r = 0$. Even though $\phi \propto \log r$ for large $r$, both the Kretschmann scalar and $\mathcal{R}$ vanish as $r \rightarrow \infty$. Since the mass, charge, and curvature all remain finite as $r \rightarrow \infty$, the solution (3.2) is well-behaved there. Second, (3.2) has no smooth transition to the $Q = 0$ case because of (3.3). However, no extremal limit exists. Finally, it is straightforward to check that the causal structure of (3.2) is exactly the same as the Schwarzschild case. Thus the singularity is spacelike.

Strings do not couple to the Einstein metric $g_{ab}$ but rather to the string metric $\tilde{g}_{ab} = e^{2\phi} g_{ab}$, where the action (1.1) reads

$$S = \int d^4 x \sqrt{-\tilde{g}} e^{-2\phi} (\mathcal{R} + 4(\nabla \phi)^2 - F^2).$$  

(3.4)
For the solution (3.1), the string metric is (after rescaling $r \rightarrow \sqrt{\frac{2}{\gamma}} Q r$)

\[
ds^2 = -\frac{r^2}{\gamma^4} \left( 1 - \frac{2\sqrt{\gamma}^2 M}{Q r} \right) dt^2 + \frac{1}{\left( 1 - \frac{2\sqrt{\gamma}^2 M}{Q r} \right)} dr^2 + r^2 d\Omega^2 .
\] (3.5)

This string metric shares many of the same properties with the Einstein metric (3.2). The event horizon is located at $r_h = \frac{2\sqrt{\gamma}^2 M}{Q}$. Again, the only curvature singularity is located at $r = 0$ and there is no extremal limit. However, the string coupling $e^{2\phi} \propto r$ is blowing up for large $r$, so string loop corrections should be important there.

Now consider the magnetically charged version of (3.1) in string metric. The magnetically charged solution can be obtained by taking $\phi \rightarrow -\phi$ and leaving the Einstein metric unchanged. In the string metric, the magnetically charged solution is

\[
ds^2 = -\frac{2Q^2}{\gamma^4} \left( 1 - \frac{4M}{r} \right) dt^2 + \frac{2Q^2}{r^2 \left( 1 - \frac{4M}{r} \right)} dr^2 + 2Q^2 d\Omega^2 .
\] (3.6)

The metric (3.6) is a well-known solution. The angular part of the metric has become a cylinder with radius $\sqrt{3}Q$. The metric is equivalent to taking the extremal limit of the magnetically charged black hole of [2] in coordinates that retain the event horizon. It is also equivalent to the direct product of the two-dimensional black hole [15] and a cylinder.

3.2. General Maxwell coupling

It is straightforward to generalize the solution (3.1) to arbitrary coupling $a$. Using the ansatz (2.5) in (2.4), we find the solution

\[
U(r) = \frac{r^{\frac{1}{2} + a^2}}{\gamma^2} \left( 1 - \frac{2(1 + a^2)M}{a^2 r} \right) ,
\] (3.7a)

\[
N = \frac{a^2}{1 + a^2} ,
\] (3.7b)

\[
\phi(r) = -\frac{1}{2a} \log \left( \frac{Q^2 (1 + a^2)}{\gamma^2} \right) + \frac{a}{1 + a^2} \log r .
\] (3.7c)

where $M$ is the quasilocal mass (2.8). It is easy to check that (3.7) reduces to (3.1) when $a = 1$. If $a^2 \rightarrow \infty$, the metric reduces to the Schwarzschild black hole. Again, (3.7) has no smooth transition to the $Q = 0$ case. It has an event horizon at

\[
r_h = \frac{2M(1 + a^2)}{a^2} ,
\] (3.8)
and a singularity at \( r = 0 \). The scalar curvature for (3.7) is

\[
R = \frac{2a^2r \frac{1 - a^2}{1 + a^2}}{(1 + a^2)^2 \gamma^2} \left( 1 - \frac{2(1 + a^2)M}{a^2 r} \right),
\]

(3.9)

which again vanishes for large \( r \). We believe that they are the first examples of black hole solutions in four-dimensional general relativity or string gravity which have charge without having an extremal limit.

The solution (3.7) is a special case of the general solutions to the equations of motion (2.4) which have appeared in [5] (for \( a = 1 \)) and [6]. Both these works overlooked the special solution that has a regular horizon but is not asymptotically flat. For example, setting \( n = 4 \) and \( A = 1 \) in eqs. (B8-10) of [6] leads to (3.7). When \( A \neq 1 \), the solution of [6] is asymptotically flat, and that is the solution considered in [6].

The behavior of (3.7) for large \( r \) depends on the value of \( a \). Consider a radial null geodesic parameterized by \( \lambda \). Along the geodesic, we have \( 0 = -U(r) \dot{\lambda}^2 + \dot{r}^2 / U(r) \), where \( \dot{\lambda} \) represents differentiation with respect to \( \lambda \). Thus \( t = \pm \int 1/U(r)dr + \text{const} \) along the geodesic. If \( t \to \infty \) as \( r \to \infty \), then it takes infinite affine parameter to reach \( r = \infty \). This is the case for \( a^2 \geq 1 \). For this case, the causal structure is like the Schwarzschild causal structure. If \( t \to \text{const} \) as \( r \to \infty \), then it takes finite affine parameter to reach \( r = \infty \), and we must extend our coordinates beyond there. This happens when \( a^2 < 1 \). Though it is impossible to integrate \( 1/U(r) \) for general \( a \), special values such as \( a^2 = \frac{1}{3} \) do yield to analysis. They have a causal structure similar to anti-de Sitter space.

For a metric of the form (2.2), the temperature at a horizon \( r_h \) is given by

\[
T = \frac{1}{4\pi} U'(r_h). \tag{3.10}
\]

For the solution (3.7), the temperature is

\[
T = \frac{1}{4\pi \gamma^2} r_h^{\frac{1 - a^2}{1 + a^2}}. \tag{3.11}
\]

Thus, the black hole temperature decreases as the mass decreases for \( a^2 < 1 \), is independent of mass for \( a^2 = 1 \), and increases as the mass decreases for \( a^2 > 1 \).
4. Exact solutions for $V(\phi) = 2\Lambda e^{2b\phi}$

In this section, we consider the action (1.1) with a Liouville-type potential

$$V(\phi) = 2\Lambda e^{2b\phi}.$$  \hfill (4.1)

We are unable to solve (2.4) in general with this $V$, but using the ansatz (2.5) in (2.4) gives solutions for special values of $\Lambda$ and $b$. In particular, three types of exact electrically charged solutions exist.

(i) Using the notation of (2.2), (2.5), and (2.6), the first solution is

$$U_1(r) = r^{2\alpha^2} \frac{1 + a^2}{(1 - a^2)^2} - \frac{2(1 + a^2)M}{\gamma^2 r} + \frac{Q^2(1 + a^2)e^{2a\phi_0}}{\gamma^4 r^2},$$  \hfill (4.2a)

$$N = \frac{1}{1 + a^2},$$  \hfill (4.2b)

$$\phi_1 = -\frac{a}{1 + a^2},$$  \hfill (4.2c)

$$\Lambda = -\frac{a^2}{\gamma^2(1 - a^2)} e^{-\frac{2a\phi_0}{a}},$$  \hfill (4.2d)

$$b = \frac{1}{a}.$$  \hfill (4.2e)

Note that this solution does not exist for the string case where $a = 1$, and the limit $a \to 0$ gives the standard Reissner-Nordstrom solution. The parameter $M$ in (4.2a) is the quasilocal mass from (2.8). As before, the only singularity is at $r = 0$.

The solution (4.2) is qualitatively somewhat different than (3.7). Eq. (4.2) has a well-defined $Q = 0$ limit, and has horizons at

$$r_{\pm} = (1 - a^2)M \left(1 \pm \sqrt{1 - \frac{Q^2 e^{2a\phi_0}}{\gamma^2(1 - a^2)M^2}}\right).$$  \hfill (4.3)

If $a^2 < 1$, then (4.2) has two horizons, and an extremal limit when

$$Q_{\text{ext}}^2 e^{2a\phi_0} = (1 - a^2)\gamma^2 M^2.$$  \hfill (4.4)

If $Q^2 > Q_{\text{ext}}^2$, then no solution exists with a smooth horizon. Using (3.10), the temperature at the outer horizon is

$$T = \frac{(1 + a^2)M}{2\pi \gamma^2 r_+^{1 + a^2}} \left(\frac{Q^2 e^{2a\phi_0}}{\gamma^2 M} - r_+ - \frac{Q^2 e^{2a\phi_0}}{\gamma^2 M}\right),$$  \hfill (4.5)
which vanishes in the extremal limit. On the other hand, if \( a^2 > 1 \), then (4.3) has just one positive root, which exists for any \( Q^2 \).

The causal structures of the black holes in (4.2) is difficult to obtain even in the uncharged case since it is generally impossible to calculate \( \int 1/U(r)dr \) for general \( a \). When \( a = 0 \), the causal structure is the Reissner-Nordstrom one. As another example, take the \( a^2 = \frac{1}{2} \), \( Q = 0 \) case. One can then perform the integration and see that the causal structure is exactly the same as the Schwarzschild one. When \( 0 < Q^2 < Q_{\text{ext}}^2 \), it can be deduced that the causal structure is the Reissner-Nordstrom one. Causal structures of other values of \( a \) within the range \( 0 < a^2 < 1 \) can be constructed similarly, and are qualitatively the same as the Reissner-Nordstrom solution. For \( a^2 > 1 \), the solution (4.2) has a naked singularity at \( r = 0 \) and a cosmological horizon at the single horizon \( r_h \), beyond which \( U_1 \) is negative and \( r \) behaves like a time coordinate.

At first glance, the chargeless action (1.1) with a potential (4.1), is identical to the dimensionally reduced action of [16,17], which has a internal space with \( l \) dimensions. In [17], it was shown that there is no positive mass Liouville black hole solution with a dilaton of the form (2.6). The apparent contradiction can be resolved by noting that the black hole solutions in (4.2) correspond to \( l < 0 \), a negative number of internal dimensions.

(ii) The second solution is

\[
U_2(r) = r^2 \left( 1 + a^2 \right) \left( -1 + \frac{2Q^2 e^{2a\phi_0}}{a^2 \gamma^2} - \frac{2(1 + a^2)M}{a^2 \gamma^2 r} \right) \tag{4.6a}
\]

\[
N = \frac{a^2}{1 + a^2}, \tag{4.6b}
\]

\[
\phi_1 = \frac{a}{1 + a^2}, \tag{4.6c}
\]

\[
\Lambda = \frac{e^{2a\phi_0}}{(1 - a^2) \gamma^2} - \frac{(1 + a^2)Q^2 e^{4a\phi_0}}{(1 - a^2) \gamma^4}, \tag{4.6d}
\]

\[
b = -a. \tag{4.6e}
\]

The \( a = 1 \) limit of (4.6) again is ill defined. Setting \( \Lambda = 0 \) in (4.6d) and solving for \( \phi_0 \) recovers (3.7). The solution (4.6) has a single event horizon at

\[
r_h = \frac{2M(1 - a^2)}{a^2 \left( -1 + \frac{2Q^2 e^{2a\phi_0}}{\gamma^2} \right)}. \tag{4.7}
\]

Regular horizons only exist when the right-hand-side of (4.7) is positive. Thus the extremal limits of (4.6) correspond to \( r_h \rightarrow 0 \) or \( r_h \rightarrow \infty \). When a horizon exists, the causal structure
of (4.6) depends on $a$ in the same manner as the $V = 0$ case. The temperature of (4.6) at $r_h$ is
\[
T = \frac{1 + a^2}{4\pi(1 - a^2)\gamma^2} \left( \frac{2Q^2e^{2a\phi_0}}{\gamma^2} \right) \frac{1 - a^2}{1 + a^2} \quad (4.8)
\]

(iii) The third solution is
\[
U_3(r) = r^{\frac{2}{1+a^2}} \left( \frac{1}{\gamma^2} - \frac{2(1 + a^2)M}{a^2\gamma^2r} + \frac{(1 + a^2)^{\frac{2a^2+1}{2}}}{(1 - 3a^2)a^2\gamma^2r} r^{-\frac{2(1-a^2)}{1+a^2}} \right) \quad (4.9a)
\]
\[
N = \frac{a^2}{1 + a^2},
\]
\[
\phi_1 = \frac{a}{(1 + a^2)},
\]
\[
\phi_0 = -\frac{1}{2a} \log \left( \frac{Q^2(1 + a^2)}{\gamma^2} \right),
\]
\[
b = -\frac{1}{a}.
\]

We have chosen $M$ in (4.9) to be the mass (2.8) when $\Lambda = 0$. The solution (4.9) is closely related to (3.7), as can be seen by setting $\Lambda = 0$ in (4.9). Because of (4.9d), the solution (4.9) only exists for nonzero $Q$. Unlike the previous two solutions, (4.9) does exist for $a = 1$. Notice that if $a^2 > 1$, then $U_3 \to -\Lambda \times \infty$ as $r \to \infty$. If $a^2 < 1$, then $U_3 \to +\infty$ as $r \to \infty$.

The spacetimes associated with the solutions (4.9) exhibit a variety of possible causal structures depending on the values of $a$ and $\Lambda$. We can obtain the causal structure for all possible values of $a$ and $\Lambda$ by finding the roots of $U_3(r)$, noting that this problem reduces to finding the intersection of a curve linear in $r$ with a variable power of $r$. Unfortunately, because of the nature of the exponents of $r$ in (4.9), it is not possible to find explicitly the locations of the horizons where $U_3(r_h) = 0$ for arbitrary $a$.

First consider $a = 1$, where a single event horizon exists at
\[
r_h = \frac{4M}{1 - 4Q^2\Lambda},
\]

From (4.10), we see that a regular horizon only exists for $4Q^2\Lambda < 1$. At this horizon, the temperature is
\[
T = \frac{1}{4\pi\gamma^2(1 - 4Q^2\Lambda)},
\]

which approaches zero in the extremal limit $4Q^2\Lambda = 1$, and is independent of $M$. 

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When $\Lambda > 0$ and $a^2 < \frac{1}{3}$ or $a^2 > 1$, there can be zero, one, or two horizons, depending on the relative magnitudes of $M$ and $\Lambda$. One horizon occurs at the extremal limit when

$$
\Lambda = \frac{a^2 \gamma \frac{2(1-a^2)}{a^2} \, r_{\text{ext}}^{\frac{2(1-a^2)}{a^2}}}{(1 + a^2)^{\frac{1+a^2}{a^2}} Q^{\frac{2}{a^2}}} \equiv \Lambda_{\text{ext}},
$$

(4.12)

where

$$
r_{\text{ext}} = \frac{(1 - 3a^2)(1 + a^2)}{a^2(1 - a^2)} M.
$$

(4.13)

is the location of the extremal horizon. If $\Lambda > \Lambda_{\text{ext}}$, then no horizons exist. Finally, if $\Lambda < 0$ or $\frac{1}{3} < a^2 < 1$, then (4.9) has a single horizon. The possible cases are listed in table 1, with the equality signs denoting an extremal limit.

**Table 1:** The possible horizons for eq. (4.9)

<table>
<thead>
<tr>
<th>Solution (4.9)</th>
<th>$\Lambda &lt; 0$</th>
<th>$\Lambda &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 &lt; a^2$</td>
<td>$O$</td>
<td>$(C, O)$ $\Lambda &lt; \Lambda_{\text{ext}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(C=O)$ $\Lambda = \Lambda_{\text{ext}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B $\Lambda &gt; \Lambda_{\text{ext}}$</td>
</tr>
<tr>
<td>$\frac{1}{3} &lt; a^2 &lt; 1$</td>
<td>$O$</td>
<td>$O$</td>
</tr>
<tr>
<td>$a^2 &lt; \frac{1}{3}$</td>
<td>$O$</td>
<td>$(O, I)$ $\Lambda &gt; \Lambda_{\text{ext}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(O=I)$ $\Lambda = \Lambda_{\text{ext}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>N $\Lambda &lt; \Lambda_{\text{ext}}$</td>
</tr>
</tbody>
</table>

*Notation: O—Outer black hole horizon  I—Inner black hole horizon  B—Cosmological singularity  N—Naked singularity  C—Cosmological horizon*

5. **Exact solutions for** $V(\phi) = 2\Lambda_1 e^{2b_1 \phi} + 2\Lambda_2 e^{2b_2 \phi}$

The final type of potential that we will study is the sum of two Liouville type terms:

$$
V(\phi) = 2\Lambda_1 e^{2b_1 \phi} + 2\Lambda_2 e^{2b_2 \phi}.
$$

(5.1)

This generalizes further the potential (4.1). If $b_1 = b_2$, then (5.1) reduces to (4.1), so we will not repeat these solutions. Requiring $b_1 \neq b_2$ again gives three classes of solutions.
(a) The first solution is identical to the solution (4.2), except for the modifications

\[ U(r) = U_1(r) - \frac{\Lambda_2(1 + a^2)^2 e^{2a\phi_0}}{(3 - a^2)^2} r^{\frac{2}{1 + a^2}}, \quad (5.2a) \]

\[ b_2 = a. \quad (5.2b) \]

Notice that for \( a = 1 \) only, \( b_1 = b_2 \), which is the previous solution (4.2).

To analyze the causal structures associated with this case, we can proceed as before, noting that the roots of \( U(r) \) may be found by finding the intersection of a quadratic with a variable power. The results are given in table 2.

**Table 2:** The possible horizons for eq. (5.2)

<table>
<thead>
<tr>
<th>Solution (5.2)</th>
<th>( \Lambda_2 &lt; 0 )</th>
<th>( \Lambda_2 &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 &lt; a^2 )</td>
<td>( C )</td>
<td>( C )</td>
</tr>
<tr>
<td>( 1 &lt; a^2 &lt; 3 )</td>
<td>( \text{(C, O, I)} ) ( \text{(C, O=I) or (C=0, I)} ) ( \text{C or (C=0=I)} )</td>
<td>( C )</td>
</tr>
<tr>
<td>( a^2 &lt; 1 )</td>
<td>( \text{(O, I)} ) ( \text{(O=I)} ) ( \text{N} )</td>
<td>( \text{(C, O, I)} ) ( \text{(C, O=I) or (C=0, I)} ) ( \text{C or (C=0=I)} )</td>
</tr>
</tbody>
</table>

We see from table 2 that there is now the possibility of having spacetimes with three horizons, whose causal structure resembles that of Reissner-Nordstrom-de Sitter spacetime. The extremal cases in table 2 can be approached in a variety of ways, depending upon the relative magnitudes of \( M, Q \) and \( \Lambda_2 \). We leave the derivation of the extremal values of these quantities as an exercise for the reader.

The general form of the temperature at the outer event horizon for these solutions is

\[ T = \frac{r_O}{2\pi\gamma^2} \left( r_O^2 - M(a^2 - 3)r_O - 2Q^2e^{2a\phi_0} \right) \quad (5.3) \]

where \( r_O \) denotes the location of the outer horizon when it exists. In general \( r_O \) must be solved for numerically, although closed form solutions can be obtained for special values of \( a \).
(b) The second solution is identical to the solution (4.6), except for the modifications

\[
U(r) = U_2(r) + \frac{\Lambda_2(1 + a^2)^2 \epsilon^{\frac{2a_0}{a} \frac{2a^2}{1 + a^2}}}{a^2(1 - 3a^2)} r^{\frac{2a}{1 + a^2}},
\]

\[
b_2 = -\frac{1}{a}.
\]

Again, when \( a = 1, b_1 = b_2. \) For arbitrary \( a, \) the extra power of \( r \) in (5.4a) makes it impossible to find explicitly the positions of the horizons, but we can determine the number and type of horizons. The possible causal structures are given in table 3, where we have used \( \hat{Q} \equiv \frac{1 + a^2}{(1 - a^2)^2} \left( \frac{2Q^2}{\gamma^2} e^{2a_0} - 1 \right). \) The extremal cases in table 3 occur for certain values \( M = M_{\text{ext}} \) of the mass parameter; the explicit derivation is similar to the previous cases and we shall not reproduce it here.

The temperature of the outer horizon in this case is given by

\[
T = \frac{r_O^{-2} \frac{1}{1 + a^2}}{2\pi \gamma^2} \left[ \left( \frac{2Q^2 e^{2a_0}}{\gamma^2} - 1 \right) r_O - \frac{3a^2 - 1}{a^2} M \right]
\]

where again \( r_O \) denotes the location of the outer horizon (when it exists) and must be solved for numerically except for special values of \( a \) and/or \( Q. \)

**Table 3:** The possible horizons for eq. (5.4)

<table>
<thead>
<tr>
<th>Solution (5.4)</th>
<th>( \Lambda_2 &lt; 0 )</th>
<th>( \Lambda_2 &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^2 &lt; \frac{1}{3} )</td>
<td>N</td>
<td>O</td>
</tr>
<tr>
<td>( \frac{1}{3} &lt; a^2 &lt; 1 )</td>
<td>(C, O)</td>
<td>(C=O)</td>
</tr>
<tr>
<td>( 1 &lt; a^2 )</td>
<td>O</td>
<td>O</td>
</tr>
</tbody>
</table>

(c) Finally, a third class of solutions exists that has no counterpart in the solutions with \( V \) of the form (4.1). These solutions exist for \( 0 < N < 1 \) with \( N \neq \frac{1}{2}, \) and satisfy
where \( r_0 \) is a constant related to the mass.

The possible causal structures for this case depend upon the relative values of \( a, b_1, \) \( r_0 \) and \( \dot{Q} \equiv 2Q^2e^{2a\phi_0}(1 - 2N \pm 2a\sqrt{N(1 - N)})^{-1}(1 - 3N \pm a\sqrt{N(1 - N)})^{-1}r^4 \). The derivation is similar to the previous cases and the results are given in table 4. Despite the appearance of the additional parameter \( N \), there are at most two horizons for any solution in this class. We leave the derivation of the temperature and extremal cases as an exercise.

**Table 4:** The possible horizons for eq. (5.6)

<table>
<thead>
<tr>
<th>Solution (5.6)</th>
<th>( \dot{Q} &lt; 0 )</th>
<th>( \dot{Q} &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2ab_1 &gt; 1 + b_1^2 )</td>
<td>( b_1 &lt; 1 )</td>
<td>( b_1 &gt; 1 )</td>
</tr>
<tr>
<td>( 1 + b_1^2 &gt; 2ab_1 &gt; 0 )</td>
<td>( \dot{Q} &gt; 0 )</td>
<td>( \dot{Q} &lt; 0 )</td>
</tr>
</tbody>
</table>
6. Solutions in \( n \) dimensions

The solutions given in the previous sections are valid in \( n = 4 \) dimensions, but can be generalized to dimensions \( n \geq 4 \). Using the action (1.2), we can derive the equations of motion (following [6])

\[
R_{ab} = \frac{4}{n-2} \left( \partial_a \phi \partial_b \phi + \frac{1}{4} g_{ab} V \right) + 2 \epsilon^a_{\phantom{a}b} \chi \left( F_{ac} F_b^c - \frac{1}{2(n-2)} g_{ab} F^2 \right), \quad (6.1a)
\]

\[
0 = -\partial_a \left[ \sqrt{-g} \epsilon^a_{\phantom{a}b} e^{-\frac{4 \phi}{n-2}} F_{ab} \right], \quad (6.1b)
\]

\[
\nabla^2 \phi = \frac{n-2}{8} \frac{\partial V}{\partial \phi} - \frac{a}{2} e^{-\frac{4 \phi}{n-2}} F_{ab} F^{ab}. \quad (6.1c)
\]

We can again choose a coordinate system satisfying (2.2). The Maxwell field can be chosen to be an isolated electric charge satisfying (6.1b)

\[
F_{tr} = e^a_{\frac{4 \phi}{n-2}} \frac{Q}{R^{n-2}}, \quad (6.2)
\]

which generalizes (2.3). With the coordinate system (2.2) and the Maxwell field (6.2), the equations of motion (6.1) reduce to [6]

\[
\frac{1}{R^{n-2}} \frac{d}{dr} \left[ R^{n-2} U \frac{d \phi}{dr} \right] = \frac{n-2}{8} \frac{dV}{d\phi} + a e^a_{\frac{4 \phi}{n-2}} \frac{Q^2}{R^{2(n-2)}}, \quad (6.3a)
\]

\[
\frac{1}{R} \frac{d^2 R}{dr^2} = -\frac{4}{(n-2)^2} \left( \frac{d\phi}{dr} \right)^2, \quad (6.3b)
\]

\[
\frac{1}{R^{n-2}} \frac{d}{dr} \left[ U \frac{d}{dr} \left( R^{n-2} \right) \right] = (n-2)(n-3) \frac{1}{R^2} - V - 2 e^a_{\frac{4 \phi}{n-2}} \frac{Q^2}{R^{2(n-2)}}. \quad (6.3c)
\]

6.1. Solutions with \( V = 0 \) for arbitrary \( n \)

First consider solutions of (6.3) where the potential vanishes, \( V(\phi) = 0 \). Making the ansatz (2.5) \( R(r) = r^N \), we find a solution of (6.3) with

\[
U(r) = \frac{((n-3)^2 + a^2)^2}{\gamma^2 (n-3 + a^2)^2} r^{\frac{2(n-3)^2}{(n-3)^2 + a^2}} - \frac{4 ((n-3)^2 + a^2)}{a^2 (n-2) \gamma^{n-2}} M r^{\frac{(n-3)((n-3)^2 + a^2)}{(n-3)^2 + a^2}}, \quad (6.4a)
\]

\[
N = \frac{a^2}{(n-3)^2 + a^2}, \quad (6.4b)
\]

\[
\phi(r) = -\frac{n-2}{4a} \log \left( \frac{2Q^2(n-3+a^2)}{(n-2)(n-3)^2 \gamma^{2(n-3)}} \right) + \frac{a(n-2)(n-3)}{2 ((n-3)^2 + a^2)} \log r, \quad (6.4c)
\]
where $M$ is the mass as defined by (2.8). The metric is independent of $Q$, but because of the $\phi$ dependence on $Q$, solutions only exist for nonzero values of $Q$. The solution (6.4) has an event horizon at

$$r_h = \left( \frac{4M(n-3+a^2)^2}{a^2(n-2)((n-3)^2+a^2)\gamma^{n-1}} \right)^{\frac{(n-3)^2+a^2}{(n-2)((n-3)^2+a^2)}}. \quad (6.5)$$

The temperature is therefore

$$T = \frac{(n-3)((n-3)^2+a^2)}{4\pi\gamma^2(n-3+a^2)} r_h. \quad (6.6)$$

Temperature decreases as mass decreases for $a^2 < (n-3)^2$, is independent of mass for $a^2 = (n-3)^2$, and increases as mass decreases for $a^2 > (n-3)^2$. The causal structure for the solution (6.4) is similar the solution (3.7) when $n = 4$, except that the boundary between Schwarzschild-like causal structure and anti-de Sitter-like causal structure is at $a^2 = (n-3)^2$.

6.2. Solutions with $V(\phi) = 2\Lambda e^{2b\phi}$

Solutions to (6.3) also exist for nonzero $V$. As in four dimensions, the simplest case is

$$V(\phi) = 2\Lambda e^{2b\phi}. \quad (6.7)$$

Using the ansatz (2.5), solutions exist only for certain values of $\Lambda$ and $b$. There are now three types of solutions to (6.3).

(i) The first solution has the form

$$U_1(r) = r^{\frac{2a^2}{1+a^2}} \left( \frac{(1+a^2)^2(n-3)}{(1-a^2)^2(n-3+a^2)} - \frac{4(1+a^2)M}{(n-2)\gamma^{n-2}} r^{-\frac{(n-3)^2+a^2}{1+a^2}} + \frac{2Q^2(1+a^2)^2}{(n-2)(n-3+a^2)\gamma^{2(n-2)}} e^{\frac{4a^2}{n-2}r - \frac{2(n-3+a^2)}{1+a^2}} \right), \quad (6.8a)$$

$$N = \frac{1}{1+a^2}, \quad (6.8b)$$

$$\phi_1 = -\frac{a(n-2)}{2(1+a^2)}, \quad (6.8c)$$

$$\Lambda = -\frac{(n-2)(n-3)a^2}{2\gamma^2(1-a^2)} e^{-\frac{4a^2}{n-2}}, \quad (6.8d)$$

$$b = \frac{2}{a(n-2)}. \quad (6.8e)$$
Note that this solution does not exist for the string case where $a = 1$. The solution (6.8) has horizons given by

\[
\frac{n-3+a^2}{r_{1+3+a^2}^2} = \frac{2(1-a^2)(n-3+a^2)M}{(1+a^2)(n-2)(n-3)\gamma^n} \left(1 \pm \sqrt{1 - \frac{(1+a^2)^2(n-2)(n-3)Q^2 e^{\frac{4a\phi_0}{n-2}}}{2(1-a^2)(n-3+a^2)^2\gamma^2M^2}}\right)
\]  

(6.9)

Thus, as in the $n = 4$ case (4.2), when $a^2 < 1$, (6.8) has two horizons, and an extremal limit when

\[
Q_{\text{ext}}^2 e^{\frac{4a\phi_0}{n-2}} = \frac{2(1-a^2)(n-3+a^2)^2\gamma^2}{(n-2)(n-3)(1+a^2)^2} M^2.
\]  

(6.10)

Again, the temperature at the outer horizon is

\[
T = \frac{(n-3+a^2)M}{\pi(n-2)\gamma^{n-2}r_+^{2(n-3+a^2)} e^{\frac{4a\phi_0}{n-2}}} \left(1 + \frac{(n-3)(n-3+a^2)Q^2 e^{\frac{4a\phi_0}{n-2}}}{(n-3)\gamma^2M} \right)
\]  

(6.11)

which vanishes in the extremal limit. If $a^2 > 1$, then (6.8) has a single horizon.

(ii) The second solution has the form

\[
U_2(r) = r^{\frac{2(n-3)^2}{r^{1+3+a^2}}} e^{-\frac{(n-3)^2+2a^2}{2(n-3+a^2)-(n-3)^2-a^2} \gamma^2} \left(1 + \frac{2Q^2 e^{\frac{4a\phi_0}{n-2}}}{(n-3)\gamma^2} \right)
\]  

(6.12a)

\[
N = \frac{a^2}{(n-3)^2 + a^2},
\]  

(6.12b)

\[
\phi_1 = \frac{a(n-2)(n-3)}{2((n-3)^2 + a^2)},
\]  

(6.12c)

\[
\Lambda = \frac{(n-2)(n-3)^3}{2((n-3)^2 + a^2)\gamma^2 e^{\frac{4a\phi_0}{(n-3)(n-3+a^2)}}} \left(1 + \frac{(n-3)(n-3+a^2)Q^2 e^{\frac{4a\phi_0}{n-2}}}{(n-3)^2 + a^2} \gamma^2 (n-2)\right),
\]  

(6.12d)

\[
b = -\frac{2a}{(n-2)(n-3)}.
\]  

(6.12e)

Setting $\Lambda = 0$ in (6.12d) and solving for $\phi_0$ recovers (6.4). The solution (6.12) has a single event horizon at

\[
\frac{(n-3)(n-3+a^2)}{(n-3)^2 + a^2} = \frac{4M(n-3+a^2)((n-3)^2-a^2)}{a^2(n-2)((n-3)^2+a^2)\gamma^n} \left(1 + \frac{2Q^2 e^{\frac{4a\phi_0}{n-2}}}{(n-3)^2 + a^2} \gamma^2 (n-3)\right).
\]  

(6.13)
Regular event horizons only exist when the right-hand-side of (6.13) is positive. Thus the extremal limits of (6.12) correspond to \( r_h = 0 \) or \( r_h = \infty \). The temperature is

\[
T = \frac{(n-3)((n-3)^2 + a^2)}{4\pi((n-3)^2 - a^2)} \left( -1 + \frac{2Q^2 e^{\frac{2a}{n-2}}}{(n-3)^2(n-3)} \right)^{\frac{(n-3)^2-a^2}{(n-3)^2+a^2}} \tag{6.14}
\]

(iii) The third solution has the form

\[
U_3(r) = \frac{((n-3)^2 + a^2)^2}{\gamma^2(n-3 + a^2)^2} r^{\frac{2(n-3)^2}{n-3 + a^2}} - \frac{4((n-3)^2 + a^2)M}{a^2(n-2)} r^{\frac{2(n-3)^2}{n-3 + a^2}} + \frac{2\Lambda Q}{a^2\gamma} \frac{2(n-3)^2 + a^2}{(n-3)2 + a^2} \frac{(n-3)^2 + a^2}{(n-3)2 + a^2} \frac{2(n-3)^2}{a^2} \frac{2(n-3)^2}{(n-3)^2 + a^2} r^{\frac{2\Lambda Q}{a^2}}.
\]

\[
N = \frac{a^2}{(n-3)^2 + a^2},
\]

\[
\phi_1 = \frac{a(n-2)(n-3)}{2((n-3)^2 + a^2)},
\]

\[
\phi_0 = -\frac{n-2}{4a} \log \left( \frac{2Q^2(n-3 + a^2)}{(n-2)(n-3)^2\gamma^2(n-3)} \right),
\]

\[
b = -\frac{2(n-3)}{(n-2)a}.
\]

Setting \( \Lambda = 0 \) in (6.15) easily reduces to (6.4). Because of (6.15d), the solution (6.15) only exists for nonzero \( Q \). Unfortunately, because of the nature of the exponents of \( r \) in (6.15), it is not possible to find explicitly the locations of the horizons where \( U_3(r_h) = 0 \).

6.3. Solutions with \( V(\phi) = 2\Lambda_1 e^{2b_1\phi} + 2\Lambda_2 e^{2b_2\phi} \)

Solutions also exist for

\[
V(\phi) = 2\Lambda_1 e^{2b_1\phi} + 2\Lambda_2 e^{2b_2\phi}.
\]

Restricting to \( b_1 \neq b_2 \) reveals three classes of solutions.

(a) The first solution is identical to the solution (6.8), except for the modifications

\[
U(r) = U_1(r) - \frac{2\Lambda_2(1 + a^2)^2 e^{\frac{2a}{n-2}}}{(n-2)(n-1-a^2)} r^{\frac{2\Lambda_2}{n-2}}.
\]

\[
b_2 = \frac{2a}{n-2}.
\]

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(b) The second solution is identical to the solution (6.12), except for the modifications

\[
U(r) = U_2(r) + \frac{2 \Lambda_2 ((n - 3)^2 + a^2)^2 e^{-\frac{4(n-3)\delta_p}{(n-2)}}}{a^2(n-2)((n-3)^2 - (n-1)a^2)} r^{(n-3) + a} \quad \text{, (6.18a)}
\]

\[
b_2 = -\frac{2(n-3)}{a(n-2)} \quad \text{. (6.18b)}
\]

(c) Finally, a third class of solutions exists that has no counterpart in the solutions with \( V \) of the form (6.7). These solutions exist for \( 0 < N < 1 \) with \( N \neq \frac{1}{2} \), and satisfy

\[
U(r) = \frac{n-3}{(2N-1)(1-4N+nN)\gamma^2} r^{2(1-N)} - r_0 r^{1-N(n-2)}
\]

\[
+ \frac{2Q^2 e^{\frac{4\delta_p}{2N-1}}} {\left(1-N(n-2) \pm 2a \sqrt{N(1-N)}\right) \left(1-N(n-1) \pm a \sqrt{N(1-N)}\right) \gamma^{2(n-2)}} \quad \text{, (6.19a)}
\]

\[
b_1 = \mp \frac{2N}{(n-2)\sqrt{N(1-N)}} \quad \text{, (6.19b)}
\]

\[
\Lambda_1 = \frac{(N-1)(n-2)(n-3)e^{\frac{4\delta_p}{(n-2)\sqrt{N(1-N)}}}}{2(2N-1)\gamma^2} \quad \text{, (6.19c)}
\]

\[
b_2 = \frac{2a}{n-2} \mp \frac{2N}{\sqrt{N(1-N)}} \quad \text{, (6.19d)}
\]

\[
\Lambda_2 = -\frac{Q^2 \left(1-N \pm a \sqrt{N(1-N)}\right)}{\left(1-N(n-1) \pm a \sqrt{N(1-N)}\right) \gamma^{2(n-2)}} e^{\pm \frac{4\delta_p}{\sqrt{N(1-N)}}} \quad \text{, (6.19e)}
\]

\[
\phi_1 = \pm \frac{1}{2}(n-2)\sqrt{N(1-N)} \quad \text{, (6.19f)}
\]

where \( r_0 \) is a constant.

7. Conclusions

To sum up, we have obtained several one-parameter families of non-asymptotically flat (anti-) de Sitter electrically charged \( S\bar{S}\bar{S} \) black hole solutions for the action (1.1) with a zero potential, Liouville potential and a two-term exponential potential. In the \( V(\phi) = 0 \) case, this family of charged black holes (which includes the string theoretic case) has two unique properties: (i) they do not have inner horizons, and (ii) \( Q \) can take any nonzero value without causing the event horizon to vanish or become singular. The former property can be found in the black holes in [1] but the second property has no previous
analogue in general relativity or string gravity. The magnetically charged string black hole has an additional property: it is a product of the 2D black hole with a constant two sphere, in terms of the string metric. For the Liouville black holes, we derived three families of solutions. For the first family (4.2), the solutions have outer and inner horizons if the extremal limit (4.4) between $Q^2$ and $M^2$ is not violated. It has no well-defined $a = 1$ limit. For the second family (4.6), the solution has only one event horizon. Again, the limit $a = 1$ is not well-defined. For the third family (4.9), which has a well-defined $a = 1$ limit, the solutions have possible outer, inner, or cosmological horizons. Three different families of SSS solutions are also obtained in the case of a two-term exponential potential. The first two families (5.2) and (5.4) are the modifications of (4.2) and (4.6) respectively. The third family has no counterpart to the solutions for the single potential (4.1). The conditions of existence of outer, inner and cosmological horizons for the three families of solutions are summarized in tables 2, 3 and 4 respectively. The Hawking temperature of all the above black hole solutions are also computed. Finally, we generalized all the above solutions to $n$ dimensions.

We close by commenting on further possible extensions of our work. It is worthwhile to mention that the black hole solutions (3.7) can be interpreted as black $p$-branes in $(4 + p)$ dimensions which are solutions to the $(4 + p)$-dimensional Einstein-Maxwell (or with a $p$-form) action, provided that $a = \sqrt{\frac{p}{p+2}}$ (the string case $a = 1$ implies a diverging $p$) [18]. Using these solutions as examples, it would be interesting to see whether curvature singularities in the four dimensional solutions can be completely resolved or become much milder, by viewing the solutions as reductions of higher-dimensional objects, with or without serious modification in higher-dimensional gravitational actions or metrics.

Note that asymptotically flat rotating charged black hole solutions were discussed and investigated in [19] for the action (1.1) with $V(\phi) = 0$. No explicit solutions have yet been found except for $a = \sqrt{3}$ [20] and for $a = 1$ when the string three-form $H_{abc}$ is included [21]. It would be interesting to add angular momentum to the string black hole solutions we obtained in section 3 and 4 to see if it is possible to extend the family of solutions we obtained to the rotating case. Generalization of all the solutions derived in this paper to arbitrary $n \geq 4$ with both electric and magnetic charge present is another problem of interest. One can also attempt to construct any asymptotically flat or (anti-) de Sitter black holes for the two-term exponential potential. We note that a partially closed-form chargeless black hole solution for action (1.1) with a more elaborate potential than (5.1) has been obtained in [22] with an exponentially decaying dilaton. Finally, we
note that using the results in $2 + 1$ dilaton gravity in [12], Maki and Shiraishi derived several static solutions to Einstein equations coupled with antisymmetric tensor fields in $(2 + l + 1)$ dimensions. The solutions describe a product of a $(2+1)$-dimensional circularly symmetric spacetime and an $l$-dimensional internal maximally symmetric manifold, and they can be applied to various supergravity models [23]. One may consider a similar work in $(3 + l + 1)$ dimensions by using the results in the present paper. We intend to relate further details elsewhere.

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References


