Dimensional continuation without perturbation theory

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A formula is proposed for continuing physical correlation functions to non-integer numbers of dimensions, expressing them as infinite weighted sums over the same correlation functions in arbitrary integer dimensions. The formula is motivated by studying the strong coupling expansion, but the end result makes no reference to any perturbation theory. It is shown that the formula leads to the correct dimension dependence in weak coupling perturbation theory at one loop.

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A question of interest, given the importance of dimensional regularization in obtaining physical predictions from gauge theories, is the following: What is the meaning of dimensional regularization beyond the Feynman diagrams of weak coupling perturbation theory? It seems difficult to find a non-perturbative definition of dimensional regularization, at least while preserving translation invariance.

I show in this paper that there is a simple dimensional continuation which, while motivated by strong coupling perturbation theory, gives a formula that is not perturbative in character. It is, of course, well-known that strong coupling expansions, or weak coupling perturbative expansions, can all be continued in dimension without difficulty. The novelty here is that the formula presented does not need perturbation theory of any sort for its statement.

The strong coupling origins of this formula are classical—recall that in the work of Fisher and Gaunt, it was shown how to expand strong coupling series for large \( d \). This required a grouping of strong coupling graphs in a geometric manner that is rather unnatural from the point of view of continuum field theory. Nevertheless, this grouping of strong coupling graphs, recalled below, is such that the contribution from each separate group of graphs can be extracted from the sum of all the graphs, provided one knows the sum of all the graphs in every dimension smaller than or equal to the integer dimension of interest. It is then a simple manner to write a dimensional continuation formula that involves an infinite summation over terms, with each term a finite sum over quantities at integer dimensions. Due to the infinite sum, the formula has a ‘transcendental’ character, probably unavoidable for a formula valid for arbitrary dimensions.

While the formula is motivated by strong coupling perturbation theory, I show that it gives the correct answer by performing an explicit computation in weak coupling field theory. This calculation gives a different insight into the manner in which the analytical continuation is working.

I shall first explain the grouping of strong coupling graphs suggested by Fisher and Gaunt. I shall then give the formula, which is almost obvious from the strong coupling expansion. Finally, I show that the formula gives the correct continuation to non-integer dimensions for a weak coupling one-loop calculation of a \( \phi^3 \) field theory correlation function.

Consider an Ising model on a \( d \)-dimensional hypercubic lattice. For a connected spin-spin correlation function, e.g., \( d^{-1} \sum_{j=1}^{d} \langle S(0)S(n\hat{e}_j) \rangle \), \( \hat{e}_j \) are the basis vectors for the lattice) the strong coupling, or high temperature, expansion can be organized as a sum over paths that go from 0 to the point \( n\hat{e}_j \) on the lattice. For any such path, there is a hyperplane spanned by the vectors corresponding to the steps in the path, \( d_{\text{min}} \). It follows
that the correlation function can be written as

\[
d^{-1} \sum_{j=1}^{d} \langle S(0) S(n \hat{e}_j) \rangle = \sum_{D=1}^{d} C_d^D \ F_{D}^{\text{irred}},
\]

where \( F_{D}^{\text{irred}} \) is defined to be the contribution from paths that span a fixed hyperplane of dimension \( d_{\text{min}} = D \), and the combinatorial coefficient

\[
C_d^D = \frac{\Gamma(d+1)}{\Gamma(D+1) \Gamma(d-D+1)}
\]

counts the number of \( D \)-dimensional hyperplanes in \( d \) dimensions. As it stands, this is not particularly useful, since we have no obvious way of computing the quantity \( F_{D}^{\text{irred}} \). However, it is evident from the vanishing of \( C_d^D \) if \( d < D \), that we can replace the upper limit in the sum by infinity. So, if we could compute \( F_{D}^{\text{irred}} \) for integer \( D \), our analytic continuation would be

\[
d^{-1} \sum_{j=1}^{d} \langle S(0) S(n \hat{e}_j) \rangle = \sum_{D=1}^{\infty} C_d^D \ F_{D}^{\text{irred}}.
\]

Now, given a sum of the form

\[
a_d = \sum_{D=1}^{d} C_d^D b_D,
\]

it is easy to check that

\[
b_d = \sum_{D=1}^{d} (-)^{d-D} C_d^D a_D.
\]

It follows that our desired analytical continuation is just

\[
d^{-1} \sum_{j=1}^{d} \langle S(0) S(n \hat{e}_j) \rangle = \sum_{D=1}^{\infty} C_d^D \sum_{k=1}^{D} (-)^{D-k} C_k^D \left[ \sum_{l=1}^{k} \langle S(0) S(n \hat{e}_l) \rangle \right].
\]

We can now state the general formula for dimensional continuation of suitable physical quantities:

\[
F_d = \sum_{D=1}^{\infty} C_d^D (-)^D \sum_{k=1}^{D} (-)^k C_k^D F_k;
\]

in words, for a physical quantity, \( F \), that can be described in a dimension-independent way, and which has the same engineering dimensions in all dimensions, given the value of
\( F \) at integer dimensions, the value at an arbitrary complex dimension can be computed. When \( d \) is an integer, it is easy to check that we recover the identity \( F_d = F_d \).

To develop an intuition for how this analytical continuation is working, I shall now compute the dimensional continuation of the two-point function at one-loop in \( \phi^3 \) theory. This is a computation in weak coupling perturbation theory, and therefore completely independent of the strong coupling motivations for the formula. The calculation must be done with a regularization which works in any number of dimensions. This requires rapid decay of propagators with increasing momenta, or working with a lattice regularization. Working on a hypercubic lattice in \( D \) dimensions,

\[
G(p) = \left( m^2 + \frac{2}{a^2} \sum_{j=1}^{D} (1 - \cos ap_j) \right)^{-1}
\]

is the lattice Green function. The one-loop correction to the two-point function is

\[
\Gamma^{(2)}_{1\text{-loop}, D}(p) = -\frac{1}{2} g^2 a^{D-6} \int_{-\pi/a}^{\pi/a} \frac{d^D q}{(2\pi)^D} G(q) G(p + q),
\]

which can be written as

\[
-\frac{1}{2} g^2 a^{D-6} \int_{-\pi/a}^{\pi/a} \frac{d^D q}{(2\pi)^D} \int_0^\infty da \int_0^\infty d\beta \exp \left[ -a G^{-1}(q) - \beta G^{-1}(p + q) \right].
\]

Suppose that \( p = (p_1, \ldots) \), then we have

\[
\Gamma^{(2)}_{1\text{-loop}, D}(p) = -\frac{1}{2} g^2 a^{D-6} \int da d\beta e^{-a+p_1(m^2+2D/a^2)} I_0 \left( \frac{2(a + \beta)}{a^2} \right)^D \hat{I}(\alpha, \beta, p_1),
\]

where \( I_0 \) is a Bessel function, and

\[
\hat{I}(\alpha, \beta, p_1) \equiv \int_{-\pi/a}^{\pi/a} \frac{dq_1}{2\pi} \exp \left[ \frac{2(\alpha + \beta)}{a^2} \cos(aq_1) + \frac{4\beta}{a^2} \sin(a(p_1/2 + q_1)) \sin(ap_1/2) \right].
\]

This quantity is now analytic in \( D \), so we don’t need the analytic continuation formula given—however, the question we are attempting to answer is whether this same analytic continuation will arise from the dimensional continuation formula given above.
As advertised the dimensional continuation formula gives back the expected answer, even powers of a number.

The point of this simple example is, however, that it shows that the seemingly complicated dimensional continuation formula is essentially just a formula for defining arbitrary powers of a number, \( x \), as a series in its integer powers, by expanding it in a Taylor series about 1, in integer powers of \((1-x)\), and then expanding each integer power of \((1-x)\) as a polynomial in \( x \). Contrast this with the strong coupling derivation.

How unique is this dimensional continuation? Let us investigate different ways of continuing \( x \) to arbitrary complex powers, e.g. let \( \lambda \) be a complex number and consider the sum

\[
f_\lambda(d) = \sum_{D=1}^{\infty} C_D^d \sum_{k=1}^{D} (-1)^{D-k} C_k D^D \exp(\lambda(d-k))
\]

\[
e^{\lambda d} \sum_{D=1}^{\infty} C_D^d \sum_{k=1}^{D} (-1)^{D-k} C_k D^D (xe^{-\lambda})^k
\]

\[
e^{\lambda d} (xe^{-\lambda})^d = x^d.
\]

Thus, for functions that grow at most geometrically with dimension, we get back the same function for any choice of \( \lambda \), consistent with \(|1-xe^{-\lambda}|<1\). (Recall that according to Carlson’s theorem\cite{3}, if two functions agree on integers, and separately satisfy the bound \(|f_i(d)| < \exp(c|d|)\), for \( R d > 0 \), with \( c < \pi \), then they define the same analytic function.)

One may wonder if the double summation in the continuation formula is really necessary. It would be much more pleasant if one could work with a series of the form

\[
f(d) = \sum_{D=1}^{\infty} f(D) c(d,D),
\]
with some coefficients $c(d, D)$. Indeed, if one looks at the finite sub-sum occurring in each term of the infinite sum, we see that using the vanishing of the combinatorial coefficients $C_k^j$ for $k > j$, we can write this as an infinite sum as well:

$$F_d = \sum_{D=1}^{\infty} C_d^D (-1)^D \sum_{k=1}^{\infty} (-1)^k C_k^D F_k,$$

Now, if we ignore the fact that we are rearranging terms in a double summation with terms that have alternating signs, we arrive at a formula of the desired type, with the unfortunate property that $c(d, D) = \delta_{d,D}$, in other words, the dimensional continuation is not analytic. It would therefore appear that the double sum is necessary, and that the finiteness of the nested summation is necessary for the analyticity of the formula.

A possible application of this formula is to gauge theory. The only gauge-invariant regulators known are dimensional regularization, and lattices. It would be of some interest to study a dimensionally continued lattice formula in the limit as the cutoff goes to zero, or as the dimension approaches 4. A more interesting possibility would be to interpret the infinite summation in terms of embedding $d$-dimensional Euclidean space in an infinite-dimensional space, with an appropriate $d$-dependent measure, such that integration with respect to this measure would lead to the dimensional continuation formula given in this paper. While I have no evidence that this can be achieved, it would be one approach to making concrete non-perturbative sense of dimensional continuation for field theories.

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References