Quantum KAM Technique and Yang-Mills Quantum Mechanics

Igor Halperin
Technion - Israel Institute of Technology
Department of Physics
Haifa, 32000, Israel
e-mail address: bigor@techunix.technion.ac.il

Abstract:
We study a quantum analogue of the iterative perturbation theory by Kolmogorov used in the proof of the Kolmogorov-Arnold-Moser (KAM) theorem. The method is based on sequent canonical transformations with a "running" coupling constant $\lambda, \lambda^2, \lambda^4, \text{etc.}$ The proposed scheme, as its classical predecessor, is "superconvergent" in the sense that after the nth step, a theory is solved to the accuracy of order $\lambda^{2n-1}$. It is shown that the Kolmogorov technique corresponds to an infinite resummation of the usual perturbative series. The corresponding expansion is convergent for the quantum anharmonic oscillator due to the fact that it turns out to be identical to the Padé series. The method is easily generalizable to many-dimensional cases. The Kolmogorov technique is further applied to a non-perturbative treatment of Yang-Mills quantum mechanics. A controllable expansion for the wave function near the origin is constructed. For large fields, we build an asymptotic adiabatic expansion in inverse powers of the field. This asymptotic solution contains arbitrary constants which are not fixed by the boundary conditions at infinity. To find them, we approximately match the two expansions in an intermediate region. We also discuss some analogies between this problem and the method of QCD sum rules.
1 Introduction

The remarkable progress made in a last few decades in the study of dynamical systems in classical mechanics has stimulated a renewed interest and revision of fundamentals of quantum mechanics. In particular, a number of efforts has been undertaken to elucidate possible manifestations of a (semi-) quantum chaos which, due to the correspondence principle, is expected to persist in this or that form in quantum mechanics, and to translate the concepts of non-linear classical mechanics to the quantum language (see e.g. [1] for a review). One may hope, at the same time, that powerful calculation schemes of classical mechanics, being respectively transformed, could yield new methods supplementing more traditional tools of quantum physics such as the perturbation theory.

One such an approach is the Kolmogorov superconvergent method of classical mechanics [2] which has been the working instrument in the proof of the celebrated KAM theorem [3]. In a shortened form, the main Kolmogorov’s idea can be introduced as follows. Imagine we have added some non-integrable perturbation $\varepsilon V_1(I, \theta)$ to a integrable Hamiltonian $H_0(I)$ which is the function of the action variables only. It has been known for a long time [4] that successive canonical transformations $(I, \theta) \rightarrow (I_1, \theta_1) \rightarrow \cdots \rightarrow (I_n, \theta_n)$ are able to lower the formal order of the perturbation by one order of the small parameter $\varepsilon$ at every step $\varepsilon V_1 \rightarrow \varepsilon^3 V_2 \rightarrow \cdots \rightarrow \varepsilon^n V_n$. However, this procedure generally results in a diverging asymptotic series applicable only at fairly short time intervals. The divergence originates in a possibility for an existence of rational (and close to rational) resonant relations among the frequencies $\omega_i$ of the unperturbed motion $\sum_i m_i \omega_i = 0$ with a set of the integer $m_i$’s. This is the so-called small denominators problem (we refer the interested reader to the classical textbooks [5], [6] and the excellent review paper [7] on the whole subject of the classical perturbation theory and the KAM theorem). As any irrational number can be approximated to an arbitrary accuracy by rational numbers, one can readily imagine the scale of difficulties to be met with. It was Kolmogorov [2] who has managed to show the first time that successive canonical transformations may be chosen such that the order of the perturbation is increased by the square of the proceeding one for each step: $\varepsilon V_1 \rightarrow \varepsilon^3 V_2 \rightarrow \varepsilon^4 V_3 \rightarrow \cdots \rightarrow \varepsilon^{n+1} V_n$. This phenomenon of superconvergence occurs as a result of a clever redefinition of what is to be named "unperturbed" and "perturbing" Hamiltonian at each iteration step. This idea actually dates back to the well known Newton method for finding the root of the equation $f(x) = 0$. Instead of expanding the function into the Taylor series around a suspected point $x_0$, the Newton method suggests a simple first order iterative procedure of the form $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Thus, the value about which the expansion is made is moved closer to the true root at each iteration, that results in the superconvergence of the Newton method (more on the analogy between the Newton method and the Kolmogorov approach can be found in [6]). It turns out that owing to the superconvergence, the new perturbation theory is able to overcome the effect of near-to-resonance small denominators. Consequently, we end up with the convergent procedure, provided the initial conditions are adjusted at each step such that not to get just on a resonance surface [3],[5],[6],[7]. It has been shown [3] how to estimate from below the smallness of the resonant denominators that represent a set of frequencies of motion of a measure zero. The KAM theorem [2],[3] states that most non-resonant invariant tori are only slightly deformed by small smooth Hamiltonian perturbations and "form the majority in the sense that the measure of the complement of their union is small when the perturbation is small" [5].
In this paper we study a quantum analogue of the Kolmogorov technique. We find that, besides of the obvious aesthetic attractiveness (the superconvergence in the above sense), the novel scheme possesses other interesting properties. In particular, it will be shown that the constructed superconvergent procedure corresponds to an infinite resummation of the usual perturbative series and in some (non-trivial) cases yields a convergent quantum iterative scheme. Another observation is that the superconvergent method can be of use in cases when the usual perturbation theory fails. To exemplify this property, we work out so-called Yang-Mills quantum mechanics [8]. Besides being a model for an infra-red behavior of the YM fields [9], this system is of particular interest due to the fact that its classical counterpart possesses strong chaotic properties [10]. Note in this respect that the most popular nowadays approach to the quasiclassical quantization of classically chaotic systems, based on a summation of periodic orbit [1], may be not the best way in view of its formidable complexity (the number of periodic orbits grows exponentially with the energy in a system with hard chaos). Sometimes it can be expected that the corresponding purely quantum system is much easier to solve. Then, given a quantum solution, the quasiclassical limit can be followed up. Finally, we mention that though our presentation is restricted by quantum mechanical examples, the formalism may hopefully be extended to the field theory.

The presentation is organized as follows. In Sect.2 we solve the standard textbook problem of building the quantum mechanical perturbation theory in a somewhat unusual manner which elucidates the quantum superconvergence phenomenon. The practical inconvenience of this approach turns out to be similar to that of the classical perturbation theory based on canonical transformations (see below). Then in Sect.3 we present an equivalent but more convenient "non-coordinate" approach based on the so-called Lie transformation technique (see e.g [6]). In this form, the method admits (at least, formally) a generalization to many-dimensional cases. Its additional advantage is a formal equivalence between the classical and quantum perturbation theory. The Kolmogorov technique within the Lie transforms is further outlined. The proposed scheme is applied in Sect.4 to the classical anharmonic oscillator and the reconstruction of the perturbative series is demonstrated. The corresponding problem for the quantum anharmonic oscillator is studied in Sect.5. The obtained series is shown to be equivalent to the well known Padé series, thus providing convergent expansions for both small and strong couplings. We check by direct calculations that a re-expansion of the Kolmogorov series back into the standard power series reproduces the known results for the ground state energy. Finally, we apply in Sect.6 the superconvergent method to physically interesting YM quantum mechanics. Avoiding the small denominators to lowest orders is demonstrated. We show that the modified perturbation theory yields a controllable approximation to the behavior of the wave function near the origin. In a large fields region, we make use of another approximation and obtain an asymptotic adiabatic expansion in inverse powers of a field. This solution contains arbitrary constants which are to be determined from a matching with the small distance expansion. We make the approximate matching in two ways. The first one is essentially the least square matching over a "stability region" similar to the procedure used in the QCD sum rules method [11]. The second approach is a matching of the wave functions and their derivatives in a distribution sense (see below). Both methods yield similar results. The ground state energy is calculated with an accuracy of order 3.5 % while the accuracy of matching the wave functions is worse, of order 15 %. Sect.7 contains a brief summary and several concluding remarks.
2 Superconvergent perturbation theory

Let us suppose we know the solution to the unperturbed problem

\[ H_0 \phi_n^{(0)} = E_n^{(0)} \phi_n^{(0)} \]  

(1)

and we try to solve the perturbed problem

\[ (H_0 + \lambda W_0) \Psi_n = E_n \Psi_n \]  

(2)

We would like to solve this standard textbook’s problem by a nonstandard method following as close as possible the Kolmogorov’s idea of superconvergence (see Sect. 3 for the corresponding classical treatment). To this end, let us look for a solution in the form of a unitary transform

\[ \Psi_n = (e^{-\lambda S_1} \phi_n^{(1)})_n = \phi_n^{(1)} - \lambda \sum_{m \neq n} \langle n^{(1)} | S_1 | m^{(1)} \rangle \phi_m^{(1)} + \ldots \]  

(3)

where \( S_1 \) is an anti-hermitean operator \( S_1^* = -S_1 \). We try to choose \( S_1 \) in such way that the resulting equation for \( \phi_n^{(1)} \) will be in some sense simpler than the initial equation (2). We obtain

\[ e^{\lambda S_1} (H_0 + \lambda W_0) e^{-\lambda S_1} \phi_n^{(1)} = E \phi_n^{(1)} \]  

(4)

or

\[ (H_0 + \lambda (W_0 + [S_1, H_0]) + \lambda^2 W_2 + \lambda^3 W_3 + \lambda^4 W_4 + O(\lambda^5)) \phi_n^{(1)} = E \phi_n^{(1)} \]  

(5)

where

\begin{align*}
W_2 &= [S_1, W_0] + \frac{1}{2} [S_1, [S_1, H_0]] \\
W_3 &= \frac{1}{2} [S_1, [S_1, W_0]] + \frac{1}{3!} [S_1, [S_1, [S_1, H_0]]] \\
W_4 &= \frac{1}{3!} [S_1, [S_1, [S_1, W_0]]] + \frac{1}{4!} [S_1, [S_1, [S_1, [S_1, H_0]]]]
\end{align*}

(6)

Let us require that the operator \( W_0 + [S_1, H_0] \) has only diagonal matrix elements in the basis of the eigenfunctions of the operator \( H_0 \):

\begin{align*}
\langle n^{(0)} | W_0 + [S_1, H_0] | m^{(0)} \rangle &= 0, \ n \neq m \\
\langle n^{(0)} | [S_1, H_0] | n^{(0)} \rangle &= \langle n^{(0)} | W_0 | n^{(0)} \rangle
\end{align*}

(7)

It follows from (7) that

\[ \langle n^{(0)} | S_1 | m^{(0)} \rangle = \frac{\langle n^{(0)} | W_0 | m^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}}, \ n \neq m \]  

(8)

If we omit the terms \( O(\lambda^2) \) and higher, we thus obtain the usual formulas of the first order perturbation theory. Indeed, to this accuracy the exact solution to the equation (6) is just the unperturbed initial wave function: the equation

\[ H_1 \phi_n \equiv (H_0 + \lambda (W_0 + [S_1, H_0])) \phi_n = E_n \phi_n \]  

(9)

has the solution

\[ \phi_n = \phi_n^{(0)}, \ E_n = E_n^{(0)} + \lambda \langle n^{(0)} | W_0 | n^{(0)} \rangle \]  

(10)
and thus the solution to our problem to the given accuracy is

$$\Psi_{n}^{(1)} = \phi_{n}^{(1)} + \lambda \sum_{m \neq n} \frac{\langle m^{(1)}|W_{n}|n^{(1)} \rangle}{E_{n}^{(1)} - E_{m}^{(1)}} \phi_{m}^{(1)}$$

(11)

Let us now try to make more precise calculations. The difference from the textbook’s solution comes due to the fact that we retain the prescription (8). Then there is nothing to prevent one treating the Hamiltonian (9) as the new unperturbed Hamiltonian in the new perturbation problem

$$\left( H_{1} + \lambda^{2}W_{2} + \lambda^{3}W_{3} + \lambda^{4}W_{4} + O(\lambda^{5}) \right)\phi^{(1)} = E\phi^{(1)}$$

(12)

It is very important to notice that the order of the perturbation in (12) is $\lambda^{2}$ (and higher powers). The term $O(\lambda)$ hidden in the Hamiltonian $H_{1}$ is no more the small parameter. This is in a sense “another $\lambda$” as it is now a part of the exactly solvable Hamiltonian $H_{1}$. One may temporary call it e.g. $\rho$ in order not to confuse it with the real small parameter $\lambda^{2}$. As we will see in a moment, this redefinition leads to an infinite resummation of the Rayleigh - Schrodinger perturbation series.

We would like now to make again a canonical transformation such that the transformed operators $W_{2}, W_{3}$ etc. become diagonal in the basis of the eigenstates of the Hamiltonian $H_{1}$. Obviously, this could be done with the choice $\phi^{(1)} = exp(-\lambda^{2}S_{2})\phi^{(2)}$, then the term $W_{2}$ is diagonalized. However, the term $W_{3}$ can be also diagonalized at the same step. To this end, we have to choose

$$\phi^{(1)} = e^{-\lambda^{2}S_{2}-\lambda^{3}S_{3}}\phi^{(2)}, \ (S_{2}^{+} = -S_{2}, S_{3}^{+} = -S_{3})$$

(13)

Let us show that with this choice we arrive at independent equations for the operator $S_{2}, S_{3}$ which are as simple as eq. (7). (an attempt to diagonalize also the term $\lambda^{4}W_{4}$ would break this property). We obtain the following equation for the wave function $\phi^{(2)}$:

$$\left( H_{1} + \lambda^{2}(W_{2} + [S_{2}, H_{1}]) + \lambda^{3}(W_{3} + [S_{3}, H_{1}]) \right)$$

$$+ \lambda^{4}(W_{4} + [S_{2}, W_{2}] + \frac{1}{2}[S_{2}, [S_{2}, H_{1}]] + O(\lambda^{5}))\phi^{(2)} = E\phi^{(2)}$$

(14)

Proceeding analogously to (7) (and taking into account (10)), we obtain

$$\langle n^{(1)}|S_{2}|m^{(1)} \rangle = \frac{\langle n^{(1)}|W_{2}|m^{(1)} \rangle}{E_{n}^{(1)} - E_{m}^{(1)}}, \ \langle n^{(1)}|S_{3}|m^{(1)} \rangle = \frac{\langle n^{(1)}|W_{3}|m^{(1)} \rangle}{E_{n}^{(1)} - E_{m}^{(1)}}, \ n \neq m$$

(15)

(The tacit assumption we have done here is that the new perturbation problem in not degenerate. If this is not the case, our procedure terminates, and the degenerate perturbation theory must be used. However, the degeneracy is not possible in the one-dimensional case, provided the operators $H_{1}, W_{i}$ are self-adjoint.)

These canonical transformations can be continued. For example, at the third step the terms up to $\lambda^{7}$ will be diagonalized in respect to the eigenbasis of the Hamiltonian

$$H_{2} = H_{1} + \lambda^{2}W_{2} + \lambda^{3}W_{3}$$

(16)

where $W_{2}, W_{3}$ are the operator in front of the powers $\lambda^{2}, \lambda^{3}$ in (14). What we have obtained is just the quantum version of the superconvergent procedure by Kolmogorov (see
Sect.1 and Sect.3). A few comments are now in order.

1) Transferring the averaged terms from the perturbation to the new unperturbed Hamiltonian is the most non-trivial part of our scheme. This is the precise analogue of the Newton method and the Kolmogorov approach in classical mechanics. The only difference is in the averaging procedure. We will see in the next sections that this quantum averaging can be done formally identical to the classical one.

2) The suggested method corresponds to the infinite resummation of the usual perturbative series. Let us illustrate this obvious observation on the example of the $O(\lambda^4)$ results. We obtain from (13), (14):

$$\phi_n^{(1)} = \phi_n^{(0)} + \lambda^2 \sum_{m \neq n} \frac{\langle m^{(0)} | W_2 + \lambda W_3 | n^{(0)} \rangle}{E_n^{(1)} - E_m^{(1)}} + \ldots$$ (17)

We would like to emphasize once again that while we expand the nominator to the given accuracy, retaining of all powers of $\lambda$ in the denominator is completely legitimate, as they are coming from the exactly solvable problem with the Hamiltonian $H_1$. Let us also show that rational functions of the coupling constant appear in the formulas for the energy levels starting from the third step. The equation to be solved is

$$(H_2 + \lambda^4 \hat{W}_4 + \ldots)\phi^{(2)} = E\phi^{(2)}$$ (18)

Proceeding as previously, we substitute $\phi^{(2)} = \exp (-\lambda^4 S_4 + \ldots)\phi^{(3)}$. Then the corrections to the energy of order $O(\lambda^4)$ are expressed in terms of the matrix elements of the operator $\hat{W}_4$ (see (14)). Thus we immediately observe an appearance of $\lambda$ in denominators owing to (15). In effect, we get automatically a Padé-type series for the energy levels. It will be shown in Sect.5 that the corresponding series is convergent for the quantum anharmonic oscillator.

3) It can happen that at some step our equation for the operators $S_i$ will not possess anti-hermitean solutions, but instead will give raise to some symmetric operators. Then a care must be taken to define self-adjoint extensions of operators at hand since otherwise the eigenvalues would not be preserved by a new canonical transformation.

4) The practical inconvenience of the method as it stands is related to the necessity to re-calculate at every step the matrix elements from one to another basis. This problem is quite similar to an analogous problem in the classical perturbation theory [6] where the generating function is a function of both the old and new variables while the new Hamiltonian must be expressed as a function of the new variables only. An elegant solution to this problem consists in proceeding to a "non-coordinate" description of the canonical transformations which is called the Lie transformation method (see Sect.3). More important is the fact that in such a form, the method admits a straightforward generalization to an arbitrary dimensional case.


3 Superconvergence with Lie transforms

The classical perturbation theory reduces the formal order of the perturbation by virtue of successive canonical transformations chosen such that to eliminate the phases (this procedure is to be done with preventing secular terms, see [5], [6]). A generating function $S(I_1, \theta, t)$ contains the old ($\theta$) and the new ($I_1$) variables, thus a relation between the new Hamiltonian $H(I_1, \theta, t)$ and the old Hamiltonian $H(I, \theta, t)$ appears also in a mixed representation:

$$R(I_1, \theta, t) = H(I, \theta) + \frac{\partial S(I_1, \theta, t)}{\partial t}$$  \hspace{1cm} (19)

where

$$I = \frac{\partial S}{\partial \theta} \text{ and } \theta_1 = \frac{\partial S}{\partial I_1}$$  \hspace{1cm} (20)

Eq. (19) is just the relation between the values of two Hamiltonians at corresponding points in the phase space. In order to convert it into a relation between the functions, one has to solve the functional equations (20). This is not very convenient when one goes beyond the leading order. An elegant modified version has been proposed by Hori and Deprit (see [6], [12] and references therein). In this version the transformation from the old canonical variables $x = (p_1, q_1)$ to the new canonical $\bar{x} = (p_1, \bar{q}_1)$ is given by a Lie generator (rather than generating function) $w(x, \lambda) = w_1 + \lambda w_2 + \ldots$ such that the transformation $x \rightarrow \bar{x}(x, \lambda)$ is just the shift by the "time" $\lambda$ along the trajectories of the system with the "Hamiltonian" $w$:

$$\frac{d\bar{x}}{d\lambda} = [\bar{x}, w]$$  \hspace{1cm} (21)

(we use the same marking for the Poisson bracket in (21) as for commutators in Sect.2 because, as will be clear soon, the formal structure of the corresponding classical transformation turns out to be identical to that of the quantum problem). One further introduces the Lie operator $L$, simply related to the vector field $X_w$ associated with the Hamiltonian $w$:

$$L = [w, \ ] = \sum_k \left( \frac{\partial w}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial w}{\partial p_k} \frac{\partial}{\partial q_k} \right) \equiv -X_w$$  \hspace{1cm} (22)

and the evolution operator $T$ which transforms any function $g$ at the new point $\bar{x}(x, \lambda)$ into another function $f$ at the original point $x$:

$$f[x] = g[\bar{x}(x, \lambda)] \iff f = Tg \ ; \ x = T\bar{x}$$  \hspace{1cm} (23)

The operator $T$ is exactly what is needed in order to convert (19) into a functional equation since (23) means, in particular, that formally

$$\Pi = T^{-1} H$$  \hspace{1cm} (24)

(we are only interested here in time-independent problems. For non-autonomous systems, the form (24) is incorrect, see [6], [12]). The operator solution for $T$ can be readily deduced from the above formulas; from (21) and (23) we obtain

$$\frac{dT}{d\lambda} = -TL \iff T = \exp[- \int^\lambda L(\lambda')d\lambda']$$  \hspace{1cm} (25)
To see the formal similarity of the presented formulas with those of quantum canonical transformation [13], let us retain for simplicity only the leading term of the power expansion $L = L_1 + \lambda L_2 + \ldots$ in (24). Then (see (35) below)

$$\bar{x} = T^{-1} x = (1 + \lambda L_1 + \frac{1}{2} \lambda^2 L_1^2 + \ldots) = e^{\lambda L_1} x$$

(26)

The inverse transformation is

$$x = T \bar{x} = (1 - \lambda L_1 + \frac{1}{2} \lambda^2 L_1^2 - \ldots) \bar{x} = e^{-\lambda L_1} \bar{x}$$

(27)

The Hamiltonian equations of motion written via the Hamiltonian vector field $X_H$

$$\frac{dx}{dt} = X_H x \equiv - \sum_k \left( \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} \right) x$$

(28)

now read (we suppose $\lambda$ to be independent of time)

$$e^{-\lambda L_1} \frac{d\bar{x}}{dt} = X_H e^{-\lambda L_1} \bar{x}$$

(29)

or

$$\frac{d\bar{x}}{dt} = X_{\bar{H}} \bar{x}$$

(30)

where

$$X_{\bar{H}} = e^{\lambda L_1} X_H e^{-\lambda L_1} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [L_1, [L_1, \ldots, [L_1, X_H]]]$$

(31)

(the brackets mean the commutation operation when stand with vector fields)

Using the identity $[X_w, X_H] = -X_{[w, H]}$ [5], we obtain from (31) the transformed Hamiltonian

$$\bar{H} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [w, [w, \ldots, [w, H]]]$$

(32)

The formal identity with the quantum mechanical formulas becomes quite transparent. All formulas of this section will be true in the quantum case, provided one substitutes "Poisson brackets" — "quantum commutators" and "functions on the phase space" — "operators in the Hilbert space".

To get the perturbative series, one further expands

$$w = \sum_{n=0}^{\infty} \lambda^n w_{n+1} ; \quad L = \sum_{n=0}^{\infty} \lambda^n L_{n+1}$$

$$T = \sum_{n=0}^{\infty} \lambda^n T_n ; \quad T^{-1} = \sum_{n=0}^{\infty} \lambda^n T_{n}^{-1}$$

$$H = \sum_{n=0}^{\infty} \lambda^n H_n ; \quad \bar{H} = \sum_{n=0}^{\infty} \lambda^n \bar{H}_n$$

(33)

where

$$L_n = [w_n, \ ] ; \quad T_0 = 1 ; \quad T^{-1}_0 = 1$$

(34)

Then a simple algebra leads to the recursion relation (see [6] )

$$T_{n}^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} T_{m}^{-1}$$

(35)
and, as the consequence of (24), \( n > 0 \)

\[
D_n w_n = n(R_n - H_n) - \sum_{m=1}^{n-1} (L_{n-m} R_m + m T_{n-m}^{-1} H_m),
\]

(36)

where

\[
D_n = [ , H_n] \quad \text{and} \quad R_n = H_n \tag{37}
\]

To the fourth order, the above equation yields

\[
D_0 w_1 = \bar{R}_1 - H_1 \tag{38}
\]

\[
D_0 w_2 = 2(\bar{R}_2 - H_2) - L_1(\bar{R}_1 + H_1) \tag{39}
\]

\[
D_0 w_3 = 3(\bar{R}_3 - H_3) - L_1(\bar{R}_2 + 2H_2) - L_2(\bar{R}_1 + \frac{1}{2}H_1) - \frac{1}{2}L_1^2 H_1 \tag{40}
\]

\[
D_0 w_4 = 4(\bar{R}_4 - H_4) - (\frac{1}{3}L_3 + \frac{1}{6}L_1 L_2 + \frac{1}{3}L_2 L_1 + \frac{1}{6}L_1^2) H_1 - L_3 \bar{R}_1 - (L_2 + L_1^2) H_2 - L_1 R_3 - 3L_1 H_3 - L_1 R_2 \tag{41}
\]

The standard (von Zeipel's) perturbation theory is reproduced in this way as follows.\ \ Given \( H_1 \), one chooses \( \bar{R}_1 \) such that secularities in the r.h.s. of (38) are eliminated, and then finds \( w_1 \). At second order, one substitutes the found \( w_1 \) into (39) and looks for a \( w_2 \) which eliminates secularities in the r.h.s. of (39), etc.

As has been stressed in [6], the Lie transforms formalism is more convenient for elucidating the superconvergence than the standard method of canonical transformations. The only difference from the above described line of reasoning is a different choice of the Lie generators \( w_i \).

The principal rule is exactly parallel to the method used by us in Sect.2: successive new Hamiltonians are obtained by successive Lie transforms while equations for different \( w_n \) and \( w_m \) \( ( \text{the superscript} \ (i) \ \text{will generally indicate the number of the iteration}) \) must be independent of each other. At each step, chosen as many as possible independent \( w^{(i)} \) 's, all remaining \( w^{(i)} \) 's are set equal to zero. We describe two steps of the Kolmogorov scheme following [6].

At the first step, there is only one equation (38). To kill the secular term, we set \( \bar{R}_1 = < H_1 > \) \(< > \) here denotes averaging over the "fast" angle variables. In Sect. 4 we will, however, follow another averaging procedure \}. Then \( w_1 \) 's is determined from

\[
D_0 w_1 = -\{H_1\} \quad ; \quad w_1 = 0, \quad i > 1 \tag{43}
\]

where \{ \} denotes the oscillating part. All others \( R_i^{(1)} \) are given by (39 - 42) with the constraint (43). One finds

\[
\bar{R}_2^{(1)} = \frac{1}{2}[w_1^{(1)}, (\bar{R}_1^{(1)} + H_1)] \tag{44}
\]

\[
\bar{R}_3^{(1)} = \frac{1}{3}[w_1^{(1)}, (\bar{R}_2^{(1)} + \frac{1}{2}[w_1^{(1)}, H_1])] \tag{45}
\]

One can show that the general recursion formula reads \( n > 2 \)

\[
R_{n+1}^{(1)} = \frac{2}{n+1}[w_1^{(1)}, (R_n^{(1)} - \frac{1}{2n}[w_1^{(1)}, R_{n-1}^{(1)}])] \tag{46}
\]
At the second step, we absorb the averaged (i.e., solvable) part $\lambda < H_1 >$ into the zero
order part of the new "old Hamiltonian" $H^{(1)}$:

$$
H^{(1)}_0 = \tilde{H}^{(1)}_0 + \lambda < H_1 > \\
H^{(1)}_i = 0 \\
H^{(1)}_i = \tilde{H}^{(1)}_i , \; i > 1
$$

(47)

After the second transformation $w^{(2)}$, we obtain a new Hamiltonian $\tilde{H}^{(2)}$ with $\tilde{H}^{(2)}_0 = H^{(1)}_0$.
As there is no first-order perturbation, we may choose $w^{(2)}_i = 0$. Thus, we obtain from (39 -42) two independent equations for $w^{(2)}_3$:

$$
D^{(1)}_0 w^{(2)}_3 = 2(\tilde{H}^{(2)}_3 - H^{(1)}_3) \\
D^{(1)}_0 w^{(2)}_3 = 3(\tilde{H}^{(2)}_3 - H^{(1)}_3)
$$

(48)

(49)

while

$$
D^{(1)}_0 w^{(2)}_4 = 4(\tilde{H}^{(2)}_4 - H^{(1)}_4) - [w^{(2)}_3, (\tilde{H}^{(2)}_3 + H^{(1)}_3)]
$$

(50)

where the new time derivative is calculated in respect to the zero order new "old" Hamiltonian:

$$
D^{(1)}_0 = [ \; , H^{(1)}_0 ]
$$

(51)

The equation (48),(49) are solved simultaneously choosing $\tilde{H}^{(2)}_3$ and $\tilde{H}^{(2)}_3$ to eliminate
secularities in $w^{(2)}_3$ and $w^{(2)}_3$, as has been done in (43). All $w^{(2)}_i$, $i > 3$ are set equal to
zero. Then the next terms in the transformed Hamiltonian are

$$
\tilde{H}^{(2)}_4 = H^{(1)}_4 + \frac{1}{4}[w^{(2)}_2, \tilde{H}^{(2)}_2 + H^{(1)}_2] \\
\tilde{H}^{(2)}_n = H^{(1)}_n + \frac{1}{n} \sum_{m=1}^{n-1} (I^{(2)}_{n-m} \tilde{H}^{(2)}_m + mT^{(2)}_{n-m} H^{(1)}_m)
$$

(52)

(53)

At the third step, the new "old Hamiltonian" is built analogously to (47):

$$
H^{(2)}_0 = \tilde{H}^{(2)}_0 + \lambda^2 < H^{(1)}_2 > + \lambda^3 < H^{(1)}_3 > \\
H^{(2)}_i = 0 , \; i = 2 , 3 \\
H^{(2)}_i = \tilde{H}^{(2)}_i , \; i > 3
$$

(54)

This iterative procedure can be continued. It can be shown by induction [6] that at the
nth step, the terms from $i = 2^n$ to $i = 2^{n+1} - 1$ are diagonalized. This completes the
description of the Kolmogorov method within the Lie transforms formalism.

It should be mentioned that, while being extremely important theoretically (for the proof
of the KAM theorem), the Kolmogorov method did not get much practical applications.
The only known to the author example has been given by Chirikov [7] who has shown
how to make the first Kolmogorov's step in the problem of the classical one-dimensional
pendulum using the usual formalism of canonical transformations. But, as we have seen
in Sect.2, really interesting phenomena within the Kolmogorov method start only from
the second step. It is also worth noting that the notion of superconvergence has been used
in a different content in [13] in treating the Birkhoff-Gustavson normal form.
4 Classical anharmonic oscillator

For this one-dimensional classical problem, the use of the perturbation theory is nothing but a formal exercise. Indeed, the corresponding equation of motion can be solved in terms of elliptic functions that indicates that a perturbative series is convergent. Obviously, this convergence is related to the absence of resonances in any one-dimensional system. The perturbative series in this case is a formal expansion of the Hamiltonian in powers of the harmonic oscillator Hamiltonian. A final answer is usually expressed in the form of an expansion of the frequency of non-linear oscillations in powers of the energy. We will study this problem merely for the sake of comparison and working out reliable tools for treating the corresponding quantum problem.

We start with the Hamiltonian

\[ H(p, x) = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2 + \lambda \omega^2 x^4 \]  

(55)

It is very convenient to proceed from the canonical variables \((p, x)\) to the new canonical complex-conjugate variables

\[
\begin{align*}
    z &= \frac{1}{\sqrt{2\omega}} (\omega x + ip) \\
    \bar{z} &= \frac{1}{\sqrt{2\omega}} (\omega x - ip)
\end{align*}
\]

(56)

Then

\[
\begin{align*}
    H &= H_0 + \lambda H_1 \\
    H_0 &= \omega \bar{z} z \\
    H_1 &= \frac{1}{4} (z + \bar{z})^4 = \frac{1}{4} (z^4 + 4z^3 \bar{z} + 6z^2 \bar{z}^2 + 4z \bar{z}^3 + \bar{z}^4)
\end{align*}
\]

(57)

In the complex variables the Poisson bracket takes the form

\[
[f, g] = -i \left\{ \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} \right\}
\]

(58)

The advantage of proceeding to \((z, \bar{z})\) and not to the action-angles variables is three-fold: (1) it makes quantum mechanical formulas written in the second quantized form very similar to classical ones; (2) simple explicit formulas for the Lie generators to all orders can be given; and (3) the averaging procedure in the complex variables becomes much simpler in comparison with averaging in the action-angle variables. Instead of integration over the angles, the averaging is simply given by picking up terms with equal powers of \(\bar{z}\) and \(z\) in a product. This procedure is obviously equivalent to the integration over the angle variables.

Proceeding along the line of Sect. 3, we obtain at the first step

\[
\begin{align*}
    \bar{R}_{0}^{(1)} &= H_0 = \omega \bar{z} z \\
    \bar{R}_{1}^{(1)} &= <H_1> = \frac{3}{2} \bar{z}^2 z^2
\end{align*}
\]

(59)

Solving eq. (43) yields

\[
\begin{align*}
    w_{1}^{(1)} &= \frac{1}{4\omega} \left( \frac{1}{4} \bar{z}^4 + 2 \bar{z}^3 z - 2 \bar{z} z^3 - \frac{1}{4} z^4 \right)
\end{align*}
\]

(60)
The transformed Hamiltonian is calculated according to (44),(45) (46):

\[
R_2^{(1)} = \frac{1}{8\omega}\left[-34\xi^3 \bar{z}^3 + (\xi^6 - 6\xi^5 z - 33\xi^4 z^2 + (c.c.))\right]
\]

(61)

\[
R_3^{(1)} = \frac{1}{96\omega^3}\left[2250\xi^3 z^4 + (33\xi^6 + 72\xi^5 z + 348\xi^4 z^2 + 1272\xi^3 z^3 + (c.c.))\right]
\]

(62)

\[
R_4^{(1)} = \frac{3}{256\omega^3}\left[-5244\xi^3 z^5 + (37\xi^6 + 10\xi^5 z - 435\xi^4 z^2 - 1760\xi^3 z^3
- 4210\xi^2 z^4 + (c.c.))\right]
\]

(63)

where (c.c.) means the complex conjugation. The lengthy intermediate calculations have been made with the help of the Mathematica package of analytic calculations. Proceeding to the second step, the new "old" Hamiltonian becomes

\[
H_{^{1}}^{(1)} = \omega \xi z + \frac{3}{2} \xi^3 \bar{z}^2
\]

\[
H_{^{1}}^{(1)} = 0
\]

\[
H_{^{1}}^{(1)} = R_{i}^{(1)}, \quad i > 1
\]

(64)

We choose

\[
R_{2}^{(1)} = < H_{2}^{(1)} > = \frac{17}{4\omega}\xi^3 \bar{z}^3
\]

(65)

\[
R_{3}^{(1)} = < H_{3}^{(1)} > = \frac{375}{16\omega}\xi^4 \bar{z}^4
\]

(66)

The equations (48),(49) are of the form

\[
[w_{n}^{(2)}, H_{0}^{(1)}] = \omega [w_{n}^{(2)}, \xi z + \frac{3\lambda}{2\omega} \xi^3 \bar{z}^2] = -n\{H_{n}^{(1)}\}
\]

(67)

To solve these functional equations, we use the following formula

\[
[w, H_{0}^{(1)}] = i\omega (m - n) \xi^n \bar{z}^m
\]

(68)

which can be checked by the direct calculation. Then answers for \(w_{2}^{(2)}, w_{3}^{(2)}\) read

\[
w_{2}^{(2)} = i \frac{1}{4\omega} \frac{1}{1 + \frac{3\lambda}{\omega} \xi^3 \bar{z}^2} \left[\frac{1}{6} \xi^6 - \frac{3}{2} \xi^5 z - \frac{33}{2} \xi^4 z^2 + (c.c.)\right]
\]

(69)

\[
w_{3}^{(2)} = i \frac{1}{48\omega^3} \frac{1}{1 + \frac{3\lambda}{\omega} \xi^3 \bar{z}^2} \left[\frac{33}{8} \xi^8 + 12 \xi^7 z + 87 \xi^6 z^2 + 636 \xi^5 z^3 + (c.c.)\right]
\]

(70)

We give the final form of the Hamiltonian to the \(O(\lambda^7)\) accuracy (Here \(I\) stands for the product \(\bar{z} z\) ) :

\[
H_{^{[3]}} = \omega I + \frac{3}{2} \lambda I^2 - \frac{17}{4} \lambda^2 \omega^2 I^3 + \frac{375}{16} \lambda^3 \omega^3 I^4
\]

(71)

\[
- \frac{15}{320} \lambda^4 \omega^4 I^5 \left(1 + \frac{3}{2} \omega I\right)^2
\]

\[
+ \frac{1}{320} \lambda^5 \omega^4 I^6 \left(1 + \frac{3}{2} \omega I\right)^3
\]

\[
- \frac{1}{512} \lambda^6 \omega^4 I^7 \frac{4239854 + 3380451 \frac{1}{2} I + 9391282 \frac{1}{4} \omega I + 99162117 \frac{1}{4} \omega^2 I^2 + 22951350 \frac{3}{8} \omega^3 I^3 + 22951350 \frac{3}{8} \omega^3 I^3}{(1 + \frac{3}{2} \omega I)^4}
\]
Obviously, at the next steps we will get the successive zero order Hamiltonians in the form of rational functions of $\frac{\tilde{z}}{I}$. The corresponding $w^{(i)}$s can be calculated using the following generalization of (68):

$$\left[ \frac{\tilde{z}^m \hat{z}^n}{P(\tilde{z})}, Q(\tilde{z}) \right] = -i(n-m)\tilde{z}^m \hat{z}^n \frac{Q'(\tilde{z})}{P(\tilde{z})}$$

where $P(y), Q(y)$ are arbitrary functions, $P(y) \neq 0$ and the derivative $Q'(y)$ exists. To solve the equation

$$\left[ w(\tilde{z}, z), Q(\tilde{z}) \right] = \frac{\tilde{z}^m z^n}{P_1(\tilde{z})}$$

appearing in higher orders, we have to choose in (72)

$$P(\tilde{z}) = Q'(\tilde{z}) P_1(\tilde{z})$$

The equations (72-74) in principle solve the problem of finding the corresponding Lie generators $w^{(i)}$ at any order. There are a few lessons to be learnt from the above calculation. In the second quantization representation, which is the relevant arena for building a corresponding quantum scheme, one can expect somewhat analogous to (71) expression for an approximate quantum Hamiltonian, the action $I$ being substituted by the particle number operator $N = a^\dagger a$, i.e. a kind of the operator Padé expansion. The obtained Hamiltonian has to be checked for being self-adjoint. Furthermore, to proceed beyond the first step, quantum substitutes of eq. (68), (72) are needed.

5 Quantum anharmonic oscillator

The quantum anharmonic oscillator is the standard test problem for any new quantum field theory - oriented method. A main feature, that the anharmonic oscillator shares with field - theoretical problems, is the zero radius of convergence of the Rayleigh-Schrödinger perturbation series [14] which prevents an extension of perturbative results into the strong coupling region. There has been a number of attempts to build strong coupling expansions meant to be generalizable to the field theory, refs. [15] may probably be mentioned among the most interesting ones.

Intuitively, the divergence of the perturbative series can be understood as a result of the inability to reproduce a behavior in the vicinity of the essential singularity at the origin (the Dyson singularity) in terms of an expansion in powers of the coupling constant. However, it is well known that the convergence can be gained after the perturbative series is resumed, the most popular resummation methods are due to Padé and Borel, see e.g. [16] for a review. In this section we would like to argue that the quantum superconvergence leads to the uniform convergence of the corresponding series for both small and strong couplings due to the fact that our results have the form of the (operator and functional) Padé series.

For the quantum version of the Hamiltonian (55) (with $\omega = 1$) we introduce the creation and annihilation operators:

$$a = \frac{1}{\sqrt{2\hbar}}(\hat{x} + i\hat{p}) \; , \; \bar{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x} - i\hat{p})$$

(75)
and the particle number operator $N = \bar{a}a$ satisfying the commutation relations

$$\left[a, \bar{a}^{\dagger}\right] = n\bar{a}^{n-1}, \quad \left[a^{\dagger}, \bar{a}\right] = na^{n-1}$$  \hspace{1cm} (76)

The unperturbed and perturbing Hamiltonians are

$$H_0 = \hbar(\bar{a}a + \frac{1}{2})$$  \hspace{1cm} (77)

$$H_1 = \frac{\hbar}{4} \left[3(1 + 4\bar{a}a + 2\bar{a}^3a^2) + (\bar{a}^4 + 4\bar{a}^3a + 6\bar{a}^2 + (\hbar c.))\right]$$

where (\hbar c.) means the hermite conjugation. Proceeding analogously to the classical case, we obtain $H_1^{(1)} = <H_1>$ and then

$$w_1^{(1)} = \frac{\hbar}{4}(2\bar{a}^4 + 2\bar{a}^3a + 3\bar{a}^2 - (\hbar c.))$$  \hspace{1cm} (78)

The transformed Hamiltonian, according to (44-46), is

$$H_2^{(1)} = \frac{\hbar^3}{16}[-42 - 288\bar{a}a - 306\bar{a}^3a^2 - 6\bar{a}^3a^3 + \ldots]$$

$$H_3^{(1)} = \frac{\hbar^4}{16}[333 + 3582\bar{a}a + 6291\bar{a}^3a^2 + 3000\bar{a}^3a^3 + 375\bar{a}^4a^4 + \ldots]$$  \hspace{1cm} (79)

$$H_4^{(1)} = \frac{\hbar^5}{256}[-26649 - 390366\bar{a}a - 977562\bar{a}^3a^2 - 741636\bar{a}^3a^3 - 196640\bar{a}^4a^4 - 15732\bar{a}^5a^5 + \ldots]$$

where we have explicitly shown only the diagonal parts of the operators, the complete forms are rather lengthy. Proceeding to the second step, we add the diagonal part of the perturbation $H_1$ to the unperturbed Hamiltonian $H_0$ to get the new "unperturbed Hamiltonian", just as we have done in the classical treatment:

$$H_0^{(1)} = \hbar(N + \frac{1}{2}) + \frac{3\lambda\hbar^2}{4}(1 + 2N + 2N^2)$$

$$H_1^{(1)} = 0$$

$$H_i^{(1)} = H_i^{(1)}, \quad i > 1$$  \hspace{1cm} (80)

At the second step, the quantum analogue of eq. (67) can be written as

$$[N + kN^2, \ w_2^{(2)}] = \frac{2(1 - k)}{\hbar}\{H_2^{(1)}\}$$  \hspace{1cm} (81)

where we have denoted

$$k = \frac{3\lambda\hbar}{1 + \frac{3\lambda\hbar}{2}}$$  \hspace{1cm} (82)

The quantum version of eq. (68) reads

$$[N + kN^2, \ \frac{1}{1 - k(n - m) + 2kN}\bar{a}^m] = (n - m)\bar{a}^m$$  \hspace{1cm} (83)

The above formula yields the following answer for the Lie generator $w_2^{(2)}$

$$w_2^{(2)} = \frac{(1 - k)\hbar^2}{8}\left[\frac{1}{3} \frac{1}{1 - 6k + 2kN}\bar{a}^6 - \frac{15}{2} \frac{1}{1 - 4k + 2kN}\bar{a}^4 - \frac{171}{2} \frac{1}{1 - 2k + 2kN}\bar{a}^2 \right. $$

$$- \frac{132}{1 - 2k + 2kN}\bar{a}^3a - \frac{3}{1 - 4k + 2kN}\bar{a}^5a - \frac{33}{1 - 2k + 2kN}\bar{a}^4a^2 - (\hbar c.)]$$  \hspace{1cm} (84)

13
The result for the Lie generator $w_3^{(2)}$ has a similar structure. To calculate higher order terms of the transformed Hamiltonian, one needs a few more commutators which can be obtained as the simple consequences of the Hausdorff identity and Schwinger exponential representation of operators ($\alpha$ and $\beta$ are arbitrary $c$-numbers):

$$[\hat{a}^\alpha \hat{a}^\beta, \frac{1}{\alpha + \beta N}] = \left( \frac{1}{\alpha - \beta(n - m) + \beta N} - \frac{1}{\alpha + \beta N} \right) \hat{a}^\alpha \hat{a}^\beta$$

(85)

$$[a^\beta, \hat{a}^\alpha] = n! \hat{a}^{\alpha - m} \sum_{l=m, m-1}^{m-1} \frac{m!}{l! (m - l)! (n - m + l)!} \hat{a}^l$$

(86)

Using these commutators, one can easily check that the operator $w_2^{(2)}$ is indeed anti-hermitean. As the resulting expression for the second-iteration Hamiltonian is rather lengthy, we will retain below only the terms possessing non-zero vacuum expectation values:

$$H^{(3)} = \frac{\hbar}{2} + 3\lambda \hbar^2 - \frac{21}{8} \lambda^2 \hbar^3 + \frac{333}{16} \lambda^3 \hbar^4 - \frac{26649}{256} \lambda^4 \hbar^5 + \frac{\lambda^4 \hbar^5 (1 - k)}{1} \left[ \frac{29241}{16} \frac{1}{1 + 2k + 2kN} + \frac{675}{2} \frac{1}{1 + 4k + 2kN} + \frac{30}{1 + 6k + 2kN} \right]$$

$$+ \frac{444609}{512} \lambda^5 \hbar^6 + \frac{\lambda^5 \hbar^6 (1 - k)}{4} \left[ \frac{208791}{1 + 4k + 2kN} + \frac{32}{3} \right]$$

$$+ \frac{14985}{8} \frac{1}{1 + 4k + 2kN} - \frac{315}{1 + 6k + 2kN} \right] + \ldots$$

(87)

One can easily see that the perturbative expansion for the ground state energy [14]

$$E_\beta = \frac{\hbar}{2} + 3\lambda \hbar^2 - \frac{21}{8} \lambda^2 \hbar^3 + \frac{333}{16} \lambda^3 \hbar^4 - \frac{30885}{128} \lambda^4 \hbar^5 + \frac{916731}{256} \lambda^5 \hbar^6 + \ldots$$

(88)

is reproduced from our formula by re-expanding the rational fractions. For example, about 25% of the perturbative result at the $\lambda^5$ order is due to the expansion of our $O(\lambda^4)$ fraction. It seems plausible to suggest that the Kolmogorov series reproduces the usual perturbative result at all orders, though we do not get a general proof.

Heuristically, one can expect the convergence of the Kolmogorov series (87) for the ground state energy due to the fact that, the higher is the order of the perturbative coefficient, the larger number of fractions will contribute to its value. This statement can be given a formal proof. The resulting from our procedure fractions can be reorganized into rational functions of the coupling constant. A convenient choice is e.g., to collect into a single rational function all fractions resulting from one Kolmogorov’s step. Let us suppose that at some step we have obtained the answer in the form

$$P_{N_1, j_1}(\lambda) \equiv f^{[N_1, N_1, j_1]}(\lambda) = R^{[N_1, N_1, j_1]}(\lambda)$$

(89)

where $R^{[N]}(Q^{[M]})$ is a polynomial of degree $N$ ($M$), (note that in our formulas $j_1 > 0$) and at the next step the answer is of the form

$$P_{N_1, j_2}(\lambda) \equiv f^{[N_2, N_2, j_2]}(\lambda) \equiv R^{[N_1, N_2, j_2]}(\lambda)$$

(90)

Let us prove that $j_1 = j_2$. Suppose this were not true. As $P_{N_1, j_1}$, $P_{N_2, j_2}$ can be thought as the representatives of two series of the Padé approximants, there exist the limits [16]

$$f_{j_1}(\lambda) = \lim_{N \to \infty} f^{[N, N_1, j_1]}(\lambda) \quad f_{j_2}(\lambda) = \lim_{N \to \infty} f^{[N, N_2, j_2]}(\lambda)$$

(91)
and the convergence is uniform in the cut $\lambda$ plane. However, $f_{j_1} = f_{j_2}$ since a measure giving the perturbative coefficients in the form of its moments is unique for the anharmonic oscillator problem [16]. Then it means that the series $f^{[N,N+j_1]} - f^{[N,N+j_2]}$ converges uniformly to 0. As $j_1$, $j_2 > 0$, this is only possible when $j_1 = j_2$ (at very large $\lambda$ $f^{[N,N+j]} \sim \lambda^j$). The fact that $j_1 = j_2$ means that the Pade series obtained from Kolmogorov procedure is diagonal and therefore converges uniformly to the true eigenvalue [17]. (Note that we have explicitly checked that all poles resulting from the Kolmogorov series are located at real negative $\lambda$ for any energy level and the resulting Hamiltonian is self-adjoint.)

We would like to end up this section with a few comments. At any order, the eigenfunctions of a resulting effective Hamiltonian are the harmonic oscillator wave functions in proper coordinates. The proposed approach is essentially the method of a free field representation. Given the needed accuracy, the interaction is transferred to such canonical transformation that yields the free (i.e. harmonic oscillator) Hamiltonian. The perturbed wave function in the initial coordinates is given by the inverse transformation of the basis. The theory of linear canonical transformations in the Bargmann-Fock space is well studied [18]. As we deal with non-linear canonical transformations, more sophisticated methods are needed. We feel that a method suggested recently [19] can be of use here. Note also that the corresponding treatment in a many-dimensional case provides a non-linear extension of the famous Bogolyubov transformations.

The results obtained so far can be summarized as follows. In all cases where the usual perturbation theory is applicable, the superconvergent methods corresponds to a Pade-type resummation of the perturbative series. In the above studied case, the convergence of the Kolmogorov expansion is the consequence of general theorems on the Pade approximation [16]. In other problems, convergence properties have to be tested separately. Instead of discussing this topic, in the next section we will show how a similar re-organization of the perturbation theory allows one to treat approximately a more complicated system with two degrees of freedom, where the usual perturbation theory cannot be applied.

6 Yang-Mills quantum mechanics

Yang-Mills (YM) classical mechanics has been proposed [10] as a toy model for a study of non-perturbative nonlinear YM waves in the Minkowsky space. There one starts with fields in the SU(2) YM theory which depend only on the time (this is the reasonable assumption in the infra-red). Then, after the gauge fixing $A_a^i = 0$, the classical YM theory turns out to be essentially equivalent to the dynamical system with the Hamiltonian [10]

$$H = \frac{1}{2} \sum_{a=1}^{3} \left( \frac{df^a}{dt} \right)^2 + \frac{1}{2} \sum_{a=1}^{3} \sum_{b \neq a} (f^a)^2 (f^b)^2$$

(92)

where the dynamical variables $f^a$ parametrize the gauge fields $A_a^i$ ($O_a^i$ is a time-independent orthogonal matrix): $A_a^i = O_a^i f^a(t) / g$ (no sum over $a$) and $O_a^i O^{ai} = \delta^{ai}$

(93)

A simplified system obtained from (92) by fixing $f^3 = 0$ is still highly non-trivial and commonly called YM classical mechanics:

$$H_{YM} = \frac{1}{2} (p_1^2 + p_2^2) + \lambda x_1^2 x_2^2$$

(94)
In obtaining the Hamiltonian (94) leads to a very complicated chaotic motion [10]. It has been believed for a long time that the system (94) is ergodic [10]. More recently, small stability islands have been found [20]. There also has been a considerable interest in a quantum system with the Hamiltonian (94) [8], [9], [21], [22], [23] (note that (94) is not the homogeneous-space model of the quantum YM theory, but still is closely related to the latter [8]). The fact that a spectrum is discrete has been proved in [21]. Different methods, based on the quasiclassical and quantum adiabatic approximations [22] and summation of adiabatically stable periodic orbits [23], have been proposed for finding the spectrum and wave functions.

It is obvious beforehand that the usual perturbation theory is hopelessly inapplicable for treating the system (94). No matter how \( \lambda \) is small, the perturbation results in a reconstruction of the spectrum from continuous to the discrete one, the huge effect. On contrary, a modified perturbation scheme a la Kolmogorov can be proposed which allows one to build a controllable approximation to the WF near the origin, where the adiabaticity breaks down (see below).

Let us consider the "perturbation" in the second quantization representation (75) (we set \( \hbar = 1 \) and use \( z_i, \bar{z}_i \) to denote the quantum annihilation and creation operators):

\[
\lambda z_i^2 \bar{z}_i^2 = \lambda \frac{1}{4} [1 + 2N_1 + 2N_2 + 4N_1 N_2] + \lambda \frac{1}{4} \{ z_i^2 \bar{z}_i^2 + z_i^2 \bar{z}_i^2 \\
\quad + (1 + 2N_2) z_i^2 + (1 + 2N_1) \bar{z}_i^2 + (h.c.) \} \tag{95}
\]

In obtaining (95), we have used the non-commutativity of the quantum operators, that has allowed one to extract from the "perturbation" the part which is actually not a perturbation as it is diagonal in the oscillator basis. The expression denoted by \([...]\) in (95) must be treated as a part of an unperturbed Hamiltonian, if we intend to end up with a discrete spectrum. The fact that the "unperturbed Hamiltonian" is of the same order \( \lambda \) as the remaining "perturbation" is not an obstacle since after the first step the perturbation is only of order \( \lambda^2 \) (plus a small part of order \( \lambda \)). An additional problem arises due to the fact that both the unperturbed Hamiltonian and perturbation are degenerate so that the degenerate perturbation theory also cannot be used. We propose the following simple trick to handle this problem: let us just make both the unperturbed Hamiltonian and perturbation non-degenerate. We choose to transfer into the unperturbed Hamiltonian the terms \( 1 + 2N_1 + 4N_1N_2 \) while the term \( 2N_2 \) is retained as the part of the perturbation. The concrete choice \( N_1 \rightarrow N_2 \) does not matter as the difference may be thought as the one to be repaired by higher order corrections. In other words, we sacrifice the exact symmetry of the problem in favor of only approximate one. (It is known that such procedure can yield wrong results for the large distance behavior, but, as will be shown below, in this region a different approximation becomes operative). As will be seen in a moment, this suggestion indeed leads to avoiding small denominators at the lowest order. Moreover, in such form the Hamiltonian is suitable for working out the Kolmogorov scheme, as the new perturbation contains the part which has a non-zero average (the \( 2N_2 \) term).

Adding to (95) the kinetic term operators, we arrive at the Hamiltonian

\[
\begin{align*}
H &= H_0 + \varepsilon H_1 \\
H_0 &= \omega_1 N_1 + \omega_2 N_2 + \mu N_1 N_2 + \alpha \\
H_1 &= \sum_{i, j = 0} (2N_2 + \bar{z}_i^2 \bar{z}_j^2 f_{im} + \bar{z}_i^2 \bar{z}_j^2 f_{jm} + \bar{z}_i^2 \bar{z}_j^2 \bar{z}_i^2 \bar{z}_j^2 + \bar{z}_i^2 \bar{z}_j^2 g_{im} \bar{z}_j^2 + \bar{z}_i^2 \bar{z}_j^2 \bar{z}_i^2 \bar{z}_j^2 )
\end{align*} \tag{96}
\]
where

$$\omega_1 = \frac{1}{2} + \frac{\lambda}{2}, \quad \omega_2 = \frac{1}{2}, \quad \mu = \lambda$$

$$\alpha = \frac{1}{2} + \frac{\lambda}{4}, \quad \varepsilon = \frac{\lambda}{4}, \quad \gamma = 1 - \frac{1}{\lambda} \quad (97)$$

and the only non-zero \( f'k, g'k \) are

$$f_{02} = \gamma + 2N_1, \quad f_{20} = \gamma + 2N_2, \quad f_{22} = 1, \quad g_{22} = 1 \quad (98)$$

(Note that the Hamiltonian of so-called Yang-Mills-Higgs quantum mechanics [8], which differs from (94) by adding the mass term \( \frac{1}{2}\omega^2(x_1^2 + x_2^2) \), can be treated along with the same line with a proper redefinition of parameters. The interest in the corresponding classical system has been recently renewed in connection with a study of non-abelian collective excitations of the quark-gluon plasma [24].)

To find the first step Lie generator \( w_1^{(1)} \), it is convenient to look for a solution in a form analogous to (96) [25]:

$$w_1^{(1)} = \sum_{n,m=0} \left( \varepsilon^{n} \varepsilon^{m} a_{nm}^{(1)} - (a_{nm}^{(1)})^* \varepsilon^{n} \varepsilon^{m} + \varepsilon^{n} c_{nm}^{(1)} \varepsilon^{m} - z_1^{-n} (c_{nm}^{(1)})^* \varepsilon^{m} \right) \quad (99)$$

We further substitute (96),(99) into eq. (43) and equate the like powers of the quantum operators. The comparison of the coefficients provides

$$a_{nm}^{(1)} = f_{nm}[n\omega_1 + m\omega_2 + \mu(nN_2 + mN_1 + nm)]^{-1} \quad (100)$$

$$c_{nm}^{(1)} = g_{nm}[n\omega_1 - m\omega_2 + \mu(nN_2 - mN_1)]^{-1} \quad (101)$$

Note that the denominator in (101) never goes to zero. For the only nonvanishing \( g_{22} = 1 \) we obtain

$$c_{22}^{(1)} = [\lambda + 2\lambda(N_2 - N_1)]^{-1} \quad (102)$$

which does not turn to infinity at any eigenvalues of the operators \( N_1, N_2 \). Using the machinery of the proceeding section, it is not difficult to calculate the next term in the approximate quantum Hamiltonian (again, we retain only the terms yielding non-zero vacuum expectation values):

$$H^{(2)} = \omega_1 (N_1 + N_2) + \mu N_1 N_2 + \alpha + \varepsilon^2 \left[ \frac{(N_1^2 - N_1)(N_2^2 - N_2)}{2\omega_1 + \omega_2 - 4\mu + \mu(N_1 + N_2)} \right. \left. - \frac{4 + 6(N_1 + N_2) + 9N_1 N_2 + 2(N_1^2 + N_2^2) + 3N_1 N_2(N_1 + N_2) + N_1^2 N_2^2}{\omega_1 + \omega_2 + 2\mu + \mu(N_1 + N_2)} \right. \left. + \frac{(N_1^2 - N_1)(N_2^2 + 3N_2 + 2)}{\omega_1 - \omega_2 + 2\mu + \mu(N_2 - N_1)} \right] + \varepsilon^3 \quad (103)$$

One can check by the term-by-term inspection that zeros do not appear in the denominators in eq. (103). It can even be further speculated, similarly to the line of reasoning of Ref. [25], that small denominators do not show up at any order of the iterative procedure. For the ground state energy we obtain from (103)

$$E_0 = \frac{1}{2} \left( 1 + \frac{\lambda}{2} - \frac{\lambda^2}{4} \frac{1}{1 + \frac{\lambda}{2}} \right) + \varepsilon^3 \quad (104)$$
or, setting \( \lambda = \frac{1}{2} \),

\[
E_0 \approx \frac{1}{2} \left( 1 + \frac{1}{4} - \frac{1}{36} \right) \approx 0.611
\]

(105)

This number is to be compared with the result of the numerical calculation [22] which reads \( E_0 \approx 0.590 \). One sees that our accuracy is rather good, of order 3.5%.

Let us now turn to the study of the approximate wave functions (WF) (here we restrict ourselves by the first order formulas). The unperturbed Hamiltonian \( H_0 \) (96) has the eigenfunctions

\[
\phi_{s}^{(0)} = |n_1 n_2 \rangle , \quad s \equiv (n_1, n_2)
\]

or, in the coordinate space

\[
\phi_{(n_1, n_2)}^{(0)}(x_1, x_2) = \frac{(\omega_1 \omega_2)^{1/4}}{\sqrt{\pi^2 n_1! n_2!}} e^{-\omega_1 x_1^2 + 2 \omega_2 x_2^2} H_{n_1}(x_1 \sqrt{\omega_1}) H_{n_2}(x_2 \sqrt{\omega_2})
\]

(107)

where \( H_n(z) \) stands for the Hermite polynomial. After a simple algebra we obtain the first order WF

\[
\Psi_{\{n_1, n_2\}}^{(1)} = \phi_{(n_1, n_2)}^{(0)} + \frac{1}{2} \varepsilon \left[ \frac{\sqrt{n_1(n_1-1)n_2(n_2+1)} - \omega_1 - \omega_2 + \mu(n_1 + n_2 - 2)}{\omega_1 - \omega_2 + \mu(n_2 - n_1 - 2)} \phi_{(n_1-2, n_2+2)}^{(0)}
\]

(108)

The following from (108) asymmetry of the ground state WF (GSWF) is evidently an artifact of the asymmetry chosen in constructing the perturbation theory. As the GSWF must possess a highest possible symmetry, the final expression must be symmetrized in respect to the replacement \( (N_1 \leftrightarrow N_2) \)

\[
\Psi_{0}^{(1)} = \frac{1}{2} \left[ e^{-\omega_1 x_1^2 + 2 \omega_2 x_2^2} [1 - \frac{\lambda}{8} (2 \gamma x_1^2 + 2 \gamma x_2^2)
\]

(109)

\[+ \frac{(1 + \lambda) (x_1^2 - 1)(x_2^2 - 1)}{1 + \frac{\lambda}{1 + \lambda}}] + (x_1 \leftrightarrow x_2)]

The obtained formula demonstrates that our expansion is controllable in a vicinity of the origin. With moving away from the origin, the correction constitutes about 50% of the unperturbed value at \( x_1 \approx 2.0 \), if one moves along the lone \( x_2 = 0 \), or \( x_1 \sim x_2 \sim 3.0 \) for the radial escape. (Note that the validity of the ansatz for the wave function for small \( x, y \) in the form \( \exp \times polynomial \) can be checked by the direct substitution of such an ansatz into eq. (110)).

To get approximate WF’s in a large fields region, we need another approximation. To this end, we make use of an intrinsically true idea that, when one coordinate is large,
the variables in the Schrödinger equation get adiabatically separated [22]. Note, however, that the method used in Ref. [22] does not allow one to estimate corrections to the adiabatic separability, which is the question of great importance if we are going to match approximately two different expansions.

Let us write down the Schrödinger equation (in what follows, we set \( x_1 \equiv x, \ x_2 \equiv y \))

\[
- \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi(x, y) + \lambda x^2 y^2 \Psi(x, y) = E \Psi(x, y)
\]

(110)

and make in the region \( x > 0 \) the point canonical transformation to the new adiabatic variables (we are indebted to L. Frankfort for this suggestion):

\[
x \to u = x \\
y \to \tau = \sqrt{xy}
\]

(111)

The Schrödinger equation then transforms into

\[
\frac{1}{2} \left[ \frac{\partial^2}{\partial u^2} + u \left( \frac{\partial^2}{\partial \tau^2} - \frac{2\lambda}{\hbar^2} \tau^2 \right) \right] \Phi(u, \tau) + E \Phi(u, \tau) = \left( \frac{1}{8} \frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\partial}{\partial u} \right) \Phi(u, \tau)
\]

(112)

where \( \Phi(u, \tau) = a^{-\frac{\lambda}{2}} \Psi(u, \tau a^{-\frac{\lambda}{2}}) \), the multiplier is due to the Jacobian of the transformation (111). Note that in eq. (112) the variables get adiabatically separated in the region of very large \( u \) provided

\[
\left| \frac{1}{4} \frac{\partial^2 \Phi}{\partial \eta^2} \right| \ll \left| u \frac{\partial^2 \Phi}{\partial \eta^2} \right|
\]

(113)

In the asymptotic region we neglect in the zero approximation the r.h.s. and look for a solution in the form

\[
\Phi_\infty = \sum_{n=0}^{\infty} \xi_n(\tau) \eta_n(u)
\]

(114)

where \( \xi_n(\tau) \) satisfies

\[
\left( \frac{1}{2} \frac{\partial^2}{\partial \tau^2} - \frac{\lambda}{\hbar^2} \tau^2 \right) \xi_n(\tau) = -\varepsilon_n \xi_n(\tau)
\]

(115)

i.e. the \( \tau \)-motion is the usual harmonic oscillator. The resulting equation for \( \eta_n(u) \) is

\[
\frac{d^2 \eta}{du^2} - (2n + 1) \omega \eta + 2E \eta = 0
\]

(116)

where \( \omega = \sqrt{2\lambda} \). The only physically accepted solution (damped as \( u \to \infty \)) is given in terms of the Airy function [26]:

\[
\eta_n(u) = \text{const} \ Ai \left[ (2n + 1) \omega \right]^{1/3} \left( u - \frac{2E}{(2n + 1) \omega} \right)
\]

(117)

However, we are only allowed to retain the asymptotics of this formula, thus

\[
\Phi_\infty(u, \tau) \simeq \sum_{n=0}^{\infty} A_n e^{-\frac{\lambda}{2} \tau^2} H_n(\tau \sqrt{\omega}) \frac{1}{u} e^{-\frac{2}{3} \varepsilon_n (u - \beta_n)^{3/2}} , \ u \to \infty
\]

(118)
where $A_n$ are the normalization coefficients and

$$
\alpha_n = \sqrt{(2n+1)\omega}, \quad \beta_n = \frac{2E}{(2n+1)\omega}
$$

(119)

Let us now come back to eq. (112) and try to calculate corrections. There is no room in (112) for the standard Green functions method because of the non-separability of the variables at arbitrary $u, \tau$. To proceed, we substitute into (112) the ansatz

$$
\Phi(u, \tau) = \phi_0(u, \tau)\Omega(u, \tau)
$$

(120)

and try to find the form of $\Omega(u, \tau)$ in a vicinity of $u \to \infty$. The boundary conditions on $\Omega(u, \tau)$ are

$$
\Omega(u, \tau) \to 1, \quad u \to \infty, \quad \tau \text{ fixed}
$$

(121)

$$
|\Omega(u, \tau)| < e^{\frac{a}{2} \tau^2}, \quad \tau \to \infty, \quad u \text{ fixed}
$$

(122)

The equation for $\Omega(u, \tau)$ reads (in what follows we denote $\Phi = \Phi_0$, $\Phi_\tau = \frac{\partial \Phi}{\partial \tau}$, etc.)

$$
\left( \frac{\tau}{u} \Phi_\tau - \frac{1}{4} \frac{\tau}{u^2} \Phi_\tau \right) \Omega + \left( 2\Phi_u + \frac{\tau}{u} \Phi_\tau \right) \Omega_u + \frac{\tau}{u} \Phi \Omega_u \tau
$$

$$
+ \left( 2\Phi_\tau + \frac{\tau}{u} \Phi_u - \frac{1}{4} \frac{\tau}{u^2} \Phi \right) \Omega_\tau + \Phi \Omega_{uu} + \Phi_\tau \Omega_{\tau\tau} = 0
$$

(123)

According to the boundary conditions, we look for a solution in the form

$$
\Omega(u, \tau) = 1 + u^{-1/2} f_1(\tau) + u^{-1} f_2(\tau) + u^{-3/2} f_3(\tau) + \ldots
$$

(124)

We further substitute this expansion into (122) and equate the like powers of $u$. The resulting equations for $f_1$, $f_2$, $f_3$ read

$$
2\xi_0^s(\tau) f_1'(\tau) + \xi_0(\tau) f_1''(\tau) = 0
$$

(125)

$$
2\xi_0^s(\tau) f_2'(\tau) + \xi_0(\tau) f_2''(\tau) = 0
$$

(126)

$$
- \alpha_n \tau \xi_0^s(\tau) + 2\xi_0^s(\tau) f_3'(\tau) + \xi_0(\tau) f_3''(\tau) = 0
$$

(127)

Let us first concentrate on the equations (125), (126). It is easy to see that the non-trivial $\tau$-dependent solutions are excluded due to the boundary condition (122). However, arbitrary constants are not fixed by the boundary condition at the infinity. (The acceptable solution for eq. (127) is a bit more complicated: $f_3(\tau) = C_3 + \alpha_n \int_0^\tau dt \xi_0^{-2}(t) \int_0^\tau dz z \xi_0'(z) \xi_0(\tau)$). This means that our method is only giving a large-$u$ parametrization of the wave function. Recalling the Jacobian and returning to the original variables, we finally arrive at the GSWF of the form (we retain here only the $n = 0$ term from (118))

$$
\Psi(x, y) = A e^{-i \frac{\alpha}{2} x y^2 - i \frac{\alpha}{2} (x^2 - y^2)^{3/2}} \left( 1 + \frac{C_1}{x^{1/2}} + \frac{C_2}{x} + \frac{\Pi(\tau, C_3)}{x^{3/2}} + \ldots \right)
$$

(128)

where $A, C_1, C_2, C_3$ are some constants and $\Pi(\tau, C_3) = C_3 + \frac{1}{2} \int_0^\tau \int_0^\tau \Gamma(\frac{3}{2}, \alpha_0^2 t^2) \Gamma(\frac{3}{2}, \alpha_0^2 \tau^2)$ stands for the incomplete Gamma function (we will omit this term in what follows). Note that the obtained WF is in no sense an analytic continuation of (109). This is in the drastic contrast with one-dimensional quantum mechanics where the mathematically rigorous matching of the WKB wave functions can be given [26]. We would like to mention that the appearance of arbitrary constants in asymptotic large distance solutions is well
known in hydrodynamics [27]. It is also known that small- and large-distance expansions can be analytically matched only if the small distance expansion has the infinite radius of convergence [28], which is not the case in our problem. Therefore, one has to rely on an approximate (or numerical) matching. In this respect, the situation is reminiscent of what one encounters in the QCD sum rules method (provided we discuss the matching problem for wave functions and not for Green functions). There two different asymptotic expansions, each of them being defined in its own range, are numerically matched in some intermediate region [11]. It may be useful to continue a bit this formal analogy. Our asymptotic adiabatic expansion is a method of parametrization of the large distance (fields) behavior and thus is analogous to the expansion over hadron states [11], the parameters $A, C_1, C_2$ being the "hadron masses". The non-perturbative Kolmogorov technique substitutes the operator product expansion at small distances. Probably a bit deeper analogy with quantum chromodynamics lies in the fact that the non-integrability of the system (94) blocks an explicit construction of "quasiparticle" weekly interacting operators in the intermediate region. It is not yet, however, clear whether the non-integrability or just the mathematical complexity of real QCD prevents one building hadron operators in terms of the fundamental quark and gluon fields.

Inspired by this analogy, we have tried to find the coefficients $A, C_1, C_2$ by the numerical matching of the form (128) with the WF (109) at moderate $x \sim 1 - 2$ in the vicinity of the line $y = 0$. Though there exists a variety of matchings in three dimensions, the simplest method one can suggest is a minimization of the difference between the two wave function over the usual normalization measure with the integration done over the "stability slice", i.e. a continuous limit of the usual least square method. As such procedure can yields jumps of the derivatives, it must actually be supplemented by some smoothing operation on the obtained functions. Within the least square method, this problem is just ignored, which is partially justified by the fact that we make the matching in the exponentially suppressed region where derivatives are of order of the functions themselves.

We have found that there exists the "stability plateau" ranged from $x \simeq 1.5$ to $x \simeq 2.5$. With the matching in the intervals of $x = [1.5, 2]$ or $x = [2, 2.5]$, the coefficients $A, C_1, C_2$ change not more than by $10\%$. The found coefficients are

$$A = 2.50 \pm 0.15, \quad C_1 = -1.55 \pm 0.10, \quad C_2 = 0.90 \pm 0.10$$

(129)

with the measure parameter for the interval $x = [1.5, 2]$

$$M \equiv \frac{\int_{1.5}^{2} dx \int dy (\Delta \Psi)^2}{\int_{1.5}^{2} dx \int dy \Psi^2} \simeq 11\%$$

(130)

where $\Delta \Psi$ stands for the difference of the WF's (109) and (128). The measure parameter over the interval $x = [2, 2.5]$ is a bit worse, of order $13\%$. The main defect of the above matching is the fact that it deals solely with the adjustment of the wave function without imposing any condition on the derivatives. As the wave functions are known only approximately, one cannot impose the condition of the continuity on the derivatives of the wave functions. However, in view of the approximate character of our solutions, in the intermediate region it seems reasonable to proceed to the description of the wave functions in terms of the distributions rather than the usual functions. This means that we introduce some smoothing which characterizes the actual accuracy of our calculations. Instead of the function at a point $x_0$, we get the "function" at some small region $\Omega$ containing the point $x_0$:

$$\tilde{f}(\Omega) = \int_{x \in \Omega} f(x) \phi(x) dx,$$

(131)
where \( \phi(x) \) is a differentiable function which is very strongly peaked at \( x_0 \) and vanishes outside the region \( \Omega \). Then a precise meaning to the notion of matching two distributions \( f_1 \) and \( f_2 \) can be given. The distributions \( f_1, f_2 \) coincide in an open set \( \Omega \) when

\[
\int_{\Omega} f_1 \phi = \int_{\Omega} f_2 \phi
\]

for every infinitely differentiable function \( \phi \) whose support lies in \( \Omega \) [29]. The analogous conditions hold for the derivatives of \( f_1, f_2 : \langle f_1, \phi' \rangle = \langle f_2, \phi' \rangle \). For the purpose of the numerical matching, the test functions \( \phi \) should be strongly peaked at the stability plateau \( x = [1.5, 2.5] \) whose existence is indicated by the least square matching. In our opinion, such matching in the distribution sense is adequate to the approximate character of both solutions. Note that a more ambitious way of solving the Schrodinger equation in the intermediate region would be a search for a test function \( \phi \) satisfying the differential equation, i.e. a solution in the form of the integral transformation.

For the approximate matching in the distribution sense, we have chosen the test function in the form of the smoothed \( \delta \) function

\[
\phi(x, y) = \frac{nm}{\pi} \exp[-n^2(x - x_0)^2 - m^2 y^2]
\]

and varied the parameters \( n, m, x_0 \) in the vicinity of the values \( n = m = 1 \) and \( x_0 = 2 \). Then the parameters \( A, C_1, C_2 \) have been determined from the matching of the integrated wave functions and the their partial derivatives over \( x \) (the matching of the partial derivatives over \( y \) is the triviality when one chooses the test function (133) and neglects the term \( \Pi(\tau, C_3) \) in (128)). As in this case there are only two equations for three unknown parameters, we have taken one of them as the known one from (129) and found the other ones. We have found the same values of the parameters with the accuracy quoted in (129). We believe that the consistency between the two methods of the matching confirms the reliability of the results (129). Still, as we have seen, these two methods are not quite independent. Another method of a matching could be a search for global conservation laws, that would allow one to find the parameters \( A, C_1, C_2 \) or some relations among them. Unfortunately, we have not succeeded in this attempt.

## 7 Concluding Remarks

In this paper we have studied a quantum extension of the Kolmogorov iterative technique. We have found explicit solutions to functional equations determining the high orders Lie generators and Hamiltonians in both the classical and quantum cases and tested the whole approach on a few problems. For YM quantum mechanics, we saw that the Kolmogorov technique for the small distance regime must be supplemented by the adiabatic large distance approximation, thus the matching problem arises. As the whole affair is still in its infancy, one can hardly say beforehand whether it is able to suggest a reliable calculation scheme for more realistic problems of quantum mechanics and quantum field theory. Nevertheless, as we have seen, the KAM technique possesses a few very attractive features. The success in the construction of the convergent iterative scheme for the quantum anharmonic oscillator may allow one to hope that this technique could also be helpful in more complicated problems, provided the method is easily generalizable to arbitrary dimensional
cases. The hope to overcome the Dyson instability is due to the non-linear character of the KAM technique. Moreover, because of its recurrent definition, this method seems to be ideally suited for an analytical computerization. The need in the computer's help is seen already at first Kolmogorov's steps. We want to emphasize, however, that these difficulties are only technical and may be easily handled. On the other hand, the KAM technique is essentially non-perturbative and suggests the iterative non-linear extension of the Bogolyubov transformations method, when applied to infinite dimensional systems. Thus, it can be of some interest for clarifying a connection between perturbative and non-perturbative effects in quantum field theory.

The proposed technique could be further developed in a few possible directions. One of them is working out a theory of non-linear canonical transformations [19], which would allow one to calculate approximate wave functions directly from the "non-coordinate" Lie transforms formalism. Another problem is an improvement of the matching procedure and the use of a more accurate smoothing technique analogous to that used in classical mechanics. Also a search for global conservation laws seems to be rather important for the matching problem. The matching procedure done in this paper is very crude and, in particular, leads to an admixture of excited states to the ground state wave function. Surely enough, the matching must be improved to make this method work. Such an improvement would be especially important for the matching large- and small- distance solutions for highly excited states, where chaotic properties are expected to manifest themselves in full. It would also be rather interesting to reformulate the present approach in the path integral formalism and apply it for a calculation of Green functions. Hopefully, this could be done in the coherent states representation of the path integral. In such a form, this method could be directly confronted with the Operator Product Expansion, that might lead to a deeper understanding the phenomenological success of the QCD sum rules method [11]. As for the problem of calculation of the spectrum, the generalization of the present approach is quite straightforward. In particular, we hope that this method may be extended to calculate the mass spectrum in the quantum YM theory with periodic boundary conditions [30]. Another possible direction is a clarification of a connection of the present approach with the quasiclassical quantization of periodic orbits which hopefully could be useful for the quantum chaos studies. In any case, a lot of work is needed before the suggested method proves or disproves its power.

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Note added When this paper has been completed, I became aware of the recent paper [31] where similar ideas were developed and applied to the quantum anharmonic oscillator.
References