ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE

CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

MATHEMATICS, MACHINES & LOGIC

by

P.B. Sheridan

GENEVA

1963
Propriété littéraire et scientifique réservée pour tous les pays du monde. Ce document ne peut être reproduit ou traduit en tout ou en partie sans l'autorisation écrite du Directeur général du CERN, titulaire du droit d'auteur. Dans les cas appropriés, et s'il s'agit d'utiliser le document à des fins non commerciales, cette autorisation sera volontiers accordée.

Le CERN ne revendique pas la propriété des inventions brevetables et dessins ou modèles susceptibles de dépôt qui pourraient être décrits dans le présent document ; ceux-ci peuvent être librement utilisés par les instituts de recherche, les industriels et autres intéressés. Cependant, le CERN se réserve le droit de s'opposer à toute revendication qu'un usager pourrait faire de la propriété scientifique ou industrielle de toute invention et tout dessin ou modèle décrits dans le présent document.

© Copyright CERN, Genève, 1970

Literary and scientific copyrights reserved in all countries of the world. This report, or any part of it, may not be reprinted or translated without written permission of the copyright holder, the Director-General of CERN. However, permission will be freely granted for appropriate non-commercial use. If any patentable invention or registrable design is described in the report, CERN makes no claim to property rights in it but offers it for the free use of research institutions, manufacturers and others. CERN, however, may oppose any attempt by a user to claim any proprietary or patent rights in such inventions or designs as may be described in the present document.
Introduction.

In the course of this brief discussion, we hope to sketch, in broad outlines, the development within the past century or so—especially, the past half-century—of a new direction and methodology in mathematics: what Prof. Hao Wang of Oxford University has felicitously termed "mechanical mathematics". The earliest glimmerings of this new approach are actually to be found in the writings of Gauss and the subsequent refinements of concept and methodology can be traced in the works of Kronecker, Poincaré, L.E.J. Brouwer and, most recently, E.W. Beth, A. Heyting, K. Gödel, J. Herbrand, G. Gentzen, S.C. Kleene, A.A. Markov, A. Turing and E. Post, amongst others. The most stunning results are, doubtless, those of Gödel.

What the new mathematics is can best be described by first briefly reviewing the sorts of problems in the foundations of contemporary mathematics which gave rise to it.

Late in the 19-th century there gradually took place what we today term the arithmetisation of analysis, viz., the development of a theory of the real numbers—which is, after all, the corpus of analysis—as entities constructible solely and ultimately from the natural numbers by iterating certain basic set-theoretic procedures. This arithmetisation was accomplished principally by Weierstrass, Dedekind and Cantor.

Probably the most well-known construction to the student of analysis is that of Dedekind, who represents the reals as cuts ("Schnitte") in the class of rationals. The student of modern (abstract) algebra (or analysis) may be familiar with the Cantor construction of reals as equivalence classes of Cauchy-convergent sequences of rationals. At any rate, what was common to these approaches was a representation of the reals by infinite sets of rationals having certain axiomatically specified properties, and the continuum of real numbers was to be conceived of as the set of all such sets. Thus, was Cantor motivated to develop a general theory of sets...
of which the Weierstrass-Dedekind-Cantor constructions would constitute 
models and so to which all of its logically valid results would apply.

Unfortunately for the extremely plausible-seeming theory which 
resulted from Cantor's researches, disaster lurked just around the corner. 
Cantor's theory of transfinite ordinals and cardinals very soon gave rise 
to certain now celebrated paradoxes. E.g., the so-called Cantor paradox 
(discovered by Cantor himself, in 1899) in the theory of transfinite cardinals which, briefly, runs as follows. Consider the set \( U \) of all sets 
and form from \( U \) the set \( S \) of all its subsets - i.e., the set of all those 
sets \( A \) such that, every member of \( A \) is a member of \( U \). By the general theory, the cardinal number of \( S \) is strictly greater than the cardinal number of \( U \).

But, since \( U \) is the set of all sets and \( S \) is itself a set of 
sets, then \( S \) is a subset of \( U \). Hence, again by the general theory, the 
cardinal number of \( S \) cannot exceed the cardinal number of \( U \). But, within 
the theory, the relations of equality and order between cardinals are shown 
to be mutually exclusive. Hence, within the framework of the Cantor set 
theory arises the paradox that for the set \( U \) of all sets we may prove both 
that the cardinal of \( S \) exceeds that of \( U \) and that it doesn't.

Another paradox stemming out of the Cantor theory and discovered 
independently by Russell and Zermelo just past the turn of the century is 
a now famous one involving the set of all those sets which are not members 
of themselves. Call this set \( S \). The question now arises: is \( S \) a member 
of itself? Suppose it is. Then \( S \) belongs to the set of all those sets 
which are not members of themselves - thus, \( S \) cannot be a member of itself, 
contradicting our initial assumption about \( S \). So far, no paradox and, by 
the above reductio ad absurdum argument, we are forced to conclude that \( S \) 
is not a member of itself. But let us carry the argument one step further. 
Since \( S \) is not a member of itself - as just shown - then it is not a member 
of the set of all those sets which are not members of themselves, i.e., 
\( S \) is not a set which doesn't belong to itself, i.e., \( S \) is a set which is 
a member of itself! Thus, we have established both that \( S \) does and does 
not belong to itself and so, indeed, have a paradox.
Since these classic paradoxes of Cantor and Russell first appeared there have issued a seemingly endless number of "logical" and "semantical" paradoxes (to use C.F. Ramsey's terminology) of a similar ilk: the barber in a village who shaves all those and only those who don't shave themselves (does he shave himself?); the paradox of impredicability (due to Russell); the Epimenides paradox ("All Cretans are liars", says the Cretan philosopher); and so on, and on. (The reader who is interested further will find an extremely lucid exposition of these and other paradoxes by Prof. W.V. Quine in the April, 1962 issue of Scientific American). The nature of the difficulty common to all of these antinomies consists in what Poincaré termed impredicative definitions. When a set $S$ and a given entity $\sigma$ are so defined that $\sigma$ is a member of $S$ but, and at the same time, the definition of $\sigma$ depends on that of $S$, we say that the definition of $\sigma$ or of $S$ is an impredicative one. Thus, in the Cantor paradox, the set $U$ of all sets contains as a member the set $S$ whose definition depends on $U$. In the Russell paradox, the set $U$ of all sets is separated into two subsets: those sets which contain themselves as members and those sets (constituting $S$) which don't; then we place the set $S$ so defined by this partition of $U$ back into $U$ to ask ourselves in which part it properly belongs. And so on, with all the other logical antinomies which one may care to analyse.

Russell later furnished essentially the same explanation as Poincaré in the form of his "vicious circle principle", viz., no set can contain any entity as member whose definition depends solely on the given set, i.e., whose existence presupposes the given set.

Roughly speaking, what this entails for set theory is that "too large" sets must be excluded from mathematical consideration in order to avoid at least the known antinomies. The modern programme to reconstruct set theory, circumscribed by as few restrictions as possible to preclude the paradoxes, goes under the name of axiomatic set theory and its course of development can be traced through the work of Zermelo, Fraenkel, Skolem, von Neumann and Hilbert and Bernays.
The Three Schools

Generally speaking, there arose, subsequent to the foundational crises attendant upon the discovery of these set-theoretic paradoxes, three principal schools of mathematical thought: the logistic school of Russell and Whitehead; the intuitionistic school of Brouwer; and the formalistic school of Hilbert.

Russell's programme was the Leibnizian one of reconstructing all of classical mathematics on a purely logical foundation; hence the term "logicism" as applied to this view. Thus, all mathematical definitions and substitution and inference rules were to be reducible to their formal logical counterparts. And it was the monumental effort of Russell and Whitehead to furnish the logistic underpinning for classical mathematics which culminated in the celebrated "Principia Mathematica" of 1910-13. To forestall the antinomies, Russell excluded impredicatively defined sets and properties by creating a ramified theory of types, which, in effect, hierarchically classifies the primitive entities, properties of these entities, properties of properties of entities, etc., such that all properties are excluded which do not belong to one of these logical types.

An analogous treatment for relations (many-place predicates) as well as classes is carried out. To exclude impredicatively defined properties within a type, each type above the 0-th (that of the primitive entities) is further sub-classified into orders (this is the ramification). Unfortunately for the logistic programme, the further separation of types into orders prevents the complete reconstruction of classical analysis, since much of the latter abounds with impredicative definitions, e.g., the least upper bound of an upper bounded set of reals in the sense, e.g., of Dedekind. To get around this difficulty, Russell laid down his axiom of reducibility which says, in effect, that for any property within a type belonging to an order above the 0-th there exists a co-extensive property (i.e., one having the same extension, or domain of validity) whose order
is 0. Thus, by the axiom of reducibility, for every impredicative definition within a given type and order there corresponds an equivalent predicative one. And it is precisely this axiom (together with the so-called "axiom of infinity", postulating, in effect, the existence of a completed, infinite totality) which formed the key-stone of an increasingly shaky edifice. The whole structure finally collapsed into a heap with the publication in 1931 of K. Gödel's paper, "On Formally Undecidable Propositions of Principia Mathematica and Related Systems". Of which, more anon.

What made the whole system of Principia suspect to many mathematicians from the very beginning was its nonconstructive character. Specifically, the axiom of reducibility postulates the existence of properties without in any way furnishing a clue as to how such properties can be mathematically constructed from the given ones, upon whose existence, indeed, they are supposed to depend. This is not unlike postulating a function whose value for any given value of its argument variable one is unable to compute: such a function would be suspect to not a few mathematicians. Similarly, with the axiom of infinity: there is no reason at all to accept the existence of a postulated infinite collection of mathematical objects without there also being furnished an explicit prescription, or algorithm for the generation of its elements - or at least a finitary method for determining of a given entity whether or not it belongs to the totality in question.

Finally, as the intuitionists pointed out, the logistic programme is open to question on the grounds that logic itself already subsumes certain mathematical notions. E.g., the notion of recursion on the natural numbers is contained in the logicians very formulation of type theory. The final death-blow to the logistic programme was delivered some three decades after the publication of its bible, when Gödel succeeded in constructing undecidable propositions within the formal system of Principia and demonstrated, as a corollary, that any attempt to achieve a consistency proof within any full number-theoretic formal system (including that of Principia) by methods formalisable within the system was doomed to failure. Of which, again more anon.
It was the father of contemporary mathematical intuitionism, L.E.J. Brouwer, who first seriously challenged the blind application of classical (Aristotelian) logic to the mathematics of infinite sets, arguing that the rules of this logic stemmed initially and solely from the mathematics of finite sets, hence were not so sacred as to be independent of context of application. The most striking renunciation of a long-revered classical logical rule, stemming out of this view, is that of the so-called "law of the excluded middle". This law asserts the seemingly self-evident notion that, for any proposition \( P \), either \( P \) holds or \( \neg P \) holds - i.e., either \( P \) is the case or \( \neg P \) is the case. Either a marble in this sack is blue or no marble in this sack is: and one or the other of these assertions is finitely verifiable, simply by examining each marble in turn until we have seen them all. In a situation such as this, an algorithm is available for checking any existential assertion one cares to make about the given collection of objects. But, in the event we are dealing with an infinite collection, argues Brouwer, we can no longer "see them all" and so, the excluded middle principle no longer generally holds. Thus, if \( P \) is now the proposition: there exists a member of the set \( S \) having the property \( \pi \), then \( \neg P \) is equivalent to: no member of \( S \) has the property \( \pi \). For the classicist this, in turn, is a priori equivalent to: every member of \( S \) has the property \( \neg \pi \). For the intuitionists, on the other hand, nothing further can be said, short of constructively establishing that given any member of \( S \), it does, indeed, possess the property \( \neg \pi \). Specifically, what the intuitionist demands is that both \( S \) and \( \pi \) be so-definable that given any element \( \sigma \) of \( S \) one can establish according to strictly finitary methods (e.g., by means of a terminating computer programme) that \( \sigma \) does possess the property in question.

Again, an existential statement, such as that, "there exists a member of \( S \) having the property \( \pi \)" has significance to the intuitionists only in so far as one can actually exhibit at least one particular member of \( S \) which, by strictly finitary methods, can be verified to possess the property \( \pi \). The situation is similar, I suppose, to that in physics when, for the purpose of logically completing a theory the existence of a previously unobserved particle is postulated, or inferred from the other parts.
of the theory. Short of actually verifying the existence of the particle by finitary methods in the laboratory, any and all statements made about the particle in question are at best ideal statements and it is only when some set of real statements can be made - such as that a certain measured track on a certain bubble-chamber photograph is the trace of the postulated particle in the chamber, etc. - that the physicist is absolutely convinced, and will henceforth accept the previously inferred ideal statements about the properties and behaviour of that particle as real.

Hence, the classical nonconstructive or indirect existence proofs are rejected by the intuitionists virtually out-of-hand. We are all familiar with such arguments from our studies of analysis and analytic topology.

"Suppose", the classical argument would begin, "for all σ in S, not-ψ (σ)". Then he deduces from this assumption a contradiction. Both classicalist and intuitionist would agree in concluding, "not for all σ in S does not-ψ (σ) hold". But the classicalist goes one step further and asserts that, "therefore, there exists a σ in S such that ψ (σ) holds". It is here that the classicalist and intuitionist have their great falling-out. "Show me", says the intuitionist, "or at least furnish me with a recipe or algorithm by which I (or my descendants, if it should take all that time) can eventually produce such a σ".

As may appear from the foregoing discussion, wherein the classical and intuitionist schools essentially differ is over their respective notions of the infinite - i.e., of infinite sets or collections. The former views the infinite as a completed entity quite independently of any process of construction; the latter, on the other hand, views the infinite as purely and solely constructive. It is the existential-axiomatic viewpoint versus the constructive-genetic, as Hilbert pointed out in his "Foundations of Geometry".

All of which brings us to the third major school of mathematical thought, that of formalism. The challenge of salvaging the content of classical mathematics while, at the same time, meeting the intuitionist critique of its methodology was one towards which Hilbert directed his major
efforts. The crowning of these efforts was the two-volume "Foundations of Mathematics" jointly written with Bernays during the period 1934-39.

Briefly, the formalist programme was to reconstruct classical mathematics as a purely formal axiomatic theory and then prove the resulting theory— in terms acceptable to the intuitionists (hence, also, to the classi-
cists) — to be consistent, i.e., to be exempt from the possibility that some formal statement A together with its negation not-A would both be derivable as theorems from the axioms by means of the explicitly postulated rules of inference. It was Hilbert's genius to realise that what was required to settle the entire classicist-intuitionist issue over the foundations was an absolute consistency proof. Let us see what is technically involved here.

Prior to the elaboration of the formalist programme, the approach to consistency proofs for mathematical theories was as follows. One dis-
covered or constructed or borrowed a model of the given theory whose consis-
tency has already been or can readily be, separately established, or whose consistency is a necessary condition for that of the theory in question. Thus, were the given theory inconsistent then, in the model a corresponding contradiction would arise by corresponding inferences from corresponding mathematical objects. Mathematical history abounds with consistency proofs, or model-theoretic reductions of consistency of this character.

Descartian analytic geometry constitutes a reduction of the con-
sistency of geometric theories to that of the theory of real numbers.;
Lobatschevskian non-Euclidean geometry was shown to have a model on certain types of surfaces in Euclidean space. However, neither the consistency of a theory of the reals nor of the system of plane-Euclidean geometry has, in turn, itself been established in terms of any finitarily constructed model (and, in view of the Gödel results on consistency proofs for systems of these types, the prospect isn't terribly rosy). An absolute proof for certain classes of projective geometries was, in fact, obtained by O. Veblen and W.H. Bussey by means of models in which the points of the axiomatic theory received a representation in the model by a finite class of objects.
Hilbert himself, in his "Principles of Mathematical Logic", furnished model-theoretic reductions of the consistency of Boolean propositional logics and of pure first-order predicate logics (the latter constituting an extension of the former by means of the addition of axioms and inference rules governing the use of universal and existential operators, or "quantifiers"). The reductions Hilbert effected, in these cases, were to the residues modulo a suitably specified integer. But an absolute proof of the consistency of classical number theory, analysis and set theory by means of a model is, as Hilbert observed, virtually doomed to failure from the very outset, since it is not at all clear that any proposed model of one of these systems would not itself lead to a theory which in turn had been previously reduced model-theoretically to one of these systems.

I.e., model construction, in this instance, would carry us in vicious circle fashion right back to where we started.

Thus, another and more direct approach, completely effective and employing only finitary methods must be discovered or devised, and it was Hilbert's great achievement to have conceived and developed the basic principles of such an approach. This approach goes today under the name of metamathematics, or proof theory. What it consists in, we shall now attempt to describe in as non-technical a fashion as possible.

Any mathematical or scientific theory begins and, usually, "takes place", as it were, on an informal, intuitive level. On this level, the content of the theory consists mainly in what Hilbert termed real statements - i.e., assertions of directly verifiable propositions regarding the fundamental objects of study: what these objects are and what their interrelationships. Thus, "either the reading on this galvanometer is 1.3 volts or the solution in that jar is acidic"; again "if x, y and z have the respective values 2.5, -3 and 1 then the set of three equations is satisfied"; etc. These are just examples of statements which assert directly verifiable propositions, i.e., real statements. In distinction, Hilbert characterises as ideal those statements which are not real, in the above sense. We are all familiar with the device of adjoining such statements to the corpus of an informal theory based originally on real statements only, when a certain sort of completeness is desired. Thus, the "point" and "line" at infinity
of projective geometry; the adjoining of 0 and the negative integers to
the natural numbers in order that linear integer equations in one variable
shall always have solutions; the extension of arithmetic successively through
the rationals, the irrationals and the complex numbers in order, again, that
certain types of equations shall have solutions. In other words, the ideal
elements of a theory are those which go beyond what is directly given and
intuitively known — usually, in practise, far beyond — in order to serve some
theoretical end, some "simplifying idealisation", as Hilbert put it. The
classical mathematical notion of infinite sets as completed totalities is
just such a "simplifying idealisation" whose purpose is to extend the scope
of applicability of Aristotelian logic to infinite sets.

Now, it is not long after the inception of an informal theory —
comprising both real and ideal statements — that formalisation begins to take
place. To this end, the statements and rules of the theory are reconstructed
and expressed in terms of a specially developed formalism or, when this is
possible, within the framework of an already existing formalism. It is intend-
ed that inferences in the informal theory shall be reflected in the formal
toery via explicitly and precisely specified formal symbolic manipulations.
Usually, however, formalisation is achieved only partially, not all of the
primitive terms of the informal theory which are essential to the deduction
of its theorems, or of its true statements, being completely specified in
the axioms; nor, again, all the rules of inference given explicit expression.
In other words, a great deal is often implicit, or assumed. Once, however,
such strictly explicit formalisation is achieved, the resulting formal theory
can then itself be made an object of study. And it is the study of formal
systems as such — quite apart from any principal interpretations which these
systems may have as instruments of informal deductive reasoning — that is
termed "metamathematics", or "proof theory". The metamathematics consists
in the description of the formal system, completely independently of any
meaning which might originally have been attached to the assertions which
have received formalisation, as well as the analysis of the structural prop-
ties of the system, e.g., under what conditions expressions of the system are
well-formed, which well-formed expressions are (or have the form of) axioms,
under what conditions a finite sequence of well-formed expressions constitutes a proof from the axioms or a deduction from given well-formed expressions, etc. It cannot be too much stressed that semantical considerations as regards the formal system itself must never figure into the metamathematical analyses; since, were there ever need to appeal to the meaning of a symbol in order to understand its rôle in a formal context, then it can only be because of an incomplete formalisation: some unstated assumption about the rôle of that symbol in the theory has been made.

The most interesting— and, with regard to mathematical theories, most difficult—problem for metamathematics is the so-called decision problem; viz., what are the (finitarily) necessary and sufficient conditions that a well-formed expression, or formula, of the formal system is a theorem. This is a statement of the decision problem for provability, the corresponding question regarding validity (or "truth") being termed the decision problem for validity. The distinction here indicated is not a trivial one since, generally, the two questions are not equivalent (and are only equivalent for relatively trivial systems such as Boolean propositional logics). In fact, Gödel, in the incompleteness theorem, constructs a valid formula of the system of Principia which is not provable if the system is consistent.

Now, the point to be stressed here is that, unlike the formal theory the metatheory proceeds intuitively and informally according strictly to direct evidence ("the evidence directly before our eyes", as it were) and without any reference what-so-ever to explicitly stated rules or procedures. Formalisation of the metatheory itself may be carried out as a separate programme, but it must be realised that the resulting system is a formalisation of real statements only—such as that, "the second and third occurrences of letters above the line are respectively occurrences of 'a' and 'b'; or that, "there are exactly three occurrences of the logical symbol '→' and one occurrence of the variable 'x' in this expression"; etc. Whereas, the formal system, which is itself the object of study,
contains formalisations of assertions of the original informal theory which are ideal as well as real.

Results and Directions.

Now, what are some of the principal results of contemporary metamathematics and what are the directions of further research which are indicated?

For the question of consistency, as early as 1930 M. Pressburger proved the consistency and completeness of as well as developed a finitary decision procedure for, a formal system of number theory which excluded the axioms for multiplication. W. Ackermann and J. von Neumann, several years earlier, demonstrated metamathematically the consistency of a formal system for number theory with a restricted axiom of induction. The resulting system, though consistent, is not complete, the well-ordering of sets of natural numbers being unprovable, for example.

A striking metamathematical result of Gödel's, which follows as a corollary from his incompleteness theorem of 1931, is that, if a number-theoretic formal system is consistent, then there is no consistency proof for it which can itself be formalised within the system! Thus, a metamathematical consistency proof for such a system must inevitably involve methods unformalisable within the system itself. What this entails is that, the usual methods of classical mathematics are not adequate to settling the ever-burning question of consistency and that the only hope of ever settling this question lay in the application of yet deeper metamathematical techniques of analysis. Of recent years, and as a result of the advent of increasingly more powerful high-speed electronic digital computers, mounting efforts have been directed towards the application of machines in the service of metamathematical research. Since all proof and decision-theoretic procedures of metamathematics must be effective (i.e., algorithmic), in nature, the motivation behind these efforts is obvious. Theorem-proving by machine, for example, has been a particularly fertile and promising area.
The basic question of whether, in fact, there exists a set of metamathematical techniques for effectively obtaining direct proofs of provable formulas in pure first-order predicate logics is settled in the affirmative by a remarkable normal form theorem of G. Gentzen (1934-35). What Gentzen did, was to set up a new formal system for first-order predicate logic which, by strictly finitary methods, he was able to show to be proof-theoretically equivalent to the classical or intuitionist formulations (depending, respectively, on variants of two of the postulates of his system). The beauty of Gentzen-type formalisms is that, not only can proofs be constructed in a purely deterministic, i.e., mechanical, fashion, but that important consistency and decision-theoretic results can be obtained for number-theoretic formal systems with relative ease. As a rather stunning example of the power of these formalisms, an IBM 704 computer was programmed several years ago by Prof. Hao Wang to construct normal form proofs of theorems of first-order predicate logics. When turned loose on all 400-odd theorems of Principia Mathematica in the realm of predicate logics with the equality relation, the following results were obtained. The whole set of over 200 theorems on the propositional parts of the Principia system contained in the first 5 chapters were proved in less than three minutes of central computer time (some 35 additional minutes being required for input and the printing out of detailed proofs generated by the computer). More than 150 of the more complicated theorems in the predicate logic with equality contained in the next 5 chapters require slightly more than an hour of 704 time.

We might note, at this point, that though Gentzen-type formalisms, such as Wang used, are extremely powerful as the basis of machine theorem-proving programs, they cannot form the basis of any general proof or dis-proof procedure for first-order logics, since, as A. Church showed in 1936, these logics are (effectively) undecidable—though, to be sure, they contain very many (effectively) decidable sub-logics, not a few of which are of considerable mathematical importance.

Before concluding, we should like to mention other research along similar or related lines which has recently taken place and is currently in progress.
Within the past year, H. Gelernter of IBM Research developed a 709 programme (based on techniques arising out of the work of E.L. Post on combinatorial systems) for proving theorems in plane Euclidean geometry—a programme which produced the proofs of several remarkably deep theorems which it had been presented with for the first time.

Professors M. Davis and Hilary Putnam, of Yeshiva and Princeton Universities, respectively, have devised rather powerful techniques for deciding statements in propositional logics, utilizing a 704 Computer.

Finally, we should like to mention the work of Prof. B. Dreben of Harvard University who has been engaged for a number of years in developing highly sophisticated combinatorial-analytic techniques for deciding satisfiability questions for first-order logics. Whereas Wang's point of departure has been the Gentzen normal form theorem, Dreben's is the Gödel-Herbrand completeness theorem (or, more exactly, its dual for satisfiability). This theorem has a corollary to the effect that a well-formed formula of first-order predicate logic is valid if and only if there exists a positive integer n such that, the disjunction (logical "or-ing" together) of the first n of an effectively specifiable set of well-formed formulas of propositional logic is valid. What makes Dreben's approach to the problem particularly attractive, then, is that it reduces, via the Gödel-Herbrand completeness theorem, the question of decidability for predicate logics to that of pattern-recognition for (quantifier-free) propositional logics. Dreben recognises that the completeness theorem is also a representation theorem which permits of exploitation by highly developed combinatorial-hence, machine—techniques. Dreben also foresee the unification of all previously developed decision-theoretic techniques by these methods so that, eventually, the known (and, hopefully, some of the as-yet unknown) decidable sub-logics may be completely specified in a single "Hauptsatz".

Promising work along lines similar to those adopted by Prof. Dreben has recently been reported by Prof. Dag Prawitz and his associates at the University of Stockholm.
We should like to add, in conclusion, that particularly fruitful applications of metamathematical techniques stemming out of the theory of recursive functions and combinatorial systems have been made in the design of automatic programming languages and their translators for digital computers, e.g., FORTRAN, ALGOL, LISP, etc.

P. Sheridan,
Data Handling Division,
CERN.
Bibliography


