DISC-LOADED DEFLECTING WAVEGUIDE

by

H. G. Hereward, M. Bell
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PART I

1. Introduction

The radio-frequency particle separator being built at CERN\(^{(1a)}\) will use a disc-loaded circular waveguide structure (see Fig. 1) to carry the deflecting R.F. field, and this Report gives the theoretical basis of a programme which computes the properties of such a waveguide. Other papers on the same subject have appeared\(^{(1b-f)}\). The results of these computations will be given in another report\(^{(14)}\).

The waveguide resembles that of the usual electron linear accelerator, but whereas the electron linacs operate with electromagnetic fields which have cylindrical symmetry, here the appropriate modes vary with cos \(\Theta\) or sin \(\Theta\), where \(\Theta\) is the azimuthal angle. The analysis is, however, carried out in a manner very similar to one used for linacs\(^{(2)}\).

Series expansions for the fields in the axial region and in the slots are given in Section 2a, and the matching between these two series is carried out in 2b. This gives the formal mathematical solution in terms of infinite matrices. In practice one cannot compute with infinite matrices, and we have worked with a finite number of modes in each region, chosen to suit the storage capacity of the Mercury computer.

This method was used, rather than the more powerful variational methods\(^{(3-6)}\) that give an upper and a lower bound for the frequency, for several reasons. We were not primarily concerned with getting a very accurate value of the frequency (or the dimensions for given frequency), for this is rather easily checked by the construction and measurement of a low-power model or prototype. Of more interest is the variation during tests.

The final waveguide design will have a small radius, rather than sharp corners, on the inside edges of the discs: it is very complicated to allow for this in the computations, so appeal to a model at some stage is almost essential anyway.
of stored energy, losses etc. with choice of aperture and disc spacing; this information is needed in order to make an efficient choice of the main waveguide parameters, and a large amount of work would be necessary to obtain it from models. The method we have used is probably accurate enough for this purpose, and is relatively straightforward to programme for a computer. Some idea of the inaccuracy associated with the finite number of modes used can be obtained by calculating also with smaller numbers, and the programme includes provision for this.

Evidence that this type of waveguide structure would be suitable and probably reasonably efficient for particle deflection was already obtained by a study of it in the "smooth approximation"\(^{(7)}\), i.e. in the limit where the disc spacing is small compared with both the wavelength and the aperture. This is equivalent to the present method limited to two modes in the axial region and one in the slots.

The programme described in Part I works in rationalised M.K.S. units, but some of the data and results are expressed in more convenient units such as millimetres, megawatts, etc. It assumes a free-space wavelength of 0.10m and a phase-velocity equal to the velocity of light \(c\): these are not severe limitations, as it is relatively easy to "scale" the results to any other wavelength and the computed value of the dispersion tells us what frequency change will be necessary to cover the very small range of phase-velocities near \(c\) that are of interest. The spacing of the discs and their thickness, together with the radius \(a\) of the hole, are fed into the computer on a data tape, and the programme first computes the radius of the guide, \(b\), by a method described in Section 3. In Sections 4a, b, c, expressions are obtained for the stored energy, the losses, and the power flow. The group velocity (= energy velocity) can also be determined for the programme (Section 5).

When it was observed experimentally (e.g. at CERN by P. Braham), that some of the dispersion curves for the guide displayed the feature of having two possible guide wavelengths \(\lambda_g\) for a given frequency, it was decided to calculate the dispersion curves over a range of \(\lambda_g\) from 0 up to the \(\pi\)-mode cut-off for the structure. This phenomenon has been met before (e.g. ref. 13). The calculations for \(v \neq c\) are described in Part II of this Report.
2a Series Expansions for the Fields

It is convenient to use the Hertz vectors $\pi$ and $\pi^*$ (8) to derive the fields. Assuming a time dependence $e^{i\omega t}$ ($\frac{\partial}{\partial t} = i\omega$), the fields are then given by

$$E_r = \frac{\partial^2 \pi_z}{\partial r \partial z} - \frac{\mu i \omega \partial \pi_z^*}{r \partial \Theta}$$

$$E_\Theta = \frac{1}{r} \frac{\partial^2 \pi_z}{\partial z \partial \Theta} + i\mu \omega \frac{\partial}{\partial r} (\pi_z^*)$$

$$E_z = \frac{\partial^2 \pi_z}{\partial z^2} + \frac{\omega^2}{c^2} \pi_z$$

$$H_r = \frac{i\epsilon \omega \partial \pi_z}{r \partial \Theta} + \frac{\partial^2 \pi_z^*}{\partial z \partial r}$$

$$H_\Theta = -i\epsilon \omega \frac{\partial}{\partial r} (\pi_z) + \frac{1}{r} \frac{\partial^2 \pi_z^*}{\partial z \partial \Theta}$$

$$H_z = \frac{\partial^2 \pi_z^*}{\partial z^2} + \frac{\omega^2}{c^2} \pi_z$$

The required fields inside the slots ($r > a$) are found by adding together all modes, with the required dependence on $\cos \Theta$ or $\sin \Theta$, having a resonance between the slot walls. For the slot centred on $z = 0$ we put therefore
\[ \Pi_z = \frac{C L \left( kr \right)}{k^2 L_1 \left( ka \right)} \cos \theta - \sum_{m=2,4 \ldots}^{\infty} \frac{C}{Y_m^2 M_1} \frac{M \left( Y_r \right)}{Y_m^2 M_1} \cos g_m z \cos \theta \]

\[ - \sum_{m=1,3,5 \ldots}^{\infty} \frac{C m}{Y_m^2 M_1} \frac{M \left( Y_r \right)}{Y_m^2 M_1} \sin g_m z \cos \theta \]

\[ \Pi_z^* = - \sum_{m=2,4 \ldots}^{\infty} \frac{D_m}{Y_m^2 M_0} \frac{N \left( Y_r \right)}{Y_m^2 M_0} \sin g_m z \sin \theta \]

\[ - \sum_{m=1,3,5 \ldots}^{\infty} \frac{D_m}{Y_m^2 M_0} \frac{N \left( Y_r \right)}{Y_m^2 M_0} \cos g_m z \sin \theta \]

where \( \omega = \) R.F. frequency in radians per second

\[ k = \frac{2 \pi}{\lambda} = \frac{\omega}{c} \text{ where } \lambda \text{ is the free-space wavelength} \]

\[ g_m = \frac{m \pi}{d} \text{ where } d \text{ is the slot width} \]

\[ Y_m^2 = \frac{g_m^2}{d^2} - k^2 \]

\[ L_1 \left( kr \right) = J_1 \left( kr \right) + CY_1 \left( kr \right) \text{ where the constant } C \text{ is chosen so that } \]

\[ J_1 \left( kb \right) + CY_1 \left( kb \right) = 0 \]

\[ M_1 \left( Y_r \right) = C \left( Y_r \right) + K \left( Y_r \right) \text{ where again the constant } C \text{ is chosen to make } \]

\[ M_1 \left( Y_b \right) = 0 \]

\[ N_1 \left( Y_r \right) = C \left( Y_r \right) + K \left( Y_r \right) \text{ where in this case } C \text{ is chosen to make } \]

\[ N_1 \left( Y_b \right) = 0 \]

These expressions (2.1) and all expressions for the fields that follow should be taken to depend on the time by a factor \( e^{j \omega t} \) which we do not write down.
The fields in the slot are then:-

(using \( \epsilon \omega z_0 = \frac{1}{2} \mu \omega = k \))

\[
E_r = \sum_{m=2,4,\ldots}^{\infty} \frac{C_m g_m M'(Y_r)}{\frac{1}{m}(Y_m)} \sin g_m z \cos \Theta
\]

\[
- \sum_{m=1,3,5,\ldots}^{\infty} \frac{C_m g_m M'(Y_r)}{\frac{1}{m}(Y_m)} \cos g_m z \cos \Theta
\]

\[
+ \frac{ik}{r} \sum_{m=2,4,\ldots}^{\infty} \frac{D_m N(Y_r)}{\frac{1}{m}(Y_m)} \sin g_m z \cos \Theta
\]

\[
+ \frac{ik}{r} \sum_{m=1,3,5,\ldots}^{\infty} \frac{D_m N'(Y_r)}{\frac{1}{m}(Y_m)} \sin g_m z \cos \Theta
\]

\[
E_\theta = - \frac{1}{r} \sum_{m=2,4,\ldots}^{\infty} \frac{C_m g_m M(Y_r)}{\frac{1}{m}(Y_m)} \sin g_m z \sin \Theta
\]

\[
+ \sum_{m=1,3,5,\ldots}^{\infty} \frac{C_m g_m M(Y_r)}{\frac{1}{m}(Y_m)} \cos g_m z \sin \Theta
\]

\[
- \frac{ik}{r} \sum_{m=2,4,\ldots}^{\infty} \frac{D_m N'(Y_r)}{\frac{1}{m}(Y_m)} \sin g_m z \sin \Theta
\]

\[
- \frac{ik}{r} \sum_{m=1,3,5,\ldots}^{\infty} \frac{D_m N'(Y_r)}{\frac{1}{m}(Y_m)} \cos g_m z \sin \Theta
\]

\[\text{(2.2)}\]
\[ E_z = \frac{C}{\mu_0} \frac{L}{\nu^2} (kr) \cos \Theta \]
\[ + \sum_{m=2,4,\ldots}^{\infty} \frac{C_m}{\mu_0} \frac{M(Y_m)}{M(Y_n)} \cos g_m z \cos \Theta \]
\[ + \sum_{m=1,3,\ldots}^{\infty} \frac{C_m}{\mu_0} \frac{M(Y_m)}{M(Y_n)} \sin g_m z \cos \Theta \]

\[ Z_H = -\frac{i}{\mu_0} \frac{L}{\nu^2} (kr) \sin \Theta \]
\[ + \sum_{m=2,4,\ldots}^{\infty} \frac{i k}{\mu_0} \frac{C_m}{\nu^2} \frac{M(Y_m)}{M(Y_n)} \cos g_m z \sin \Theta \]
\[ + \sum_{m=1,3,\ldots}^{\infty} \frac{i k}{\mu_0} \frac{C_m}{\nu^2} \frac{M(Y_m)}{M(Y_n)} \sin g_m z \sin \Theta \]
\[ - \sum_{m=2,4,\ldots}^{\infty} \frac{D_m g_m}{\mu_0} \frac{N'(Y_m)}{N'(Y_n)} \cos g_m z \sin \Theta \]
\[ + \sum_{m=1,3,\ldots}^{\infty} \frac{D_m g_m}{\mu_0} \frac{N'(Y_m)}{N'(Y_n)} \sin g_m z \sin \Theta \]
\[ Z_{\Theta} = -i e^{i(kr)} + \sum_{m=2,4,\ldots}^{\infty} \cos \frac{m}{r} + i d \sum_{m=1,3,\ldots}^{\infty} \sin \frac{m}{r} \]

\[ \Theta = \sum_{m=2,4,\ldots}^{\infty} \frac{D_{m}}{N_{m}^{'}} \cos \frac{m}{r} + i d \sum_{m=1,3,\ldots}^{\infty} \sin \frac{m}{r} \]

In the axial region, \( r \ll a \), we use travelling waves in the \( z \) direction:

\[ \Pi_{z} = \sum_{\text{alln}} -\frac{A_{n}}{\chi_{n}} \frac{I_{1}(\chi_{n}r)}{I_{1}(\chi_{n}a)} \cos \Theta \cos \Theta \]

\[ \Pi_{z}^{*} = \sum_{\text{alln}} \frac{B_{n}}{\chi_{n}^{2}z_{0}} \frac{I_{1}(\chi_{n}r)}{I_{1}(\chi_{n}a)} \sin \Theta \sin \Theta \]

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where \( \chi_n^2 = \beta_n^2 - k^2 \)

and \( \beta_n = \beta_o + n \frac{2\pi}{D} \), where D is the distance between centres of adjacent slots and \( \beta_0 \) D is the phase change per slot. We find

\[
E_r = \sum_n \frac{i\beta_n}{\chi_n} A_n \frac{I'(\chi_n r)}{I(\chi_n a)} \cos \Theta \cdot e^{-i\beta_n z}
\]

\[
+ \frac{1}{r} \sum_n \frac{B_n}{\chi_n^2} \frac{I(\chi_n r)}{I'(\chi_n a)} \cos \Theta \cdot e^{-i\beta_n z}
\]

\[
E_\theta = - \sum_n \frac{iA_n \beta_n}{\chi_n} \frac{I'(\chi_n r)}{I(\chi_n a)} \sin \Theta \cdot e^{-i\beta_n z}
\]

\[
- \frac{1}{r} \sum_n \frac{B_n}{\chi_n^2} \frac{I'(\chi_n r)}{I(\chi_n a)} \sin \Theta \cdot e^{-i\beta_n z}
\]  \hspace{1cm} (2.3)

\[
E_z = \sum_n \frac{A_n}{\chi_n} \frac{I(\chi_n r)}{I'(\chi_n a)} \cos \Theta \cdot e^{-i\beta_n z}
\]

\[
Z_{HR} = \frac{ik}{r} \sum_n \frac{A_n}{\chi_n^2} \frac{I(\chi_n r)}{I'(\chi_n a)} \sin \Theta \cdot e^{-i\beta_n z}
\]

\[
+ \frac{i}{r} \sum_n \frac{B_n}{\chi_n} \frac{I'(\chi_n r)}{I(\chi_n a)} \sin \Theta \cdot e^{-i\beta_n z}
\]

\[
Z_{H\theta} = ik \sum_n \frac{A_n}{\chi_n} \frac{I'(\chi_n r)}{I(\chi_n a)} \cos \Theta \cdot e^{-i\beta_n z}
\]

\[
+ \frac{i}{r} \sum_n \frac{B_n}{\chi_n^2} \frac{I'(\chi_n r)}{I(\chi_n a)} \cos \Theta \cdot e^{-i\beta_n z}
\]

\[
Z_{Hz} = \sum_n \frac{B_n}{\chi_n} \frac{I(\chi_n r)}{I'(\chi_n a)} \sin \Theta \cdot e^{-i\beta_n z}
\]
We are interested in phase velocities equal to the velocity of light, i.e. \( \beta_0 = k \) and \( \chi_0 = 0 \), and this gives rise to a difficulty with the \( n = 0 \) terms in the above fields, for most of them become infinite as \( \chi_0 \) tends to zero. It is easy to obtain one \( n = 0, \beta_0 = k \) mode with \( E_z = H_z = 0 \) and the other fields finite: we note that for \( \chi_0 r \) small

\[
I_1(\chi_0 r) \sim \frac{1}{2} \chi_0 r
\]

and that the denominator \( \chi_0^2 Z \frac{d}{dn} I_1'(\chi_0 n) \) in the terms of \( \pi_z^* \) is only a convenient normalisation that we are at liberty to change. For the \( n = 0 \) term in \( \pi_z^* \) we therefore take:

\[
\Pi_{z0}^* = -\frac{B}{kz_0} r \sin \Theta e^{-ikz}
\]

This and \((2.0)\) give us the field components

\[
\begin{align*}
E_r &= i B_0 \cos \Theta e^{-ikz} \\
E_\phi &= i B_0 \sin \Theta e^{-ikz} \\
E_z &= 0 \\
Z_0 H_r &= i B_0 \sin \Theta e^{-ikz} \\
Z_0 H_\phi &= i B_0 \cos \Theta e^{-ikz} \\
H_z &= 0
\end{align*}
\]

This mode is the \( \nu = c \) T.E.M. plane-polarized wave; it has the property that the electric and magnetic forces on a particle moving at \( \nu = c \) with it exactly cancel.

In a similar way one may obtain a finite \( n = 0 \) mode from an electric Hertz vector \( \pi_{z0} \), but this turns out to be some constant multiple of \((2.4)\), so does not provide us with a second independent zero mode. The use of Hertz vectors and \((2.0)\) commits us to expansions
in terms of waves that are TM and TE with respect to the \( z \) axis, and these both tend to the same TEM wave as the phase velocity in the \( z \) direction tends to \( c \). There are several ways of obtaining expressions for a \( v = c \) deflecting (not TEM) mode: we use one of the orthodox methods of dealing with the situation where two solutions of a linear differential equation cease to be distinct at particular values of the parameters; due to d'Alembert (9). It is convenient first to put

\[
\Pi_{z_0} = \frac{A}{ka\chi_0} \left( I_0(\chi_0 r) \cos \Theta \ e^{-i\beta z} \right)
\]

\[
\Pi_{z_0}^* = -\frac{A}{ka\chi_0} \left( I_1(\chi_0 r) \sin \Theta \ e^{-i\beta z} \right)
\]

and obtain from (2.0) a set of fields

\[
E_r = A_o \left\{ \frac{-i\beta}{ak} I'(\chi_0 r) + \frac{i}{a\chi_0 r} I(\chi_0 r) \right\} \cos \Theta \ e^{-i\beta z}
\]

\[
E_\Theta = 0 \text{ etc.}
\]

which constitutes a valid solution for \( k \neq \beta_0, \chi_0 \neq 0 \), but vanish in the limit \( k \to \beta_0 \). We obtain a second \( k = \beta_0 \) solution by applying d'Alembert's method to these fields; differentiating with respect to \( k \) (with \( \beta_0 \) constant, so that \( \frac{\partial}{\partial k} = -\frac{k}{\chi_0} \frac{\partial}{\partial \chi_0} \)) and then putting \( k \to \beta_0, \chi_0 \to 0 \). This yields:

\[
E_r = i A_o \left( \frac{1 + k^2 r^2}{2k} \right) \cos \Theta \ e^{-ikz}
\]

\[
E_\Theta = -i A_o \left( \frac{1 - k^2 r^2}{2k} \right) \sin \Theta \ e^{-ikz}
\]

\[
E_z = A_o \sum_a \cos \Theta \ e^{-ikz}
\]

(2.8)
\[ Z_0 h_z = -iA_0 \left( \frac{1 + k^2 r^2}{2ka} \right) \sin \theta e^{-ikz} \]

\[ Z_0 h_\theta = -iA_0 \left( \frac{1 - k^2 r^2}{2ka} \right) \cos \theta e^{-ikz} \]

\[ Z_0 h_r = -A_0 \frac{r}{a} \sin \theta e^{-ikz} \]

We could equally well have derived a second \( n = 0 \) solution from a \( \pi_z \) or a \( \pi_z^* \) above, rather than from (2.6); this would have given us, in place of (2.8), some linear combination of (2.4) and (2.8).

There is, however, a certain convenience in having our \( \nu = c \) modes expressed as (2.4) which gives no deflecting force on a \( \nu = c \) particle, and (2.8) in which the electric and magnetic deflecting forces are equal. Furthermore, (2.4) and (2.8) are orthogonal in respect of power flow and stored energy, so the linear combination of them that is most efficient in producing a deflection from a given amount of stored energy is in fact (2.8) alone.

Using (2.3) for \( n \neq 0 \) with (2.4) and (2.6) for \( n = 0 \) we have the complete expansion of the fields in the axial region:

\[ E_r = i \left[ A_0 \left( \frac{1 + k^2 r^2}{2ka} \right) + B_0 \right] \cos \theta e^{-ikz} \]

\[ + \sum_{n \neq 0} \frac{i \beta}{r} A \frac{I'(x_r)}{r \chi_n} \cos \theta e^{-i \beta_n z} \]

\[ + \sum_{n \neq 0} \frac{1}{r \chi_n} B \frac{I(x_r)}{I'(x_a)} \cos \theta e^{-i \beta_n z} \]

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\[ E_\theta = -i \left[ A_o \left( \frac{1 - \frac{k^2 r^2}{2}}{2ka} \right) + B_o \right] \sin \Theta \ e^{-ikz} \]

\[ -i \sum_{n \neq 0} A_n \frac{\beta_n}{\chi_n^2} \frac{I(\chi_n r)}{I_1(\chi_n r)} \sin \Theta \ e^{-i\beta_n z} \]

\[ -ik \sum_{n \neq 0} B_n \frac{\chi_n I'(\chi_n r)}{\chi_n I_1(\chi_n r)} \sin \Theta \ e^{-i\beta_n z} \]

\[ E_z = A_o \frac{r}{a} \cos \Theta \ e^{-ikz} \]

\[ + \sum_{n \neq 0} A_n \frac{I(\chi_n r)}{I_1(\chi_n r)} \cos \Theta \ e^{-i\beta_n z} \]

\[ Z_{\theta r} = i \left[ -A_o \left( \frac{1 + \frac{k^2 r^2}{2}}{2ka} \right) + B_o \right] \sin \Theta \ e^{-ikz} \]

\[ + ik \sum_{n \neq 0} A_n \frac{\beta_n}{\chi_n^2} \frac{I(\chi_n r)}{I_1(\chi_n r)} \sin \Theta \ e^{-i\beta_n z} \]

\[ + i \sum_{n \neq 0} B_n \frac{\beta_n}{\chi_n} \frac{I'(\chi_n r)}{I_1(\chi_n r)} \sin \Theta \ e^{-i\beta_n z} \]

\[ Z_{\theta \phi} = i \left[ -A_o \left( \frac{1 - \frac{k^2 r^2}{2}}{2ka} \right) + B_o \right] \cos \Theta \ e^{-ikz} \]

\[ + ik \sum_{n \neq 0} A_n \frac{\chi_n I'(\chi_n r)}{I_1(\chi_n r)} \cos \Theta \ e^{-i\beta_n z} \]

\[ + i \sum_{n \neq 0} B_n \frac{\beta_n}{\chi_n^2} \frac{I(\chi_n r)}{I_1(\chi_n r)} \cos \Theta \ e^{-i\beta_n z} \]

(2.8)*
\[ Z_{H_z} = -A \sum_{n \neq 0} \frac{I_n}{\chi_n} \sin \theta e^{-ikz} \]
\[ + \sum_{n \neq 0} B_n \frac{I_n(\chi_n r)}{I_n(\chi_n)} \sin \theta e^{-i\beta_n z} \] 

(2.8)*

We now have series expansions for the fields in both the slots and the axial region, and must match them over their common surface.

2b Matching of the Tangential Fields at the Slot Mouth

Consider the electric fields first. In the slots, at \( r = a \)

\[ \frac{E_{\theta}}{\sin \theta e^{iwt}} = -\frac{1}{a} \sum_{m=2,4,6,\ldots} \frac{C_{m}}{Y_{m}^2} \sin g_{m} z + \frac{1}{a} \sum_{m=1,3,5,\ldots} \frac{C_{m}}{Y_{m}^2} \cos g_{m} z \]

\[ -ik \sum_{m=2,4,6,\ldots} \frac{D_{m}}{Y_{m}} \cos g_{m} z - ik \sum_{m=1,3,5,\ldots} \frac{D_{m}}{Y_{m}} \cos g_{m} z \] 

(2.9)

and in the axial region

\[ = -i \left[ A_0 \left( \frac{1 - k^2 r^2}{2ka} \right) + B_0 \right] e^{-ikz} \]

\[ -i \sum_{n \neq 0} A_n e^{ikz} -i\beta_n \sum_{n \neq 0} B_n e^{-i\beta_n z} \]

(2.10)

Expression (2.10) must equal (2.9) for \(-\frac{d}{2} < z < \frac{d}{2}\) and equal 0 for \(\frac{d}{2} < |z| < \frac{D}{2}\). We multiply by \(e^{i\beta_n z}\) or \(e^{ikz}\) and integrate over \(\frac{D}{2} < z < \frac{D}{2}\), obtaining

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\[ -i \left[ A_0 \frac{1-k^2\alpha^2}{2} + B_0 \right] D \]

\[ = \frac{2i}{\alpha} \sum_{m=2,4,\ldots} \infty C_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^m \sin \frac{k d}{2}}{(g_m^2 - k^2)} \]

\[ + \sum_{m=1,3,\ldots} \infty C_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^{m-1} \cos \frac{k d}{2}}{(g_m^2 - k^2)} \tag{2.11} \]

\[ - 2k \sum_{m=2,4,\ldots} \infty D_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^m \sin \frac{k d}{2}}{(g_m^2 - k^2)} \]

\[ - 2ik \sum_{m=1,3,\ldots} \infty D_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^{m-1} \cos \frac{k d}{2}}{(g_m^2 - k^2)} \]

and for \( n = 0 \)

\[ -i A_n \frac{\beta_n}{\ell \chi_n^2} D - ik \frac{B_n}{\ell \chi_n^2} D \]

\[ = \frac{2i}{\alpha} \sum_{m=2,4,\ldots} \infty C_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^m \sin \frac{\beta_n d}{2}}{(g_m^2 - \beta_n^2)} \]

\[ + \sum_{m=1,3,\ldots} \infty C_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^{m-1} \cos \frac{\beta_n d}{2}}{(g_m^2 - \beta_n^2)} \tag{2.12} \]

\[ - 2k \sum_{m=2,4,\ldots} \infty D_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^m \sin \frac{\beta_n d}{2}}{(g_m^2 - \beta_n^2)} \]

\[ - 2ik \sum_{m=1,3,\ldots} \infty D_m \frac{\ell_m^2}{Y_m^2} \frac{(-1)^{m-1} \cos \frac{\beta_n d}{2}}{(g_m^2 - \beta_n^2)} \]

6987/p
Now consider $E_z'$.

At $r = a$, in the slots

$$
\frac{E_z}{\cos \theta e^{jwt}} = C_0 + \sum_{m=2,4,\ldots}^{\infty} C_m \cos \theta_m z + \sum_{m=1,3,\ldots}^{\infty} C_m \sin \theta_m z \quad (2.13)
$$

and in the axial region

$$
\frac{E_z}{\cos \theta e^{jwt}} = A_0 e^{-ikz} + \sum_{n=0}^{\infty} A_n e^{-i \beta_n z} \quad (2.14)
$$

Again (2.14) = (2.13) for $-\frac{d}{2} < z < \frac{d}{2}$ and $= 0$ for $\frac{d}{2} < |z| < \frac{D}{2}$.

Multiplying by $e^{i \beta_n z}$ and integrating over $-\frac{D}{2} < z < \frac{D}{2}$, we find

$$
A_n D = 2C_0 \frac{\sin \theta_n \frac{d}{2}}{\beta_n} \quad (\text{including } n=0)
$$

$$
-2 \sum_{m=2,4,\ldots}^{\infty} \frac{C_m (-1)^m \beta_n \sin \beta_n \frac{d}{2}}{(\theta_m^2 - \beta_n^2)} + 2i \sum_{m=1,3,\ldots}^{\infty} \frac{C_m (-1)^{m-1} \beta_n \cos \beta_n \frac{d}{2}}{(\theta_m^2 - \beta_n^2)} \quad (2.15)
$$

Also at $r = a$, in the slots

6987/p
\[ \frac{Z_0 H_0}{\cos \theta, e^{iwt}} = -i C \frac{L'(ka)}{L_0(k)} + ik \sum_{m=2,4}^{\infty} \frac{C}{M_m} \frac{M'(Y_n)}{M_1(Y_n)} \cos g_m z \]

\[ + ik \sum_{m=1,3}^{\infty} \frac{C}{M_m} \frac{M'(Y_n)}{M_1(Y_n)} \sin g_m z \]  

\[ \frac{1}{D_m} \frac{N(Y_n)}{N_1(Y_n)} \cos g_m z \]

\[ \frac{1}{D_m} \frac{N(Y_n)}{N_1(Y_n)} \sin g_m z \]  

and in the axial region

\[ \frac{Z_0 H_0}{\cos \theta, e^{iwt}} = i \left[ -A_0 \left( \frac{1 - k^2 a^2}{2ka} \right) + B_0 \right] e^{-ikz} \]

\[ + ik \sum_{n \neq 0}^{\infty} \frac{A}{\chi_n} \frac{I'(\chi_n)}{I_1(\chi_n)} e^{-i\beta_n z} \]

\[ + i \sum_{n \neq 0}^{\infty} \frac{B}{\chi_n^2} \frac{I(\chi_n)}{I_1(\chi_n)} e^{-i\beta_n z} \]  

The magnetic fields need be equated only over their common region

\[ |z| < \frac{d}{2} \], since on the surface \( \frac{d}{2} < |z| < \frac{d}{2} \), the boundary condition will be satisfied by the surface current. We multiply \( \cos g_m z \) or \( \sin g_m z \)

and integrate over \( -\frac{d}{2} \leq z \leq \frac{d}{2} \), obtaining
\[
\begin{align*}
\frac{2i}{k} & \left[ -A_0 \left( 1 - \frac{k^2 a^2}{2} \frac{1}{2ka} \right) + B_0 \right] \sin k \frac{d}{2} \\
+ 2ik \sum_{n=0}^{\infty} & \frac{A_n}{\chi_n} \frac{I'(\chi_n)}{I_n(\chi_n)} \sin \beta_n \frac{d}{2} \\
& + 2i \sum_{n=0}^{\infty} \frac{B_n}{\chi_n^2} \frac{I(\chi_n)}{I'(\chi_n)} \sin \beta_n \frac{d}{2} \\
= -i & \frac{C_0}{L'(ka)} \frac{L'(ka)}{L_1(ka)} \\
\text{Also for } m = 2, 4, \ldots \\
\frac{ik}{Y_m} & \frac{M'(Y_m)}{M(Y_m)} \frac{d}{2} - \frac{i}{a} \frac{g_m}{N(Y_m)} \frac{N(Y_m)}{N'(Y_m)} \frac{d}{2} \\
= 2ik & \left[ -A_0 \left( 1 - \frac{k^2 a^2}{2} \frac{1}{2ka} \right) + B_0 \right] \sin k \frac{d}{2} (-1)\frac{m}{2} \\
& \frac{1}{(k^2 - g_m^2)} \\
& \frac{2i}{\chi_m} \frac{I'(\chi_m)}{I_n(\chi_m)} \sin \beta_n \frac{d}{2} (-1)\frac{m}{2} \\
& \frac{B_n}{\chi_n^2} \frac{I(\chi_n)}{I'(\chi_n)} \sin \beta_n \frac{d}{2} (-1)\frac{m}{2} \\
& \frac{1}{(\beta_n^2 - g_m^2)} \\
\text{and for } m = 1, 3 \ldots
\end{align*}
\]
\[
-2k \left[ -A \frac{\left( 1 - \frac{k^2 a^2}{2} \right)}{2ka} + B \right] \cos \frac{\theta}{2} \left( -1 \right)^{\frac{m-1}{2}} \\
\left( k^2 - g_m^2 \right)
\]

\[
-2k \sum_{n \neq 0} \frac{A_n}{\chi_n} I_0^\prime (\chi_n a) \frac{\cos \beta_n}{2} \left( -1 \right)^{\frac{m-1}{2}} \left( \beta_n^2 - g_m^2 \right)
\]

\[
2 \sum_{n \neq 0} \frac{B_n}{\chi_n^2} I_1 (\chi_n a) \frac{\cos \beta_n}{2} \left( -1 \right)^{\frac{m-1}{2}} \left( \beta_n^2 - g_m^2 \right)
\]

We must also match $H_z$.

At $r = a$, in the slots

\[
\frac{Z}{\sin \theta} e^{-iwt} = \sum_{m=2,4,\ldots} D_m \frac{N(Y_m)}{N'(Y_m)} \sin g_m z + \sum_{m=1,3,\ldots} D_m \frac{N(Y_m)}{N'(Y_m)} \cos g_m z
\]

and in the axial region

\[
\frac{Z}{\sin \theta} e^{-iwt} = -A_0 e^{-ikz} + \sum_{n \neq 0} B_n \frac{I_0'(\chi_n a)}{\chi_n} e^{-i\beta_n z}
\]

On multiplying by $\sin$ or $\cos g_m z$ and integrating from $-\frac{d}{2}$ to $\frac{d}{2}$ we find for $m$ even
\[
D_m \frac{N(Y_m)}{N'(Y_m)} \frac{a^d}{2} = \frac{2iA(-1)^m \sin k \frac{a}{2}}{(k^2 - \xi_m^2)} \\
- 2i \sum_{n=0}^{\infty} B_n \frac{I_i(\chi_n)(-1)^m \sin \beta_n \frac{a}{2}}{I'_i(\chi_n) (\beta_n^2 - \xi_m^2)}
\]

and for \( m \) odd

\[
2A(-1)^{\frac{m-1}{2}} \frac{\cos k \frac{a}{2}}{(k^2 - \xi_m^2)} \\
- 2 \sum_{n=0}^{\infty} B_n \frac{I_i(\chi_n)(-1)^{\frac{m-1}{2}} \cos \beta_n \frac{a}{2}}{I'_i(\chi_n) (\beta_n^2 - \xi_m^2)}
\]

\( C_m \) for \( m \) odd and \( D_m \) for \( m \) even are imaginary.

Let us write \( C_m(m \text{ even}) = C'_m, \quad D_m(m \text{ even}) = iD'_m \)
\( C_m(m \text{ odd}) = iC'_m, \quad D_m(m \text{ odd}) = D'_m \)

Then, from the matching we have found the following relationships

\[
A_n = -\frac{2\beta_n}{D_n} \sum_{m=0,1,2,3}^{\infty} \frac{C'_m}{(\xi_m^2 - \beta_n^2)} \left\{ (-1)^{\frac{m}{2}} \sin \beta_n \frac{a}{2}, \quad m \text{ even} \right\} \\
\quad \left\{ (-1)^{\frac{m-1}{2}} \cos \beta_n \frac{a}{2}, \quad m \text{ odd} \right\}
\]

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\[ B_o = -\frac{A}{2ka} \left(1 - \frac{k^2 a^2}{2}\right) \]

\[- \frac{2}{aD} \sum_{m=1, 2, 3, \ldots} C^m \frac{g^2_m}{\gamma_m^2(g^2_m - k^2)} \times \left\{ (-1)^{m/2} \sin \left(\frac{k}{2}\right), \quad m \text{ even} \right\} \left\{ (-1)^{m+1/2} \cos \left(\frac{k}{2}\right), \quad m \text{ odd} \right\} \]

\[+ \frac{2k}{D} \sum_{m=1, 2, 3, \ldots} D^m \frac{g^2_m}{\gamma_m^2(g^2_m - k^2)} \times \left\{ (-1)^{m/2} \sin \left(\frac{d}{2}\right), \quad m \text{ even} \right\} \left\{ (-1)^{m+1/2} \cos \left(\frac{d}{2}\right), \quad m \text{ odd} \right\} \]

\[ B_n = -\frac{\beta_n}{k a \chi_n} \]

\[- \frac{2\chi_n}{a k D} \sum_{m=1, 2, 3, \ldots} C^m \frac{g^2_m}{\gamma_m^2(g^2_m - \beta_n^2)} \times \left\{ (-1)^{m/2} \sin \left(\frac{\beta_n}{2}\right), \quad m \text{ even} \right\} \left\{ (-1)^{m+1/2} \cos \left(\frac{\beta_n}{2}\right), \quad m \text{ odd} \right\} \]

\[+ \frac{2\chi_n}{D} \sum_{m=1, 2, 3, \ldots} D^m \frac{g^2_m}{\gamma_m^2(g^2_m - \beta_n^2)} \times \left\{ (-1)^{m/2} \sin \left(\frac{d}{2}\right), \quad m \text{ even} \right\} \left\{ (-1)^{m+1/2} \cos \left(\frac{d}{2}\right), \quad m \text{ odd} \right\} \]

\[ D_m' = \frac{N'(Y) a_{g_m}}{N(Y) a_{(k^2 - g_m^2)}} \times \left\{ (-1)^{m/2} \sin \left(\frac{k}{2}\right), \quad m \text{ even} \right\} \left\{ (-1)^{m+1/2} \cos \left(\frac{k}{2}\right), \quad m \text{ odd} \right\} \]

\[- \frac{1}{d} \sum_{n=0}^{1} \frac{B_n}{N(Y) a_{(g_m^2 - \beta_n^2)}} \times \left\{ (-1)^{m/2} \sin \left(\frac{\beta_n}{2}\right), \quad m \text{ even} \right\} \left\{ (-1)^{m+1/2} \cos \left(\frac{\beta_n}{2}\right), \quad m \text{ odd} \right\} \]
\[ C_0' = \frac{-2L}{kdL_1'(ka)} \left[ \frac{A}{2ka} \left( \frac{1 - k^2a^2}{2} \right) + B_0 \right] \sin \frac{k d}{2} \]

\[ - \frac{2L}{L_1'(ka)} \sum_{n \neq 0} \frac{A_n I'(\chi_n a)}{\chi_n^2 I'_1(\chi_n a)} \sin \frac{\beta_n d}{2} \]

\[ - \frac{2L}{a} \sum_{n \neq 0} \frac{B_n I'(\chi_n a)}{\chi_n^2 I'_1(\chi_n a)} \sin \frac{\beta_n d}{2} \]

\[ C_m' = D_m' \frac{\xi_m}{a Y_k N_1'(Y_m a) M_1'(Y_m a)} \left[ - \frac{A}{2ka} \left( \frac{1 - k^2a^2}{2} \right) + B_0 \right] \times \left\{ \frac{(-1)^m}{\cos \frac{k d}{2}}, \text{ m even} \right\} \]

\[ + \frac{L}{d} \frac{Y_m}{(k^2 - \xi_m^2) M_1'(Y_m a)} \sum_{n \neq 0} \frac{A_n I'(\chi_n a) \beta_n}{\chi_n^2 I'_1(\chi_n a) (\beta_n^2 - \xi_m^2)} \times \left\{ \frac{(-1)^m}{\cos \frac{\beta_n d}{2}}, \text{ m even} \right\} \]

\[ + \frac{L}{d} Y_m \sum_{n \neq 0} \frac{B_n \beta_n^2 I'_1(\chi_n a)}{\chi_n^2 I'_1(\chi_n a) (\beta_n^2 - \xi_m^2)} \times \left\{ \frac{(-1)^m}{\cos \frac{\beta_n d}{2}}, \text{ m even} \right\} \]

Write \( A \) for the vector \((A_0, B_0, A_{-1}, B_{-1}, A_1, B_1, A_2, \ldots, \ldots)\)

and \( C \) for the vector \((C'_0, D'_0, C'_{-1}, D'_{-1}, C'_1, D'_1, C'_2, \ldots, \ldots)\)
The matching procedure has produced $A = \mathcal{G}C$, and $\mathcal{C} = \mathcal{F}A$ where $\mathcal{G}$ and $\mathcal{F}$ are matrices whose elements are the coefficients of the various matching equations above.

Hence $\mathcal{C} = \mathcal{F}G_C$

or $(\mathcal{F}G_C - 1)\mathcal{C} = 0 \tag{2.22}$

For this to be possible the matrix $(\mathcal{F}G_C - 1)$ must have zero determinant. The elements of $\mathcal{F}$ and $\mathcal{G}$ are functions of $\omega$, $a$, $b$, $d$, $D$ and one must fix four of these and calculate the fifth to satisfy this condition. $\mathcal{F}$ and $\mathcal{G}$ are of course infinite matrices, and in practice the numerical work may be considerable before a sufficiently accurate solution has been found.

3a Method of Solution

The secular determinant equation is normally regarded as an equation to be solved for $\omega$ with the dimensions fixed, but from a practical point of view we are interested in fixing $\omega$, $a$, $d$, $D$ and finding the value of $b$ which gives the correct phase velocity. The parameter $b$ also has considerable advantages from the computational point of view:

1) it affects only the slot-modes, leaving the ones in the beam region unchanged in form

2) we have written down the series for the modes in such a way that only the matrix $\mathcal{F}$ is affected by a change in $b$, $\mathcal{G}$ being calculable from $\omega$, $a$, $d$, $D$, and remaining fixed while one searches for the correct $b$

3) for practical values of $d$, all the slot-modes are highly evanescent except for the $m = 0$ electric mode, and the properties at $r = a$ of these evanescent modes are very little affected by the value of $b$.

Thus the only quantity strongly sensitive to $b$ appearing in any of our matrix elements is the factor $\frac{L'(ka)}{L(ka)}$, which appears right across the first row of the matrix $\mathcal{F}$. 

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It is therefore possible to proceed by the following method of successive approximation, which converges rapidly.

We can put

$$F' = L \tilde{F}$$  \hspace{1cm} (3.1)

where $L$ is the diagonal matrix

$$
\begin{pmatrix}
L'(ka) & 0 & 0 & \ldots \\
L'(ka) & 0 & 1 & \ldots \\
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots 
\end{pmatrix}
$$  \hspace{1cm} (3.2)

Then the elements of $F'$ are all rather insensitive to $b$, and the programme evaluates them using some approximate value of it, say $b_0$. This amounts to using the approximate $b_0$ in all the evanescent $m \neq 0$ slot modes, while leaving $b$ as an unknown in the $m = 0$ slot mode.

We can now write (2.22) as

$$L \tilde{C} = \tilde{F}' \tilde{G} \tilde{C}$$  \hspace{1cm} (3.3)

and subtract $\tilde{C}$ from both sides to obtain

$$
\begin{pmatrix}
L'(ka) \\
L'(ka) - 1
\end{pmatrix}
\begin{pmatrix}
C' \\\n0, 0, 0, \ldots
\end{pmatrix}
= (\tilde{F}' \tilde{G} - 1) (C'_0, D'_1, \ldots)
$$  \hspace{1cm} (3.4)
Therefore
\[ m_{11} \left\{ \frac{L'(ka)}{L_1(ka)} - 1 \right\} = 1 \] (3.5)

where \( m_{11} \) is the top left-hand corner element of
\[ \left( F_g C - l_2 \right)^{-1} \]
and so
\[ \frac{L'(ka)}{L_1(ka)} = 1 + \frac{1}{m_{11}} \]
\[ = B, \text{ say.} \] (3.6)

The programme performs the matrix operations and obtains a
value for \( m_{11} \) and hence a value, \( B \), for \( \frac{L'(ka)}{L_1(ka)} \). It is then
a matter of solving a simple transcendental equation for \( b \), which gives
a better estimate, say \( b_1 \). The above process is then repeated
(begging again at the evaluation of \( F_g' \)) using \( b_1 \), and so on until
sufficient accuracy is obtained.*

It is worth remarking that the equation
\[ \frac{L'(ka)}{L_1(ka)} = B \]
is of precisely the same form as the equation obtained for matching average
wave impedance across the slot mouth if the problem is treated in smooth
approximation, i.e. with a field of given properties in the cylindrical
region and only the \( m = 0 \) electric mode in the slots. The difference
in our case is that our matrix operations have enabled us to find a
value of \( B \) that takes account of the other modes.

* In practice each iteration adds 3 or 4 significant figures to the
result.
The transcendental equation for \( b \) is the boundary condition:

\[
J_1'(kb) + C Y_1(kb) = L_1(kb) = 0,
\]

subject to the coefficient \( C \) being determined by the matching condition at \( r = a \), (3.6), which can be written:

\[
C = \frac{J'(ka) - BJ(ka)}{BY_1(ka) - Y'_1(ka)} \quad (3.8)
\]

So we have

\[
\frac{J(kb)}{Y_1(kb)} + \frac{J'(ka) - BJ(ka)}{BY_1(ka) - Y'_1(ka)} = 0 \quad (3.9)
\]

The programme solves this for \( b \) by a Newton-Raphson iteration, making use of the fact that the derivative with respect to \( b \) of the left-hand side of (3.9) is

\[
\frac{\partial}{\partial b} \left( \frac{J(kb)}{Y_1(kb)} \right) = k \frac{J(kb)Y'(kb) - J'(kb)Y(kb)}{Y_1^2(kb)}
\]

\[
= - \frac{2}{\pi b Y_1^2(kb)} \quad (3.10)
\]
The Stored Energy

The programme also evaluates the stored energy. The stored energy density when working in MKS units in free space is

$$\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 = \frac{1}{2} \epsilon_0 \left( E^2 + Z_0^2 H^2 \right)$$

where $$\frac{1}{2} \epsilon_0 = \frac{10^{-9}}{2\pi}$$

We calculate the time average of the stored energy over a metre of the guide. When working with complex fields the stored energy density is

$$\frac{1}{4} \epsilon \left( EE^* + Z_0^2 HH^* \right)$$

In evaluating the stored density we will normalise the amplitudes of the modes by multiplying all the coefficients by $$\frac{k\alpha}{A_o^*}$$. This sets the new value of $$A_o$$ at $$k\alpha$$, giving a total deflection force equivalent to unity electric (peak) deflecting field. For convenience we multiply by $$10^7$$ giving $$10^7$$V/m standard field. We therefore calculate

$$\frac{10^5}{144\pi} \left( EE^* + Z_0^2 HH^* \right)$$

$$= \frac{2}{\pi} W(EE^* + Z_0^2 HH^*)$$ where $$W = \frac{10^5}{288}$$

Performing the integration over $$r$$ and $$\theta$$, we find the following contributions

1) $$A_0^2 \frac{a^2}{k^2 a^2} \left[ 1 + k^2 a^2 + \frac{k^4 a^4}{12} \right] \left[ a^2 \left[ 1 + k^2 a^2 + \frac{k^4 a^4}{12} \right] \right]$$

since $$A_o$$ has been set equal to $$ka$$

and $$4Wa_0^2 B_0^2$$

$$E_y = Z_0 H_x = \frac{1}{2} A_0 e^{-1kz}$$ at $$r = 0$$, from (2.8)

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The $A_B$ items vanish

2) Using

$$\int_0^a I_1^2(\chi_n r) r dr = \frac{1}{2} \left[ \left( r^2 + \frac{1}{\chi_n^2} \right) I_1^2(\chi_n r) - r^2 I_1^2(\chi_n r) \right]_0^a$$

and

$$\int_0^a \left( \chi_n^2 I_1'(\chi_n r) + \frac{I_1^2(\chi_n r)}{r^2} \right) r dr$$

$$= \left[ \frac{\chi_n^2}{2} \left( r^2 I_1^2(\chi_n r) + \frac{2r}{\chi_n} I_1(\chi_n r) I_1'(\chi_n r) \right) - \left( r^2 + \frac{1}{\chi_n^2} \right) I_1^2(\chi_n r) \right]_0^a$$

to find

$$\sum_{n \neq 0}^{WA} \frac{2a \left( 1 + \frac{2k^2}{\chi_n^2} \right) I_1'(\chi_n a)}{\chi_n} + \frac{2k^2}{\chi_n^2} \left( a^2 - \frac{I_1'(\chi_n a)}{I_1(\chi_n a)} \right)$$

$$+ \left( a^2 + \frac{1}{\chi_n^2} \right) I_1^2(\chi_n a)$$

and

$$\sum_{n}^{WB} \frac{2k^2}{\chi_n^2} \left( a^2 - \left( a^2 + \frac{1}{\chi_n^2} \right) \frac{I_1(\chi_n a)}{I_1'(\chi_n a)} \right) + \frac{2a}{\chi_n} \left( 1 + \frac{2k^2}{\chi_n^2} \right) I_1(\chi_n a)$$

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3) Also since

\[ \int_{0}^{a} I_{n}(\chi_{n}r)I'_{n}(\chi_{n}r)dr = \frac{1}{2\chi_{n}} I_{n}^{2}(\chi_{n}a) \]

we have

\[ W \sum_{n} 8A_{n} B_{n} \frac{\beta_{n} \ k_{n}}{\chi_{n}^{4}} I_{n}^{2}(\chi_{n}a) \]

This completes the stored energy in the axial region.

In the slots,

4) since

\[ \int_{0}^{b} L_{n}^{2}(kr)rd\gamma = \frac{1}{2} \left[ r^{2} L_{n}^{12}(kr) + (r^{2} - \frac{1}{k^{2}}) L_{n}^{2}(kr) \right]_{a}^{b} \]

and

\[ \int_{a}^{b} \left[ L_{n}^{12}(kr) + \frac{L_{n}^{2}(kr)}{k^{2}r^{2}} \right]rdr \]

\[ = \frac{1}{2} \left[ r^{2} L_{n}^{12}(kr) + \frac{2r}{k} L_{n}^{1}(kr) L_{n}^{1}(kr) + (r^{2} - \frac{1}{k^{2}}) L_{n}^{2}(kr) \right]_{a}^{b} \]

we have (remembering that \( L_{n}^{1}(kb) = 0 \))

\[ 2 \ W \sum_{D} C_{n}^{2} \left[ b^{2} L_{n}^{12}(kb) - a^{2} L_{n}^{12}(ka) - \frac{1}{k} L_{n}^{1}(ka) - (a^{2} - \frac{1}{k^{2}}) \right] \]
The integrals involving $I_1$ quoted in (2) for the $A_m^2$ terms are valid also for $M_1(Y_m)$ (there is also a lower limit in this case,) and since $M_1(Y_m) = 0$ we have

\[
5) \quad \sum_{m} \frac{1}{2} \int \frac{dW}{D_m} C \left[ \frac{2k^2}{Y_m^2} \left( \frac{b^2 + 2(Y_m)N_1^2(Y_m)}{M_1^2(Y_m)} \right) + \left( a^2 + \frac{1}{Y_m^2} \right) \right]
\]

\[+ \left( 1 + \frac{2k^2}{Y_m^2} \right) \left( -\frac{2a}{Y_m \cdot M_1(Y_m)} \right) \]

Using the same integrals with $N_1(Y_m)$ for $I_1$, and since $N_1(Y_m) = 0$, we find

\[
6) \quad \sum_{m} \frac{1}{2} \int \frac{dW}{D_m} C \left[ \frac{2k^2}{Y_m^2} \left( -a^2 - \frac{1}{Y_m^2} \right) \right] \left( \frac{b^2 + \frac{1}{Y_m^2}}{N_1^2(Y_m)} \right) + \frac{2k^2}{Y_m^2} \left( -\frac{2a}{Y_m \cdot N_1(Y_m)} \right) \]

and using

\[
\int_a^b \left( \frac{M_1(Y_m)N_1(Y_m) + M_1(Y_m)N_1'(Y_m)}{M_1(Y_m)} \right) \, dr
\]

\[= \frac{1}{Y_m} \left[ \frac{M(Y_m)N(Y_m)}{M(Y_m)} \right]^b_a = \frac{-1}{Y_m} M(Y_m)N(Y_m)
\]

we find

\[
7) \quad \sum_{m} \frac{4W}{Y_m^2} C \left[ D_m \left( -\frac{1}{M(Y_m)} \right) \right] \int \frac{dW}{D_m} \left( \frac{N(Y_m)}{N_1'(Y_m)} \right)
\]

This concludes the expressions for the stored energy.
4b R.F. Losses

We require the time average power dissipation per metre at our standard amplitude. The losses are worked out for one cell and then multiplied by the number of cells per metre.

The instantaneous surface dissipation in watts per square metre is

$$H^2 R$$

where $R$ is the surface resistivity of copper at the frequency used, in ohms per square, and $H$ is the surface current density in amps per metre, numerically equal to the magnetic field in M.K.S. units. We are working with a free-space wavelength of 0.1 metre. (The results can be scaled if necessary). For pure copper at this $\lambda_o$,

$$R = 1.428 \times 10^{-2} \text{ ohms per sq.}$$

We therefore calculate

$$\frac{10^{14} \times \text{ (number of cells per metre)} \times 10^{-6} (1.428) 10^{-2} \int (Z H)^2 dS}{(376.7)^2} \text{ averaged surfaces}$$

which gives the answer in $\text{MW}$

$$= 10.063 \times \text{(number of cells/m.)} \int (Z H)^2 dS$$

which with complex fields

$$= \frac{2W}{\pi} \frac{1}{2} \int (Z H)(Z H^*) dS \text{ where } W = \frac{\pi}{2} (10.063) \times \text{(number of cells/m.)}$$

Consider first the losses on the outer cylindrical wall at $r = b$.

For the loss right across one slot we require
\[
\begin{align*}
\mathcal{W} \int_{\pi}^{2\pi} \int_{\theta=0}^{2\pi} & \left( \mathcal{H}_{\Theta} \mathcal{H}_{\Theta}^* + \mathcal{H}_{Z} \mathcal{H}_{Z}^* \right) \, \bar{b} \, \bar{a} \, \bar{z} \\
& = \mathcal{W} b \left[ \frac{dC}{d} \left\{ \left( \frac{L'(kb)}{L'(ka)} \right)^2 \right\} + k^2 \sum_m C_m^2 \frac{1}{Y_m^2} \left( \frac{M'(Yb)^2}{M(Ya)} \right) \right] \\
& + \frac{1}{b^2} \sum_m D_m^2 \frac{1}{Y_m^4} \left( \frac{N(Yb)}{N'(Ya)} \right)^2 \\
& - \frac{2k}{b} \sum_m D_m C_m^2 \frac{1}{Y_m^3} \frac{M'(Yb)}{M(Ya)} \frac{N(Yb)}{N'(Ya)} \\
& + \sum_m D_m^2 \left( \frac{N(Yb)}{N'(Ya)} \right)^2
\end{align*}
\]

On the disc faces we want

\[
\begin{align*}
\mathcal{W} b \int_{\theta=0}^{2\pi} \left( \mathcal{H}_{\Theta} \mathcal{H}_{\Theta}^* + \mathcal{H}_{Z} \mathcal{H}_{Z}^* \right) \, dS \quad \text{on each disc}
\end{align*}
\]

\[
\begin{align*}
& = \mathcal{W} b \int_{r=a}^{b} \left( \mathcal{H}_{\Theta} \mathcal{H}_{\Theta}^* + \mathcal{H}_{Z} \mathcal{H}_{Z}^* \right) \, r \, d\Theta \, dr \quad \text{when including two disc faces}
\end{align*}
\]
The following contributions to this arise

\begin{align*}
(1) \quad & 2W C'_{20}^2 \left[ \frac{1}{L_i^2(ka)} \int_a^b L_i^2(kr) \, rdr + \frac{1}{k L_i^2(ka)} \int_a^b \frac{L_i^2(kr)}{r} \, dr \right] \\
& = W C'_{20} \left[ b^2 \frac{L_i^{12}(kb)}{L_i^2(ka)} - a^2 \frac{L_i^{12}(ka)}{L_i^2(ka)} - \frac{2a L_i'(ka)}{k L_i'(ka)} - \left( a^2 - \frac{1}{k^2} \right) \right]
\end{align*}

\begin{align*}
(2) \quad & 2W k^2 \sum_{m=0}^{\infty} C_m^2 \frac{1}{M_1^2(Y_m)} Y_m^2 \left[ \int_a^b M_1^2(Y_m r) \, rdr + \frac{1}{Y_m} \int_a^b \frac{M_1(Y_m r)}{r} \, dr \right] \\
& = W k^2 \sum_{m=0}^{\infty} C_m^2 \frac{1}{Y_m^2} \left[ b^2 \frac{M_1^2(Y_m b)}{M_1^2(Y_m a)} - a^2 \frac{M_1^2(Y_m a)}{M_1^2(Y_m a)} - \frac{2a M_1(Y_m a)}{Y_m M_1(Y_m a)} + \left( a^2 + \frac{1}{Y_m^2} \right) \right]
\end{align*}

\begin{align*}
(3) \quad & 2W \sum_{m=0}^{\infty} \frac{D_m^2}{N_1^2(Y_m) Y_m^2} \left[ \frac{1}{Y_m} \int_a^b \frac{N_1(Y_m r)}{r} \, dr + \int_a^b \frac{N_1(Y_m r)}{r} \, rdr \right] \\
& = -W \sum_{m=0}^{\infty} \frac{D_m^2}{Y_m^2} \left[ a^2 - \left( a^2 + \frac{1}{Y_m^2} \right) \frac{N_1(Y_m a)}{N_1^2(Y_m a)} \right]
\end{align*}
(4) \(-\frac{L_{WC}'}{2} \sum_{m=2,4,6} \frac{C_m'(-1)^\frac{m}{2}}{Y_m M_1 M_1} \left[ \frac{1}{Y_m} \int_a^b L_{1}(kr) M_1 M_1 \right] \right. \\
+ k \int_a^b \left. L_1'(kr) M_1'(Y_m) \right) r \, dr \right\] \\
= \frac{L_{WC}'}{2} \sum_{m=2,4,6} \frac{C_m'(-1)^\frac{m}{2}}{Y_m^2} \left[ ka \frac{L_1'(ka)}{L_1(ka)} + \frac{k^2}{g_m} \left\{ \frac{-ka}{L_1'(ka)} + \frac{Y_a}{M_1'(Y_a)} \right\} \right] \\

(5) \frac{C'}{L_1(ka)} \sum_{m=2,4,6} \frac{D_m g_m(-1)^\frac{m}{2}}{Y_m N_1'(Y_a)} \left[ \frac{1}{k} \int_a^b L_1'(kr) N_1'(Y_m) \right. \right. \\
+ \frac{1}{Y_m} \left. \int_a^b L_1'(kr) N_1'(Y_m) \right) \left. \right. \right. \left. \right. \right. \right. \\
= \frac{L_{WC}'}{2} \sum_{m=2,4,6} \frac{D_m g_m(-1)^\frac{m}{2}}{k Y_m^2} \frac{N_1(Y_a)}{N_1'(Y_m)} \left[ -\frac{1}{k} \int_a^b L_1'(kr) N_1'(Y_m) \right. \right. \\
+ \frac{1}{Y_m} \int_a^b L_1'(kr) N_1'(Y_m) \right) \left. \right. \right. \right. \right. \\
= \frac{L_{WC}'}{2} \sum_{m=2,4,6} \frac{D_m g_m(-1)^\frac{m}{2}}{k Y_m^2} \frac{N_1(Y_a)}{N_1'(Y_m)}
\[(5)_{a}\]  \\[ \sum_{m=2,4,6} \frac{C'D'}{m} \frac{g_s}{m} \frac{(1)^{m+s}}{2} \left[ \frac{1}{Y_m} \int_{Y_m}^{b} M(Y, r) N(Y, r) \, dr \right] \\[+ \frac{1}{Y_s} \int_{Y_s}^{b} M'(Y, r) N'(Y, r) \, dr \] \\

= \frac{W}{2} \sum_{m=2,4,6} \frac{C'D'}{m} \frac{g_s}{m} \frac{(1)^{m+s}}{2} \frac{N(Y, a)}{Y_m^2 Y_s^2} \frac{1}{Y_s} \int_{Y_s}^{b} M'(Y, r) N'(Y, r) \, dr \\

and (b) the \(C'D'_{m,m} \) terms for \(m\) and \(m'\) odd are the same with 
\((-1)^{m+m'-2} \) replacing \((-1)^{m+m'} \) \\

We still have the losses on the disc edge to consider.
We require

\[
\frac{2W}{\pi} \int \int (a_o^2 H_o^2 + a_z^2 H_z^2) \, ds \\
= \frac{2W}{\pi} \int \int \left( a_o^2 H_o + a_z^2 H_z \right) \, d\Theta, dz \text{ per cell.} \\
\Theta=0 \ Z=Z \]

We find the following contributions

1) \[Wa_2 \int_{D-d} \left[ -A_0 \left( \frac{1}{2} - \frac{k^2 v^2}{2k_0 a} \right) + B_0 \right] + WaA_2 (D-d) \]

2) \[Wa \sum_{n=0}^{\infty} a_n \frac{I'(X, a)}{I_1 (X, a)} \left( \frac{1}{n} \right)^2 (D-d) \]

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3) \[ W_a \sum_{n \neq 0} B_n^2 \left[ \frac{\beta_n^2}{\alpha^2 \lambda_n^4} \frac{I^2(X_a)}{I_1'(X_a)} + \frac{I^2(X_a)}{I_1'(X_a)} \right] \] (D-d)

4) \[ 4W_a \left[ -\frac{a}{2} \left( 1 - \frac{k^2 a^2}{2} \right) + B \right] \sum_n \frac{k A_n I'(X_a)}{X_n I_1(X_a)} \int_0^D \frac{\cos \left( \frac{2\pi n z}{D} \right)}{2} dz \]

\[ = -2W_a \left[ -\frac{a}{2} \left( 1 - \frac{k^2 a^2}{2} \right) + B \right] \sum_n \frac{k A_n I'(X_a)}{X_n I_1(X_a)} \frac{D}{\pi n} \sin \left( \frac{\pi n d}{D} \right) \]

5) \[ -2W_a \left[ -\frac{a}{2} \left( 1 - \frac{k^2 a^2}{2} \right) + B \right] \sum_n \frac{I(X_a)}{I_1'(X_a)} \frac{B \beta_n D}{\pi n} \sin \left( \frac{\pi n a^2}{D} \right) \]

\[ + 2W_a \frac{A}{\alpha} \sum_{n \neq 0} \frac{I(X_a)}{I_1(X_a)} \frac{D}{\pi n} \sin \left( \frac{\pi n d}{D} \right) \]

6) \[ 2W(D-d) \kappa \sum_{n \neq 0} \frac{A}{\alpha} \frac{I(X_a)}{I_1(X_a)} \frac{1}{(n-s)} \sum_{s \neq 0} \frac{B \beta_s I(X_a)}{X_s^2 I_1'(X_s)} \times \left\{ \begin{array}{l} \frac{1}{\pi} \sin \left( \frac{\pi D}{D-d} (n-s) \right) \text{ if } n \neq s \\ \frac{D}{\pi} \sin \left( \frac{\pi D}{D-d} (n-s) \right) \text{ if } n = s \end{array} \right. \]

This concludes the R.F. losses.

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4c Power Flow

The power flow in the $z$-direction is

$$\text{(EXH)}_z = E_r H_z - E_z H_r$$

We evaluate this power flow across a plane, $z = \frac{D}{2}$, i.e., at the middle of a disc. Since we are using complex fields we want $\frac{1}{2} \Re \left( \text{EXH}^* \right)_z$ integrated over $r = 0$ to $a$ and $\Theta$ from 0 to $2\pi$.

The following terms arise

1) \[ \frac{1}{2} \int_0^{2\pi} \cos^2 \Theta \, d\Theta \int_0^a \left[ A_0 \left( \frac{1 + k^2 r^2}{2ka} \right) + B_0 \right] \left[ -A_0 \left( \frac{1 - k^2 r^2}{2ka} \right) + B_0 \right] r \, dr \]

\[ + \frac{1}{2} \int_0^{2\pi} \sin^2 \Theta \, d\Theta \int_0^a \left[ A_0 \left( \frac{1 - k^2 r^2}{2ka} \right) + B_0 \right] \left[ -A_0 \left( \frac{1 + k^2 r^2}{2ka} \right) + B_0 \right] r \, dr \]

\[ = \frac{\pi}{2} \left[ -\frac{A_0^2}{4k^2} \left( 1 + \frac{k^4 a^4}{12} + B_0^2 a^2 \right) \right] \]

2) \[ \sum_{n \geq 0} \sum_{s \neq 0} \frac{1}{2} \int_0^{2\pi} \cos^2 \Theta \, d\Theta \int_0^a \frac{A_n^s \beta \, k \, I_1^r \, I_0^s}{X_n \, X_s \, I_1^r \, I_0^s} \left( -1 \right)^{n-s} r \, dr \]

\[ + \sum_{n \geq 0} \sum_{s \neq 0} \frac{1}{2} \int_0^{2\pi} \sin^2 \Theta \, d\Theta \int_0^a \frac{A_n^s \beta \, I_1^r \, I_0^s}{X_n \, X_s \, I_1^r \, I_0^s} \left( -1 \right)^{n-s} r \, dr \]

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Using \[ \int_{0}^{a} \left[ \frac{I_1'(x_n r) I_1'(x_s r)}{r X_n X_s} + \frac{1}{r X_n X_s} I_1'(x_n r) I_1(x_s r) \right] \, dr \]

\[ = \frac{1}{(X_s^2 - X_n^2)} \left[ X_a I_1'(x_n) I_1(x_s) - X_a I_1'(x_s) I_1(x_n) \right] \text{ when } s \neq n \]

and \[ a \frac{I_1'(x_n) I_1'(x_s) - \frac{1}{2} \left( a^2 + \frac{1}{X_n^2} \right) I_1^2(x_n) - a^2 I_1^2(x_s) \right] \text{ when } n = s \]

we find \[ \sum_{n \neq 0}^{\pi A^2 \beta^2 (-1)^{n-s}} \sum_{s \neq 0}^{2n X_n X_s} \left[ \frac{a}{X_n I_1'(x_n)} - \frac{1}{I_1'(x_s)} \left( a^2 + \frac{1}{X_s^2} \right) - a^2 \right] \left( -1 \right)^{n-s} \]

3) \[ \sum_{n \neq 0}^{1} \int_{0}^{2\pi} \cos^2 \Theta \, d\Theta \int_{0}^{a X_s X_s r} \frac{B B_k^2 \beta^2}{X_n^2 X_s^2} I_1'(x_n) I_1'(x_s) \left( -1 \right)^{n-s} \, dr \]

\[ + \sum_{n \neq 0}^{1} \int_{0}^{2\pi} \sin^2 \Theta \, d\Theta \int_{0}^{a X_s X_s r} \frac{B B_k^2 \beta^2}{X_n^2 X_s^2} I_1'(x_n) I_1'(x_s) \left( -1 \right)^{n-s} \, rdr \]

Using the same integral as in (2), we find \[ \sum_{n \neq 0}^{\pi k B^2 \beta^2 (-1)^{n-s}} \sum_{s \neq 0}^{2n X_n X_s} \left[ \frac{a}{X_n I_1'(x_n)} - \frac{1}{I_1'(x_s)} \left( a^2 + \frac{1}{X_s^2} \right) - a^2 \right] \left( -1 \right)^{n-s} \]

\[ \text{ when } n = s \]
4 \sum \frac{1}{2} \int_0^{2\pi} \int_0^a \left[ A_o \left( \frac{1 + \frac{k^2r^2}{2}}{2ka} \right) + B_o \right] k \frac{A_n}{X_n} \frac{I'_n(X_nr)}{I_n(X_n)} (-1)^n dr \\
+ \left[ -A_o \left( \frac{1 - \frac{k^2r^2}{2}}{2ka} \right) + B_o \right] \frac{A_n}{X_n} \frac{I'_n(X_nr)}{I_n(X_n)} (-1)^n \right] rdr \\
+ \sum \frac{1}{2} \int_0^{2\pi} \int_0^a \left[ A_o \left( \frac{1 - \frac{k^2r^2}{2}}{2ka} \right) + B_o \right] k \frac{A_n}{X_n} \frac{I(X_nr)}{I'_n(X_n)} (-1)^n dr \\
+ \left[ -A_o \left( \frac{1 + \frac{k^2r^2}{2}}{2ka} \right) + B_o \right] \frac{A_n}{X_n} \frac{I(X_nr)}{I'_n(X_n)} (-1)^n \right] \text{dr}

Using \[ \int_0^a \left[ I'_n(X_nr) - \frac{I(X_nr)}{X_n} \right] dr = \frac{a}{X_n} I'_n(X_na) \]

and \[ \int_0^a \left[ r^2 I'_n(X_nr) - \frac{r^2}{X_n} I(X_nr) \right] dr \]

\[ = \frac{a^3}{X_n} I(X_na) - \frac{4a^2}{X_n^2} \left[ I'_n(X_na) - \frac{1}{X_n} I(X_n) \right] \]

we find

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\[
\frac{\pi A_0}{2ka} \sum_{n \neq 0} \frac{(-1)^n A_n}{X_n} \left[ \frac{a}{X_n} (k - \beta_n) + \left\{ \frac{a^3}{X_n} - \frac{4a^2}{X_n^2} \frac{I(X_n)}{I_1(X_n)} - \frac{1}{X_n} \right\} \frac{k^3}{2} + \frac{\beta_n k^2}{2} \right] \\
\frac{\pi}{2} \sum_{n \neq 0} (-1)^n \frac{B_n}{X_n} \frac{a}{X_n} (k + \beta_n)
\]

\[
\sum_{0}^{2\pi} \int_{0}^{\theta} \cos^2 \theta \, d\theta \int_{0}^{a} \left[ A_0 \left( \frac{1 + \frac{k^2r^2}{2}}{2ka} \right) + B_0 \right] \frac{B_n}{X_n} \frac{I(X_n)}{I_1(X_n)} (-1)^n \, dr \\
\sum_{0}^{2\pi} \int_{0}^{\theta} \sin^2 \theta \, d\theta \int_{0}^{a} \left[ A_0 \left( \frac{1 - \frac{k^2r^2}{2}}{2ka} \right) + B_0 \right] \frac{B_n}{X_n} \frac{I'(X_n)}{I_1(X_n)} (-1)^n \, rdr
\]

Using the same two integrals as in (4) we find

\[
\frac{\pi A_0}{2ka} \sum_{n \neq 0} \frac{(-1)^n B_n}{X_n} \left[ \frac{a}{X_n} (\beta_n - k) \frac{I(X_n)}{I_1(X_n)} \\
- \left\{ \frac{a^3}{X_n} \frac{I(X_n)}{I_1(X_n)} - \frac{4a^2}{X_n^2} \left( 1 - \frac{1}{X_n} \frac{I(X_n)}{I_1(X_n)} \right) \frac{\beta_n k^2}{2} + \frac{k^3}{2} \right\} \\
+ \frac{\pi}{2} \sum_{n \neq 0} (-1)^n \frac{B_n}{X_n} \frac{a}{X_n} (\beta_n + k) \frac{I(X_n)}{I_1(X_n)} \right]
\]

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\[ \sum \frac{1}{2} \int_{0}^{2\pi} \cos^2 \Theta \, d\Theta \int_{0}^{a} \left[ \frac{\beta^{\prime} A}{n} \frac{I_n^{(1)}}{I_n^{(1)}} \frac{B}{s} \frac{I_s^{(1)}}{I_s^{(1)}} \frac{(-1)^{n-s}}{X_n^{1}} \frac{I_n^{(1)}}{I_n^{(1)}} \frac{B}{s} \frac{I_s^{(1)}}{I_s^{(1)}} \frac{(-1)^{n-s}}{X_n^{1}} \right] \, dr \\
+ \sum \frac{1}{2} \int_{0}^{2\pi} \sin^2 \Theta \, d\Theta \int_{0}^{a} \left[ \frac{\beta^{\prime} A}{n} \frac{I_n^{(1)}}{I_n^{(1)}} \frac{B}{s} \frac{I_s^{(1)}}{I_s^{(1)}} \frac{(-1)^{n-s}}{X_n^{2}} \frac{I_n^{(1)}}{I_n^{(1)}} \frac{B}{s} \frac{I_s^{(1)}}{I_s^{(1)}} \frac{(-1)^{n-s}}{X_n^{2}} \right] \, dr \\
\text{Using} \quad \int_{0}^{a} \left[ \frac{I_n^{(1)}}{I_n^{(1)}} \frac{I_n^{(1)}}{I_n^{(1)}} + \frac{I_n^{(1)}}{I_n^{(1)}} \frac{I_n^{(1)}}{I_n^{(1)}} \right] \, dr \\
= \frac{1}{X_n^{1}} \frac{I_n^{(1)}}{I_n^{(1)}} \frac{I_n^{(1)}}{I_n^{(1)}} \\
\text{we find} \quad \sum_{n \neq 0} \frac{\pi}{2} \frac{(-1)^{n-s}}{X_n^{2}X_n^{2}} \frac{(k^2 + \beta^{\prime} s)}{n_s^{n}} \frac{A}{s} \frac{B}{s} \frac{I_s^{(1)}}{I_s^{(1)}} \frac{I_n^{(1)}}{I_n^{(1)}} \\
\text{including } n = s .
\]

This concludes the power flow.
5. **Group Velocity**

The group velocity and the energy velocity can be shown to be equal for an unattenuated electromagnetic wave in a periodic structure. (See e.g. reference 10). We estimate this velocity in the calculations in two ways, the agreement between them being a measure of the accuracy with which the problem has been solved. One way will be to divide the power flow by the stored energy per unit length. This will be immediately available from the printed results of the programme. The other estimate will be calculated as follows. Let us keep the ratio of disc-thickness to disc-spacing constant. The parameters of the structure will then be:

- the hole radius \(a\)
- the outer radius \(b\)
- the disc-spacing \(D\) (or discs per metre) \(\text{(5.1)}\)
- the free-space wavelength \(\lambda_o\) (or frequency)
- the guide-wavelength \(\lambda_g\)

We want to know

\[
\frac{\delta \lambda_g}{\delta \lambda_o}
\]  \(\text{(5.2)}\)

with \(a, b, D\) fixed.

The eigenvalue problem produces one relationship between the 5 parameters. In the programme it calculates \(b\) with all the others given.

The programme has \(\lambda_o\) and \(\lambda_g\) permanently fixed equal to one another and to 0.1 m. From its results, by tabulating differences or graph plotting, one can obtain

\[
\frac{\delta b}{\delta a} \quad D, \lambda_o, \lambda_g \text{ fixed} \quad \text{(5.3)}
\]

\[
\frac{\delta b}{\delta D} \quad a, \lambda_o, \lambda_g \text{ fixed} \quad \text{(5.4)}
\]
If we express the eigenvalue condition as

\[ b = b(a, D, \lambda_0, \lambda_g) \]  

and put

\[ \frac{\partial b}{\partial a} = W \frac{\partial a}{\partial a} + X \frac{\partial D}{\partial a} + Y \frac{\partial \lambda}{\partial a} + Z \frac{\partial \lambda}{\partial a} + Z \frac{\partial \lambda}{\partial \lambda} \]  

(5.6)

then the scaling law is

\[ 1 = W + X + Y + Z \quad (5.7) \]

Now (5.3) and (5.4) will enable us to obtain \( W \) and \( X \) from the results of the programme. (5.7) provides one relation between the four quantities. We therefore need one more piece of information. We will obtain \( Y \) from a perturbation formula. We will know, from the programme, the total stored energy in the system, and we will also arrange to know the difference between the electric and magnetic stored energy densities at the wall \( r = b \).

Consider a small shift of the conducting wall at \( r = b \), parallel to itself, in a direction to increase the cavity volume.

Let \( \Delta \frac{1}{2} \int_V (E^2 - Z_0^2 H^2) \, dV \)  

(5.8)

represent the associated increase in \( \frac{1}{2} \int_V (E^2 - Z_0^2 H^2) \, dV \) calculated from the known old fields at \( r = b \) and the volume increase.

We take (5.8) to be an average over time and length, and per metre length of guide.

Let \( \frac{1}{2} \int_V (E^2 + Z_0^2 H^2) \, dV \)  

(5.9)

be the former total time-average stored energy in the guide per metre length.
The Slater perturbation theorem \((11,12)\) states that the frequency shift associated with this \(b\)-increase, with all other dimensions and \(\lambda \) fixed, will be given by

\[
\frac{\Delta \omega}{\omega} = \frac{\frac{1}{2} \int V (E^2 - Z_o^2 H^2) \, dV}{\frac{1}{2} \int V (E^2 + Z_o^2 H^2) \, dV}
\]

or alternatively

\[
\frac{\Delta \lambda}{\lambda_o} = -\frac{\frac{1}{2} \int V (E^2 - Z_o^2 H^2)}{\frac{1}{2} \int V (E^2 + Z_o^2 H^2) \, dV}
\]

This enables \(Y\) to be calculated. \((5,7)\) now allows \(Z\) to be calculated, and \(-\) is \(\frac{\partial \lambda}{\partial \lambda_o}\) which is what we want.
Phase Velocities not equal to \( c \).

We now consider phase velocities different from \( c \). In the slot the fields are as written in Part I, equations (2.2). In the axial region we must take the two cases separately. For phase velocity \( v < c \), we have

\[
\beta_0 > k
\]

and can put

\[
\chi_0^2 = \beta_0^2 - k^2
\]

Then the \( n = 0 \) mode, as well as those with \( n \neq 0 \), has the form given in Part I (2.3). If, on the other hand, we have \( v > c \), then

\[
\beta_0 < k
\]

In this case we define \( \chi_0 \) by

\[
\chi_0^2 = k^2 - \beta_0^2
\]

and take the \( n = 0 \) term in \( \Pi_{\pm} \) to be

\[
\frac{A_0}{\chi_0^2} \frac{J_0(\chi_0 r)}{J_0(\chi_0 a)} \cos \Theta e^{-i\beta_0 z}
\]

and in \( \Pi_{\pm}^* \) to be

\[
\frac{B_0}{\chi_0^2 Z_0} \frac{J_1(\chi_0 r)}{J_1(\chi_0 a)} \sin \Theta e^{-i\beta_0 z}
\]

with the \( n \neq 0 \) terms remaining as in Part I.
We then have the following expressions for the fields in the axial region.

\[ E_r = \frac{1}{\sqrt{\gamma}} \frac{J'(X_r)}{J_1(X_0)} \cos \Theta - \beta z \]

\[ - \frac{ik}{r} \frac{J(X_r)}{J_1(X_0)} \cos \Theta - \beta z \]

\[ + \sum_{n \neq 0} \frac{i \beta}{X_n} \frac{I'(X_n)}{I_n} \cos \Theta_n - \beta z \]

\[ + \frac{ik}{r} \sum_{n \neq 0} \frac{B_n}{r} \frac{I(X_n)}{I_n} \cos \Theta_n - \beta z \]

\[ E_\Theta = \frac{1}{\sqrt{\gamma}} \frac{J'(X_r)}{J_1(X_0)} \sin \Theta - \beta z \]

\[ + \frac{ik}{\sqrt{\gamma}} \frac{J(X_r)}{J_1(X_0)} \sin \Theta - \beta z \]

\[ - \sum_{n \neq 0} \frac{A_n}{r} \frac{I(X_n)}{I_n} \sin \Theta_n - \beta z \]

\[ - \sum_{n \neq 0} \frac{B_n}{r} \frac{I'(X_n)}{I_n} \sin \Theta_n - \beta z \]
\[ E_z = A_0 \frac{J_1(\chi r)}{J_0(\chi a)} \cos \theta_e - i \beta_0 z + \sum_{n \neq 0} A_n \frac{I_1(\chi r)}{I_1(\chi a)} \cos \theta_e - i \beta_n z \]
\[ Z_{0r} = - \frac{ik A_0}{r} \frac{J_1(\chi r)}{\chi^2 J_1(\chi a)} \sin \theta_e - i \beta_0 z \]
\[ - i B_0 \frac{\beta_1 J_0(\chi r)}{\chi \chi_0 J_1(\chi a)} \sin \theta_e - i \beta_0 z \]
\[ + \frac{ik}{r} \sum_{n \neq 0} A_n \frac{I_1(\chi r)}{\chi^2 I_1(\chi a)} \sin \theta_e - i \beta_n z \]
\[ + i \sum_{n \neq 0} B_n \frac{\beta_n I_0(\chi r)}{\chi \chi_0 I_1(\chi a)} \sin \theta_e - i \beta_n z \]
\[ Z_{0\theta} = - \frac{ik A_0}{\chi \chi_0 J_1(\chi a)} \cos \theta_e - i \beta_0 z \]
\[ - i \frac{B \beta_0}{r} \frac{J_0(\chi r)}{\chi \chi_0 J_1(\chi a)} \cos \theta_e - i \beta_0 z \]
\[ + \frac{ik}{r} \sum_{n \neq 0} A_n \frac{\beta_n I(\chi r)}{\chi \chi_0 I(\chi a)} \cos \theta_e - i \beta_n z \]
\[ + i \sum_{n \neq 0} B_n \frac{\beta_n I(\chi r)}{\chi^2 I(\chi a)} \cos \theta_e - i \beta_n z \]
\[ Z_0 H_0 = B_0 \frac{J(x_0 r)}{J(x_0 a)} \sin \theta_0 e^{-i \beta_0 z} \]

\[ + \sum_{n \neq 0} B_n \frac{I(x_0 r)}{I(x_0 a)} \sin \theta_0 e^{-i \beta_n z} \quad \text{where} \]

\[ X_0^2 = k^2 - \beta_0^2 \]

\[ X_n^2 = \beta_n^2 - k^2, \ n \neq 0 \]

We now match the tangential components of the fields at \( r = a \) as in Part I. On putting, as before

\[ C_m (m \text{ even}) = C_m' \quad D_m (m \text{ even}) = iD_m' \]

\[ C_m (m \text{ odd}) = iC_m' \quad D_m (m \text{ odd}) = D_m' \]

we find the following relationships (2) (1.2)

\[ A_n^{(\text{all } n)} = \frac{2 \beta_n}{B} \sum_{m} \frac{C_m'}{(\beta_n^2 - \xi_m^2)} \times \frac{(-1)^m}{\sin \frac{\beta_n d}{2}}, m \text{ even} \]

\[ \frac{(-1)^{m-1}}{\cos \frac{\beta_n d}{2}}, m \text{ odd} \]

\[ B_0 = -A_0 \frac{\beta_0}{X_0 k} \]

\[ \frac{2 \chi_0}{\alpha d} \sum_{m=1,2} \frac{C_m \xi_m^2}{\gamma_m^2 (\beta_0^2 - \xi_m^2)} \times \frac{(-1)^m}{\sin \frac{\beta_0 d}{2}}, m \text{ even} \]

\[ \frac{(-1)^{m-1}}{\cos \frac{\beta_0 d}{2}}, m \text{ odd} \]

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\[ B_n = - \frac{A_n}{\beta_n} \frac{\beta}{X_a k} \]

\[ + \frac{2\chi_n}{m \cdot m} \sum_{m=1,2,\ldots} \frac{D_m g_m}{m^2} (\beta_n^2 - e_m^2) Y_m (\beta_n^2 - e_m^2) \] \[ \times (-1)^{m-1} \cos \beta_n \frac{d}{2}, \text{ m odd} \]

\[ - \frac{2\chi_n}{m \cdot m} \sum_{m=1,2,\ldots} \frac{D_m g_m}{m^2} (\beta_n^2 - e_m^2) Y_m (\beta_n^2 - e_m^2) \] \[ \times (-1)^{m} \sin \beta_n \frac{d}{2}, \text{ m even} \]

\[ + \frac{2\chi_n}{m \cdot m} \sum_{m=1,2,\ldots} \frac{C_m g_m}{m^2} (-1)^{m-1} \cos \beta_n \frac{d}{2}, \text{ m odd} \]

\[ - \frac{2\chi_n}{m \cdot m} \sum_{m=1,2,\ldots} \frac{D_m g_m}{m^2} (-1)^{m} \sin \beta_n \frac{d}{2}, \text{ m even} \]

and

\[ C_0' = \frac{2L (ka) \beta}{d L (ka)} \beta \] \[ \times \left[ \begin{array}{c} \frac{kA J_0^1 (X_a)}{C_0} + \frac{B \beta J_0^1 (X_a)}{X_0^2 a J_1 (X_a)} \\ X_0^2 a J_1 (X_a) + X_0^0 a J_1 (X_a) \end{array} \right] \]

\[ \beta \leq k, \left( \frac{\sin \beta}{\beta^2} \right) \]

\[ - \frac{2L (ka) k}{L (ka) d} \sum_{n \neq 0} \frac{\beta A \beta A' (X_a)}{n \cdot n} \sin \beta_n \frac{d}{2} \]

\[ - \frac{2 L (ka) \beta^2}{a L (ka) d} \sum_{n \neq 0} \frac{B \beta \beta \beta' (X_a)}{n \cdot n} \sin \beta_n \frac{d}{2} \]
\[ C^m = D^m \frac{g_m}{a \gamma_{m}} \frac{N(Y_a)}{N(Y_a)} \frac{M(Y_a)}{M(Y_a)} \]

\[ - \frac{4}{d} \frac{Y \beta_o}{M(Y_a)} \frac{M(Y_a)}{M(Y_a)} \times \frac{k}{\beta_o^2} \left( -l^2 \sin \beta_{o2}, \text{ m even} \right) \]

\[ \frac{k}{\beta_o^2} \left( -l^2 \cos \beta_{o2}, \text{ m odd} \right) \]

\[ + \frac{4}{d} \frac{M(Y_a)}{M(Y_a)} \frac{\sum_{n \neq 0} A_1 I(X_a) \beta_n}{\sum_{n \neq 0} X_n I(X_a) (\beta_n^2 - g_m^2)} \times \frac{l}{(-l^2 \sin \beta_{n2}, \text{ m even})} \]

\[ \frac{4}{kd} \frac{M(Y_a)}{M(Y_a)} \frac{B_{n2} \beta_{n2} I(X_a)}{\sum_{n \neq 0} X_n^2 I(X_a) (\beta_n^2 - g_m^2)} \times \frac{l}{(-l^2 \cos \beta_{n2}, \text{ m odd})} \]

\[ D^m = \frac{4}{d} \frac{N(Y_a)}{N(Y_a)} \frac{B_0}{I(X_a)} \frac{\beta_o < k}{\beta_o > k} \frac{g_m}{(-l^2 \sin \beta_{o2}, \text{ m even})} \]

\[ \frac{4}{d} \frac{N(Y_a)}{N(Y_a)} \frac{\sum_{n \neq 0} B_n I(X_a) \beta_n}{\sum_{n \neq 0} X_n I(X_a) (\beta_n^2 - g_m^2)} \times \frac{g_m}{(-l^2 \cos \beta_{n2}, \text{ m odd})} \]

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In the same way as for \( v = c \), two programmes were written, one for \( v < c \), and one for \( v > c \), which calculated the outer radius \( b \) for given \( \omega \), \( a \), \( D \), and \( d \). In this way, for a given guide, dispersion curves were drawn and compared with experimental values. These will appear in another report, which will contain only computational results.

In general there are two branches in the dispersion plots. The end-points, that is, the places where \( \beta_0 = 0 \), \( \beta_n = -\beta_n \), and \( \chi_n = \chi_{-n} \), are of some interest. In the simple theory (ref. 7), where only the coefficients \( A_0, B_0, \) and \( C_0 \) are used (and in addition usually \( d \) is assumed to be small) these end points correspond to a magnetic mode which does not penetrate into the slots (the frequency given by \( J'_1(ka) = 0 \)) and an electric mode (the frequency given by \( J'_1(kb) = 0 \) for infinitely thin discs). We now consider the characteristics of these end-points in the many term theory. Since the wave is no longer a travelling wave the field components in one cell must be odd or even functions of \( z \). The two possible modes are defined by

\[
(1) \quad E_z', H_r', H_\theta \quad \text{even}
\]
\[
\text{and} \quad H_z', E_r', E_\theta \quad \text{odd}
\]

\[
(2) \quad E_z', H_r', H_\theta \quad \text{odd}
\]
\[
\text{and} \quad H_z', E_r', E_\theta \quad \text{even}.
\]

Mode (1), which is the electric-like one corresponds, in terms of our coefficients, to

\[
B_0 = 0
\]
\[
A_n = A_{-n} \quad , \quad E_n = -B_{-n}
\]

and, in the slots, only \( C_1, D_2, C_2, D_4, C_4 \ldots \) etc. appear.

Mode (2), the magnetic-like one corresponds to

\[
A_0 = 0
\]
\[
A_n = -A_{-n} \quad , \quad B_n = B_{-n}, \quad \text{and in the slots only}
\]
\[
D_1, C_1, D_3, C_3 \ldots \quad \text{etc. appear.}
\]
If we plot the coefficients, on a branch tending towards mode (2), as a function of $\beta_0$ as we approach $\beta_0 = 0$, we find $D_1$, $C_1$, $D_2$, $C_2$, etc. and $A_n$, $B_n$ rise sharply. This is because we have normalised the coefficients in the programme to $C_0 = 1$, and $C_0 \to 0$ for mode (2).

At the end-points we write as before $A = \mathcal{G} \mathcal{C}$ and $\mathcal{C} = \mathcal{K}A$ where, for mode (1) $A$ is the vector $(A_0', A_1', B_1', A_2', B_2', \ldots)$ and $\mathcal{C}$ is the vector $(C_0', D_2', C_2', D_4', C_4', \ldots)$. Since $A_n = A_{-n}$, and $B_n = -B_{-n}$ we need only half the number of $A$'s and $B$'s as before.

The elements of $\mathcal{K}$ are given by

$$A_0 = C_0 \frac{d}{D}$$

$$A_n = \frac{2\beta_n}{D} \sum_{m=0,2,4,\ldots} \frac{C_m}{(\beta_n^2 - g_m^2)} \frac{(-1)^m}{\sin \beta_n \frac{d}{2}}$$

$$B_n = -A_n \frac{\beta_n}{\chi_{n,ak}}$$

$$+ \frac{2\chi_n}{akD} \sum_{m=2,4,\ldots} \frac{C_m g_m}{Y_m (\beta_n^2 - g_m^2)} \frac{(-1)^m}{\sin \beta_n \frac{d}{2}}$$

$$- \frac{2\chi_n}{D} \sum_{m=2,4,\ldots} \frac{D_m g_m}{(\beta_n^2 - g_m^2)Y_m} \frac{(-1)^m}{\sin \beta_n \frac{d}{2}}$$

and those of $\mathcal{G}$ by

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\[ C' \equiv \frac{L (ka)}{L (ka)} A \frac{J' (ka)}{J (ka)} \]

\[ - \frac{4L (ka)}{L (ka)} k \sum_{n=1,2,3} \frac{\beta_n A_n}{X_n} I' (X_n) \frac{\sin \beta_n d}{2 (\beta_n^2 - g_n^2)} \]

\[ - \frac{4L (ka)}{ad L (ka)} \sum_{n=1,2,3} \frac{B_n^2}{X_n} I' (X_n) \frac{\sin \beta_n d}{2 (\beta_n^2 - g_n^2)} \]

\[ C'_m = D'_m \frac{g_m}{aY_k} \frac{N (Y_a) M (Y_a)}{N (Y_a) M (Y_a)} \]

\[ \sum_{n=1,2} \frac{A_n}{X_n} I' (X_n) \frac{\beta_n (-1)^{m/2} \sin \beta_n d}{2 (\beta_n^2 - g_m^2)} \]

\[ \sum_{n=1,2,3} \frac{B_n \beta_n \sin \beta_n d}{2 (\beta_n^2 - g_m^2)} \]

\[ D'_m = - \frac{N (Y_a)}{d N (Y_a)} \sum_{n=1,2} \frac{I (X_n)}{I (X_n)} \frac{g_m (-1)^{m/2} \sin \beta_n d}{2 (\beta_n^2 - g_m^2)} \]
For mode (2), $A$ is the vector $(B_0, A_1, B_1, A_2, B_2, \ldots)$ and $G$ is the vector $(D_1, C_1, D_2, C_2, \ldots)$ and the elements of $G$ are given by

$$B_0 = \frac{2}{aD} \sum_{m=1,3,} \frac{C_m}{Y_m} (-1)^{m-1} \frac{m-1}{2}$$

$$- \frac{2k}{D} \sum_{m=1,3,} \frac{D_m}{Y_m g_m} (-1)^{m-1} \frac{m-1}{2}$$

$$A_n = \frac{2\beta_n}{D} \sum_{m=1,3,} \frac{C_m}{(\beta_n^2 g_m^2)} \cdot \frac{m-1}{2} \cos \frac{\beta_n d}{2}$$

$$B_n = -A_n + \frac{\beta_n}{\chi_{ak}} \frac{n}{n}$$

$$+ \frac{2X_n}{akD} \sum_{m=1,3,} \frac{C_m g_m^2}{Y_m} \cdot \frac{m-1}{2} \cos \frac{\beta_n d}{2}$$

$$- \frac{2X_n}{D} \sum_{m=1,3,} \frac{D_m g_m}{(\beta_n^2 g_m^2) Y_m} \cdot \frac{m-1}{2} \cos \frac{\beta_n d}{2}$$

$(2)(1.4)$
and the elements of $\mathcal{F}$ by

$$D_m = \frac{4}{a} \frac{N}{N_m} \frac{\mathcal{J}}{\mathcal{J}_m} \frac{(-1)^{m-1}}{\partial}$$

$$- \frac{8}{d} \frac{N}{N_m} \frac{I}{I_m} \frac{1}{1} \frac{\varepsilon}{\varepsilon_m} (-1)^{m-1} \cos \beta \frac{d}{2}$$

$$n = 1, 2, \ldots$$

$$C_m = D_m \frac{\varepsilon}{\varepsilon_m} \frac{N}{N_m} \frac{M}{M_m}$$

$$\frac{8}{d} \frac{M}{M_m} \frac{1}{1} \frac{\mathcal{I}}{\mathcal{I}_m} \frac{\beta}{\beta} (-1)^{m-1} \cos \beta \frac{d}{2}$$

$$n = 1, 2, \ldots$$

$$+ \frac{8}{d} \frac{M}{M_m} \frac{1}{1} \frac{\mathcal{I}}{\mathcal{I}_m} \frac{\beta}{\beta} (-1)^{m-1} \cos \beta \frac{d}{2}$$

$$n = 1, 2, \ldots$$

Two programmes were written which calculate the value of

$\Delta (\mathcal{F}, \mathcal{G} - 0)$ for a finite number of terms for given $a$, $b$, $d$, $D$ and $\omega$.

We calculated this value in particular cases for various $\omega$, everything else remaining fixed. When the value of $\omega$ to give $\Delta = 0$ had been found, the programme then found the values of the coefficients in the following way.
For mode (1) we have

$$C = |\begin{vmatrix} g_1 & g_2 & g_3 \\ \end{vmatrix}|$$

Let $P$ be the matrix

$$\begin{bmatrix}
0, 0, 0, 0, 0, \\
0, 1, 0, 0, 0, \\
0, 0, 1, 0, 0, \\
0, 0, 0, 1, 0, \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Then $|\begin{vmatrix} g_1 & g_2 & g_3 \\ P_1 & P_2 & P_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{vmatrix}|^{-1} = |\begin{vmatrix} D_1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{vmatrix}|$

We normalise $D^i$ to be 1, and hence find the other coefficients.

For mode (2) the same is done with $C_0^i$ instead of $D^i$ being made equal to 1.

The fields for mode (1) are given in the axial region by

$$E_r = \sum_{n=1,2,\ldots} \frac{2\beta_n}{\lambda_n} A_i^{(n)} I_i^{(X_n)} \cos \theta \sin \beta_n z$$

$$+ 2k \sum_{n=1,2,\ldots} \frac{B_n}{r \lambda_n} I_i^{(X_n)} \cos \theta \sin \beta_n z$$
\[
E_\theta = - \sum_{n=1,2} \frac{2A_n \beta_n}{n} \frac{I_0(X_n a)}{I_n(X_n a)} \sin \Theta \sin \beta_n z
- 2k \sum_{n=1,2} \frac{B_n I'_0(X_n r)}{X_n I'_1(X_n a)} \sin \Theta \sin \beta_n z
\]

\[
E_z = A_0 \frac{J(kr)}{J_0(ka)} \cos \Theta + \sum_{n=1,2} \frac{2A_n I_1(X_n r)}{I_n(X_n a)} \cos \Theta \cos \beta_n z
\]

\[
Z_{o H_r} = - i \frac{A_0 \frac{1}{J_1(ka)}}{r k} \sin \Theta
+ \frac{2ik}{r} \sum_{n=1,2} \frac{A_n}{X_n} \frac{I_1(X_n r)}{I_n(X_n a)} \sin \Theta \cos \beta_n z
+ 2i \sum_{n=1,2} \frac{B_n I'_1(X_n r)}{I_1(X_n a)} \sin \Theta \cos \beta_n z
\]

\[
Z_{o H_\theta} = -iA_0 \frac{1}{J_1(ka)} \cos \Theta
+ 2ik \sum_{n=1,2} \frac{A_n I'_1(X_n r)}{X_n I'_n(X_n a)} \cos \Theta \cos \beta_n z
+ \frac{2i}{r} \sum_{n=1,2} \frac{B_n \beta_n}{X_n^2} \frac{I_1(X_n r)}{I_n(X_n a)} \cos \Theta \cos \beta_n z
\]
\[ Z_{H_{Z}} = 2i \sum_{n=1,2,..} B_{n} \frac{I(x_{r})}{I(x_{a})} \sin \theta \sin \beta_{n} \]

and in the slots by

\[ E_{r} = \sum_{m=2,4..}^{\infty} C_{m} g_{m} \frac{M_{1}^{\prime}(Y_{r})}{M_{1}^{\prime}(Y_{a})} \cos \theta \sin g_{m} \]

\[ - k \sum_{m=2,4..}^{\infty} D_{m} g_{m} \frac{N_{1}^{\prime}(Y_{r})}{N_{1}^{\prime}(Y_{a})} \cos \theta \sin g_{m} \]

\[ E_{\theta} = \frac{1}{r} \sum_{m=2,4..}^{\infty} C_{m} g_{m} \frac{M_{1}^{\prime}(Y_{r})}{M_{1}^{\prime}(Y_{a})} \sin \theta \sin g_{m} \]

\[ + k \sum_{m=2,4..}^{\infty} D_{m} g_{m} \frac{N_{1}^{\prime}(Y_{r})}{N_{1}^{\prime}(Y_{a})} \sin \theta \sin g_{m} \]

\[ E_{z} = C_{0} \frac{L_{1}(kr)}{L_{1}(ka)} \cos \theta + \sum_{m=2,4..}^{\infty} C_{m} g_{m} \frac{M_{1}^{\prime}(Y_{r})}{M_{1}^{\prime}(Y_{a})} \cos \theta \cos g_{m} \]

\[ Z_{H_{r}} = - \frac{iC_{0}}{kr} \frac{L_{1}(kr)}{L_{1}(ka)} \sin \theta \]

\[ + \frac{i k}{r} \sum_{m=2,4..}^{\infty} C_{m} g_{m} \frac{M_{1}^{\prime}(Y_{r})}{M_{1}^{\prime}(Y_{a})} \sin \theta \cos g_{m} \]

\[ - i \sum_{m=2,4..}^{\infty} D_{m} g_{m} \frac{N_{1}^{\prime}(Y_{r})}{N_{1}^{\prime}(Y_{a})} \sin \theta \cos g_{m} \]
\[ Z_0 H_\theta = -iC_o \frac{L'_0}{L'_1} \cos \theta \]
\[ + ik \sum_{m=2,4} \frac{C_m}{Y_m} \frac{N(\rho r)}{1 - \frac{1}{m}} \cos \theta \cos g_m z \]
\[ - \sum_{m=2,4} \frac{D_m}{Y_m} \frac{g_m}{N(\rho a)} \cos \theta \cos g_m z \]
\[ Z_0 H_z = i \sum_{m=2,4} \frac{D_m}{N(\rho a)} \frac{N(\rho r)}{1 - \frac{1}{m}} \sin \theta \sin g_m z \]

The fields for mode (2) are given, in the axial region by

\[ E_r = -\frac{iB_o}{r} \frac{J_1(\rho r)}{J_1(\rho a)} \cos \theta \]
\[ + 2i \sum_{n=1,2} \frac{\beta_n}{\chi_n} \frac{A_n}{I_n(\rho a)} \cos \theta \cos \beta_n z \]
\[ + 2ik \sum_{n=1,2} \frac{B_n}{r \chi_n^2} \frac{I_n(\rho r)}{I_n(\rho a)} \cos \theta \cos \beta_n z \]
\[
\begin{align*}
E_\Theta &= \frac{iB}{\pi} \frac{J'(kr)}{J_1(ka)} \sin \Theta \\
&- 2i \sum_{n=1,2,\ldots} \frac{\lambda_n \beta_n I_n(X_n^r)}{r \chi_n^2 I_1(X_n^a)} \sin \Theta \cos \beta_n z \\
&- 2ik \sum_{n=1,2,\ldots} \frac{B_n I_n(X_n^r)}{\chi_n I_1(X_n^a)} \sin \Theta \cos \beta_n z \\
E_Z &= -2i \sum_{n=1,2,\ldots} \frac{\lambda_n I_n(X_n^r)}{I_1(X_n^a)} \cos \Theta \sin \beta_n z \\
Z_{0H_r} &= +\frac{2k}{r} \sum_{n=1,2,\ldots} \frac{\lambda_n I_n(X_n^r)}{\chi_n^2 I_1(X_n^a)} \sin \Theta \sin \beta_n z \\
&+ 2 \sum_{n=1,2,\ldots} \frac{B_n \beta_n I_n(X_n^r)}{\chi_n I_1(X_n^a)} \sin \Theta \sin \beta_n z \\
Z_{0H_\Theta} &= +2r \sum_{n=1,2,\ldots} \frac{\lambda_n I_n(X_n^r)}{\chi_n I_1(X_n^a)} \cos \Theta \sin \beta_n z \\
&+ 2 \sum_{n=1,2,\ldots} \frac{B_n \beta_n I_n(X_n^r)}{\chi_n^2 I_1(X_n^a)} \cos \Theta \sin \beta_n z \\
Z_{0H_z} &= \frac{J'(kr)}{J_1(ka)} \sin \Theta \\
&+ 2 \sum_{n=1,2,\ldots} \frac{B_n I_n(X_n^r)}{I_1(X_n^a)} \sin \Theta \cos \beta_n z
\end{align*}
\]
and in the slots by

\[
E_r = -i \sum_{m=1,3,\ldots} \frac{C_m g_m M(Y_r) M(Y_a)}{Y_m} \cos g_m, \cos \Theta
\]

\[
+ \frac{ik}{r} \sum_{m=1,3,\ldots} \frac{D_m g_m M(Y_r) N(Y_a)}{Y_m} \cos g_m, \cos \Theta
\]

\[
E_\Theta = \sum_{m=1,3,\ldots} \frac{C m g_m M(Y_r) M(Y_a)}{Y_m^2} \cos g_m, \sin \Theta
\]

\[
- \frac{ik}{r} \sum_{m=1,3,\ldots} \frac{D m g_m M(Y_r) N(Y_a)}{Y_m} \cos g_m, \sin \Theta
\]

\[
E_z = i \sum_{m=1,3,\ldots} \frac{C_m g_m M(Y_r) M(Y_a)}{Y_m} \sin g_m, \cos \Theta
\]

\[
Z_{0r} = - \frac{k}{r} \sum_{m=1,3,\ldots} \frac{C_m g_m M(Y_r) M(Y_a)}{Y_m^2} \sin g_m, \sin \Theta
\]

\[
+ \sum_{m=1,3,\ldots} \frac{D_m g_m N(Y_r) N(Y_a)}{Y_m} \sin g_m, \sin \Theta
\]
\[ Z_{0H\theta} = -k \sum_{m=1,3} \frac{c_m Y_m}{M_m Y_m} \frac{M_m}{(Y_a)_m} \sin g_m z \cos \theta \]

\[ + \frac{1}{r} \sum_{m=1,3} \frac{D_m (Y_r)}{N_m (Y_a)_m} \sin g_m z \cos \theta \]

\[ Z_{0H\zeta} = \sum_{m=1,3} \frac{D_m (Y_r)}{N_m (Y_a)_m} \cos g_m z \sin \theta \]

(2)(1.8)
(cont.)

Two programmes were written to calculate the fields for the two modes for several values of \( r \) and \( z \), using the values of the coefficients found by the earlier programmes.

(b) Discussion of a Radio Frequency Particle Separator for the CERN proton synchrotron by W. Schnell, CERN 61-5.


(e) The Deflecting Mode in the Circular Iris-loaded Waveguide, BNL-AGS, HH-5 by H. Hahn.

(f) Mode Identification in the Iris-loaded Waveguide of an r.f. Particle Separator BNL-AGS, HH/HJH 2 by H. Hahn and and H. J. Halama.


9) e.g. Ordinary Differential Equations, by E. L. Ince, p 136. Dover Publications, Inc.


