THE HIGH INTENSITY MUON BEAM WITH LOW PION CONTAMINATION AT THE CERN SYNCHRO-CYCLOTRON

by

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FOREWORD

The present report is an enlarged version of an internal note published in 1957. In that note, the elements for designing a muon channel were exposed with a view to possible application at the CERN Synchro-cyclotron.

In the meantime, the CERN channel has been designed in detail, constructed, and put into operation. A brief summary of its performance was published in 1960. Since then, a large number of experiments have been performed using this facility. Other laboratories have constructed similar devices, or are considering doing so. Under these circumstances we thought it might be useful for those interested in understanding the performance of an existing channel or in the design of a new one to have at their disposal a complete and up-to-date version of the report. It is intended for physicists who, like ourselves, know very little about beam dynamics. Accelerator specialists will find many of our considerations trivial and clumsy.

Since the publication of our first note, design methods have changed by the much wider use of computers. It has become less important to insist upon more or less good approximations which allow the problem to be treated by analytic methods. We have presented here, in the first place, the general considerations on which our evaluations of the channel performance are based. Then we have treated a particular approximation, the short-lens approximation, which gives a qualitative understanding of the functioning of the device. Finally, results of recent computations are given and confronted with the experimental findings and, in some cases, with the short-lens approximation.
We have restricted ourselves to the description of a channel for momenta of a few hundred MeV/c. Many considerations can, of course, be generalized to channels in the GeV/c region. In fact, the short-lens approximation is more valid in these applications. Fronteau has elaborated computer methods, which avoid some of the simplifications we have used (infinitely long channel, strict periodicity, uniformly filled ellipses). He has applied them to the design of a channel in the GeV/c region. The authors would like to point out that in writing this report they are acting as the secretaries of a group, of changing composition, who designed, built and tested the muon channel. Under these circumstances it is perhaps fair to indicate briefly, for each section, those persons responsible for the results used therein, with a reference to partial publications.

Sections I to VI outline the theory of the channel proper. The general ideas and their application in the short-lens approximation are due to Citron. For these sections, Øverås had considered the influence of finite lens length and fringe fields. All these calculations are not reported here, since they have been largely superseded by computer programmes treating the long lens case. These programmes are also due to Øverås. The experimental data about the lenses needed for the early calculations were provided by Septier on the basis of model measurements. The recent computations use information from field and floating wire measurements on the real lenses, carried out by Braunsreuther, Chabaud, Delorme and van Gulik.

Section VII, treating the matching of the channel to the accelerator, is due mainly to Øverås. The data about the motion in the SC were obtained by floating wire measurements due to Citron, Farley, Michaelis and Øverås, and by a computer programme prepared by Farley.

Section VIII describes the analysing magnet. The considerations are originally those of M. Morpurgo, who also did the engineering design both of the lenses and of the analysing magnet. The present, more general formulation is due to Øverås.
Section IX presents some experimental findings about the channel performance. They were obtained by Citron, Delorme, Fries, Heintze, Michaelis, Øverås and Shtcherbakov\textsuperscript{9}).

Finally we must mention that this report would not have appeared without the competent assistance of Miss M. Hanney and Miss C. Petit (computing and graph drawing), Mrs. S. Trentini (figures), and Mrs. K. Wakley (typing, English, etc.).

* * *
I. **INTRODUCTION**

The pion beams from accelerators contain a certain admixture of muons. This admixture usually is of the order of $5-30\%^{10}$. If one is interested in the muons, one often wants to improve the $\mu/\pi$ ratio. One method, first employed by Swanson and Campbell$^{11}$, makes use of the fact that the muons have a wider momentum spectrum than the pions. Therefore, by putting an analysing magnet into the beam and by "de-tuning" this magnet off the momentum peak found for the pions, one obtains a beam, in which the percentage of muons goes up as the de-tuning increases, at the expense of absolute $\mu$ flux. Typical figures are $1\text{ cm}^{-2}\text{ sec}^{-1}$ with a $\pi$ contamination of $<10^{12}$. In the following a system is described which allows us to obtain a muon beam with both a high intensity and a high purity.

II. **KINEMATICS**

Since we shall obtain our muons from the reaction

$$\pi \rightarrow \mu + \nu,$$

(II.1)

we shall first list some properties of this reaction which are relevant to the design of our system.

The mass ratio of the $\mu$ and $\pi$ is

$$a = \frac{m_\mu}{m_\pi} = \frac{105.653 \text{ MeV}}{139.6 \text{ MeV}} = 0.7571.$$  

(II.2)

The velocity of the muon in the rest system of the pion, in terms of light, $c$, is

$$\beta = \frac{1 - a^2}{1 + a^2} = 0.2712.$$  

(II.3)

For the corresponding $\mu$ momentum in the $\pi$ system (and consequently for the maximum transverse momentum in the lab. system) we find
\[
\bar{p} = \frac{1 - \frac{c^2}{2\alpha}}{\mu} \quad \text{m} \quad c = 29.78 \text{ MeV/c}, \quad \text{(II.4)}
\]

and for the total energy

\[
\bar{E} = \frac{1 + \frac{c^2}{2\alpha}}{\mu} \quad \text{m} \quad c^2 = 109.82 \text{ MeV} = m \frac{c^2}{\mu} + 4.17 \text{ MeV}. 
\]

We shall illustrate the transformation into the laboratory system for the special case of a \( \pi \) momentum of 400 MeV/c (284 MeV kinetic energy), which is typical for one mode of operation of the system actually used. The total energy in terms of the rest energy is in this case

\[
\gamma_{\pi} = 3.03, \quad \text{(II.5)}
\]

and the corresponding velocity

\[
\beta_{\pi} c = 0.944 \text{ c}. \quad \text{(II.6)}
\]

The extreme \( \mu \) momenta in the laboratory system (corresponding to forward resp. backward emission in the \( \pi \) system) in terms of the \( \pi \) momentum are

\[
\begin{align*}
\max p_{\mu} &= \frac{\beta_{\pi} \pm \bar{\beta}}{\beta_{\pi}(1 + \bar{\beta})} p_{\pi} \quad \text{(II.7)} \\
\min p_{\mu} &= \frac{\beta_{\pi} - \bar{\beta}}{\beta_{\pi}(1 + \bar{\beta})} p_{\pi} 
\end{align*}
\]

For \( \beta_{\pi} = 1 \) this yields 1.0 resp. 0.574 time \( p_{\pi} \). For the \( \beta_{\pi} \) given in Eq. (II.6): 1.013 resp. 0.560 \( p_{\pi} \), or

\[
\begin{align*}
\max p_{\mu} &= 405 \text{ MeV/c} \quad (313 \text{ MeV kin.}) \\
\min p_{\mu} &= 224 \text{ MeV/c} \quad (142 \text{ MeV kin.}). \quad \text{(II.8)}
\end{align*}
\]

Between these limits the differential energy spectrum is strictly flat. This is a consequence of the isotropy of the decay in the \( \pi \) system. The differential momentum spectrum has a very slight slope. It is given by
\[ N(p)dp = \frac{(1+\beta)}{2\beta p_\pi} \beta \mu dp \mu. \]  \hspace{1cm} (II.9)

We are also interested in the laboratory angle of emission \(\theta\). If we express it as a function of laboratory energy, it is given by

\[ \cos \theta = \frac{E \mu - E m c^2}{p_\pi p_\mu c^2}, \]  \hspace{1cm} (II.10)

where the \(E\) stand for total energies.

\[ \sin \theta_{\text{max}} = \frac{1}{\alpha} \frac{p_\pi}{p_\mu} \]  \hspace{1cm} (II.11)

gives the maximum laboratory angle, which is 99 mrad or 5.65° in our example.

Equation (II.10) is plotted in Fig. II.1. Note the steep rise of \(\theta\) at the low-energy end of the distribution.

The decay length \(\lambda\) of the pions follows from the mean life

\[ \tau_\pi = 25.6 \text{ nsec} : \]

\[ \lambda = \frac{\tau_\pi}{\tau_\pi} = \frac{\tau_\pi}{m_\pi p_\pi} = 5.50 \frac{m_\pi}{p_\pi} \]  \hspace{1cm} (II.12)

(\(\lambda\) in cm for \(p_\pi\) in MeV/c.)

We find 22.0 m for \(p_\pi = 400\) MeV/c.

In the pion rest system the muons are fully polarized longitudinally with respect to their direction of emission. If we denote right-handed helicity as positive, then in this system the polarization of the muon is \(-e/|e|\).

In the laboratory system the degree of longitudinal polarization is given \(^{13}\) by

\[ \cos \phi_{\text{pol}} = \frac{e}{|e|} \frac{(1-\beta)E_\mu - E_\pi}{\beta p_\mu c}. \]  \hspace{1cm} (II.13)
III. PRINCIPLE OF THE SYSTEM

From Eqs. (II.7) and (II.8) we see that monochromatic pions yield a wide flat spectrum of muons, with momenta between roughly 0.6 and 1.0 times the π momentum.

Let us admit such monochromatic pions into a decay path. We place a momentum analyser at the end of the path which selects a momentum band below the π momentum. We will then, in principle, obtain pure muons (Fig. III.1). In reality we need not start with monochromatic pions, for from the width of the spectrum it follows that pions of many momenta can contribute muons in a given momentum band. As we are interested in a high intensity, we shall accept all these pions. We only have to introduce a lower momentum limit for the pions at the entrance of the system. This limit has to be sufficiently far above the momentum band admitted by the second analyser to make quite sure that pions which are admitted at the entrance and which have not decayed do not pass the second analyser.

So our system (Fig. III.2) will consist of:

i) a first momentum analyser I which passes a very wide band of pion momenta, but has a sharp lower limit of the passband;

ii) a decay path;

iii) a second momentum analyser II of which the resolution is, in principle, determined by experimental requirements.

In order to obtain many muons, we want a decay path not too short compared to the decay length. The large values for the decay length [Eq. (II.12)] and the decay angle [Eq. (II.11)] suggest that the decay path has to be fitted with focusing devices. This becomes more evident from an inspection of Fig. II.1. Small decay angles occur only for the top energy (forward decay) and for the lowest energy (backward decay). If we would require our μ momentum band to be near the π momentum, only a few π energies could contribute, and no magnetic selection of π and μ mesons would be possible. If we would choose it at the largest possible distance from the π momentum, again only a narrow range of pion energies will contribute.
Furthermore, the steepness of the curve at the low-energy end indicates that angles up to half the maximum angle are included as soon as the momentum band of the second analyser has a total width of 3% of the muon momentum. For a width of 10%, angles up to 0.8 times the maximum angle occur. So it will be necessary to design the focusing system for angles near the maximum one. As we will see, this will at the same time permit acceptance of more incident pions.

Figures III.3 and III.4 show the system as it materialized, and we shall now treat in more detail first the focusing system along the decay path, then the entrance analyser which is provided by the magnetic field of the synchro-cyclotron, and finally the second momentum analyser.

IV. THE FOCUSING CHANNEL

As we have seen above, the decay channel has to have focusing properties. Since we want to focus pions over a wide range of momenta as well as muons of lower momenta, originating in various positions in the channel, the focusing properties must not depend too critically on momentum.

As will be shown, the principle of alternating gradient focusing permits the designing of such a wide-band focusing device. Two versions of such alternating gradient devices have originally been considered for our purpose. One, first proposed by Krienen, uses one single "helical lens", i.e. a very long quadrupole lens twisted through several turns. The second system consists of a large number of relatively short quadrupole lenses, every lens being turned by 90° with respect to the preceding one. The calculated performance of the two systems is similar. The helical lens provides more efficient focusing, but is more difficult to manufacture, and is a single purpose device. The elements of a channel composed of individual quadrupole lenses can be used for building another channel for another momentum band, at different lens spacing, or for conventional focusing purposes. For these reasons the preference has finally been given to the short quadrupole system. We shall now treat the beam dynamics in this system.
IV.1 The quadrupole lens

Consider in an x-y plane the orthogonal hyperbolae, which have the x and y axes as asymptotes, and of which the four points nearest to the origin are at a distance \( a \) from it (Fig. IV.1). Let these hyperbolae represent the boundaries of four pole tips, in the shape of hyperbolic cylinders, made out of material of infinite permeability and producing a magnetic field strength of \( B \) gauss at the four points mentioned above.

Assume the sign of \( B \) to be such that the pole tip in the first quadrant is a south pole (\( B < 0 \)). The field between the poles will be described by

\[
B_x = By/a \tag{IV.1}
\]

\[
B_y = Bx/a .
\]

In reality a lens will not be "ideal" in the sense described above. The field will then have the shape given by Eq. (IV.1) only up to a certain radius \( < a \). In this case we shall still use the description of Eq. (IV.1) for the central part, but with \( B/a \) corresponding to the gradient observed there. The field at the pole tip may then be different from \( B \). (See section V.4.)

If a particle of charge \( e \), transversal mass \( m \), and velocity \( v \) travels in the z direction (i.e. out of the paper), the equation of its transversal motion will be (dots denote time derivations)

\[
m\ddot{x} - e[\dot{v} \times \vec{B}]_x = 0 \tag{IV.2}
\]

\[
m\ddot{y} - e[\dot{v} \times \vec{B}]_y = 0 .
\]

Substituting

\[
[\dot{v} \times \vec{B}]_x = v_y B_z - v_z B_y
\]

and

\[
[\dot{v} \times \vec{B}]_y = v_x B_z - v_z B_z \tag{IV.3}
\]

with \( v_z \approx v \) (small angles), \( B_z = 0 \), and the values of \( B_x \) and \( B_y \) found above, we have
\[ \ddot{x} + \frac{eB}{mva} \dot{x} = 0 \tag{IV.4} \]

\[ \ddot{y} - \frac{eB}{p a} \dot{y} = 0, \]

so the \( x \) and \( y \) motions are independent in these co-ordinates and in small-angle approximation.

Introducing \( z \) as an independent variable instead of the time \( t \) via \( z = v_z t \), using \( \frac{dv_z}{dt} \approx \frac{dv}{dt} = 0 \), and denoting derivatives with respect to \( z \) by strokes, we obtain

\[ x'' + k^2 x = 0 \tag{IV.5} \]
\[ y'' - k^2 y = 0 \]

with

\[ k^2 = \frac{eB}{mva} = \frac{eB}{pa} = \frac{1}{\rho a}, \tag{IV.6} \]

where

\[ p = mv \tag{IV.7} \]
\[ \rho = \frac{p}{eB} = \frac{10^4}{\frac{2}{3} B} \text{[cm]}, \tag{IV.8} \]

(\( p \) in MeV/c, \( B \) in gauss). \( \rho \) is the radius of curvature of particles of momentum \( p \) in a field \( B \).

Note that the two Eqs. (IV.5) differ only by the sign before \( k^2 \), i.e. before the product \( eB \). In this subsection IV.1 we shall assume \( eB > 0 \), in which case \( k \) is real and the \( x \) motion is focusing.*

In a general beam transport system, \( k^2 \) may vary continuously (in most cases periodically) with \( z \). If \( k^2 \) is only depending on \( z \) (not on \( x \) and \( y \)), we shall call the system linear.

For a constant \( k^2 \) and initial conditions \( x_0, x_0', y_0, y_0' \) the solutions of Eqs. (IV.5) are

\[ x = x_0 \cos kz + \left( \frac{1}{k} \right) x_0 \sin kz \tag{IV.9} \]
\[ y = y_0 \cosh kz + \left( \frac{1}{k} \right) y_0 \sinh kz, \]

*) In the sections on channel theory we shall, when no confusion can arise, use \( k^2 \) for \( |k^2| \) and \( B \) for \( |B| \).
and their derivatives

\[ x' = -kx_0 \sin kz + x_0' \cos kz \]
\[ y' = ky_0 \sinh kz + y_0' \cosh kz . \]

(IV.10)

\( x' \) is a measure for the velocity in the \( x \) direction. For constant total velocity and small angles, \( x' \) is proportional to the \( x \) component of the momentum. As co-ordinate and momentum are canonical variables, \( x \) and \( x' \) thus subtend a phase plane. The same is true for \( y \) and \( y' \). As the momentum in the \( z \) direction is considered to be constant (small angles), and as the \( x \) and \( y \) motion were shown to be independent, the \( x-x' \) plane and the \( y-y' \) plane, which are projections of phase space, each have the general properties of phase space: for example, the Liouville theorem applies in both planes.

The linear relations of Eqs. (IV.9) and (IV.10) give the transformation from one such phase plane to another with a different \( z \). They can be represented by their "transfer" matrices. For a lens of length \( S \) these transfer matrices become for the focusing (\( x \) direction

\[
\begin{pmatrix}
\cos kS & (1/k) \sin kS \\
-k \sin kS & \cos kS
\end{pmatrix}
\]

(IV.11)

and for the defocusing (\( y \) direction

\[
\begin{pmatrix}
\cosh kS & (1/k) \sinh kS \\
k \sinh kS & \cosh kS
\end{pmatrix}
\]

(IV.12)

It is seen that the determinant of these matrices is one.

Any linear beam transport system can be described, to any degree of approximation, by a sufficiently large number of such lenses in succession, with suitably chosen values of \( k \) and \( S \) for each lens (including drift spaces, where \( k = 0 \)). The determinant of such a general system is the product of the determinants of all its lenses, and thus also equal to one.

*) Neglecting the difference between the "total" and the "conjugate" momentum.
In deriving Eqs. (IV.11) and (IV.12) we have assumed that the magnetic field is given by Eq. (IV.1) for $0 < z < S$. If we want the equation to describe the lens completely, we must assume that the field vanishes outside these limits (ideal long lens). For a real lens, this will not be the case; there will be stray fields. However, in so far as such a lens is considered linear, its action can be described by the general matrix

$$ L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \quad (IV.13) $$

with

$$ \|L\| = 1. \quad (IV.14) $$

If the lens has, moreover, a plane of symmetry $z = \text{const}$, we also have

$$ L_{11} = L_{22}. \quad (IV.15) $$

So only two elements of the matrix $L$ are independent. Thus, by comparing the elements of Eq. (IV.13) with Eqs. (IV.11) or (IV.12), we can always find a suitable pair of parameters $k$ and $S$ to describe the general lens. We suffer no loss in generality by using the picture of a long ideal lens. This description is, of course, only valid for points outside the lens. For the motion inside the lens, Eqs. (IV.9) and (IV.10) prescribe a particular trajectory, whereas a transfer matrix such as Eq. (IV.13) leaves this question completely open.

The way in which the parameters $k$ and $S$ are related to the actual field and dimensions of a real lens will be described in Section VI.1.

If the lens is short compared to the wavelength $2\pi/k$, i.e. if $kS << 1$ or

$$ S << \sqrt{\alpha \rho}, \quad (IV.16) $$
we can develop the trigonometric and hyperbolic functions and break off after the first term. The matrices then become

\[
\begin{pmatrix}
1 & S \\
-k^2 S & 1
\end{pmatrix}
\]  
(IV.17)

The focusing action is now given by the single parameter

\[
\delta = k^2 S = \frac{S}{p_0}.
\]  
(IV.18)

The determinant of Eq. (IV.17) is 1 to the order $\delta S$. The lateral drift inside the lens given by the matrix element $S$ can, to the order $\delta S$, be included in the preceding or following drift space. Then the matrix

\[
\begin{pmatrix}
1 & 0 \\
-\frac{1}{\delta} & 1
\end{pmatrix}
\]  
(IV.19)

describes an infinitely thin focusing (defocusing) lens, which gives an angular kick to the particles passing it, proportional to the distance from the axis at which they pass. It can easily be shown that $1/\delta$ is the focal length of the lens. For $p = 400$ MeV/c and $B = 10$ kgauss, we find $\rho = 133$ cm. The condition in Eq. (IV.16) for the short-lens approximation then reads: $S << 36.5$ cm for $a = 10$ cm. For an $S = 10$ cm long lens, we would find $1/\delta = 133$ cm.

### IV.2 The Channel

We will now study the behaviour of a periodic arrangement of many equal lenses, with $B$ alternating in sign, separated by equal spaces. Many of the properties of such a periodic structure can be expressed in terms of the transfer matrix $M$ through one period of the structure

\[
\begin{pmatrix}
x(z+\ell) \\
x'(z+\ell)
\end{pmatrix} = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}\begin{pmatrix}
x(z) \\
x'(z)
\end{pmatrix},
\]  
(IV.20)

which we can also write as

\[
\hat{x}(z+\ell) = M\hat{x}(z).
\]  
(IV.21)

Here $\ell$ is the length of one period.
Before treating a more general system, we shall calculate this transfer matrix for the special case where the lenses can be described by the short-lens approximation. In this case, we obtain simple expressions. Formulae obtained under this approximation will be marked with an $S$ following their number. Even if the condition in Eq. (IV.16) is not satisfied, the short-lens approximation may give a good first estimate of the performance, which has to be implemented in this case by estimates of the corrections arising from the fact that the lens is extended and has, moreover, a fringe field.

We consider thin lenses of strength $\pm \delta$, of alternating sign, and separated by drift spaces of length $L$. The period of the structure is twice the lens separation. We will, for reasons which will become apparent later, put the beginning of a period just after a lens which is focusing (in the $x$ direction, say). The transfer matrix is found by matrix multiplication:

period = focusing lens $\times$ drift space $\times$ defocusing lens $\times$ drift space,

or

$$
M = \begin{pmatrix}
1 & 0 \\
-\delta & 1
\end{pmatrix}
\begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\delta & 1
\end{pmatrix}
\begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix} =
$$

$$
= \begin{pmatrix}
1 + \delta L & L(2 + \delta L) \\
-\delta & 1 - \delta L - (\delta L)^2
\end{pmatrix} = \begin{pmatrix}
1 + \eta & L(2 + \eta^2) \\
-\delta \eta & 1 - \eta - \eta^2
\end{pmatrix} \tag{IV.22S}
$$

with

$$
\eta = \delta L = \frac{SL}{\rho_a} = k^2 SL \tag{IV.23S}
$$

In Appendix A we prove that the condition for an over-all stability of the motion is the well known

$$
+1 > \cos \mu > -1, \tag{IV.24}
$$

where $\cos \mu$ is half the trace of the transfer matrix through one period. In the short-lens approximation where

$$
\cos \mu = 1 - \eta^2 / 2
$$

$$
\sin \mu = \frac{\eta}{2} \sqrt{4 - \eta^2}, \tag{IV.25S}
$$
the condition in Eq. (IV.24) leads to the limits

\[ 0 < |\eta| < 2. \quad (\text{IV.26S}) \]

In the first limit, there will be no focusing at all. In the second, there will be over-focusing.

For our problem it is not sufficient to ensure that the stability condition in Eq. (IV.24) is satisfied; the space available for the beam is not unlimited. Let us specify that the beam must stay inside a square pipe with sides of length 2\( q \) parallel to the \( x \) and \( y \) axes (Fig. IV.1). Then we must require that during all the motion inside the pipe

\[
\begin{align*}
|x| &< q \\
|y| &< q. 
\end{align*} \quad (\text{IV.27})
\]

In order to find for which initial conditions this will be fulfilled throughout the motion, we shall have to make a more detailed study of this motion. If we start from the initial conditions \( x_0, x'_0 \) at an arbitrary point in the structure (e.g. in the middle of a focusing lens), we can obtain position and angles at any corresponding point in the periodic structure from integer powers of the transfer matrix \( M \).

The \( n^{\text{th}} \) power of the matrix \( M \) is, for \( \| M \| = 1 \),

\[
A = M^n = \frac{\sin n\mu}{\sin \mu} M - \frac{\sin (n-1)\mu}{\sin \mu} E, \quad (\text{IV.28})
\]

where \( E \) is the unit matrix and \( \mu \) follows from the definition of \( \cos \mu \) as half the trace of \( M \). (Proof in Appendix B.) The first row of \( A \) gives the lateral excursions in corresponding points of the structure, in terms of the initial conditions

\[
x = A_{11} x_0 + A_{12} x'_0. \quad (\text{IV.29})
\]

This becomes

\[
x = x_0 \left[ M_{11} \frac{\sin n\mu - \sin (n-1)\mu}{\sin \mu} + x'_0 M_{12} \frac{\sin n\mu}{\sin \mu} \right]. \quad (\text{IV.30})
\]
We now introduce a continuous variable $n$, which coincides with the number $n$ introduced above for all integer values. $x(n)$ then represents a curve, which coincides with the true trajectory at all integer values of $n$. In this sense, the motion is periodic in $n$, with a period of motion

$$\Delta n = \frac{2\pi}{\mu} = \frac{2\pi}{\arccos \left( \frac{M_{11} + M_{22}}{2} \right)}$$ \hspace{1cm} (IV.31)

$$\Delta n = \frac{2\pi}{\arccos \left( 1 - \eta^2 / 2 \right)}.$$ \hspace{1cm} (IV.32S)

This is the "wavelength" of the motion, in terms of the structure period $2L$: it is independent of the starting point and of the initial conditions. For small values of $\eta (\mu \approx \eta)$, Eq. (IV.32S) yields $2\pi / \eta$, so the wavelength is large. For $\eta = \sqrt{2}$ we find $\mu = \pi / 2$, $\Delta n = 4$. The smallest value of $\Delta n$ is reached for $\mu = \pi (\eta = 2)$, where $\Delta n = 2$. Therefore the minimum wavelength for a stable motion is $4L$.

In the short-lens approximation the projections of the true trajectory on the $x$ and $y$ planes are broken straight lines. We expect the maximum excursion to occur in focusing lenses. If, therefore, the starting point of the transfer matrix $M$ is chosen in a focusing lens, the maximum (with respect to $n$) of the periodic motion given in Eq. (IV.30) will at the same time be the maximum distance from the axis which the true trajectory will ever reach, for given initial conditions $x_0, x'_0$. Moreover, if the system is sufficiently long, and unless the ratio $\Delta n$ [Eq. (IV.31)] of the period of motion to the structure period can be exactly expressed as the ratio of small integer numbers, the maximum of the broken line trajectory will, at some stage of the motion, be very near to the maximum of the periodic motion.

For long lenses, the trajectories are no longer broken lines, but are rounded off. For reasons of symmetry we expect in this case that the maximum excursion of accepted particles takes place in the middle of a focusing lens. If, therefore, we choose the middle of a focusing lens as the starting point for the transfer matrix, the trajectory will coincide with the periodic motion in the middle of each focusing lens, and we can apply the same reasoning as for short lenses.
In a short channel these considerations would give a pessimistic picture, for we regard a particle as lost if any point of its (extrapolated) trajectory falls outside the boundaries, without investigating whether this point is reached at all before the particle has left the channel.

We must now find the maximum of Eq. (IV.30) with respect to $n$. $dx/dn = 0$ yields, since $\mu \neq 0$:

$$
\tan n \mu = \frac{(M_{11} - \cos \mu) x_0 + M_{12}x_0'}{x_0 \sin \mu}
$$

$$(\text{IV.33})$$

$$
\tan n \mu = \sqrt{\frac{2 + \eta}{2 - \eta}} \left( 1 + \frac{2x_0' L}{x_0 \eta} \right).
$$

Inserting this into the left side of Eq. (IV.30) and requiring the maximum excursion so obtained to be $q$, we find the locus of starting conditions $x_0, x_0'$ in the plane of a focusing lens, which makes particles just scrape the wall in some phase of their motion. Omitting the index we find for this locus

$$
F_0 x^2 + 2F_1 x x' + F_2 x'^2 = F_3
$$

$$(\text{IV.34})$$

with

$$
F_0 = 1 - M_{11}M_{22}
$$

$$
F_1 = (M_{11} - M_{22})M_{12}/2
$$

$$
F_2 = M_{12}^2
$$

$$(\text{IV.35})$$

$$
F_3 = q^2 \sin^2 \mu = \left( 1 - \frac{(M_{11} + M_{22})^2}{4} \right) q^2,
$$

or for short lenses

$$
F_0 = 4\eta^2
$$

$$
F_1 = 2\eta(2 + \eta)L
$$

$$
F_2 = 4(2 + \eta)L^2
$$

$$(\text{IV.36S})$$

$$
F_3 = \eta^2(2 - \eta)q^2.
$$
This is the equation of an ellipse which contains all those starting conditions in the phase plane at the middle of a thick (resp. just behind a thin) focusing lens, which will make particles just pass through the system ("focusing ellipse").

Before discussing this ellipse we shall find the boundary curves in phase planes at other $z$. Since the only physical limitation of the beam occurs in the focusing lenses, where the transversal excursions are greatest, the limitations in all other phase planes are only consequences of this one limitation:

The linear transfer equations for positions and angles brings one ellipse over into another one with the same $F_3$. Points inside a boundary ellipse lie on ellipses with smaller $F_3$, and will therefore always be transferred to points inside the transferred boundary ellipse. In other words, points on the boundary ellipse will always remain boundary points throughout the motion. The ellipse areas will be conserved according to Liouville's theorem, which is expressed by the fact that all transfer matrices have a determinant $= 1$. After one period of structure the ellipse will have recovered its original shape and position. This does not mean, however, that individual points have done so.

The motion in the phase plane is particularly simple. In drift spaces all points move on lines parallel to the $x$ axis, with velocities proportional to $x'$ and clockwise with respect to the origin. In short lenses, a reflection of the ellipses with respect to the $x$ axis takes place, clockwise for a focusing lens, anticlockwise for a defocusing lens. In ideal long lenses, the reflection is replaced by a sort of rotation, where individual points describe elliptical resp. hyperbolic trajectories. In the centre of the lenses, the ellipses are on principal axes.

We find the equation of the ellipse belonging to a general $z$ position by using the transfer matrix $N$ from co-ordinates $x_\alpha, x'_\alpha$ in this general position to co-ordinates $x_f, x'_f$ inside (resp. just after) a focusing lens. The $x_f, x'_f$ are then expressed in terms of the $x_\alpha, x'_\alpha$ and substituted into the ellipse of Eq. (IV.34). It is
often more convenient to construct the inverse matrix \(N^{-1}\), leading from the focusing lens to the general position. The inverse matrix of

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix}
\text{if } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1.
\] (IV.37)

In short-lens approximation the transfer matrix to points at a distance \(a\ell(0 \leq a \leq 1)\) behind a focusing lens is

\[
N^{-1} = \begin{pmatrix} 1 & a\ell \\ \ell & 1 \end{pmatrix},
\] (IV.38)

and to points at a distance \(a\ell\) behind a defocusing lens

\[
N^{-1} = \begin{pmatrix} 1 + \alpha & L[1 + (\eta + 1)\alpha] \\ \delta & 1 + \eta \end{pmatrix}.
\] (IV.39)

In this way we obtain for the ellipse in a general position an equation analogous to Eq. (IV.34), but with coefficients \(P_i\) given by

\[
P_0 = F_0N_{11}^2 + 2F_1N_{11}N_{21} + F_2N_{21}^2
\]
\[
P_1 = F_0N_{11}N_{12} + F_1(N_{11}N_{22} + N_{12}N_{21}) + F_2N_{21}N_{22}
\]
\[
P_2 = F_0N_{12}^2 + 2F_1N_{12}N_{22} + F_2N_{22}^2
\]
\[
P_3 = F_3.
\] (IV.40)

The \(F_i\) are defined in Eq. (IV.35), \(N_{ij}\) are the elements of \(N\).

In the short-lens approximation we find the following coefficients:

\[
P_0 = 4\eta^2
\]
\[
P_1 = 2[2 + \eta(2\alpha - 1)] \eta L
\]
\[
P_2 = 4[2 + \eta(2\alpha - 1) - \alpha^2(1 - \alpha)]L^2
\]
\[
P_3 = F_3 = \eta^2(2 - \eta)q^2,
\] (IV.41)
where the upper sign applies to positions after a focusing and the lower to positions after a defocusing lens. A case of special interest is the position just behind a defocusing lens \((\alpha = 0)\), with the following ellipse coefficients \(D_n\):

\[
D_0 = 4\eta^2 \\
D_1 = -2\eta(2-\eta)L \\
D_2 = 4(2-\eta)L^2 \\
D_3 = F_3 = \eta^2(2-\eta)q^2 ,
\]

which differ from Eq. \((IV.36S)\) only by the sign of \(\eta\) of the left-hand side of the ellipse equation (defocusing ellipse).

A point of special interest for the matching of the channel to other devices is the midpoint between lenses. Here we find for the ellipse coefficients analogous to those in Eq. \((IV.34)\) for "ideal" lenses of arbitrary length

\[
M_1 = \left(\frac{q}{k}\right)^{-1}\left\{\left(\frac{\omega}{2} + \tan \epsilon\right)^i + A\left(\frac{\omega}{2} - \cot \epsilon\right)^i\right\}
\]

\[
M_3 = \left(\frac{q}{\cos \epsilon}\right)^2
\]

with

\[
A = -\frac{(1+\omega+\tan \epsilon)^2 e^{4\epsilon} - (1-\omega+\tan \epsilon)^2 e^{4\epsilon}}{(1+\omega-\cot \epsilon)^2 e^{4\epsilon} - (1-\omega-\cot \epsilon)^2}
\]

\[
\omega = k(L-S) \\
\epsilon = kS/2 ,
\]

and for short lenses

\[
M_0 = 4\eta^2 \\
M_1 = \pm 4\eta L \\
M_2 = (8-\eta)L^2 \\
M_3 = F_3 .
\]
The upper sign applies after a focusing lens, and the lower one after a defocusing lens.

The quantity $A$ as given by Eq. (IV.43) is plotted in Fig. IV.2 as a function of $\eta = k^2SL$ for various values of $S/L$.

These admittance ellipses describe the focusing properties of the channel completely. For any plane, including the entrance plane of the channel, we can write down the two ellipses belonging to the $x$ and $y$ direction, and can decide whether the points in the $x$ and $y$ phase spaces which describe the position and angle of a particle both fall within their respective acceptance ellipses or not. The pions at the entrance will be distributed over the two phase planes in a certain way, which may also be correlated with their momentum. The number of accepted pions will normally increase if we increase the area of the ellipses. For given areas it is the task of suitable matching devices to fit the distribution of the particles in the phase planes to the shape and orientation of the ellipses.

In the following we will, for easy reference, list some properties of the ellipses described above, both for general transfer matrices and in short-lens approximation. This discussion will also yield a figure of merit for the beam transport system and its momentum dependence.

In Fig. IV.3 we recall the way in which the co-ordinates of some special points depend on the coefficients of the ellipse equation.

In the general case we will assume that the phase plane, to which the focusing ellipse described by Eq. (IV.34) refers, is in the centre of that lens and thereby in a point of symmetry of the system. This symmetry is expressed by

$$M_{11} = M_{22} = \cos \mu ,$$  \hspace{1cm} (IV.45)

which leads to the coefficients
\[ F_0 = \sin^2 \mu \]
\[ F_1 = 0 \]
\[ F_2 = M_{12}^2 \]
\[ F_3 = q^2 \sin^2 \mu. \]

The ellipse is on principal axes. The length of half-axis along the x axis, i.e. the size of widest parallel beam passing, follows from
\[ x_{\text{max}} = \sqrt{\frac{F_3}{F_0}} = q, \]

which was our starting assumption.

The intersection with the x' axis, i.e. the largest angle admitted for a point source on the axis, is
\[ K_F = x_{\text{max}}' = \sqrt{\frac{F_3}{F_2}} = \left| \frac{q \sin \mu}{M_{12}} \right|. \]

In finding from Eq. (IV.40) the ellipse referring to the middle of a defocusing lens, we can make use of a further symmetry property expressed by
\[
\begin{pmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{pmatrix}
\begin{pmatrix}
N_{22} & N_{12} \\
N_{21} & N_{11}
\end{pmatrix}
= \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{11}
\end{pmatrix}.
\]

This ellipse is also on principal axes and has
\[ x_{\text{max}} = \sqrt{\frac{F_3}{D_0}} = \left| \frac{q M_{21} \sec \frac{\mu}{2}}{2N_{21}} \right|, \]

\[ K_D = x_{\text{max}}' = \sqrt{\frac{F_3}{D_2}} = \left| \frac{q \sin \frac{\mu}{2}}{N_{12}} \right|. \]
$K_F$ and $K_D$ given by Eqs. (IV.48) and (IV.51) will be important for the discussion on the focusing of the decay products (Section VI).

For the ellipse area we find from Eq. (IV.46)

$$F = \pi \frac{q^2 \sin \mu}{|M_{t2}|} .$$  \hspace{1cm} (IV.52)

This area (without the factor $\pi$) is the one-dimensional admittance product of the system. As far as the focusing of the pions is concerned, this is the relevant figure of merit of the beam transport system.

In terms of Eq. (IV.43) this area becomes

$$F = \pi \frac{k q^2 \tan \epsilon}{\sqrt{A}} .$$  \hspace{1cm} (IV.53)

The way in which these quantities depend on the parameters of the system is more transparent in the short lens approximation. Here it proves to be more convenient for the later discussion (Section V) to choose the reference plane in the lenses, not in the middle but just after the lenses. So the symmetry conditions Eqs. (IV.45) and (IV.49) do not apply to this case. The focusing ellipse Eq. (IV.34) with the coefficients of Eq. (IV.36S) is not on principal axes.

For $x' = 0$ (beam parallel to the axis) we find

$$x = \frac{F_3}{F_0} = \left( \frac{q}{2} \right) \sqrt{2 - \eta} .$$  \hspace{1cm} (IV.54S)

The important quantity $K_F$ obtained for $x = 0$ is:

$$K_F = x' = \frac{F_3}{F_2} = \left( \frac{q \nu}{2L} \right) \sqrt{\frac{2 - \eta}{2 + \eta}} .$$  \hspace{1cm} (IV.55S)

The expression has a maximum $= 0.305 \, q/L$ for $\eta = 1.24$. For $1/\delta = 133$ cm this would mean $L = 165$ cm, and for $q = 10$ cm

$$K_F = 18.5 \text{ mrad (1.06°)} .$$
The ellipse is confined between limits

$$x_{\max} = \sqrt{F_2} \ G = q \quad \text{(IV.56)}$$

(our starting assumption), with

$$G = \frac{\sqrt{F_3}}{\sqrt{F_0 F_2 - F_1^2}} \quad \text{(IV.57)}$$

and

$$x'_{\max} = \sqrt{F_0} \ G = \frac{q \eta}{L \sqrt{2 + \eta}} \quad \text{(IV.58)}$$

This becomes $0.69 \ q/L$ for $\eta = 1.24$.

These limits give the extreme positions and angles for which particles can be made to pass for a special choice of the other starting condition, namely

$$x' = \frac{F_1 G}{\sqrt{F_2}} = \frac{q \eta}{2L} \quad \text{(IV.59)}$$

and

$$x = \frac{F_1 G}{\sqrt{F_0}} = \frac{q}{2} \sqrt{2 + \eta} \quad \text{(IV.60)}$$

respectively.

The area of the ellipse is given by

$$F = \pi \sqrt{F_3} \ G = \frac{\pi q^2 \eta}{2L} \sqrt{\frac{2 - \eta}{2 + \eta}} \quad \text{(IV.61)}$$

It has the same dependence on $\eta$ as $K_F$ in Eq. (IV.55S).

This means that the system with $\eta = 1.24$ has the optimum acceptance. The function $FL/mq^2$ is plotted against $\eta$ in Fig. IV.4. It is seen that the maximum is fairly flat. This makes it possible to design a beam transport system for a fairly wide momentum band (\eta is inversely proportional to $p$). If, for instance, one is prepared to sacrifice up to 20% in the area, one finds for $S/L = 0$ \eta limits of 0.67 and 1.7, so the extreme momenta can have a ratio 2.5.
For the defocusing ellipse we find the following special points:

\[ x' = 0, \ x = \frac{F_2}{D_0} = \left( \frac{a}{2} \right) \sqrt{2 - \eta}, \]  

(IV.62S)

as for the focusing ellipse of Eq. (IV.54S)

\[ x = 0, \ K^D = x' = \sqrt{\frac{F_3}{D_2}} = \frac{a\eta}{2L}, \]  

(IV.63S)

which gives for \( \eta = 1.24 \)

\[ K^D = 0.62 \frac{a}{L}. \]

The extreme points are now

\[ x'_{\text{max}} = \sqrt{D_2} \ G = q \sqrt{\frac{2 - \eta}{2 + \eta}} \]  

(IV.64S)

\[ x'_{\text{max}} = \frac{D_1 G}{\sqrt{D_2}} = \frac{a\eta}{2L} \sqrt{\frac{2 - \eta}{2 + \eta}} \]  

(IV.65S)

as in Eq. (IV.58S), which are reached for

\[ x' = \frac{D_1 G}{\sqrt{D_2}} = \frac{a(2 - \eta)}{2 \sqrt{2 + \eta}} \]  

(IV.66S)

\[ x = \frac{D_1 G}{\sqrt{D_0}} = \frac{a(2 - \eta)}{2 \sqrt{2 + \eta}} \]  

(IV.67S)

respectively.

Note that

\[ G = \frac{F_3}{\sqrt{F_0 F_2 - F_1^2}} = \frac{F_3}{\sqrt{D_0 D_2 - D_1^2}}. \]
V. THE FOCUSING OF THE DECAY PRODUCTS

V.1 Principle of the method

Pions are liable to decay in any position within the channel, with a random azimuth of decay. We want to know which fraction of the decay products will avoid hitting the channel walls and thus emerge from the end of the channel. We shall call this figure the trapping efficiency.

The trapping efficiency \( T = T(p_\pi, p_\mu) \) is more precisely the ratio of the number of muons in a momentum band around \( p_\mu \) emerging from the channel to the number of muons emitted into that momentum band in the channel from pions of the momentum \( p_\pi \). (V.1)

The decay has two effects in the phase space diagrams:

i) the decay angle will be represented by a shift of the points in the pion ellipses *) parallel to the \( x' \) and \( y' \) axis;

ii) the change of momentum will be reflected in a different shape and orientation of the new ellipses that will "trap" the muons.

The trapping efficiency will be the fraction of muons for which both of the points representing them in the two phase planes fall inside the muon ellipses after the decay [Fig. V.1].

Let us look at the shift in somewhat more detail. The decay will take place with a random azimuth. So in the two projections we will find \( x' \) resp. \( y' \) shifts ranging from zero to the amount corresponding to the full decay angle \( \theta \leq \theta_{\text{max}} \) given by Eq. (II.10).

Let us call the azimuth of the decay measured from the \( x-z \) plane, \( \phi \). We shall assume for a moment that the decay takes place in the middle of a lens for which the \( x \) direction is the focusing one. In the \( x \) phase plane, the \( \pi \) ellipse with its population will shift up or down by an amount equal to the projection \( \theta' \) of the decay angle \( \theta \) on the \( x \) plane, namely \( \theta \cos \phi \).

*) We call "pion ellipse" the ellipse for the pion momentum, etc.
For a given $\varphi$, the population inside the new trapping ellipses, the $\mu$ ellipses, follows immediately from the distribution before the decay. We shall tentatively assume that the pion population is uniform inside the pion acceptance ellipses. We shall have to come back to this point when we treat the matching at the entrance of the system (Section VII.5), and discuss how the deviation of the actual population from a uniform one affects our results. The one-dimensional trapping efficiency $t_F(\theta \cos \varphi)$ can then be found as the area common to the shifted $\pi$ ellipse and the unshifted $\mu$ ellipse divided by the area of the $\pi$ ellipse. If $\varphi = \pi/2$ there will be no shift of the $\pi$ ellipse. Still, $t_F(0)$ will not be unity, as the unshifted $\pi$ ellipse differs from the $\mu$ ellipse in shape and orientation. The ellipses to be used here are given by Eq. (IV.34), with coefficients from Eqs. (IV.46) or (IV.36S).

For the same azimuth $\varphi$, the one-dimensional trapping efficiency in the defocusing $y$ plane, $t_D(\theta \sin \varphi)$, can be found in an analogous manner. Here the ellipse defined by its axes in Eqs. (IV.50) and (IV.51), or the ellipse given by the coefficients Eq. (IV.42S), has to be used.

We now drop the special assumption that the decay takes place in the middle of an $x$ focusing lens and find one-dimensional trapping efficiencies $t_x(\theta \cos \varphi)$ and $t_y(\theta \sin \varphi)$ for an arbitrary decay point, using two ellipses with coefficients from Eq. (IV.40) where the matrices $N$ lead from the point of decay to the middle of the preceding focusing lens in the $x$ and the $y$ plane, respectively. In short-lens approximation we use the coefficients in Eq. (IV.41S). If the decay takes place at a distance $dL$ after an $x$ focusing lens, we will use coefficients with the upper sign to find $t_x$, with the lower sign to find $t_y$.

We have now, in principle, found one-dimensional trapping efficiencies for particles decaying under the most general conditions inside the channel. The fact that a particle is lost or trapped in the $x$ direction is totally unrelated to whether it is lost or trapped in the $y$ direction, since for our choice of co-ordinate system and the square shape of the limiting box [Eq. (IV.27)] the $x$
and y motions are decoupled. We have then to average the product of \( t_x \) and \( t_y \) over a structure period in \( z \), and over \( \varphi \), in order to find the trapping efficiency \( T(p_\pi, p_\mu) \) for the channel:

\[
T(p_\pi, p_\mu) = \frac{4}{\pi L} \int_0^{\pi/2} d\varphi \int_0^{L/2} dz \; t_x(p_\pi, p_\mu, z, \vartheta \cos \varphi) t_y(p_\pi, p_\mu, z, \vartheta \sin \varphi), \tag{V.2}
\]

where the appropriate angle \( \vartheta \), given by Eq. (II.10), is used for each couple \( p_\pi, p_\mu \). The integration can be limited to the intervals shown because of symmetry.

Multiplying the muon decay spectrum, as given by Eq. (II.9), by the trapping efficiency and integrating gives the muon yield

\[
Y(p_\pi) = \int_{p_{\min}}^{p_{\max}} dp_\mu \; T(p_\pi, p_\mu) \frac{\beta_\mu (1 + \bar{\beta})}{2\bar{\beta} p_\pi}. \tag{V.3}
\]

For a pion spectrum \( N_\pi(p_\pi) \) at the entrance of the channel the muon spectrum is given by

\[
N_\mu(p_\mu) = \int dp_\pi \; N_\pi(p_\pi) (1 - e^{-QL/5.50 p_\pi}) \frac{\beta_\mu (1 + \bar{\beta})}{2\bar{\beta} p_\pi} T(p_\pi, p_\mu), \tag{V.4}
\]

where \( Q \) is the number of quadrupoles in the channel and where Eq. (II.12) has been used.

The degree of longitudinal polarization of the beam can be found by multiplying the integrand of Eq. (V.4) by \( \cos \varphi \) as given by Eq. (II.13), and by dividing this integral by \( Y_\mu(p_\mu) \). Transversal polarization will cancel out, because of the symmetry of the system.

V.2 Details of the calculation in short-lens approximation

Although Eq. (V.2) may be easily evaluated by a computer programme which we now have, we shall discuss it in the short-lens
approximation, both for historical reasons and because the physics involved might then become more transparent.

First let us consider the effect of the momentum change. To fix the ideas we assume a change from 400 to 300 MeV/c, i.e. by a factor 0.75. According to Eq. (IV.23S), \( \eta \) will increase by a factor \( 1/0.75 = 1.33 \). If the channel is designed to have \( \eta = 1.0 \) for the pions, it will have \( \eta = 1.33 \) for the muons. The ellipses will change in three respects:

i) The area \( F \) given by Eq. (IV.61S). Between the two \( \eta \) values given above, which are on either side of the optimum value \( \eta = 1.24 \), this change will be smaller than 5%. We will disregard this change.

ii) The inclination of the "axis of vertical symmetry", i.e. the line from the origin to the point of maximum excursion (Fig. IV.3). From Eqs. (IV.56) and (IV.59S), and from Eqs. (IV.64S) and (IV.66S), it is seen that this inclination is \( \pm \eta \lambda /2 \) for the ellipses both in the focusing and the defocusing direction.

iii) The vertical extent with respect to the axis of vertical symmetry and the horizontal width of the ellipses. Here the ellipses for the focusing and the defocusing case behave differently. In the focusing case the ellipses have always the same horizontal width \( 2q \) [Eq. (IV.56)], and the vertical height with respect to the axis of vertical asymmetry is proportional to \( K_F \) [Eq. (IV.48)], which is considered as constant in \( \eta \) for the reasons given under i). For the defocusing case the ellipses change both in vertical and horizontal extent. The vertical extent goes up as \( \eta/2 \) [Eq. (IV.63S)], and the horizontal one down as Eq. (IV.64S) with increasing \( \eta \).

In the case \( \eta_\mu = \eta_\pi \) (no change of shape of the ellipses) the calculation of the common area of the shifted ellipses would be simple. It leads to the formula

\[
t = \begin{cases} 
\left( \frac{2}{\pi} \right) \arccos \theta - \theta \sqrt{1 - \theta^2} & \text{for } |\theta| \leq 1 \\
0 & \text{for } |\theta| > 1 
\end{cases} \quad \text{(V.58)}
\]
where

\[ \Theta = \frac{\theta'}{2k} . \]

\( \theta' \) is the projection of \( \theta \) on the plane under investigation, and \( K \) is the maximum angle accepted from a point source on the axis, as given in general by the expression

\[ K = \sqrt{\frac{F_3}{P_2}} , \]

where \( F_3 \) and \( P_2 \) are coefficients of the general ellipse [ Eq. (IV.40) or Eq. (IV.41S)]. Special values of \( K \) were given in Eqs. (IV.48) and (IV.51). We find from Eqs. (IV.55S) and (IV.63S)

\[ \Theta_F = \left( \frac{\theta'_F}{q \eta} \right) \sqrt{\frac{2 + \eta}{2 - \eta}} \quad \text{for a decay at a focusing lens} \quad (V.6S) \]

\[ \Theta_D = \frac{\theta'_D}{q \eta} \quad \text{at a defocusing lens} , \quad (V.7S) \]

and in the general case, at the distance \( d_L \) behind a focusing lens:

\[ \Theta = \Theta_F \sqrt{1 - a \eta + \frac{a^2 \eta^2}{2 + \eta}} . \quad (V.8S) \]

Substituting Eq. (V.8S) into Eq. (V.5S), we find that \( t \) is almost exactly a linear function of \( a \).

If \( \eta_\mu \neq \eta_\pi \), \( t_F \) can still be calculated analytically under the approximation that the area of the ellipse does not depend on \( \eta \). A single parameter \( \Phi_F \) describes the change of shape in this case

\[ \Phi_F = \frac{\eta_\mu - \eta_\pi}{2 \eta \sqrt{\frac{2 - \eta}{2 + \eta}}} . \quad (V.9S) \]

Note that the expression in the denominator is proportional to the area of the ellipse, and it is therefore consistent to neglect its dependence on \( \eta \). In the same way \( \Theta_F \) can be regarded as independent
of \( \eta \). A family of curves of \( t_F \) as a function of \( \Theta_F \), with \( \Phi_F \) as a parameter, is given in Fig. V.2.

The effect of a change in \( \eta \) on the defocusing ellipses is a more complicated one, as we have seen before. Therefore, these ellipses were actually drawn for various values of \( \eta \) and the common areas were measured with a planimeter. Graphs of \( t_D \) as a function of \( \vartheta' L/q \) have been prepared for various combinations of the two parameters \( \eta_\pi \) and \( \eta_\mu \).

We have found above for the short-lens approximation and for \( \eta_\pi = \eta_\mu \) that the \( z \) dependence is linear. We shall assume that this linearity also holds for \( \eta_\pi \neq \eta_\mu \). Then we can express \( t \) in terms of \( t_F \) and \( t_D \) as follows:

\[
\begin{align*}
t_x(\alpha) &= t_F + \alpha(t_D - t_F) \\
t_y(\alpha) &= t_D + \alpha(t_F - t_D),
\end{align*}
\]  

(V.10S)

where all the \( t \)'s are functions of \( \varphi \).

If we abbreviate as follows:

\[
\begin{align*}
t_F(\vartheta \cos \varphi) &= t_{FC} \\
t_D(\vartheta \cos \varphi) &= t_{DC} \\
t_F(\vartheta \sin \varphi) &= t_{FS} \\
t_D(\vartheta \sin \varphi) &= t_{DS},
\end{align*}
\]  

(V.11)

we obtain for the average of the product \( t_x t_y \):

\[
T_s(p_\pi, p_\mu) = \left( \frac{4}{\pi} \right) \int_0^{\pi/4} \int_0^1 \left[ t_{FC} + \alpha(t_{DC} - t_{FC}) \right] \left[ t_{DS} + \alpha(t_{FS} - t_{DS}) \right] d\alpha \, d\varphi.
\]  

(V.12S)

The intervals of integration are different here from Eq. (V.2). This is only the consequence of slightly different symmetry considerations.
The integration over $\phi$ can be carried out and yields
\[ T_\pi(p_\pi, p_\mu) = \left( \frac{2}{3\pi} \right) \int_0^{\pi/4} \left( 2\ t_{FC}t_{DS} + 2\ t_{DC}t_{FS} + t_{DC}t_{DS} + t_{FC}t_{FS} \right) d\phi. \] (V.13S)

The integration over $\phi$ has to be carried out numerically. If one of the four partial trapping efficiencies (for example, $t_{FC}$) vanishes for a certain $\phi$, a value $\alpha_0$ of $\alpha$ will exist such that
\[ t_\pi = 0 \text{ for } \alpha < \alpha_0. \] (V.14S)

Instead of the first equation (V.10S) we will then have
\[ t_\pi(\alpha) = \frac{\alpha - \alpha_0}{1 - \alpha_0} t_{DC}. \] (V.15S)

We take the value of $\alpha_0$ from the simplified theory which disregards the effects due to $\eta_\mu \neq \eta_\pi$:
\[ \alpha_0(\phi) = \frac{2 + \eta}{2 - \eta} \left( 1 - \sqrt{\frac{4}{(2 + \eta)\Theta_F(\phi)} - \frac{2 - \eta}{2 + \eta}} \right). \] (V.16S)

where $\Theta_F$ is given by Eq. (V.6S). It is not unambiguous which $\eta$ to take when using this formula. The effect is discussed below (Section V.3).

The integral then becomes:
\[ T_\pi(p_\pi, p_\mu) = \frac{2}{3\pi} \int_0^{\pi/4} t_{DC}(1 - \alpha_0)(t_{DS}(1 - \alpha_0) + t_{FS}(2 + \alpha_0)) d\phi. \] (V.17S)

Cases where more than one of the $t$'s vanish correspond to very small trapping efficiencies and are of no practical interest.

In principle this procedure has to be carried out for the $\mu$ momentum one finally wants to accept, and for all $\pi$ momenta which can contribute. Each $\pi$ momentum corresponds to a different $\eta_\pi$ and to a different decay angle $\theta$, according to Eq. (II.10). A weighted average over $\pi$ momenta has then to be taken as in Eq. (V.4).
Figure V.3 compares some short-lens results with the results of a more rigorous evaluation of ellipse areas and trapping efficiencies.

V.3 Results of the short-lens calculation and a rough approximation

If we now ask ourselves in which way the trapping efficiency depends on the design parameters, then we see that this dependence must be through the partial trapping efficiencies $t$, and through $a_0$. The partial trapping efficiencies $t_F$ were taken from curves, which had the expression $\varphi_F$ [Eq. (V.68)] as abscissa and $\Phi_F$ [Eq. (V.98)] as parameter. $\varphi_F$ contains the design parameters in the expression $\partial L/\partial q$. $\Phi_F$ contains only the $\eta$'s.

In the same way $t_D$ is taken from curves, which have $L/q$ as abscissa and the $\eta$'s as parameter.

$a_0$ depends on $\varphi_F$ and on $\eta$.

In conclusion, we find that the trapping efficiency depends primarily on the expression

$$\gamma = \frac{\varphi}{\partial L}$$

(V.188)

and to a lesser extent on the values $\eta_\mu$ and $\eta_\pi$.

For a particular choice of the $\eta$'s, namely $\eta_\mu = 1.4$, $\eta_\pi = 1.05$, Fig. V.4 gives the trapping efficiency $T_s(p_\mu, p_\mu)$ as a function of $\gamma$. The trapping efficiency first rises about linearly with $\gamma$, then levels off towards a saturation value of 66%. The reason why no trapping efficiency of 100% is reached even for a very wide system (or very small decay angles) is the momentum change, which produces a mismatch between the $\mu$ and $\pi$ ellipses. This results in losses of the order of 20% in either plane for our choice of $\eta$ and for uniform population of the ellipses. The region between $\gamma = 1$ and $\gamma = 2.2$ which is of most interest for our case has been plotted on an extended scale in Fig. V.5.

At low $\gamma$ values, where Eq. (V.178) has to be used, the uncertainty in $a_0$ mentioned above is demonstrated by using either $\eta_\mu$ or $\eta_\pi$ in Eq. (V.168). It is seen that this uncertainty is not serious at not too low trapping efficiencies.
The curve is almost a straight line in this region. It can be represented by the empirical equation

\[ T_s(p_\pi, p_\mu) = 0.23 \ (\gamma - 1.05) \quad 1.05 \leq \gamma \leq 2. \quad (V.19S) \]

Since the method outlined above is still somewhat involved, one can ask whether it is possible to obtain a rough approximation in a quicker way. One possible way would be to take the arithmetical mean of \( \vartheta_F \) and \( \vartheta_D \) [Eqs. (V.6S) and (V.7S), using \( \vartheta_\mu \) and \( \vartheta' = \vartheta \)]

\[ \vartheta = (\vartheta_F + \vartheta_D)/2, \quad (V.20S) \]

insert this into Eq. (V.5S), and use the resulting \( t \) as trapping efficiency. We will denote the trapping efficiency arrived at in this way by \( T'_s(p_\pi, p_\mu) \). It is also shown in Figs. V.4 and V.5. Since this approximation disregards the change in momentum, the saturation value is near to 100% in this case. In the region between \( \gamma = 1 \) and \( \gamma = 2.2 \) we again find an almost straight line, but with too big a slope.

Since this method is very expedient, we give in Fig. V.6 the correction \( \Delta T_s \) which has to be applied to a trapping efficiency \( T'_s \) found by the approximate method in order to obtain the result \( T_s \) of the more complete method. We can express this correction as:

\[ \Delta T_s = T_s - T'_s = -0.39 \ (T'_s - 0.095) \text{ for } T'_s < 0.25. \quad (V.21) \]

V.4 Non-linear effects

A full description of the field of a real quadrupole yields many terms in the equations of motion, which lead to non-linear transfer equations. Septier\(^{s}\) has measured the field of a lens similar to those we were considering. If he compares the actual field at a certain radius to that expected for a lens in which the gradient is strictly constant and equal to that observed near the axis, he finds that up to a radius of about 0.9\( a \) these field errors stay below 3%.
For the lenses actually used, this has also been verified a posteriori\textsuperscript{e}).
The main deviation observed at large radii is a drop of the gradient.

In order to study the effect of this particular error on the focusing in a long channel, we have assumed a drop of the field gradient in the outer part of the lens (but inside 0.9a) in accordance with the experimental findings, and we have followed the trajectories of particles in such a field. This has been done graphically in a phase plane, using short-lens approximation. Particles were always started on the optical axis, at a distance L in front of a focusing or a defocusing lens. If a particle touched the boundary (assumed at 0.9a) at or before the fifteenth pair of lenses, it was considered as lost. Its starting angle was then decreased until it was no longer lost. In this way we obtained the reduction in $K_F$ and $K_D$ [Eqs. (IV.558) and (IV.638)]. This reduction was 3% in $K_F$ and 5% in $K_D$. For a trapping efficiency $T_s^\prime$ near 0.15 this means a reduction in $T_s^\prime$ by 0.03.

We conclude from this that it is safest to keep non-linearities as small as possible.

The non-linearities of the type mentioned, essentially make the effective aperture (i.e. q) of the lenses smaller. However, since q is undetermined within rather wide limits anyway (see Section VI.1), we shall limit the discussion of non-linear effects to these remarks and consider in the following only linear motion.

VI. DESIGN PROCEDURE

VI.1 Considerations about q and S

2q was defined near Eq. (IV.27) as the side of the square pipe which is assumed to confine the beam in the channel. In actual fact, the area available to the beam is not a square, but a complicated figure bounded by the pole faces and the coils. Also the field is, in a real lens, not linear everywhere in this region (see Section V.4).

It seems natural to discuss q in terms of the radial distance to the pole tips a, and to fix lower and upper limits for it as follows:
i) **Lower limit**: The edges of the square pipe touch the pole pieces, i.e.

\[ q = a/\sqrt{2} = 0.71a \]

ii) **Upper limit**: The square pipe shall have the same area as the total field cross-section between the poles and coils, which would lead to something like \( q \approx 1.1a \).

A reasonable choice inside these limits could be to give the square pipe the same cross-section as a circular cylinder inscribed to the pole pieces. This leads to

\[ q = a\sqrt{\pi}/2 = 0.88a \]

The design was made such that the channel could work even under the very pessimistic assumption in i). When comparing observations on the channel with computed results we shall, however, consider q as a free parameter, and see if there is any q within reason that can fit the observations. There is a so much stronger reason for doing this, as the fact that the channel is not infinitely long, that we replace the long real lens with fringe field by at best a long "ideal" lens, and that the ellipses are not uniformly filled, will in some way influence the apparent value of q.

For technical reasons the ratio of the lens length and lens separation

\[ f = \frac{S}{L} \]

which we shall call the "filling factor", cannot be chosen arbitrarily large. S is related to \( S_{\text{iron}} \), the physical length of the iron of the lens, and L will have to exceed \( S_{\text{iron}} \) at least by the space needed for the coils on both sides. For the region of interest these technical considerations resulted in the condition

\[ S_{\text{iron}} \leq 0.4L \]

The relation between S and \( S_{\text{iron}} \) had to be extrapolated from fringe field measurements on other lenses, mainly those in Ref. 5). The
most usual, but somewhat arbitrary way, of accounting for a fringe field is to give to the field \( B \) the value \( B_m \) observed at the middle of the lens and determine the only parameter left, the effective field length \( S_{\text{eff}} \), from the integral of the field gradient with respect to \( z \). This would probably lead to something like

\[
S_{\text{eff}} \approx S_{\text{iron}} + a
\]

As pointed out in Section IV, after Eq. (IV.15), one can, without loss of generality when only transfer between points outside the lens is considered, interpret the matrix elements (computed from field measurements or directly measured with a floating wire) in terms of the two free parameters \( k \) and \( S \). This usually amounts very roughly to putting

\[
S \approx S_{\text{iron}} + 1.5 \ a
\]

and

\[
B \approx 0.8 \ B_m .
\]

For the purpose of the design discussion, we shall assume this value for \( B \) and

\[
f = S/L \lesssim 0.7 . \quad \text{(VI.1)}
\]

Since the maximum excursion occurs in the middle of a focusing lens (see Section IV.2), thus \textit{inside} a lens, neither of the two above-mentioned "ideal" lens models is quite adequate, but unless otherwise stated we shall stick to the latter.

\section*{VI.2 Determination of design figures in the short-lens approximation}

We shall now show how, on the basis of the theory developed in Sections IV and V, we arrive in the short-lens approximation at design figures for a given problem.
In the first place our system has to provide adequate focusing for a wide band of \( \pi \) momenta and for the \( \mu \) momentum. The focusing is determined by the parameter \( \eta \) [Eq. (IV.23S)]

\[
\eta_{\mu}\rho_{\mu} = \eta_{\pi}\rho_{\pi} = SL/a, \tag{VI.2S}
\]

or with Eq. (IV.8)

\[
\eta_{\mu}\rho_{\mu} = \eta_{\pi}\rho_{\pi} = eB \frac{SL}{a}. \tag{VI.3S}
\]

When we have chosen a representative \( \pi \) momentum \( p_{\pi} \), and the \( \mu \) momentum \( p_{\mu} \), the ratio \( \eta_{\mu}/\eta_{\pi} \) is fixed. On the other hand, neither of the \( \eta \) values should be too far from the optimum value \( \eta = 1.24 \) [see discussion of Eq. (IV.61S)]. We can, for instance, postulate

\[
\sqrt{\eta_{\mu}/\eta_{\pi}} = 1.24. \tag{VI.4S}
\]

So the quantities \( \eta \) can be considered as fixed from now on.

The obtainable field strength \( B \) is limited by the saturation of the iron. Let us assume that we choose the field as high as possible without excessive saturation phenomena. Then Eq. (VI.2S) or Eq. (VI.3S) gives an expression for the product \( SL \) as a function of the free design parameter \( a \) only.

From Eq. (V.5S) it follows that for a large trapping efficiency \( K \) should be made as large as possible. According to Eqs. (IV.55S) and (IV.63S) this calls for a small value of the lens separation \( L \), i.e. a maximum filling factor.

Assuming this maximum filling factor \( f = 0.7 \), as given by Eq. (VI.1), we have now fixed both the product and the ratio of the quantities \( S \) and \( L \). Therefore the only remaining free design parameter is \( a \):

\[
L = \frac{\eta_{\pi}\rho_{\pi}}{\sqrt{f}}, \tag{VI.5S}
\]

\[
S = \sqrt{\eta_{\rho}a f}. \tag{VI.6S}
\]
The quantity $\gamma$, defined in Eq. (V.18S), which determines the trapping efficiency, then becomes:

$$\gamma = \frac{q}{a\theta} \sqrt{\frac{af}{\rho\eta}} \quad \text{(VI.7S)}$$

The value of $q/a$, i.e. the ratio of the half-side $a$ of the square confining pipe to the pole radius, was discussed in Section VI.1.

For $p_\mu = 300 \text{ MeV/c}$, $p_\pi = 400 \text{ MeV/c}$, we choose $\eta_\mu = 1.40$, $\eta_\pi = 1.05$. We can then use the result of Section V, where the trapping efficiency $T(400,300)$ was given as a function of $\gamma$ for these values of $\eta$ (Fig. V.5). For $B = 8 \text{ kgauss}$ we obtain $\rho_\mu = 125 \text{ cm}$ [Eq. (IV.8)], and thus $\rho_\pi = 175 \text{ cm}$. The decay angle $\theta$ becomes 100 mrad, according to Eq. (II.10).

We can then convert the values of $\gamma$ used in the abscissa of Figs. V.4 and V.5 into values of $a$, using the following assumptions:

a) $q/a = 0.88$ \quad b) $q/a = 0.71$

These two scales for $a$ are included in Fig. V.4. The second one is rather pessimistic, the first one is expected to give a more realistic figure.

From Fig. V.5 it is seen that, in view of the suppressed zero of the abscissa scale, $T_\gamma(400,300) \sim \gamma^2$ in the region under discussion. This means that, keeping $q/a$ constant in Eq. (VI.7S), $T_\gamma(400,300) \sim a$.

The muon intensity depends not only on this trapping efficiency, but also on the number of pions accepted into the system. The areas of the admittance ellipses in both planes are proportional to $a$ for $q/a = \text{const}$ [Eq. (IV.52)]. As the incident pions come from nearly a point source, there exists a strong correlation between their angle and position of incidence (see Section VII.5). This means that the number of pions accepted varies only roughly as $a$ in the vertical plane. In the horizontal plane the same is true for a fixed momentum, but here we have moreover dispersion due
to the field of the synchro-cyclotron, so we expect a dependence roughly as \( a^2 \) in this plane. Then the number of pions admitted goes as \( a^3 \), and the muon intensity roughly as \( a^4 \). This a dependence only holds if the experimental apparatus which follows the system is large enough to intercept most of the \( \mu \) beam emerging from the system. If the apparatus has dimensions smaller than the emerging beam, the gain in muons falling on the apparatus which we obtain in increasing \( a \) will be only about as \( a^2 \).

The choice of \( a \) is further dictated by economical considerations. The price of a quadrupole increases roughly as \( a^2 \). Finally, there are technical considerations. The ratio \( S/2a \) should not be too small, otherwise the fringe fields become too predominant. We have finally chosen \( a = 10 \) cm. This gives a trapping efficiency of 16.5% under assumption a), and 8% under the pessimistic assumption b) [p. 36]. From Eqs. (VI.58) and (VI.68) we find

\[
\begin{align*}
a &= 10 \text{ cm} \\
L &= 50 \text{ cm} \\
S &= 35 \text{ cm}
\end{align*}
\]  

which makes \( S_{\text{iron}} \approx 35 - 1.5 \times 10 = 20 \) cm.

The length of the channel, i.e. the number of lenses, is determined by the decay length and by practical and economical considerations. We choose 24 lenses, giving a total length of 13 m. This means that 45% of the 400 MeV/c pions entering the channel will decay in it.

**VI.3 Final choice of channel parameters**

Historically the design procedure of the preceding section was followed by some calculations in which the effects of finite lens length and the fringe field were taken into account in more detail by suitable approximations. These were based on the measurements carried out by Septier\(^5\). These considerations resulted in the following final choice of parameters:

\[
\begin{align*}
a &= 10 \text{ cm} \\
L &= 55 \text{ cm} \\
S_{\text{iron}} &= 22 \text{ cm}
\end{align*}
\]
Floating wire measurements, carried out on the actual lenses once they were delivered, gave as optimum values $S = 40$ cm and the value of $k^2p = eB/a$ given in Fig. VI.5 as a function of the lens current.

VI.4 Some computed afterthoughts

With the computers now available to the authors, formulae such as Eqs. (V.2) to (V.4) can easily be evaluated for more complicated transfer matrices, and we have computed as an (admittedly belated) check of our design considerations, $Y(p_\pi)$ and $T(p_\pi, p_\mu)$ for various parameters of a channel consisting of long ideal lenses.

Figure VI.1 shows the variation of the muon yield $Y(p_\pi)$ with gradient $B/a$, lens length $S$, lens separation $L$, and the size of the confining box $q$. $Y(p_\pi)$ always increases with increasing $q$, but passes through a maximum with increasing $B/a$, $S$ or $L$. This maximum is reached earlier for low momentum, and for high values of the other two parameters. This demonstrates the overfocusing effect, which can even lead to complete instability [Eq. (IV.24)].

Figure VI.2 shows in particular the variation of $Y(400)$ with $S$, all other parameters being fixed. It is seen that the maximum is fairly flat. The value chosen, $S = 40$ cm, yields already very nearly the maximum trapping efficiency.

Figure VI.3 shows $T(400, 300)$ also as a function of $S$. The shape is similar to that of the yield $Y(400)$ in Fig. VI.2, but the trapping efficiencies are lower since, for the values of the momenta chosen, the decay angle is very near to the maximum one. (See Fig. II.1.) $T(400, 300)$ is compared to the result $T_S(400, 300)$ of the short lens approximation. The short cut procedure of Eq. (V.20S) has been used. The orders of magnitude are correct, but the short lens approximation, whose limits of validity given in Eq. (IV.16) are well passed, predicts an overfocusing effect at too low a value of $S$. It would therefore suggest an optimum lens length of 31 cm yielding a $T$ of 80% of the maximum one. The optimization formula (VI.5S) suggests a higher value $S = 55$ cm [Eq. (VI.8S)], because it takes into account that the stiffer $\pi$ mesons also have to be focused.
Figure VI.4 gives the dependence of $T(400,300)$ on the lens radius $a$. In each point the values of $L$ and $S$ are chosen according to the optimizing conditions in Eqs. (VI.5S) and (VI.6S) derived from short-lens approximation [Fig. V.3]. The field $B$ has been kept constant. The calculations with long ideal lenses and with the complete short-lens approximation are confronted. They show the same trend, but the short-lens calculation gives slightly too optimistic values.

Concerning the short-lens approximation, we can conclude that it provides a useful guide, but that its results must be checked by more elaborate calculations.

VII. MATCHING OF THE CHANNEL TO THE CYCLOTRON

VII. 1 The horizontal motion in the cyclotron

We shall first consider the horizontal motion in the median plane of the synchro-cyclotron (SC). The geometry of the channel and the cyclotron as seen from above is given in Fig. VII.1 for the case of the protons rotating anticlockwise and of negative pions, i.e. the vertical SC field component $B_y(r_t) < 0$ ("field down") and $e < 0$, so $eB_y(r_t) > 0$.

$\varphi$, $\varphi_0$, $\psi$, $\psi_0$ shall be positive in the anticlockwise direction. "Absolute" azimuths $\alpha$ are at CERN for some reason measured clockwise in degrees from "SC zero".

For $\pi^-$, $\zeta$ shall be the angle between the proton and pion directions of flight, for $\pi^+$ the angle between the pion and the opposite direction of the proton. $0 < \zeta < \pi$ means that the pion is emitted inwards from the target.

The radius $R$ (at CERN = 350 cm) is so chosen that outside it the cyclotron field is negligible. $b$ (at CERN = 50 cm) is the distance from the beginning of the channel to the intersection of the channel axis with the circle $R$. The "beginning" of the channel is conveniently defined as $\frac{1}{2}$ (a quarter of a channel period) in front of the midplane of the first lens. We shall not assume any
special "matching lens" in between the cyclotron and the channel. However, at the end of this section (VII.6) we shall make some comments on the possibility of using such a lens.

We shall first write down the relations between the channel co-ordinates and the cyclotron co-ordinates.

From Fig. VII.1 we get

\[ x' = \psi - \psi_0 + \varphi - \varphi_0 \]  \hspace{1cm} (VII.1)

and

\[ x = \frac{2R \sin \frac{\varphi - \varphi_0}{2} \cos \left( \frac{\psi + \varphi - \varphi_0}{2} \right)}{\cos (\psi - \psi_0 + \varphi - \varphi_0)} . \]  \hspace{1cm} (VII.2)

Since we shall only be concerned with

\[ |x'| \lessapprox 0.1 \text{ rad} \]
\[ |x| \lessapprox 10 \text{ cm} , \]  \hspace{1cm} (VII.3)

we may in sufficiently good approximation write

\[ x = u - u_0 \]
\[ x' = v - v_0 , \]  \hspace{1cm} (VII.4)

where in a general case

\[ u = \text{sign}(eB) R \varphi \cos \psi , \]
\[ u_0 = \text{sign}(eB) R \varphi_0 \cos \psi_0 \]  \hspace{1cm} (VII.5)

\[ v = \text{sign}(eB) (\varphi + \psi) , \]
\[ v_0 = \text{sign}(eB) (\varphi_0 + \psi_0) , \]

and

\[ \text{sign}(eB) = \frac{eB_y(r_t)}{|eB_y(r_t)|} . \]  \hspace{1cm} (VII.6)

\( u_0 \) and \( v_0 \) are thus determined by the signs of \( e \) and \( B_y(r_t) \) as well as by the target and channel positions. At CERN

\[ \varphi_0 = \frac{\pi}{180} (29^\circ - \alpha) \]  \hspace{1cm} (VII.7)

\[ \psi_0 = -\frac{\pi}{180} 11^\circ = -0.192 \text{ rad} , \]
and \( r_t \) is in the neighbourhood of 220 cm. \( \varphi = \varphi(R) \) and \( \psi = \psi(R) \) are connected to \( p \) and \( \zeta \) by (see, for example, Ref. 7, p. 19)

\[
\sin \varphi(r) = r_t \sin \varphi(r_t) - \frac{e}{p} \int_{r_t}^{r} r' B_u(r') \, dr'
\]

(VII.8)

and

\[
\varphi(r) = \int_{r_t}^{r} \frac{\tan \psi(r')}{r'} \, dr',
\]

(VII.9)

where

\[
\psi(r_t) = \text{sign} \left( eB \right) \left( \zeta - \frac{\pi}{2} \right) + \pi,
\]

(VII.10)

if \( \zeta \) is in radians.

Equation (VII.9) leads to some difficulty when \( \psi(r_t) = \pi/2 \).

To circumvent this, one may, assuming that the field \( B_u(r') \approx B_u(r_t) \) is constant inside \( r_m > r_t \), use the following equations:

\[
\sin \psi(r) = r_t \sin \psi(r_t) - \frac{r_m^2 - r_t^2}{2 \rho_t} - \frac{e}{p} \int_{r_m}^{r} r' B_u(r') \, dr'
\]

(VII.11)

and

\[
\varphi(r) = -\arctan \left( \frac{\rho_t \cos \psi(r_t)}{r_t + \rho_t \sin \psi(r_t)} \right) + \arctan \left( \frac{\rho_t \cos \psi(r_m)}{r_m + \rho_t \sin \psi(r_m)} \right) + \int_{r_m}^{r} \frac{\tan \psi(r')}{r'} \, dr',
\]

(VII.12)

where

\[
\rho_t = \frac{p}{e B_u(r_t)}.
\]

(VII.13)

With Eqs. (VII.11) and (VII.12), lines for constant \( p \) and \( \zeta \) may be computed and plotted in a \((u,v)\) diagram. Figure VII.2 gives an example of such a plot. It is seen from Eqs. (VII.8) to (VII.13) that the \( u \) and \( v \) do not depend on the signs of \( e \) and \( B_u(r_t) \), so the same plot may be used for \( \pi^- \) and \( \pi^+ \) produced by protons circulating in one direction.
or the other. However, each plot is only valid for one pair of \( r_t \) and \( R \).

Such diagrams may also be established from floating wire measurements (see ref. 7), where some "Michaelis plots" are given, i.e. lines for constant \( p \) and \( \zeta \) plotted in a \((\varphi, \varphi + \varphi)\) diagram.

The total length of the trajectory from the target to \( R \) is given by

\[
s(R) = s(r_m) + \int_{r_m}^{R} \frac{dr'}{\cos \varphi(r')} ,
\]

where

\[
s(r_m) = 2|\rho_t| \arcsin \sqrt{\frac{r_t^2 + r_m^2 - 2r_tr_m \cos \varphi(r_m)}{4 \rho_t^2}} .
\]

VII.2 The vertical motion in the cyclotron

The vertical motion may be taken into account by a transfer matrix \( V = V(r) = V(r, p, \zeta) \), which is different for each horizontal orbit and connects \( y_t \) and \( y'_t \) at the target with \( y_r \) and \( y'_r \) at \( r \):

\[
\begin{pmatrix}
y_r \\
y'_r
\end{pmatrix} = V \begin{pmatrix}
y_t \\
y'_t
\end{pmatrix} .
\]

\( V \) is given by the differential equation [differentiation with respect to the track length \( s \)]

\[
y'' + k^2(r)y = 0 ,
\]

where

\[
k^2(r) = \frac{e}{p} \left( \frac{\partial B_y}{\partial r} \right)_{y=0} \sin \varphi(r) ,
\]

and \( r \) is considered as a function of \( s \), \( p \) and \( \zeta \).
Since \((\partial B_y/\partial r)_{y=0} < 0\) in the region of interest for the SC field up [i.e. \(B_y(r_t) > 0\)], positive particles \((e > 0)\) are, in that case, deflected towards the median plane when \(\psi(r) < 0\), and vice versa for the SC field down. For the focusing of the particles it is therefore very important what \(\psi(r)\) is at the edge of the cyclotron field, where \(|\partial B_y/\partial r|\) is large.

Equations (VII.10) and (VII.8) show that \(\sin \psi(r)\) changes sign with \(eB_y\), as does \(e \partial B_y/\partial r\). For a given \(p\) and \(\zeta\), \(k^2(r)\) is therefore independent of the signs of \(e\) and \(B_y\), so \(V\) has the same "universality" as \(u\) and \(v\).

One intuitively attractive way of deriving \(V\) from Eq. (VII.17) is the following. Up to \(r_m\) it is consistent with the constant field assumption in Eqs. (VII.11) and (VII.12) to consider the vertical motion as rectilinear:

\[
V(r_m) = \begin{pmatrix}
1 & s(r_m) \\
0 & 1
\end{pmatrix}.
\]  

(VII.19)

\(R - r_m\) we divide into \(N\) intervals of length \(\Delta\), within which we assume a constant mean \(k^2\). For each interval we then get a transfer matrix of the type in Eq. (IV.11). Defining for the \((\mu+1)\)th interval

\[
\omega_{\mu+1} = \frac{\Delta}{\cos \psi(r_m + \mu\Delta)} \left\{ \frac{eB_y(r_m + \mu\Delta + \Delta) - B_y(r_m + \mu\Delta)}{\Delta} \left( \frac{\sin \psi(r_m + \mu\Delta)}{p} \right) \right\}^{1/2},
\]

(VII.20)

where all values are taken at \(y = 0\), we get by successive multiplication of the transfer matrices the recurrence formula

\[
V(\mu + 1)_{\alpha \beta} = V(\mu)_{\alpha \beta} \cos \omega_{\mu+1} - (-1)^{\alpha} \omega_{\mu+1} \left[ \frac{\Delta}{\cos \psi(r_m + \mu\Delta)} \right]^{1/2} \sin \omega_{\mu+1},
\]

(VII.21)

where \(V(\mu) = V(r_m + \mu\Delta)\), and \(\alpha' = \alpha - (-1)^{\alpha}\).

We start from \(V(0) = V(r_m)\) and end up with \(V(N) = V(R) = V(R, \zeta, p)\).
VII.3 The pion spectrum accepted by the channel

Let us assume that the target emits

\[ N_t(p, \zeta) = (\text{multiply traversing proton flux}) \times \frac{\text{target thickness in g cm}^{-2}}{\text{atomic weight}} \times \frac{6 \times 10^2}{\pi} \frac{\partial^2 \sigma}{\partial \zeta \partial p} \left( \frac{p}{e} - \frac{\zeta}{\pi} \right) \]  

(VII.22)

pions per MeV/c and steradian, considering the target as a point source.

\[ d\Omega \approx d\zeta dy'_t \]  

(VII.23)

Which of these emitted pions are accepted by the channel or any other device is determined by the horizontal and vertical accepted phase area of the device. Let this area in the horizontal plane at the beginning of the channel be limited by

\[ F_x(x_1, x_f) = 0 \]  

(VII.24)

which in cyclotron co-ordinates at radius R becomes

\[ F_x[u - u_0 + b(v - v_0), v - v_0] = 0 \]  

(VII.25)

In the vertical plane a limiting curve

\[ F_y(y_1, y_f) = 0 \]  

(VII.26)

at the beginning of the channel becomes

\[ F_y(y + by', y') = 0 \]  

(VII.27)

at the radius R.

Specializing now to the channel acceptance ellipses, whose coefficients are given by Eq. (IV.43), we get the following limits in the cyclotron co-ordinates:
\[ A_H(u - u_o)^2 + 2B_H(u - u_o)(v - v_o) + C_H(v - v_o)^2 = \left( \frac{q}{\cos \epsilon} \right)^2 \]  
(VII.28)

\[ A_V y^2 + 2B_V yy' + C y'^2 = \left( \frac{q}{\cos \epsilon} \right)^2, \]  
(VII.29)

where

\[
\begin{align*}
A_H &= A_V = M_0 \\
B_H &= + M_1 + bM_0 \\
C_H &= M_2 + 2bM_1 + b^2M_0
\end{align*}
\]

\[
B_V = - M_1 + bM_0 \\
C_V = M_2 - 2bM_1 + b^2M_0.
\]  
(VII.30)

If the first lens of the channel is vertically focusing the upper sign in Eq. (IV.43) is valid, and if defocusing the lower one is valid.

Figure VII.2 shows the horizontal ellipses for \( p = 400 \) MeV/c particles and various channel currents, and Fig. VII.3 shows the corresponding vertical ellipses. The first lens of the channel is vertically focusing. The lens length \( S = 40 \) cm and the dependence of \( k^2p \) on the channel current is taken from the lower curve of Fig. VI.5.

The \((u,v)\) diagrams were drawn from quantities computed with an existing Mercury Autocode programme written by Farley.

The limiting curve \( \Phi(p, \zeta) = 0 \) in \( p \) and \( \zeta \) may be found graphically by superimposing at a fixed channel current the ellipses for various momenta described by Eq. (VII.28) (for instance, on transparent paper) on a \( u,v \) diagram for the cyclotron, and reading off directly the \( \zeta \) value of the intersection of the ellipse and the cyclotron line for the same \( p \). Figure VII.4 shows an example of this, and Fig. VII.5 the resulting limit in \( p \) and \( \zeta \). A Fortran programme has also been written to this end.

Equation (VII.29) with \( y_t = 0 \) gives as vertical bite for \( dy_t' \)

\[ \Delta y'_t(p, \zeta) = \frac{2 \frac{q}{\cos \epsilon}}{\sqrt{A_VV_{12}^2 + 2B_VV_{12}V_{22} + C_VV_{22}^2}}. \]  
(VII.31)

The rate of decay of the pions before entering the channel is determined by the decay length [see Eq. (II.12)].
\[ \lambda = 5.50 \text{ p} \quad \text{[in cm if p in MeV/c]} \]  \quad \text{(VII.32)}

The momentum spectrum accepted by the channel is now

\[
N_{ch}^*(p) = \left[ 1 - e^{-\frac{s+b}{5.5p}} \right] \int_{\Phi(p,\zeta)=0} N_t(p,\zeta) \Delta y'(p,\zeta) d\zeta , \quad \text{(VII.33)}
\]

where \( s = s(R) \).

With this \( N_{ch}^*(p) \) the \( \mu \) spectrum is obtained from Eq. (V.4), provided the ellipses are uniformly populated (see Section VII.5). It is, of course, pessimistic to consider as lost the pions decaying before entering the channel, since some of the resulting muons will certainly come through the channel.

If \( N_t(p,\zeta), \Delta y'(p,\zeta) \) and the decay factor vary little over the acceptance area, the shape of the momentum spectrum is essentially given by \( \Delta \zeta(p) \), obtained from \( \Phi(p,\zeta) \) as indicated in Fig. VII.5.

If \( N_t(p,\zeta) \) is not known a priori, it may in principle be found by using a well-defined telescope, for which the conditions mentioned above are well fulfilled, and exploring the area of interest with it.

In fact, the shape of one single spectrum is usually not badly represented by \( \Delta \zeta(p) \). However, the comparison between different spectra is more reliable if the value at the mean \( \zeta, \zeta^\prime \), for each \( p \) of the integrand of Eq. (VII.33), is included. Even a most phenomenological \( N_t(p,\zeta) \) could at least introduce some sort of desirable drop at low and high \( p \), and at large angles. Short of anything more complete, \( N_t(p,\zeta) \) was taken as described in Appendix C. In Fig. VII.5 the spectrum \( N_{ch}^*(p) \) calculated in this way is also plotted:

\[
N_{ch}^*(p) \sim \left[ 1 - e^{-\frac{s+b}{5.5p}} \right] N_t(p,\zeta) \Delta y'(p,\zeta) \Delta \zeta(p) . \quad \text{(VII.34)}
\]
The total number of pions accepted by the channel is

\[ N_{\text{ch}} = \int_{0}^{\infty} N_{\text{ch}}(p) \, dp. \quad (\text{VII.35}) \]

### VII.4 The choice of position of the channel

The choice of the position of the channel was guided by the desire to obtain a pure and intense \( \mu^- \) beam at the highest possible momentum, say \( p_\mu \geq 300 \text{ MeV/c} \). With this in mind, approximate spectra \( N^*_\text{ch}(p) \), like that at the bottom of Fig. VII.5, were drawn for various \( u_0 \) and \( v_0 \), or \( \varphi_0 \) and \( \varphi_0 \). Chosen and fixed for ever were the values given in Eq. (VII.7). \( r_t \), \( \alpha \), and the SC field direction are variable. In particular, \( r_t = 226 \text{ cm} \), \( \alpha = 10^\circ \), and the SC field up was judged to give a good \( \pi^- \) spectrum. It is plotted in Fig. VII.6b. The corresponding muon spectrum is also indicated for \( q = 10 \text{ cm} \) and \( 12 \text{ cm} \). For this position the horizontal ellipses for \( p = 400 \text{ MeV/c} \) and various channel currents are shown in Fig. VII.2. The upper part of the ellipses exceeding, say, \( p_\pi = 500 \text{ MeV/c} \), cannot be much populated, because very few pions are produced with such high momentum at the CERN SC. Some intensity had thus been sacrificed to give high enough muon momenta.

The position mentioned above was not much used when the channel was ready, because the vibrating target that came into operation shortly after could not work with good efficiency as far out as \( r_t = 226 \text{ cm} \). Experimentally it was found that by sacrificing muon momentum (i.e. going down to \( p_\mu = 240 \text{ MeV/c} \)) one could get good intensity and purity also with the SC field down and the vibrating target at \( r_t = 215 \text{ cm} \) and, say, \( \alpha = 40.5^\circ \), which is the position of Figs. VII.4 and VII.5. In Fig. VII.6a and Fig. VII.6b, the pion and muon momentum spectra for these two positions are compared. The scale is arbitrary, but the relative size of the spectra should be correct. The total intensity is not very different in the two cases, but the muon momentum ranges are quite different.

Figure VII.7 shows the degree of longitudinal polarization of muons as function of the muon momentum.
VII.5 The pion distribution in the ellipse

In Section V.1 we made the hypothesis that the ellipses are uniformly filled by pions. In real fact we see, for example from Fig. VII.3, that the vertical ellipse is filled only along narrow strips (depending slightly on momentum) through the origin, the width of which depend on target dimensions, whereas the strips filling the horizontal ellipse (Fig. VII.2 or VII.4) are displaced with respect to the origin and therefore the various momenta together fill the whole ellipse.

So at first sight our assumption seems rather badly fulfilled in the vertical direction. During the motion in the channel, the strips will, however, rotate with respect to the ellipse, since the ellipse makes a turn in one period of structure, whereas the strip makes a turn in one period of motion as defined in Eq. (IV.31). So, averaged over a sufficiently long channel, all points inside the ellipse will be populated. But even this average population is not uniform. In the vertical ellipses, where the strip goes through the centre, the average population is higher in the middle of the ellipse. In the horizontal ellipses, all depends on the momentum spectra. If this spectrum has a peak at the momentum on which the ellipse is centred, there will also be a higher population in the middle. For a flat spectrum it will be more uniform. These effects can be studied by methods avoiding the concept of uniformly filled ellipses\(^3,4\)\(^\text{,}\) or estimated by considering a number of uniformly filled ellipses with different coefficients \(F_3\) superimposed. We believe that this effect adds essentially to the already considerable "geometrical" uncertainty in \(q\), so for the muon trapping calculations the hypothesis of uniform population has been maintained.

VII.6 The possibility of using a matching lens

The slope of the target line in Fig. VII.3 shows that the high-momentum pions are vertically diverging when they emerge from the channel. The simplest matching is therefore to start the channel as closely as possible to the cyclotron. In fact, every centimetre
that the CERN channel would move away from the cyclotron would cost of the order of 1% in intensity. For various reasons beyond the authors' control, the beginning of it is at r = 380 cm, which is not the closest mechanically possible.

Since the channel can accept quite big angles, but a rather limited space, a special matching lens, which would obviously have to be bigger than the normal channel lenses, was discussed. However, for the high momenta for which the CERN channel was designed, it was thought that the cost and difficulties in making the bigger lens strong enough was not justified by the improvements it would give. For lower momenta this last conclusion is not so valid and, in fact, in Chicago a matching lens has been put in between channel and cyclotron.

VIII. THE ANALYSING MAGNET

VIII.1 Theoretical generalities

A beam from a strong-focusing channel might suffer severe intensity losses, even in the chosen momentum band, if passed through an analysing magnet without adequate focusing properties. For the higher momenta, in particular, where the particles must pass a considerable length in the magnet in order to be sufficiently separated, some sort of strong focusing should be provided.

Out of the various possibilities, we shall consider here the motion of a particle in an analysing magnet consisting of a series of curved sections with quadrupole profile, see Fig. VIII.1. The field B alternates in direction from section to section, and the sections are alternately displaced a distance $|\Delta|$ to the right ($\Delta < 0$) or to the left ($\Delta > 0$) with respect to the equilibrium radius $r_0$. $r$ and $r_0$ shall be positive when the particle is deviated to the left. The magnet of this type, actually made at CERN, is shown in Fig. VIII.2. (Since the particles move only in one half of the quadrupole, the other half is suppressed.)

The field inside each section may, in good enough approximation, be described by
\[
B_r = \frac{B}{a} y \quad (*)
\]

\[
B_\varphi = 0 \quad (VIII.1)
\]

\[
B_y = \frac{B}{a} (r - r_0 + \Delta)
\]

We introduce

\[
\xi = r - r_0, \quad (VIII.2)
\]

assuming \( |\xi| << |r_0| \), and the linearized equations of motion are

\[
\xi'' + \frac{1-nw}{r_0^2} \xi = \frac{1-w}{r_0} \quad (VIII.3)
\]

\[
y'' + \frac{nw}{r_0^2} y = 0, \quad (VIII.4)
\]

where

\[
\xi' = \frac{d\xi}{r_0 d\varphi}, \text{ etc.} \quad (VIII.5)
\]

\( r_0 d\varphi \) is always > 0 in the direction of the particle.

\[
n = -\left( \frac{r}{B_y} \frac{dB_y}{dr} \right)_{r=r_0} = -\frac{r_0}{\Delta} = \frac{eBr_0^2}{a p_0} \quad (VIII.6)
\]

is called the field index. It is given by the geometry. When the field \( B \) is given too, \( p_0 \) is determined by Eq. (VIII.6), or more practically by

\[
p_0 \text{[in MeV/c]} = 3 \times 10^{-4} \frac{Br_0^2}{an} \left[ \text{if } B \text{ in gauss, } r_0 \text{ and } a \text{ in cm}. \right] \quad (VIII.7)
\]

\[
w = \frac{pa}{p},
\]

where \( p \) is the momentum of the considered particle.

For one section of length \( r_0 \varphi \), within which \( n \) is constant, the solutions of Eqs. (VIII.3) and (VIII.4) may be written

\(*)\) Here as elsewhere, \( y \) denotes the vertical component.
\[ \dot{\xi} = H \xi_o + \dot{R} \]  \hspace{1cm} (VIII.9)

and

\[ \dot{y} = V y_o , \]  \hspace{1cm} (VIII.10)

where

\[ \xi = \begin{pmatrix} \xi \\ \xi' \end{pmatrix} , \text{etc.} \]  \hspace{1cm} (VIII.11)

\[ H = \begin{pmatrix} \cos \varphi h & \frac{r_o}{h} \sin \varphi h \\ - \frac{h}{r_o} \sin \varphi h & \cos \varphi h \end{pmatrix} \]  \hspace{1cm} (VIII.12)

and

\[ \dot{R} = (1 - w) \begin{pmatrix} \frac{r_o}{h^2} [1 - \cos \varphi h] \\ \frac{1}{h} \sin \varphi h \end{pmatrix} , \]  \hspace{1cm} (VIII.13)

where

\[ h = \sqrt{1 - nw} . \]  \hspace{1cm} (VIII.14)

\[ \text{V is given by Eq. (VIII.12), replacing } h \text{ by} \]  \hspace{1cm} (VIII.15)

\[ \nu = \sqrt{nw} . \]

In a strong focusing system \(|n| >> 1\), so for \(w\) of order 1 we have the following situation:

For \(eB > 0\) we have \(n > 0, r_o \Delta < 0, h^2 < 0\) and \(\nu^2 > 0\),
which means vertical focusing \((V = V^+)\) and horizontal defocusing \((H = H^-)\).

For \(eB < 0\) we have \(n < 0, r_o \Delta > 0, h^2 > 0\) and \(\nu^2 < 0\),
and therefore horizontal focusing \((H = H^+)\) and vertical defocusing \((V = V^-)\).

We shall now consider a series of sections. \(r_o \varphi_\pm\) is the
length of the 1st, 3rd, etc., \(r_o \varphi_\pm\) of the 2nd, 4th, etc. \(B\) and \(\Delta\),
and hence \(n\), alternate in sign from one section to another, and
starting with a horizontally defocusing one, one gets after \(2m + \nu\)
sections \((\nu = 0 \text{ or } 1)\):
\[
\hat{\xi}(2m + \nu) = \mathcal{H}(2m + \nu)\hat{\xi}_0 + \hat{\mathcal{R}}(2m + \nu) \tag{VIII.16}
\]

and
\[
\hat{\eta}(2m + \nu) = \mathcal{V}(2m + \nu)\hat{\eta}_0. \tag{VIII.17}
\]

With
\[
\cos \mu = \frac{1}{2} \text{tr } \mathcal{H}^+\mathcal{H}^- = \cosh \left( \varphi^- \sqrt{|n|w - 1} \right) \cos \left( \varphi^+ \sqrt{|n|w + 1} \right) \sinh \left( \varphi^- \sqrt{|n|w - 1} \right) \sin \left( \varphi^+ \sqrt{|n|w + 1} \right) \frac{1}{\sqrt{(nw)^2 - 1}}, \tag{VIII.18}
\]

we have (see Appendix B)
\[
\mathcal{H}(2m + \nu) = (\mathcal{H}^-)^\nu(\mathcal{H}^+\mathcal{H}^-)^m = \frac{\sin m \mu}{\sin \mu} (\mathcal{H}^-)^\nu \mathcal{H}^+ \mathcal{H}^- - \frac{\sin (m - 1) \mu}{\sin \mu} (\mathcal{H}^-)^\nu. \tag{VIII.19}
\]

\(\mathcal{V}(2m + \nu)\) is given by similar expressions, replacing
\[
\mathcal{H}^+ \text{ by } \mathcal{V}^+ (\varphi_+ \text{ now belong to } \mathcal{V}^+ \text{)} \text{ or } \sqrt{|n|w + 1} \text{ by } \sqrt{|n|w}. \]

\[
\hat{\mathcal{R}}(2m + \nu) = (\mathcal{H}^-)^\nu \sum_{\omega=0}^{m-1} (\mathcal{H}^+\mathcal{H}^-)^\omega (\mathcal{H}^+\mathcal{R}^- + \mathcal{R}^+) + \nu \mathcal{R}^- \tag{VIII.20}
\]

Since
\[
\sum_{\omega=0}^{m-1} (\mathcal{H}^+\mathcal{H}^-)^\omega = \frac{\left[ \cos \frac{\mu}{2} - \cos \left( \frac{\mu - \frac{\mu}{2}}{2} \right) \right] \mathcal{H}^+ \mathcal{H}^- - \left[ \cos \frac{3\mu}{2} - \cos \left( \frac{3\mu - \frac{3\mu}{2}}{2} \right) \right] E}{\cos \frac{\mu}{2} - \cos \frac{3\mu}{2}} \tag{VIII.21}
\]

we may, if we introduce
\[
\mathcal{K} = \frac{[(\mathcal{H}^+\mathcal{H}^-)^{-1} - E][\mathcal{H}^+\mathcal{R}^- + \mathcal{R}^+]}{2 - 2 \cos \mu} \tag{VIII.22}
\]

and
\[
\mathcal{L} = \frac{[(\mathcal{H}^+\mathcal{H}^-)^{-1} + E][\mathcal{H}^+\mathcal{R}^- + \mathcal{R}^+]}{2 \sin \mu} \tag{VIII.23}
\]
\[ R_j(2m + \nu) = (-1)^{1+\nu+\nu_j} \left( \frac{\nu+\nu_j}{K_j + \sqrt{K_j^2 + L_j^2}} \cos \left( m \mu - \alpha_{\nu_j} \right) \right), \quad (VIII.24) \]

where

\[ \cos \alpha_{\nu_j} = \frac{\nu+\nu_j}{\sqrt{K_j^2 + L_j^2}}. \quad (VIII.25) \]

\[ 0 \leq \alpha_{\nu_j} \leq \pi \quad \text{if} \quad (-1)^{\nu_j} L_j + \frac{\nu}{\sin \mu} \left[ R_j^+ \cos \mu - (-1)^j (H_j^+ R_j^-)^j \right] \geq 0 \]

\[ \pi \leq \alpha_{\nu_j} \leq 2\pi \quad \text{if} \quad \left. \frac{\nu_{\nu_j}}{\sin \mu} \right| R_j^+ \cos \mu - (-1)^j (H_j^+ R_j^-)^j \right| \leq 0. \quad (VIII.26) \]

\( j = 1 \) gives the position and \( j = 2 \) the angle. If, instead of with a horizontally defocusing lens, we start with a focusing one, we must interchange \( H^+ \) and \( H^- \), \( R^+ \) and \( R^- \) (\( \varphi_- \) then belongs to \( H^+ \), etc.) in these formulae.

As an example, we work out Eq. (VIII.24) for the case of \( 2m + \nu \) magnet sections \( \varphi_+ = \varphi_- = 0.42 \) (neglecting fringe field), \( r_0 = 110 \, \text{cm}, \ |\Delta| = 17 \, \text{cm}, \ |n| = 6.5 \), and \( \nu = 0.9 \):

\[ \hat{R}(2m + \nu) = \begin{pmatrix} 7.26 & +7.28 \cos (1.049 m - 3.22 + 0.62 \nu) \\ -7.26 & 0.082 + 0.120 \cos (1.049 m - 0.81 - 1.05 \nu) \end{pmatrix}. \quad (VIII.27) \]

The result is plotted in Fig. VIII.3, adapting \( R' \) to the horizontal scale.

Although Eq. (VIII.27) gives the best general impression of what happens at any \( m \), in practice it may be quicker to work out \( H(2m + \nu), V(2m + \nu), \) and \( \hat{R}(2m + \nu) \) by successive numerical matrix multiplication if only a few sections are considered.
VIII.2 Matching of the magnet to the channel

We now put such a magnet at the end of the CERN channel as shown in Fig. VIII.4. When the last channel lens is horizontally focusing, the horizontal channel ellipse at a distance \( L/2 \) after midplane of the last lens is

\[
M_0 x^2 + 2 M_1 x x' + M_2 x'^2 = \left( \frac{q}{\cos \epsilon} \right)^2,
\]  

(VIII.28)

where the coefficients are given by Eqs. (IV.43), using the upper sign.

Let the distance from this position to the edge of the magnet (or rather the "edge" of the fringe field) be \( t \). Then

\[
\vec{\xi}_0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \vec{x},
\]

(VIII.29)

and \( \vec{\xi}(2m + \nu) \), which for short we denote only by \( \vec{\xi} \), is given by

\[
\vec{\xi} = \mathcal{K} \vec{x} + \vec{R},
\]

(VIII.30)

where

\[
\mathcal{K} = H(2m + \nu) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},
\]

(VIII.31)

and

\[
\vec{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} R \\ R' \end{pmatrix}.
\]

Equation (VIII.30) gives

\[
\vec{x} = \mathcal{K}^{-1} (\vec{\xi} - \vec{R}).
\]

(VIII.32)

The limiting ellipse at any position \( 2m + \nu \) is therefore

\[
P_{H_0} (\xi - R)^2 + 2 P_{H_1} (\xi - R)(\xi' - R') + P_{H_2} (\xi' - R')^2 = \left( \frac{q}{\cos \epsilon} \right)^2,
\]

(VIII.33)

where

\[
P_{H_0} = M_0 \mathcal{K}_{22}^2 - 2 M_1 \mathcal{K}_{22} \mathcal{K}_{21} + M_2 \mathcal{K}_{21}^2
\]

\[
P_{H_1} = - M_0 \mathcal{K}_{12} \mathcal{K}_{22} + M_1 (\mathcal{K}_{11} \mathcal{K}_{22} + \mathcal{K}_{12} \mathcal{K}_{21}) - M_2 \mathcal{K}_{11} \mathcal{K}_{21}
\]

(VIII.34)

\[
P_{H_2} = M_0 \mathcal{K}_{12}^2 - 2 M_1 \mathcal{K}_{11} \mathcal{K}_{12} + M_2 \mathcal{K}_{11}^2
\]
In the vertical plane the limiting ellipse is

\[ P_{V_0} y^2 + 2P_{V_1} y y' + P_{V_2} y'^2 = \left( \frac{q}{\cos \epsilon} \right)^2, \quad (\text{VIII.35}) \]

where \( P_{V_0}, P_{V_1}, \) and \( P_{V_2} \) are given by formulae similar to Eq. (\text{VIII.34})

replacing \( M_1 \) by \(-M_1\) and \( \mathcal{H} \) by

\[ y' = V(2m + \nu) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \quad (\text{VIII.36}) \]

From Eq. (\text{VIII.33}) we see that in addition to changing shape, the horizontal ellipse is shifted with respect to the equilibrium radius \( r_0 \) unless \( p = p_0 \). This makes possible the use of this device as an analysing magnet. Equation (\text{VIII.27}) or Fig. \text{VIII.3} shows that \( 2m + \nu = 3 \) such sections give \( R = 7.4 \text{ cm} \) and \( R' = 0.16 \) for \( w = p_0/p = 0.9 \), which seems quite promising, and the CERN analysing magnet was built with three sections. For the actual design of the magnet, the size of the beam for the required momentum inside the magnet is important. Figure \text{VIII.5} shows the maximum extension of the channel ellipses for

\( p = 240 \text{ MeV/c} \) particles at various positions in the magnet. Here we have used \( \varphi^- = 0.42 \text{ rad} \) and \( \varphi^+ = 0.47 \text{ rad} \) in order to allow for some fringe field in the end sections. This makes \( t = -6.5 \text{ cm} \). The largest vertical extension of the 240 MeV/c beam is seen to be 7.2 cm, and the largest horizontal one 6.2 cm for \( q = 7.1 \text{ cm} \). Figure \text{VIII.6} shows these limits relative to the magnet cross-section. It is seen that for \( q = 7.1 \text{ cm} \) the required beam moves exclusively in the empty space between two of the poles. The other half of the quadrupole could therefore be suppressed and replaced by a neutral pole, as is shown in Fig. \text{VIII.2}. This leads to a bending magnet with field lines only in one direction, which greatly simplifies the design. Some distortion of the quadrupole field will, of course, result from this suppression, which together with the uncertainty in the fringe field will make our calculations somewhat qualitative.
VIII.3 Resolution properties of the magnet

We shall now study the momentum discrimination of the magnet in some detail. It will, of course, depend on what comes after the channel. Three possibilities have been considered:

**CASE I**: Two $10 \times 10$ cm$^2$ counters, one at 30 cm and one at 80 cm from the iron pole of the magnet.

**CASE II**: A $10 \times 10$ cm$^2$ counter seen through a quadrupole doublet consisting of two lenses of the same type as those of the channel, and with the same current. The doublet is in a position which is symmetric with respect to the magnet of the last two lenses of the channel.

**CASE III**: A $10 \times 10$ cm$^2$ counter seen through a quartet consisting of two of the doublets in Case II.

The position of the $10 \times 10$ cm$^2$ counter in Cases II and III is a distance $L/2 = 27.5$ cm after the midplane of the last lens.

Figure VIII.4 illustrates the two first cases. For the channel current, $I = 850$A was used, and this was also the current in the doublet and the quartet.

Figures VIII.7 to VIII.12 show the situation in the phase planes at the end of the fringe field of the magnet. In all three cases the discrimination between $p = p_0 = 240$ MeV/c and $p = p_0/0.9 = 267$ MeV/c seems theoretically complete for $q = 7.1$ cm, and not quite so complete for $q = 10$ cm.

Figure VIII.7 shows the variation of resolving power with counter distance, and that by putting the counter close enough one accepts nearly the whole ellipse, as do the counters in Cases II and III. Vertically, too, the situation seems comparable for the three cases. The main advantages of using a doublet or a quartet after the magnet would therefore be to reduce background and to get away from the big stray field of the analysing magnet.

The dotted ellipses in Figs. VIII.11 and VIII.12 show, incidentally, that the acceptance area of four lenses is already approaching the ellipse of an infinite channel.
IX. PERFORMANCE OF THE CERN SC CHANNEL

IX.1 Modes of operation

As has already been mentioned in Section VII.4, the channel was originally designed for operation with a target at $\alpha \approx 10^\circ$ and $r_t = 226$ cm (Fig. VII.1), SC field up for high-momentum negative particles ($\approx 300$ MeV/c), or $\alpha \approx 40^\circ$, $r_t = 226$ cm, SC field down for low-momentum particles ($\approx 140$ MeV/c). In practice, a smaller target radius had to be used if a long burst was desired. It turned out that essentially the same intensity and purity could be obtained, up to momenta of about 250 MeV/c, by using target radii of about 215 cm and angles in the region of 39 to 45°; always with SC field down, for negative mesons.

Table 1 gives a summary of conditions under which measurements have been carried out. The data necessary for the reproduction of these conditions are listed. It is seen that a large variety of modes of operation has been used, as dictated by requirements of individual experiments. This makes the comparison of data obtained under these conditions somewhat difficult.

IX.2 Spectra and intensities

We have to distinguish between two types of spectra:

i) The spectrum of the particles contained in the channel (channel spectrum). This was, in general, obtained by placing a detector behind the analysing magnet and recording the counting rate as a function of the current in the analysing magnet. The curve obtained is thus the spectrum of the particles in the channel folded with the resolution of the analysing system.

ii) The spectrum of the particles emerging from the analysing magnet at fixed current (magnet spectrum). This is the resolution curve of the analysing system, weighted with the incident spectrum. It is obtained from range curves.

A typical channel spectrum is reproduced in Fig. IX.1. We observe a peak with a wide shoulder at the low-energy side. This is the shape expected for a not very wide $\pi$ spectrum (Fig. III.1). It
will be seen that the peak contains pions and muons in comparable amounts, whereas the shoulder is composed mainly of muons.

In comparing the intensity figures, the indication about the circulating current in the first column of Table 1 should be taken into account.

The highest flux figures and also the highest stop rates are observed in the peak, see line 6 of the table.

The other flux and stop rates refer to settings in the shoulder of the distribution.

The dependence of intensity on channel current for fixed analyser current is given in Fig. IX.2. There is a steep rise at low currents up to a rather broad maximum. Its position is at higher current for higher momenta. The factor "channel on/channel off" is about 30.

The counting rate depends strongly on the distance between the analysing magnet and the detector. This is illustrated in Fig. IX.3. It is, however, difficult to bring counters very near to the magnet because of its strong fringe field. For this reason there is an advantage in placing additional lenses behind the analysing magnet. They can be placed quite near to the magnet; on the other hand, their own stray field falls off so fast that counters can be put very near. In this way a high flux can be obtained under favourable background conditions. (See line 11 of Table 1.)

A typical magnet spectrum is shown in Fig. IX.4. Resolutions are between 6 and 10% half-width at half-height depending mainly on detector geometry.

IX.3 Pion and electron contamination

This has been studied by methods described elsewhere\(^9\). The examples given below do not always correspond to optimum conditions. Besides a channel spectrum, Fig. IX.1 shows the spectrum of the pions. For the \(\pi\) contamination in the shoulder, values of a few per cent are found: they are listed in Table 1. The purity depends on:
i) the position in the shoulder: near the \( \pi \) peak there is obviously a rise in the \( \pi \) contamination;

ii) the channel current (this is illustrated in Fig. IX.2): whereas the intensity depends more slowly on channel current over most of the range, the contamination is very sensitive to it; at certain channel currents there are steep peaks in the \( \pi \) contamination;

iii) the distance: at too small distances from the magnet the pions are not yet sufficiently separated from the \( \mu \) beam (Fig. IX.3). For this reason it is found that in the case where lenses are used after the analysing magnet, the \( \pi \) contamination is slightly higher. The lenses, which are very near to the magnet, intercept the wing of the \( \pi \) beam.

The magnet spectrum of the pions is shown in Fig. IX.4. This spectrum was taken near the \( \pi \) peak and therefore the \( \pi \) spectrum is unsymmetric. There is a large component having roughly the same momentum as the muons, although a wider distribution. It is probably due to pions scattered in the channel. In addition, there is a low-energy component which is assumed to be due to scattering in the bending magnet.

It is possible to contaminate the \( \mu \) beam artificially with pions without appreciably changing the magnet spectrum. For this purpose it is sufficient to place an absorber at the entrance of the channel. The thickness of the absorber is chosen such that the energy loss in it brings the pions just into the momentum band admitted by the analyser. In this way the \( \pi \) contamination can be raised to about 50%.

In the case of stopping particles, the purity figures given in the table can, of course, be multiplied by the further purification obtained by the difference in range between pions and muons. In this way effective contamination below 0.1% has been reached.

The electron contamination was measured in one case (line 7 of Table 1). It was 1%.
### Table 1

Momentum-analysed beams from the SC muon channel

- **Cyclotron magnet current**: 1720 A.
- **Target material**: beryllium.
- **Channel polarity**: last lens vertically defocusing, first lens vertically focusing.
- **Bending angle of alternating gradient magnet**: 70°.
- **Detector distances measured from pole edge of bending magnet or from the pole edge of last quadrupole when supplementary lenses behind the analysing magnet are used.**
- **Notation**:
  - **F** = fixed target; **V** = vibrating target; **p** = mean particle momentum; **Δp/p** = fractional half-width of momentum distribution at half-height; **Δθ** = half-width of angular distribution in vertical plane.

<table>
<thead>
<tr>
<th>No.</th>
<th>Int. beam (μA)</th>
<th>Target</th>
<th>Channel and Magnet</th>
<th>Detector</th>
<th>Beam</th>
<th>Composition (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Radium</td>
<td>Azimuth (degrees)</td>
<td>Thickness</td>
<td>Rad. width</td>
<td>Height</td>
<td>Lens arrangement</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>226</td>
<td>10</td>
<td>F</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
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<td>6</td>
<td>0.3</td>
<td>226</td>
<td>40</td>
<td>F</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>0.3</td>
<td>215</td>
<td>45.7</td>
<td>V</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>0.6</td>
<td>226</td>
<td>33</td>
<td>F</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>0.3</td>
<td>226</td>
<td>40</td>
<td>F</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
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<td>0.6</td>
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<td>F</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
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<td></td>
<td>215</td>
<td>40.5</td>
<td>V</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
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<tr>
<td>13</td>
<td></td>
<td>&quot;</td>
<td>&quot; &quot;</td>
<td>F</td>
<td>&quot; &quot;</td>
<td>&quot; &quot;</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>&quot;</td>
<td>&quot; &quot;</td>
<td>F</td>
<td>&quot; &quot;</td>
<td>&quot; &quot;</td>
</tr>
</tbody>
</table>
A. THEORETICAL COMMENTS TO SOME OBSERVATIONS

We shall discuss here a few typical observations on the CERN muon channel, and see if they are plausible in the light of the theory we have exposed. However, we shall not aim at a detailed quantitative agreement.

X.1 Particle flux as function of channel current

The lenses of the channel were designed for a maximum current of 2000 A, because at high momenta (> 350 MeV/c particles) the acceptance area of the channel increases up to that current, and one did not want to be excluded from profiting from this increase. The observed total particle flux, however, already reaches its maximum at about 1200 A, as shown in Fig. X.1. The curves are the theoretical number of accepted pions $N^*_\text{ch}_\pi$, calculated by integrating Eq. (VII.34). A value was obtained for $I = 0$ by considering the channel as an ordinary pipe of finite length. The shape of the two curves and their ratio is computed, but their absolute value is adjusted so as to bring them near the observed points.

The reason why a maximum is already reached at such a low current is suggested by Figs. VII.2 and VII.3. Although the total area of the pion ellipse continues to increase above 1000 A (see bottom of Fig. X.1), that part of the horizontal ellipse area which is strongly "populated" does not change very much. On the other hand, the long axis of the vertical pion ellipse deviates more and more from the "target line" (see Fig. VII.3) when $I$ increases, with a corresponding decrease in vertical acceptance $\Delta y'_t$ (see bottom of Fig. X.1).

For the muon flux the situation is somewhat more complex, since this flux depends rather strongly on the muon momentum, as is seen from Fig. X.2, where the curves are calculated with $N^*_\text{ch}_\pi(p_\mu)$ from Eq. (VII.34) as $N_\pi(p_\pi)$ in Eq. (V.4). The points are observations for $p_\mu = 270$ MeV/c.

In all these calculations, $k^2 p$ as a function of the channel current $I$ was taken from the lower curve of Fig. VI.5.
X.2 Fluctuations of the pion contamination with channel current

The top graph in Fig. X.3, which is extracted from Fig. IX.2, gives for \( p = 170 \text{ MeV/c} \) the observed pion content in the beam as a function of the channel current, and at first sight it shows somewhat surprising fluctuations.

However, a line through the origin of the phase plane at the beginning of the channel will rotate many times along the channel, and its position at the end of the channel is therefore very sensitive to the channel current. Figure X.3 shows the angle of inclination of such a line at the end of the channel as a function of the channel current. The line for \( p = 170 \text{ MeV/c} \) goes through a complete period of \( 2\pi \) in about \( \Delta I = 200 - 300 \text{ A} \). In the vertical direction the pions from a point are injected nearly on a line through the origin. Since the intensity depends on the position of the line at the end of the channel relative to the acceptance area of the device behind it, it is quite understandable that such intensity fluctuations may occur.

It is indicated in Fig. X.3 that the channel does not accept 170 MeV/c particles for channel currents stronger than \( \approx 1400 \text{ A} \). This agrees quite well with the total \((\mu + \pi)\) intensity in Fig. IX.2, taking into account the final resolution of the analysing magnet. It does not agree so well for the \( \pi \) intensity alone, and this fact would support the hypothesis that a large fraction of these pions has come through the channel with a momentum higher than 170 MeV/c, and lost some of it near the end of the channel or at the beginning of the analysing magnet, the possibility of the appropriate momentum loss being dependent on the position of the pion line. For the target position of Fig. IX.2, the channel should theoretically not accept pions from the target below about 250 MeV/c, a result which is not contradicted by Fig. IX.1. The third graph of Fig. X.3 shows the angle at the end of the channel for \( p = 250 \text{ MeV/c} \).

The graph at the bottom of Fig. X.3 shows that the period \( \Delta I \) is much longer at high momenta, so one would expect much less fluctuation there. A non-momentum analysed beam will show even less effect,
since the various momenta are "out of phase". In addition to this, in the case of the muons the decay tends to fill the whole ellipse, which makes the effect disappear completely. These expectations have not been violated by observations!

X.3 The pion and muon spectrum

The points on Fig. X.4 shows an observed $\pi+\mu$ (mainly) spectrum, taken with the CERN strong-focusing analysing magnet after the muon channel. The curves are theoretical spectra, established as follows: The accepted pion spectrum was calculated with Eq. (VII.34), and from that the trapped muon spectrum with Eq. (V.4). Since the resolution curve of the analysing magnet has about 14% width at half-height (see Fig. X.5), both the muon spectrum and the non-decayed pion spectrum were folded with a Gaussian of that width. The shape of the curves is thus calculated, but their vertical scale is adapted to bring them near the observed points. Various factors make the choice of the best value for $q$ uncertain (see Sections VI.1 and VII.5). The points in Fig. X.4 only agree with a rather big $q$, since a low $q$ value gives a too big dip in the middle. Curves for $q = 10$ and $12$ cm are drawn, using $S = 40$ cm and $k^2p$ from the lower curve in Fig. VI.5. For comparison there is also included a curve calculated with $S_{\text{eff}} = 32$ cm and $k^2p$ from the upper curve of Fig. VI.5, which for $q = 10$ cm has a somewhat shallower dip. One must not forget, of course, the possibility that the dip may have been partly filled in by muons "born" before the channel.

The curve for the pion content agrees quite well with observation (see the Table of Section IX), but one should keep in mind that the finite resolution of the analysing magnet is only one of the sources for pion contamination (see Section X.2).

X.4 The resolution of the analysing magnet

A converted range curve, giving the resolution curve of the CERN analysing magnet for $p_0 \approx 240$ MeV/c, is plotted in Fig. X.5. It is taken under conditions nearly corresponding to Case I in Section VIII.3,
see Fig. VIII.7, with the 10×10 cm² detector at 80 cm from the iron pole. \( q = 7.1 \) or 10 cm seems to give rather too narrow a resolution curve, whereas \( q = 12 \) cm would approach more the observed resolution. However, since the field and fringe field of the magnet is rather uncertain, we draw this conclusion about \( q \) with some reserve.

***
Appendix A

Proof of the stability condition $|\text{tr } M| < 2$ [Eq. (IV.24)]

$M$ is the transfer matrix through one period of structure:

$$\ddot{x}(z + \ell) = M \ddot{x}(z). \quad (A.1)$$

If

$$|\text{tr } M| \neq 2, \quad (A.2)$$

this matrix has two different eigenvalues $d_1$ and $d_2$. In this case an operator $V$ can be found that transforms $M$ into diagonal form

$$V M V^{-1} = \overline{M} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}. \quad (A.3)$$

This transformation conserves the determinant and the trace of the matrix

$$d_1 d_2 = 1 \quad (A.4)$$

$$d_1 + d_2 = \text{tr } M. \quad (A.5)$$

The transformed vector becomes

$$V \ddot{x} = \ddot{y} \quad (A.6)$$

$$\overline{M} \ddot{y} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \ddot{y}. \quad (A.7)$$

If we go through a number $N$ of periods, the resulting vector becomes

$$\ddot{y}_N = \overline{M}^N \ddot{y} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2^n \end{pmatrix} \ddot{y}. \quad (A.8)$$

For stability we require that all components of $\ddot{x}_N = V^{-1} \ddot{y}_N$ remain bounded for an arbitrarily large value of $N$. This is the case if, and only if,

$$|d_1| \leq 1$$

$$|d_2| \leq 1. \quad (A.9)$$
This, together with Eq. (A.4), yields

\[ d_1 = e^{i\mu} \quad \text{with } \mu \text{ real} \]  \quad (A.10)
\[ d_2 = e^{-i\mu} \]

Then because of Eq. (A.5)

\[ \text{tr} \ M = 2 \cos \mu \]

Since \( \mu \) is real, we have stability for

\[ |\text{tr} \ M| < 2 \] \quad (A.11)

for \( |\text{tr} \ M| > 2 \) we have instability. \quad (A.12)

In the case of \( |\text{tr} \ M| = 2 \), which was excluded by Eq. (A.2), only special initial conditions \( \dot{x} \) lead to a stable motion.
APPENDIX B

Proof of Eq. (IV.28) for the power of a unimodular matrix

We want to verify that for \( \|M\| = 1 \)

\[
M^n = \frac{\sin n\mu}{\sin \mu} M - \frac{\sin (n-1)\mu}{\sin \mu} E ,
\]

where \( \mu \) is given by

\[
2 \cos \mu = \text{tr} M ,
\]

and \( E \) is the unit matrix.

Proceeding by induction we have from Eq. (B.1)

\[
M^{n+1} = \frac{\sin n\mu}{\sin \mu} M^2 - \frac{\sin (n-1)\mu}{\sin \mu} M .
\]

Since

\[
M^2 = 2 \cos \mu M - E = \frac{\sin 2\mu}{\sin \mu} M - E ,
\]

Eq. (B.2) becomes

\[
M^{n+1} = \frac{\sin (n+1)\mu}{\sin \mu} M - \frac{\sin n\mu}{\sin \mu} E ,
\]

which has the form of Eq. (B.1) as it should.
An ad hoc formula for the pion production cross-section

In order to calculate the pion and muon spectra properly, \( N_t(p, \zeta) \) in Eq. (VII.34), which is essentially the pion production cross-section, is needed. Unfortunately it is not too well known, but in order to get at least its maximum with respect to \( p \) and \( \zeta \) in approximately the right position we used the following procedure.

In reference 16 it is found that the momentum spectrum of \( \pi^- \) produced by 660 MeV protons on Be at a laboratory angle of 24° can be described by a production cross-section in the centre of mass of the incident nucleon and a nucleon at rest in the Be nucleus which is isotropic in the centre-of-mass angle and has a momentum distribution given in Fig. C.1. We have approximated this curve by

\[
w(p') = 1.2 \ e^{-\frac{(p' - 167)^2}{7500}}
\]  

(C.1)

thereby shifting it slightly down in momentum, hoping that this will compensate somewhat for the fact that these measurements were made for a proton energy of 660 MeV rather than at the CERN SC energy of 600 MeV.

In the laboratory system we then get from this

\[
N_t(p, \zeta) \sim \sqrt{1 - w^2} \ \cos \zeta \ e^{-\frac{(p' - 167)^2}{7500}}
\]

(C.2)

where

\[
p' = p\sqrt{Q}
\]  

(C.3)

\[
Q = \sin^2 \zeta + \frac{(\cos \zeta - \frac{w}{\beta})^2}{1 - w^2}
\]  

(C.4)
\[ \beta = \frac{p}{\sqrt{p^2 + m^2}} , \quad \text{(C.5)} \]

and

\[ w = \frac{p_{\text{prot}}}{E_{\text{prot}} + m_{\text{prot}}} \quad \text{(C.6)} \]

is the laboratory velocity of the centre of mass of the nucleon-nucleon system. For 600 MeV protons

\[ w = \frac{1215}{2476} = 0.49 \]
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N. Christofilos (unpublished).

M. Morpurgo, CERN SC/142 (1957).

Laboratory angle of emission of muons from 400 MeV/c pions as a function of the muon laboratory momentum.
Momentum spectrum of muons from monochromatic pions. The band B contains in principle only muons.

**FIG. III. 1**

Principle of the system

**FIG. III. 2**
FIG. III.3
The CERN Synchro-cyclotron with the channel and the analysing magnet.
FIG. III. 4 The end of the channel outside the shielding wall and the analysing magnet.
Quadrupole lens and confining box.
FIG. IV. 2

The parameter $A$ of Eq. (IV. 43)

$$2\epsilon = kS = \sqrt{\eta S/L}$$

$$\eta = k^2 SL$$
FIG. IV. 3

Ellipse equation:
\[ P_0 x^2 + 2P_1 xx' + P_2 x'^2 = P_3 \]
\[ G = \sqrt{P_3 / (P_0 P_2 - P_1^2)} \]

Ellipse drawn for \( P_1 > 0 \)
FIG. IV. 4

F is the ellipse area

For $S/L = 0$ $K_F$ (Eq. (IV. 55S)) is proportional to $F$

For $S/L = 0$ Eq. (IV. 61S) was used

For $S/L > 0$ Eq. (IV. 53) was used
Graphical determination of the trapping efficiency
Partial trapping efficiency

:\[ \Phi_f \quad \text{Eq. (V.6S)} \]
\[ \phi_f \quad \text{Eq. (V.9S)} \]
FIG. V. 3
A check of the short-lens approximation

Decay angle $\theta$

$S = 40 \text{ cm}$
$S = 20$
$S = 4$
$S = 1$

$S = 0$

Ellipse areas

Muon momentum $p_\mu$

$150 \quad 200 \quad 250 \quad 300 \quad \text{MeV/c}$

Ellipse area

$k^2 p S = 0.3 \quad S B / \alpha = 5.88 \text{ (MeV/c)} \quad \text{cm}^{-1}$

was kept constant

$L = 55 \text{ cm}$
$a = 10 \text{ cm}$
$q = 10.6 \text{ cm}$

The curves are computer results for ideal lenses of length $S$

$S = 40$ - Short-lens approximation $T_s(300,p_\mu)$

Trapping efficiency $T(300,p_\mu)$
$T_s$ from Eqs. (V. 55) and (V. 205)

$\eta_r = 1.05$

$\eta_i = 1.40$

$\Delta = 100$ mrad
\[ \Delta T_s = T_s - T_s' \]

Correction to be applied to \( T_s' \),
from Fig. (V.4) and (V.5)

FIG. V. 6
$q = 8.8 \text{ cm}$

$B/a = 833 \text{ gauss/cm}$

$P_{\pi} = 400 \text{ MeV/c}$

$\gamma(p_{\pi})$

Muon yield vs. lens length

Yields vs. lens length

$Y(p_{\pi})$

30%

20

10

0

20 30 40 50 60 cm
\( q = 6.8 \text{ cm} \)

\( B/a = 8.33 \text{ gauss/cm} \)

\( P_{\pi} = 400 \text{ MeV/c} \)

\( P_{\mu} = 300 \text{ MeV/c} \)

\( \mathcal{N} = 100 \text{ mrad} \)

---

**Trapping efficiency as a function of lens length.**

\( \tau_s \) short lens approximation
\( T(\rho_{\pi}, \rho_{\mu}) \)

\( B = 8.33 \text{ kGauss} \)

\( p_{\pi} = 400 \text{ MeV/c} \)

\( p_{\mu} = 300 \text{ MeV/c} \)

\( \theta^* = 100 \text{ mrad} \)

\( g = 0.884 \)

\( f = 0.7 \)

Trapping efficiency as a function of lens radius. \( S \) and \( L \) chosen according to Eq. (VI.5.S) and (VI.6.S)

\( T_S \) short lens approximation
$$k^2 p = 0.3 \frac{B}{a}$$
for \( B \) in kG

\( a \) in cm

Lower curve \( \times 1.25 \)
to be used with \( S_{\text{eff}} = 32 \text{cm} \)

Together with \( S = 40 \text{cm} \)

This curve for \( k^2 p \)
represents the CERN lenses best.

**FIG. VI. 5**

Field gradient against lens current for the CERN Lenses

From ref. 6: CERN 61-12 Fig. 14
The geometry in the horizontal plane (seen from above) of the Synchro Cyclotron and the channel. SC field down. $\varepsilon < 0$
Horizontal ellipses for $p = 400 \text{ MeV}/c$ and various channel currents on a $(U,V)$ plot

SC target radius $r_t = 226 \text{ cm}$

- azimuth $\phi = 40^\circ$

- Field up

- First lens vert. foc.
  $q = 7.1 \text{ cm}$

- $E_r = 500 \text{ MeV}/c$

- $T = 500 \text{ A}$

- $1000 \text{ A}$

- $1500 \text{ A}$

- $2000 \text{ A}$

FIG. VII. 2
vertical ellipses for
$p = 400 \text{ MeV/c}$ and various
channel currents.

S.C Target radius $r = 226 \text{ cm}$
" " azimuth $\alpha = 10^\circ$
" " Field up
First lens vert. foc.
$q = 7.1 \text{ cm}$
Horizontal ellipses on a $(U, V)$ plot

SC Target radius $r_t = 215 \text{ cm}$

azimuth $\alpha = 40.5^\circ$

Field down

First lens vert. foc.
Channel current $I = 800 \text{ A}$

$q = 7.1 \text{ cm}$

FIG. VII.4
Limit of channel acceptance in the \((p, \xi)\) plane

SC Target radius \(r_t = 215\) cm

azimuth \(\alpha = 40.5^\circ\)

field down

First lens vert. foc.

Channel current \(I = 800\) A

\(a = 7.1\) cm

\(\phi(p, \xi) = 0\)

\(\Delta \xi(p)\)

The sharp corners come from a linear approximation in the computer programme

**FIG. VII. 5**
FIG. VII.6

(a.)

Pion spectra before channel
Muon spectra after channel

SC target radius \( r_t = 215 \text{ cm} \)

azimuth \( \alpha = 40.5^\circ \)

Field down

First lens vert. foc.

Channel current \( I = 800 \text{ A} \)

- \( q = 10 \text{ cm} \)
- \( q = 12 \text{ cm} \)

Momentum

200 300 400 MeV/c

(b.)

SC target radius \( r_t = 225 \text{ cm} \)

azimuth \( \alpha = 10^\circ \)

Field up

First lens vert. foc.

Channel current \( I = 1000 \text{ A} \)

- \( q = 10 \text{ cm} \)
- \( q = 12 \text{ cm} \)

Momentum

200 300 400 MeV/c
Longitudinal polarization of the muons from the channel

(= complete polarization in the direction of flight)
FIG. VIII. 1

The principle of the analysing magnet

\[ \Delta > 0 \]

\[ r_0 - \Delta \]

\[ \phi > 0 \]

\[ r \text{ and } r_0 > 0 \text{ for particles bent to the left.} \]
The CERN strong focusing analysing magnet.
Horizontal trajectory in the analysing magnet of a particle starting tangentially on the equilibrium radius $r_0$ at the beginning of a hor. def. section.

- $q_+ = q_- = 0.42$ rad
- $r_0 = 110$ cm
- $\Delta l = 17$ cm
- $|n| = 6.5$
- $w = P_0/P = 0.9$

---

**FIG. VIII. 3**

- $v = 1$ valid at end of hor. def. section
- $v = 0$ valid at end of hor. foc. section

---

- Estimated trajectory between points of computed position $R_1$ and slope $R_2$.

---

1 section $= r_0 \cdot q_+ = 46$ cm

---

sections $(2m + v)$
Maximum extension of the beam at various positions inside the CERN analysing magnet.

As beam limitation is used transformed CERN channel ellipses for $q = 7.1\text{cm}$ and $p_0 = 240 \text{ MeV}/c$.

Channel current $I = 850\text{A}$

---

$\xi, \gamma$

Hor. $w = 0.9$

Vert.

5 cm

Distance from or median plane

- End of iron

0 first section 0.47 sec. section 0.89/ third section 1.36 rad $\phi$
Cross section of the CERN magnet

Maximum extension of CERN channel ellipses or the wanted momentum $p_0 (=240 \text{ MeV}/c)$
CASE I

10 x 10 cm$^2$ counters directly after the CERN analysing magnet at 30 cm and at 80 cm from the iron pole.

Horizontal phase plane at the end of a fringe field of 5.5 cm

Channel current I = 850 A
CASE I
10 × 10 cm² counters directly after the CERN analysing magnet at 30 cm and 80 cm from the iron pole.

Vertical phase plane at the end of a fringe field of 5.5 cm p = 240 MeV/c

Channel current I = 850 A
$q = 7.1$ both for channel and for doublet
$I = 850\text{ A}$

W = 0.9
$p = 267\text{ MeV/c}$

CASE II

$10 \times 10\text{ cm}^2$ counter after a doublet
Horizontal phase plane at the end of a fringe field of 5.5 cm
 FIG. VIII. 10

Case II

10 x 10 cm$^2$ counter after a doublet

Vertical phase plane at the end of a fringe field of 5.5 cm.

$p = 240$ MeV/c

$q = 7.1$ cm both for channel and for doublet

$I = 850$ A
q = 7.1 cm both for channel and for quartet
I = 850 A

Limiting channel ellipse for the beam
Limiting acceptance ellipse if quartet replaced by a channel

CASE III
10 x 10 cm counter after a quarter
Horizontal phase plane at the end of a fringe field of 5.5 cm
CASE III.

10 x 10 cm counter after a quartet

Vertical phase plane at the end of a fringe field of 5.5 cm

p = 240 MeV/c

q = 7.1 cm both for channel and quartet

I = 850 A

- Limiting channel ellipse for the beam
- Limiting acceptance ellipse if quartet replaced by channel
Channel spectrum
SC Target radius $r_t = 2.205$ cm
" azimuth $\alpha = 40.5^\circ$
" Field down
Channel current $I = 750$ A
$Z = 80$ cm
Expt. 2/5 1961
FIG. IX. 2

SC Target radius $r_t = 220.5$ cm

"" azimuth $\alpha = 40^\circ$

Field down

Magnet current 400 A

$(170$ MeV/c$)$

$Z = 80$ cm

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Flux (particles sec$^{-1}$)$\times 10^{-3}$

$\mu + \pi$

$\pi \times 10$

Channel current

0 200 400 600 800 1000 1200 1400 1600 A
50 C Target radius $r_t = 226 \text{ cm}$
" azimuth $\alpha = 40^\circ$
" Field up
Channel current $I = 700 \text{ A}$
Magnet current $700 \text{ A} (270 \text{ MeV/c})$

Flux (particles sec$^{-1}$)$\times 10^{-3}$ $\pi^+ + \mu$

Distance between the analysing magnet and detector

**FIG. IX. 3**
Observed $\mu$ and $\pi$ spectra

$\mu$: Expt. 68, 29.11.61
$\pi$: 21, 12.3.62

SC Target radius $r = 215$ cm

azimuth $\alpha = 42.5^\circ$

Channel current $I = 800$ A
Magnet current $600$ A (240 MeV/c)

(from a range curve)
Observed and theoretical particle flux as a function of channel current

SC Target radius $r_t = 226 \text{ cm}$

azimuth $\alpha = 10^\circ$

Field up

$q = 10 \text{ cm}$

First lens vertically focusing

First lens horizontally focusing

Ellipse area for $450 \text{ MeV}$

Vertical acceptance $\Delta x$

Computed by considering the channel as a pipe of finite length

Channel current $I$
FIG. X.3

Fluctuations of pion content with channel current.

For $I > 1400 \, \text{A}$
170 MeV/c particles do not pass the channel.
Observed and theoretical momentum spectra

SC Target radius \( r_t = 215 \text{ cm} \)

azimuth \( \alpha = 44^\circ \)

Field down

Channel current \( I = 800 \text{A} \)

**FIG. X. 4**

The magnet's resolution curve is folded into the theoretical spectra.

- observed total particle spectrum with analysing magnet
  - \( \pi + \mu \) th. spectr. for \( q = 10 \text{cm}, S = 40 \text{cm}, k^2 p = 0.147 \)
  - \( q = 12 \text{cm} \)
  - \( q = 10 \text{cm}, r_{eq} = 32 \text{cm}, k^2 p = 0.184 \)

momentum

150 200 250 300 MeV/c
Observed momentum spectrum (obtained from a range curve) after the CERN channel and analysing magnet.

5C Target radius \( r_t = 215 \text{ cm} \)

Azimuth \( \alpha = 40.5^\circ \)

Field down

Channel current \( I = 800 \text{ A} \)

\( P_0 \approx 240 \text{ MeV/c} \)

Beam limiting detector: \( 10 \times 10 \text{ cm}^2 \) counter at \( 80 \text{ cm} \) from the iron pole

FIG. X.5
Pion momentum spectrum in nucleon-nucleon C.M. system measured with 660 MeV protons on Be at lab. angle 24°.

FIG. C.1

Arbitrary scale

C.M. momentum $p'$

$\frac{(p' - 167)^2}{1.2 \times 10^{-7}}$