SPECULATIONS ON EXPERIMENTAL CONSEQUENCES
OF REGGE POLES

by

W. Kummer
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INTRODUCTION

The concept of "Regge poles" recently became a widely discussed object in the theory of elementary particles. It seems therefore worthwhile not only to present the theoretical background of these new ideas, but also to speak about their possible experimental verification. At this point it must be emphasized very explicitly that in the present state one needs such experimental evidence rather urgently, because a lot of assumptions form an essential part of the theoretical development.

The point is that the existence of poles in the complex angular momentum plane was found by T. Regge \(^1\) in potential theory for a very large and reasonable class of potentials. There were several attempts postulating that certain results could be carried over to elementary particle physics \(^2\),\(^3\),\(^4\) proposing experimental tests. Only recently Amati, Rubini, and Stanghellini \(^5\) have succeeded in finding out the analogue behaviour of amplitudes in a high-energy field theoretic model. There appear, however, some essential new features which seem to make dubious several conclusions for experimental tests drawn in earlier investigations.

We shall see that measurements in the high-energy region of total and elastic cross-sections would presumably be a tool for clarifying this situation. The high-energy behaviour of these entities shows rather typical features if Regge poles rather than elementary particles according to the usual concept are involved. We shall come back to this question in detail.

In the first section we show the way Regge poles appear in potential theory where their existence can be proved. The second section deals with the rather speculative extrapolation of these results to elementary particle physics. In the first part of this section the unified view of elementary particles \(^2\) is treated, whereas in the second part experimental applications will be discussed \(\text{ref.}^{3},^{4}\)\(^7\).

This report should mainly give an introduction to the whole problem, therefore the interested reader is invited to look for further details in the original papers cited in the references.
I. POTENTIAL SCATTERING AND POLES IN THE COMPLEX ANGULAR MOMENTUM PLANE

The mathematical procedure starts at something very familiar, so to speak at the very beginning of quantum theory: the Schrödinger equation. If it depends on a central symmetric potential, we can split the wave function into a product of functions, which depend separately on the spherical co-ordinates $r, \theta, \varphi$

$$\Psi = R(r, \theta, \varphi) \ e^{i \ell \theta} \ P_\ell^m (\cos \theta) \ e^{im \varphi}$$  \hspace{1cm} (I.1)

$P_\ell^m$, $e^{im \varphi}$ are the spherical harmonics of angular momentum $\ell$, with a third component $m$. $R$ obeys ($R=1$, $M$ is the mass, $E$ the energy of the particle considered)

$$\frac{1}{2M} \frac{d^2 R}{d\epsilon^2} + \left( E - V(\epsilon) - \frac{\ell(\ell+1)}{2M \epsilon^2} \right) R = 0$$  \hspace{1cm} (I.2)

Scattering amplitude

The scattering amplitude $f(\theta, E)$ in the differential cross-section

$$\frac{d\sigma}{d\cos \theta} = \left| f(\theta, E) \right|^2$$  \hspace{1cm} (I.3)

is well known ($E = p^2/(2M)^{-1} \theta, \cos \theta = z$)

$$f(\theta, E) = \sum_{\ell=0}^{\infty} \ (2\ell+1) \ P_\ell(\cos \theta) \ P_\ell(z)$$

$$f(\epsilon, \xi) = (e^{2\epsilon \xi} - 1) / 2\epsilon \rho$$  \hspace{1cm} (I.4)

The phase-shifts $\xi$ depend of course on the potential $V(r)$. A mathematical trick due to Sommerfeld and Watson makes it possible to transform this sum into a certain integral in the complex $\ell$-plane. We show that $f(\theta, E)$ can be written
\begin{equation}
\int_{C_1} f(z) = \frac{1}{2} \int \frac{(\ell \cdot + 1) d\ell}{\nu + p \ell} \left( \ell, \varepsilon \right) \left. \ell \varepsilon \right|_{C_1} \quad (I.5)
\end{equation}

along a contour \( C_1 \) as follows

Fig. 1

According to the Cauchy theorem in the theory of functions, the integral along a closed contour in clockwise sense of an analytic function equals \( 2\pi i \) times the sum of the residu \(^(*)\) of the poles, which are included by the path. The function under our integral has clearly poles exactly at the integer values of \( \ell \) \((\sin \pi \ell = 0:\) and only these poles are surrounded by \( C_1 \), because we exclude "accidental poles" (like \( P \) in Fig.1) which may be caused by \( f(\ell, \varepsilon) \). If we develop the integral in (I.5) near an integer value \( n \) of \( \ell \)

\( \ell \approx n + x, \quad x \ll 1 \)

\(^{*})\) The coefficient of \((x-x_0)^{-1}\) is called "residuum" in a Mac Laurant series of a function \( f(x) \) at a point \( x_0 \) where the function becomes infinite.
we get
\[
(1)^n \left( \frac{2\pi i}{3\pi} \right) P_n(z) \frac{1}{3\pi i}
\]

By doing this for all integers \( n \geq 0 \) the factors of \( \frac{1}{3\pi} \) (the residue) times \(-2\pi i\) (the minus sign arises from the clockwise sense of our integration) gives then in fact back all terms of our sum formula (1.4).

We now complete the path \( C_1 \) not "closing it short" at infinity like implicitly assumed before, but closing it by means of an integral parallel to the imaginary axis and two infinitely distant quarters of a circle [Fig. 2]

![Fig. 2](image)

Using again Cauchy's theorem, we can write formally
\[
\int_{C_1} + \int_A + \int_0 - \int_{\frac{1}{\infty}} = 2\pi i \sum \text{ (Residue of integrand)}
\]
Regge now showed that for a superposition of Yukawa potentials

\[ \sqrt{\ell} = \int_{\mu}^{\infty} \frac{4\mu}{\ell} e^{-\alpha \mu} \left\{ \gamma(\mu) \mu \alpha \right\}^{\infty} \mu^{\infty}. \]

\( f(\ell, \varepsilon) \) approaches 0, if \( \ell \to \infty \), enough strongly to compensate the unagreement of \( \sin \frac{\pi}{2} \ell \) and \( P_\ell (\cos \theta) \) for complex \( \ell \to \infty \). Further one can prove quite independent of the shape of the potential that all poles of the integrand (those at \( \sin \frac{\pi}{2} \ell = 0 \) are now clearly excluded) must lie in the upper half-plane of \( \ell \) (Im: \( \ell > 0 \)). Furthermore, there is only a limited number of them. We can thus write finally for \( f(z, \varepsilon) \)

\[ f(z, \varepsilon) = \frac{-1}{z} \int_{-\infty}^{\infty} \left\{ (\frac{2\varepsilon + 1}{2\varepsilon}) \frac{\ell}{\varepsilon} \right\} f_\ell (-z) + \sum_{N=1}^{N} \frac{\alpha_n(\varepsilon)}{\omega \alpha(\varepsilon)} P_{\ell_n}(\varepsilon) (-z) \]  

(I.6)

where the residues of the integrand have numbers from 1 to \( N \). In \( n(\varepsilon) \) different functions of \( \varepsilon \) are combined.

Amplitude for bound and resonant states caused by one Regge pole

From the contribution of one pole in the complex angular plane in (I.6):

\[ f(z, \varepsilon) = \frac{\alpha_1(\varepsilon)}{\omega \alpha(\varepsilon)} P_{\ell}(\varepsilon) (-z) + \]  

(I.7)

\[ \ell_n(\varepsilon) = \alpha(\varepsilon) \]

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6.

We can project out by means of the formula

\[ \frac{1}{2} \int_{-\infty}^{+\infty} F_n(z) P_{\ell}(-z) dz = \frac{1}{\pi} \frac{\alpha \cdot \pi}{(\alpha - \ell)(\alpha + \ell + 1)} \]  \hspace{1cm} (I.8)

the contribution to one specific partial wave \( f_\ell(E) \) (\( \ell \) now integer)

\[ f_\ell(E) = \frac{1}{\pi} \frac{\alpha(E)(x(E) + \ell - \frac{1}{2})^{-1}}{\alpha(E) - \ell} \]  \hspace{1cm} (I.9)

Now if \( \alpha(E) \) for an energy \( E \) approaches very closely the integer \( \ell \), we can develop \( \alpha(E) \)

\[ \alpha(E) \approx \ell + \left( \frac{d\alpha}{dE} \right)_{E = \ell} (E - \ell) + i \left( \frac{\text{Im} \alpha}{E - \ell} \right)_{E = \ell} \]  \hspace{1cm} (I.10)

and get

\[ f_\ell(E) = \frac{1}{\pi} \frac{\alpha(E)(x(E) + \ell - \frac{1}{2})^{-1}}{\left( \frac{d\alpha}{dE} \right)_{E = \ell} (E - \ell) + i \Gamma_{\ell}/2} \]  \hspace{1cm} (I.11)

This is the well-known Breit-Wigner resonance formula with a half-width (or imaginary part of the resonance energy)

\[ \Gamma/2 = \left( \frac{\text{Im} \alpha}{E} \right)_{E = \ell} \]  \hspace{1cm} (I.12)

For \( E \approx E_0 < 0 \) it can be shown quite generally that \( \text{Im} \alpha = 0 \) and thus no imaginary part \( \text{Im} \alpha \) appears in (I.10). We then get no half-width and therefore a normal bound state. Now it can be seen clearly that for physical (integer) values of \( \ell \) such a pole gives the same effect as a pole in the complex energy
plane. But on its way it can pass near different values of \( \ell \) (integer \( \geq 0 \)) and therefore one Regge pole can be the reason for resonances in different angular momentum states \( \ell \), \( \ell = 2 \) and \( 3 \) in Fig. 4 \( _{7} \) at different energies. This is in strict contrast to the usual point of view that a resonance in one partial wave has no direct connection with the behaviour of other partial waves. From (I.9) we see that a Regge pole acts for one energy \( \left( \frac{\sqrt{E}}{\ell} \right) \) theoretically on all partial amplitudes \( \left( \frac{\sqrt{E}}{\ell} \right) \) actually the influence becomes rather small if \( \ell \) is very different from \( \text{Re} \, \alpha(n) \frac{1}{\ell} \).

---

**Movement of one pole with energy**

We already know that one single pole \( \ell(E) = \alpha(E) \) can only move around in the upper half of the \( \ell \)-plane for \( E > 0 \), and that its imaginary part vanishes for \( E < 0 \). To get further insight into its movements, it seems useful to look also for the derivative with respect to the energy. We shall give a very simple intuitive "proof". Let us take for simplicity the Schrödinger equation for a "potential hole"

\[
\sqrt{\ell} = \begin{cases} 
\sqrt{\ell} & \ell \leq \alpha \\
0 & \ell > \alpha 
\end{cases}
\]

The centrifugal term \( \ell(\ell+1)/2 Ma^2 \) may then be considered as an additional contribution to the potential. On the other hand, there must be for a resonance at a real value of the energy \( (\ell > 0) \) and an integer value of \( \ell \), a connection with something that is known for a long time in potential scattering: a so-called virtual bound state. This is not stable \( (E > 0) \) but has other properties quite similar to a real bound state. It is kept together \( (E_v \text{ in Fig. 3}) \) by the centrifugal barrier, but can get away through it with a certain probability by

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![Diagram](image)
tunnel effect. If we vary the energy in one "residuum" of (I.6), it can happen that $\ell'$ passes near an integer value so that the whole amplitude behaves like having a resonance in this value of $\ell'$ for a certain energy as discussed in the last paragraph. We may therefore expect that for an integer value of $\ell'$ Regge poles and (virtual) bound states are two expressions of one and the same thing. Now two such virtual levels will be approximately at

$$E_1 \approx \text{const} + \frac{\ell'(\ell' + 1)}{2Ma^e},$$

$$E_2 \approx \text{const} + \frac{\ell'(\ell' + 1)}{2Ma^e}.$$

If they are caused by the same Regge pole, we know ($\ell_1', \ell_2'$ are now only the physical, integer values) two points of the pole curve. Including also non-integer values, one can write for a small change in energy

$$\frac{d\left[\frac{\ell'\ell'(\ell'+1)}{2}ight]}{dE} \approx 2Ma^e \tag{1.13}$$

or

$$\frac{d\left(\ell' + \frac{1}{2}\right)^2}{d\rho^2} \approx a^2$$

which is, of course, only valid near real values of $\ell'$. We get thus

$$\frac{d\ell'}{dE} > 0 \quad \text{if} \quad \ell' > -\frac{1}{2}$$

and have so a monotonous connection between $\ell'$ and $E$ near physical values. Therefore a typical Regge trajectory of such a pole term in the $\ell'$-plane looks like in Fig. 4.
It first progresses along the real axis for $E < 0$, passing through the real bound state $\ell = 0, 1$ at $E = E_{B_1}$ and $E_{B_2}$, and turns away into the upper half-plane at $E = 0$, causing resonances in the amplitude at $E_{r,2}$ in $\ell = 2$ and $E_{r,3}$ (much weaker, because more distant from real values, where $(\sin \pi \ell)^{-1}$ can become big) in $\ell = 3$.

There is a slightly different behaviour when we admit also the appearance of an exchange potential (for the space co-ordinates) in the Schrödinger equation. We get then practically two different actions of the whole potential if the solution is symmetric or antisymmetric in the co-ordinates, i.e.,

$$f(z, E) = \frac{\beta(E)}{\pi \alpha(E)} \sum_{\nu \in \pi} \left( P_{\nu, \ell}(z) \pm P_{\nu, \ell}(\bar{z}) \right) (1.4)$$

For the plus (minus) sign the contribution therefore vanishes if $\alpha(E)$ goes through odd (even) integers, making the distance between bound states (resonances) now $\Delta \ell = 2$, if each trajectory with specific signature is now considered separately.
II. THEORY OF ELEMENTARY PARTICLES

The question now arises whether it is possible to carry the above-mentioned concepts over to the theory of elementary particles. Even in potential theory (and for the simplest models) it seems mathematically very difficult to find a typical Regge trajectory \(*\). The Bethe-Salpeter equation (as the analogue in elementary particle theory of the Schrödinger equation) has been recently investigated for a certain field theoretic model and there result in fact formulae which look rather familiar if we remember the properties of scattering amplitudes we know from potential theory.

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Chew-Frautschi diagram of elementary particles

Nevertheless, Chew and Frautschi \(2\) tried to fit the known particles (resonances included) into different Regge trajectories of different quantum states of one single underlying field. Because nobody knows exactly the imaginary part of the poles in the complex angular momentum \(\mathcal{Q}(\alpha)\) plane, in Fig. 5 \(\text{Re } \alpha\) is plotted as a function of \(t = (\text{energy})^2\) (we admit now also half-integer values of \(\alpha\)).

In order to treat all particles and resonances on the same footing, Chew and Frautschi assume that they all arise together from poles in the complex angular momentum plane \(\mathcal{Q}^\ast\). But of the known "particles", only \(N\) (Nucleon) and \(N^\ast\) \(\left(\frac{5}{2}\right\) resonance in the nucleon-pion system) have the same "internal" quantum numbers \(T\) (isotopic spin), \(B\) (baryon number), \(S\) (strangeness) and can therefore originate from the same Regge trajectory. The others could lie on trajectories which turn down before reaching a second physical value (turning upwards in the \(\mathcal{Q}^\ast\)-plane corresponds clearly to turning down in the co-ordinates of Fig. 5; \(\mathcal{Q}_\alpha\) is an assumed curve which shows this behaviour).

\(*\) e.g., for a potential hole one needs Bessel functions of complex order. The Coulomb potential \(6\) gives no short range force. It is therefore not so interesting in this respect.

\(**\) Of course, also all nucleus bound states, etc., belong to this class.
The only two points $N, N^+_3$ which can belong together, give a slope \(~1/50\ m^2_{\pi}\), which implies by (1.13) a rather plausible average radius of interaction for the bound state $\sim 1/2\ m_{\pi}$. Slopes of this order of magnitude could be expected also for other trajectories where only one point is known.

It can be seen from Fig. 5 (the hypothetical trajectory $\alpha_{\text{vac}}$ will be discussed immediately) that there is no trajectory of known particles which crosses $t=0$ with this slope at $\alpha(0)\geq 1$. In fact, there exists, independent of the Regge pole concept, the Froissart theorem 7), which says $\alpha(t)\leq 1$ for $t<0$. It was proved for a scalar particle scattering amplitude using unitarity and analyticity of the S-matrix. The Froissart theorem would therefore b
"saturated" if there exists a curve ($\alpha_{\text{vac}}$) with the same slope passing through $\alpha(0) = 1$. Chew and Frautschi now claim that their earlier postulated "principle of maximal strength" $^8$ (the forces in strong interactions are as strong as they can be under the conditions of unitarity and analyticity of the S-matrix), can be interpreted in the diagram of Fig. 5 in the following way: lower internal quantum numbers $(S,B,T)$ of trajectories imply bigger $\alpha(0) \leq 1$. Thus $\alpha_{\text{vac}}$ should have the simplest quantum numbers; those of the vacuum $(S=B=T=0)$.

If the poles in the presence of an exchange potential are separated by $\Delta \alpha = 2$ there could be $\sqrt{2}$ in Fig. 5 a "ghost" pole at $m_\pi^2 \sim 50$ and a predicted pole at $\alpha = 2$ plus sign in (I.14), which could coincide with the recently discovered $2\pi$ resonance at $1 \text{ GeV}$ ($t \sim 50 m_\pi^2$). The disagreeable ghost pole could disappear if in (I.14), $\beta$ as a function of $t$, happens to vanish there. Such a "vanishing by chance" must also be postulated for several nuclei which have an observed ground state with spin bigger than 2. We shall see that from the high-energy behaviour of cross-sections, there comes further more direct support for the properties of $\alpha_{\text{vac}}$.

If the binding energy is too weak, the curve could possibly turn down even before reaching the axis $\text{Re } \alpha = 0$. This could be the case for $\alpha_{\text{ABC}}$, the trajectory of the pole of Abashian, Booth, and Crowe ("ABC-pole") $^9$. It also has the quantum numbers of the vacuum, but could be a "secondary trajectory" of these quantum numbers. For other trajectories which belong to less strong binding forces, these secondary curves would presumably be far below $\text{Re } \alpha = 0$.

The ABC-pole could represent an example for a pole with a small imaginary and real difference with respect to a physical (integer or half-integer) value $^10$ of $\alpha$. 
**High-energy behaviour of cross-sections**

Let us start with the investigation of elastic differential cross-sections. It will be an easy matter to use afterwards the optical theorem in order to calculate from the elastic forward amplitude the total cross-sections.

The elastic scattering of two particles Fig.6-b can be looked upon also in two other different ways Fig.6-c, if we only change ingoing particles into outgoing antiparticles. In the invariant scattering amplitude $A$ (for simplicity of scalar particles with equal masses $m$), defined by

$$
\langle \hat{S}|S|\rangle = \langle \hat{f}|f\rangle + \frac{i}{(2\pi)^2} \int \frac{d^4 p}{p^+ p^-} \frac{d^4 q}{q^+ q^-} G(p) A(p, q) G(q) A(q, p) \quad (II.1)
$$

we then change nothing but the region for the two independent variables from which it depends, defined with the momentum four vectors $p_1, p_2, q_1, q_2$, for convenience, as

$$
\gamma = (p_1 - q_1)^L (p_2 - q_2)^L \quad (II.2)
$$

$$
\tau = (\lambda_1 - \lambda_2)^L = (\lambda_1 - \lambda_2) = (\lambda_2 + \lambda_2)^L. \quad (II.3)
$$

This region can be easily seen by introducing centre-of-mass variables (c.m.) in the three "channels" of Fig. 6:

*) Time-like four vectors have a positive value in our metric
In channel 6-a ("s-channel") this gives

\[ p_1 = \left( \frac{\beta}{\sqrt{p_1^2 + M_1^2}} \right) \quad \xi_1 = \left( \frac{\beta}{\sqrt{p_1^2 + M_1^2}} \right) \]

\[ p_2 = \left( \frac{\gamma}{\sqrt{p_2^2 + M_2^2}} \right) \quad \xi_2 = \left( \frac{\gamma}{\sqrt{p_2^2 + M_2^2}} \right) \]

(II.4)

and therefore

\[ s = \left( 2 \sqrt{p_1^2 + M_1^2} \right)^2 = 2M_1^2 + 2M_2 \xi_2 \gamma = 4M_1^2 \]

\[ t = -2 \frac{p_2^2}{p_1} (1 - \cos \theta_2) \leq 0 \]

(II.5)

whence in 6-b ("\bar{s}-channel")

\[ \bar{s} = \left( \frac{1}{p_1^2 + M_1^2} \right)^2 = 4M_1^4 - s - t = \]

\[ = \left( \frac{1}{\sqrt{p_1^2 + M_1^2}} \right)^2 = 4M_1^4 \]

\[ t = -2 \frac{p_2^2}{p_1} (1 - \cos \theta_2) \leq 0 \]

(II.6)
and in 6-c ("t-channel")

\[ t = (\sqrt{t} + M^2)^2 \geq 4 M^2 \]

\[ s = -2 \beta^2 \left( 1 - \cos \vartheta \right) \leq 0 \]  \hspace{1cm} (II.7)

Evidently, \( s, s', t \) are the c.m. energies in the three channels respectively, and \( t, t', s \) the momentum transfers.

We can now ask the question, which are the properties we could get for the amplitude in the \( s \)-channel, when the amplitude \( A \) behaves like having a resonance of the Regge-pole-type in one of the channels 6-b or 6-c.

This is the point where speculative extrapolations were made from the unrelativistic scattering amplitude. We shall proceed also in this way for clearness sake, but will point out that differences can possibly be obtained from more exact considerations.

Let us consider the \( t \)-channel. In order to apply (I.6), we must identify \( f \) with the c.m. scattering amplitude \( f_t \) in the \( t \)-channel, the angle \( \vartheta \) with \( \vartheta_t \) and generalize the unrelativistic dependence on the kinetic energy \( E \) to a dependence on \( t \). From (II.7), we get in (I.6)

\[ \hat{z} = \omega \cos \vartheta = 1 - \frac{E^2}{4 M^2} \]  \hspace{1cm} (II.8)

We now turn to the amplitude in the \( s \)-channel in which we are interested.

If we consider small momentum transfer \((\approx \text{small } t \leq 0)\) and high energies \((s \gg 4 M^2)\) in this channel, we see that the contribution of the integral in (I.6) is something that goes down like \( 1/\sqrt{s} \) for high energies:

\[
\int_{\frac{V_m - i \epsilon}{V}} \left( \frac{1}{\pi} \frac{1}{V} \gamma \right) \gamma \left( \frac{1}{V} \gamma \frac{1}{V} \right) = \frac{1}{\sqrt{2}} \gamma \gamma \epsilon \gamma \frac{1}{\sqrt{s}}
\]
The influence of this integral — the so-called "background integral" — is moreover entirely overshadowed by the single terms of the sum in (I.6), which contain, from the \( P_{\ell}n(t)\), factors which behave like \( z \ell n(t) \sim \frac{1}{\sqrt{\eta - t}} \) at high energies. Quite generally, the term with the highest \( \ell n(t) = x(t) \) dominates the amplitude \( A \) for \( s \gg 4M^2 \).

\[
A(s,t) = 2\pi \sqrt{|t|} \int \ell (s,i) \ell (t,\ell n) \ell P_{x(\ell)} \left( \frac{\sqrt{|t|}}{\sqrt{\eta - t}} \right)
\]

(Normalisation)

Near an integer value \( t \) of \( \eta \), which may be reached for \( t = m^2 \) (it lies, of course, by assumption in the physical range of the t-channel), we can develop

\[
\alpha(t) = \ell + \varepsilon_{\ell} (t - \eta_{\ell}^2) + \mathcal{I}_{\ell}
\]

(Norm and integral)

\[
\varepsilon_{\ell} = \left( \frac{dH_{\ell}}{dt} \right)_{t = \eta_{\ell}^2}
\]

\[
\mathcal{I}_{\ell} = \left( \int_{\eta_{\ell}^2}^{\infty} \alpha \right)_{t = \eta_{\ell}^2}
\]

and obtain therefore

\[
A(s,t) = \frac{2\pi \sqrt{|t|} \ell P_{\ell} \left( \frac{\eta_{\ell}^2}{\sqrt{\eta - t}} \right)}{t - \eta_{\ell}^2 + \mathcal{I}_{\ell} \ell}
\]

In the special cases where a peripheral approximation with exchange of an elementary particle in the usual sense, with spin \( \ell \), is justified, one gets for big \( s \) a formula similar to (II.9), but with the essential difference that \( P_{\chi(t)} \) is to be replaced by \( P_{\ell} \) (\( \ell \) constant). The dependence on \( t \) in \( P_{\chi(t)} \) is therefore typical for a Regge pole behaviour. On the other hand, near
the pole $\sum_{\text{complex Eq. (II.12)}}$ the contribution to the amplitude is the same. (II.9) can be generalized to the case of an exchange "potential" like (I.14). One obtains in the high-energy limit

$$A(s, t) = \lim_{s \to \infty} \frac{4 \gamma(s)}{\nu \nu'(s)} \left( C_{\nu \nu'}^2 \pm \frac{1}{2} \right) \sum_{\nu} \alpha(s).$$

(II.13)

In fact, Batini, Frabetti and Sanfranceschi found in their model of high-energy pion phenomena exactly the same behaviour if they neglect inelastic channels. In addition, one obtains a mathematical expression for $\gamma(t), \alpha(t)$ functions which were purely phenomenological in the inductive treatment. Besides this, however, in a refined calculation which also takes into account in some sense the influence of inelastic channels, there appears an entirely new feature which has, of course, no counterpart in potential theory where no inelastic channel exists; the contribution of a branch cut in the $\nu$-plane must be added

$$A(s, t) = \left( \frac{1}{2} \right) \gamma(t) \sum_{\nu \nu'} \frac{1}{2} \int \frac{d x}{x} x(x') \alpha(x') \sum_{\nu \nu'} \frac{1}{2} \int \frac{d y}{y} y(y') \alpha(y') + \int \frac{d x}{x} x(x') \alpha(x') \sum_{\nu \nu'} \frac{1}{2} \int \frac{d y}{y} y(y') \alpha(y').$$

(II.13')

For big $s$ this integral behaves like $\sum_{\nu \nu'} \frac{\alpha(s)}{\nu \nu'} \sim \frac{\alpha(s)}{\nu \nu'}$ and gives therefore a similar contribution as the first term.

A may be used to calculate the differential cross-section in the c.m.

$$\frac{d \sigma}{d(\nu_\nu \nu_\nu)} = \frac{1}{2 \pi s} \left| A \right|^2$$

(II.14)

Because of

$$\sum_{\nu_\nu \nu_\nu} \frac{S}{s} = 1 - \frac{2 \pi}{\nu_\nu \nu_\nu S} \simeq 1 + \frac{2 \pi}{s}$$

$$\int \frac{d \nu_\nu \nu_\nu}{s} \simeq \frac{2}{s}$$

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we get from (II.13) and (II.14) 

\[ \frac{d\sigma}{dt} = \left| \frac{\gamma(t)}{\sqrt{\pi}} \right|^2 \frac{2^\Lambda(\Lambda+1)}{2M} \frac{\frac{x(t)}{2M} + \frac{1}{2}}{\sin \pi x(t)} \approx \frac{2^{\Lambda(\Lambda+1)}}{2M} \frac{1}{\omega_{\text{lab}}} \]

\[ (\text{II.15}) \]

with

\[ \gamma(t) = \frac{\gamma(t)}{\sqrt{\pi}} \left( \frac{x(t)}{2M} + \frac{1}{2} \right) \frac{1}{2 \sin \pi x(t)} \]

\[ (\text{II.16}) \]

Calculating the total cross-section we use the optical theorem which has, in our notation, the simple form

\[ \Gamma_{\text{tot}}(s) \approx \frac{1}{s} \sum A(s,0) \]

\[ (\text{II.17}) \]

This gives, from (II.13):

\[ \Gamma_{\text{tot}} = \sum \left\{ \frac{\pi}{\sqrt{\pi}} \gamma(\omega) \left( \frac{\omega}{\sin \pi x(\omega)} \right)^2 + 1 \right\} \frac{x(\omega) - 1}{\omega_{\text{lab}}} \approx \]

\[ = \sum \left\{ \frac{\pi}{\sqrt{\pi}} \gamma(\omega)^2 \left( \frac{\omega}{\sin \pi x(\omega)} \right)^2 + 1 \right\} \frac{x(\omega) - 1}{\omega_{\text{lab}}} \]

\[ (\text{II.18}) \]

Taking into account the spin dependence in a more general scattering problem, we would have arrived finally at essentially the same shape of these formulae.
From (II.15) and (II.18), we see immediately that the experimental and theoretical rather probable constancy at very high energies of total cross-sections and of differential cross-sections in the forward direction \((t \approx 0)\) can be guaranteed if there is a Regge pole dominating all different amplitudes at very high energies, which has such a trajectory that

\[ \alpha(\omega) = 1 \]

This is exactly the property of \(\nu_{\text{vac}}\) shown in Fig. 5. With the plus sign in (II.18) \(\nu\) pole should lie on the trajectory only at \(\alpha = 2\) as discussed in connection with the Chew-Frautschi diagram, Fig. 27 the amplitude becomes purely imaginary, and

\[ \langle \gamma | T_{\sigma} | \psi \rangle = \frac{\lambda}{\pi} \gamma(\omega) \]  

(II.19)

Clearly this vacuum pole *) should have very simple quantum numbers. In this case only it can appear - in the same way - in the crossed channels of quite different reactions (e.g., the t-channel for p-p scattering is \(p^- \rightarrow p^-\), and the possible intermediate states have the quantum numbers \(B=\frac{1}{2}, 0 , T=\omega, 1\); the same thing happens for p^- scattering and also in \(p^- \rightarrow n^- p^+ -\) reactions).

If we look into the Chew-Frautschi diagram Fig. 27, we see that in fact the trajectories of all other poles obey \(\alpha(0) \leq 1\) and thus are overshadowed by the vacuum pole: they give some negative number in the power of \(E_{\text{lab}}\) in (II.15) or (II.18) and die out therefore at very high energies. Moreover, according to recent results (II.19'), all such poles may be practically swallowed up by the branch cut contribution of the highest vacuum pole if the Amati-Pubini-Stanghellini model 5) is correct.

Independent of the appearance of a vacuum pole, the differential cross-section shows rather typical features. Near the forward direction we may develop

*) - or Pomeronchuk pole - because it guarantees the similar behaviour of all total cross-sections at high energies.
\[ \lambda(t) \approx \lambda(0) + \frac{t}{\lambda'}(0) \]  

and get from (II.15):

\[ \frac{d\tilde{\tau}}{dt} = \left| \frac{\gamma(t)}{(\gamma(0))^{\lambda'(0) - 1}} \right|_M^{(\gamma(0))^{\lambda'(0) - 1}} \tilde{\tau}^{(1 - \lambda'(0))} \exp \left( \frac{\gamma(t)\lambda'(0)}{M} + \frac{\gamma(t)\lambda'(0)}{M} \right) (\Pi 21) \]

which corresponds to \( t \ll 0 \) in the s-channel! to an exponential decreasing tail of the diffraction peak \( \sqrt{\gamma(t)}(2M)^{\lambda'(0) - 1} \) is a rather slow-varying function of \( t \). Such a behaviour would be very difficult to explain from usual field theory, but seems to be compatible with experiments.

If we want to have information concerning other poles than the vacuum pole we have to look for linear combinations of cross-sections in order to subtract out the influence of the vacuum pole. In choosing specific combinations, poles with quantum numbers different from those of the vacuum may contribute.

Let us examine, for example, the total cross-section in \( p - p \) scattering. We first look for the elastic amplitudes in the forward direction. No spin complications arise there. By invariance in isospace:

\[ A = A^+ - \sum_{\lambda=1}^{3} \tilde{\tau}_{\lambda} A^- \]

where \( \tilde{\tau}_{\lambda} \) are the isotopic spin matrices and \( \tilde{\tau}_{\lambda} \) is the isotopic spin operator for the pion. The two amplitudes \( A^+ \) and \( A^- \) are connected with the amplitudes belonging to total isospin \( A^{3/2} \) and \( A^{1/2} \) by

\[ A^+ = \frac{i}{3} \left( A^1 L + L A^3 L \right) \]

\[ A^- = \frac{i}{3} \left( A^1 L - L A^3 L \right) \]
Using appropriate Clebsch-Gordan coefficients, one finds

\[ A^\pi^+ = A^\nu_i \]
\[ A^\pi^- = \frac{1}{\gamma} \left( A^{\gamma_i} + \Delta A^{\gamma_i} \right) \]

and with the optical theorem (II.17) expressing \( A^\pi^+ \), \( A^\pi^- \) in terms of \( A^+ \) and \( A^- \)

\[ \frac{2}{\gamma} \left[ \sigma(n^-p) + \sigma(n^+p) \right] = \frac{1}{\gamma} \left( J_m A^+(t = 0) \right) \]
\[ \frac{2}{\gamma} \left[ \sigma(n^-p) - \sigma(n^+p) \right] = \frac{1}{\gamma} \left( J_m A^-(t = 0) \right) \]

In the t-channel of \((\pi^N)\) scattering \((NN \to 2\pi)\), clearly \(G = 1\), because there are two pions in the final state. Furthermore, every intermediate state in this channel must be an isoscalar (T=0) in \(A^+\) but an isovector (T=1) in \(A^-\). According to Fig. 5 the poles with the comparatively biggest \(\alpha(0)\) and the correct quantum numbers are thus the vacuum pole and perhaps the ABC-pole \(^1\) (G=1, T=0, B=0, S=0) for \(A^+\), and the \(\gamma\)-pole (T=1) for \(A^-\). Therefore

\[ \sigma(n^-p) + \sigma(n^+p) = \alpha_i \epsilon_{\omega^\nu}^{\pm 1} \times \alpha(\nu_i) \]

\[ \sigma(n^-p) - \sigma(n^+p) = \epsilon_{\omega^\nu}^{\pm 1} \times \alpha(\nu_i) \]

Considerations in \(p\nu, \bar{p}\nu, \bar{p}N\) scattering along the same lines lead to similar formulae. See Table I\(^\prime\). Certain constants, e.g., \(\gamma, \phi, \alpha(0)\) appear there simultaneously in different reactions. In this way one gets a simple parametrization of the high-energy behaviour of total cross-sections.

\(^1\) Cf., however, the discussion between Eqs. (II.19) and (II.20).
### TABLE I

Regge poles contributing to various cross-sections

<table>
<thead>
<tr>
<th>Cross sections</th>
<th>Contributing Regge poles</th>
<th>Expected high-energy behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(s^-p) + \sigma(s^+p)$</td>
<td>vacuum, ABC</td>
<td>$a + b E^{-1 - \alpha ABC(0)}$</td>
</tr>
<tr>
<td>$\sigma(s^-p) - \sigma(s^+p)$</td>
<td>$\rho$</td>
<td>$c E^{-1 - \alpha \rho(0)}$</td>
</tr>
<tr>
<td>$\sigma(K^-p) - \sigma(K^+p)$</td>
<td>$\eta, \omega, \rho$</td>
<td>$d E^{-1 - \alpha \eta(0)} + e E^{-1 - \alpha \omega(0)} + f E^{-1 - \alpha \rho(0)}$</td>
</tr>
<tr>
<td>$\sigma(K^-p) + \sigma(K^+p)$</td>
<td>vacuum, ABC</td>
<td>$g E^{-1 - \alpha ABC(0)}$</td>
</tr>
<tr>
<td>$\sigma(p^+p) - \sigma(p^-p)$</td>
<td>$\eta, \omega, \rho$</td>
<td>$h E^{-1 - \alpha \eta(0)} + i E^{-1 - \alpha \omega(0)} + j E^{-1 - \alpha \rho(0)}$</td>
</tr>
<tr>
<td>$\sigma(p^+p) + \sigma(p^-p)$</td>
<td>vacuum, $\pi$, ABC</td>
<td>$k E^{-1 - \alpha \pi(0)} + l E^{-1 - \alpha ABC(0)}$</td>
</tr>
<tr>
<td>$\sigma(p^+p) - \sigma(n^-p)$</td>
<td>$\rho, \pi$</td>
<td>$m E^{-1 - \alpha \rho(0)} + n E^{-1 - \alpha \pi(0)}$</td>
</tr>
<tr>
<td>$\sigma(p^+p) + \sigma(n^-p)$</td>
<td>vacuum, ABC, $\eta, \omega$</td>
<td>$o E^{-1 - \alpha ABC(0)} + p E^{-1 - \alpha \eta(0)} - q E^{-1 - \alpha \omega(0)}$</td>
</tr>
<tr>
<td>$\sigma(K^+p) + \sigma(K^-n)$</td>
<td>vacuum, ABC, $\eta, \omega$</td>
<td>$r E^{-1 - \alpha ABC(0)} - s E^{-1 - \alpha \eta(0)} - t E^{-1 - \alpha \omega(0)}$</td>
</tr>
<tr>
<td>$\sigma(K^+p) - \sigma(K^-n)$</td>
<td>$\rho$</td>
<td>$u E^{-1 - \alpha \rho(0)}$</td>
</tr>
</tbody>
</table>

*) This table is taken from ref. 4), Table I. But, according to recent results 5), it is perhaps impossible to isolate influences of other poles from the first one \((\text{cf.}, \text{ the discussion between Eqs. (II,19) and (II,20))})\). Furthermore, the \(\eta\) -resonance has rather probably spin 0 and therefore \(\alpha_\eta(0) << \alpha_\rho(0), \alpha_\omega(0)\). Some terms \(\alpha\) therefore written in brackets.
Instead of considering the forward elastic amplitude and total cross-sections, one could make a similar treatment of the backward elastic amplitude $\sqrt{t} = -4 \frac{p_s^2}{\bar{s}}$, $\bar{s} = 0$ in the s-channel. Here Regge resonances in the $\bar{s}$-channel $\text{Fig.6-2}$ would dominate the differential cross-section. E.g., in the $\pi-p$-scattering this channel is again $\pi-p$ scattering with the following poles: the nucleon itself, the $J3$-resonance, and the further resonances $N^+_2$, $N^*_3$. One could in this way test if the nucleon is something like a Regge pole, or an elementary particle in the usual sense $^3$).

CONCLUSION

We have seen that there are several experimental consequences following from the extrapolation of the properties, found by T. Regge for poles in the complex angular momentum plane in potential scattering, to the theory of elementary particles:

a) predictions of resonances from a Chew-Frautschi diagram $\text{Fig.5}$ by extrapolation of trajectories whose slope ($\alpha'(0)$) could perhaps be determined from the tail of the diffraction peak in the region $t < 0$ $\text{Eq. (II.21)}$;

b) high-energy behaviour of differential cross-sections mainly at forward angles, milking out $\text{Eq. (II.21)}$ $^*$);

c) high-energy behaviour of total cross-sections, $\text{see Table I}$.

$^*$) For backward angles see remarks mentioned in the first paragraph of this page.
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