VERY ELEMENTARY THEORY
OF WEAK INTERACTIONS

Notes on three lectures*
by
J.S. Bell.
(Prepared by M. Bell)

* These lectures were given at the Rutherford High-Energy Laboratory, Harwell, in May 1959. They are reprinted here without change.
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GENEVA

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INTRODUCTION

It is the aim in these talks to proceed in easy stages from something the listener is supposed to know about -- de Broglie waves -- to something he is supposed to want to know about -- the current theory of weak interactions. Only the bare bones of the theory are given. Many important ideas are omitted. History is distorted.

The typical 'weak interaction' is that causing the decay of the neutron into a proton, an electron, and an antineutrino:

\[ N \rightarrow P + e + \bar{\nu}. \]  \hspace{1cm} (1)

A similar process is the decay of the \( \mu \) meson:

\[ \mu \rightarrow \nu + e + \bar{\nu}. \]  \hspace{1cm} (2)

An apparently dissimilar process is the capture of \( \mu \) mesons by the protons in nuclei

\[ P + \mu \rightarrow N + \nu. \]  \hspace{1cm} (3)

However, it is a general theoretical principle that a reaction can go in either direction, and that if it goes with a particle on one side it goes also with an antiparticle on the other; thus

\[ N + \nu \rightarrow P + \mu \]  \hspace{1cm} (4)

and

\[ N \rightarrow P + \mu + \bar{\nu} \]  \hspace{1cm} (5)

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also exist. The similarity of (1), (2), and (5) is manifest. These reactions all involve four particles, each a fermion of spin \( \frac{1}{2} \). An account of the relevant theory falls into three parts:

I. basic theory of spin \( \frac{1}{2} \) particles and antiparticles --- Dirac equation;

II. theory of interactions;

III. restrictive principles for interactions, e.g. 2-component neutrino, \( F-G \) theory, chirality invariance, etc.

I. \textbf{DIRAC EQUATION}

The starting point is the de Broglie wave. Consider particles moving freely in space. \( E \) is energy, \( \vec{p} \) is momentum. They are described by a plane de Broglie wave:

\[
\psi = e^{i \vec{k} \cdot \vec{r} - i \omega t}
\]

with \( \vec{k} = \vec{p}/\hbar, \quad \omega = E/\hbar \).

The continual appearance of \( \hbar \) and \( c \) in formulae is a nuisance which we can eliminate by a reasonable choice of units. In any case, the usual laboratory units are quite inappropriate for elementary particle physics -- one seldom sees the mass of a new particle quoted in kilograms; it is given rather as a multiple of the mass of some familiar particle, e.g. the proton, taken as standard. Let \( m_0 \) be the mass of the standard particle. We choose the unit of mass such that

\[
m_0 = 1.
\]

A convenient unit of length is then the Compton wavelength:

\[
\lambda_0 = \hbar/m_0 c = 1
\]

and of time

\[
\tau_0 = \lambda_0/c = \hbar/m_0 c^2 = 1.
\]
Then $c = \frac{\hbar}{\tau_0} = 1$ and $\hbar = \frac{\hbar}{m_0 c} = 1$. The unit of energy is $m_0 c^2 \approx 938$ MeV $\approx 1$ GeV if $m_0$ is proton mass. When we put $\hbar = c = 1$ our equations are not valid for arbitrary changes of units other than that of mass -- only this freedom remains for those addicted to dimensional analysis.

With these units

$$E^2 = c^4 m^2 + p^2 c^2$$

becomes

$$\omega^2 = m^2 + k^2.$$  \hspace{1cm} (6)

Because of this the de Broglie wave is a solution of

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - m^2 \psi$$

or

$$\frac{\partial^2 \psi}{\partial t^2} = (\omega^2 - m^2) \psi.$$  \hspace{1cm} (7)

This is the Klein–Gordon equation. A non-relativistic approximation is obtained by putting

$$\omega = (m^2 + k^2)^{1/2} = m (1 + k^2/m^2)^{1/2} \approx m + k^2/2m.$$ 

Then the de Broglie wave is

$$\psi = e^{-imt} \varphi$$

where

$$i k \cdot r - i \frac{k^2}{2m} t$$

$$\varphi = e$$

which satisfies

$$\frac{\partial \varphi}{\partial t} = i \frac{k^2}{2m} \nabla^2 \varphi,$$  \hspace{1cm} (8)

the non-relativistic Schrödinger equation. This may be more familiar in the form

$$\frac{i\hbar}{2m} \frac{\partial \varphi}{\partial t} + \frac{\hbar^2}{8m} \nabla^2 \varphi = 0.$$
In the early days of quantum theory the formalism was elaborated on the basis of the non-relativistic equation, which had great successes. However, eventually people became interested in relativistic effects and a modified equation was required in which time and space appeared on more symmetrical terms. Such, of course, is the Klein-Gordon equation, of second order in all variables. But by this time the idea had taken hold, for reasons then convincing, that the fundamental equation is necessarily of first order with respect to the time. Dirac was therefore led to look for an equation of first order with respect to all the variables. The simplest is

\[
\frac{\partial \psi}{\partial t} = \sigma_x \frac{\partial \psi}{\partial x} + \sigma_y \frac{\partial \psi}{\partial y} + \sigma_z \frac{\partial \psi}{\partial z}
\]

where the \( \sigma \) are coefficients to be determined. Dirac required that the Klein-Gordon equation be satisfied in addition—for this essentially expresses the relativistic energy momentum relation. From the above

\[
\frac{\partial^2 \psi}{\partial t^2} = \left( \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \right) \frac{\partial \psi}{\partial t}
\]

\[
= \left( \sigma_x \frac{\partial}{\partial x} + \cdots \right) \left( \sigma_x \frac{\partial}{\partial x} + \cdots \right) \psi
\]

\[
= \left[ \sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2} + \left( \sigma_x \sigma_y + \sigma_y \sigma_x \right) \frac{\partial^2}{\partial x \partial y} + \cdots \right] \psi.
\]

To satisfy the Klein-Gordon equation we need therefore

\[
\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1
\]

\[
\sigma_x \sigma_y + \sigma_y \sigma_x = 0, \text{ etc.}
\]

\( m = 0 \).

One drawback is that the mass is necessarily zero—this will be remedied later. The really novel feature is that the \( \sigma \) cannot be ordinary numbers;
they do not commute. However, they can be matrices, and a digression
is now made on this subject.

1. Matrices

A matrix is a convenient notation for expressing some familiar
ideas. Consider, for example, an electric field. This is specified
by three components

\[(E_x, E_y, E_z)\]

each of which is a function of space and time. Very often we have
to work with such sets of functions; let

\[(\psi_1, \psi_2, \ldots, \psi_n) = \psi\]

be such a set, \(\psi\) denoting the set as a whole. The \(\psi\) in the wave
equation will turn out to be a set of this kind. If \(a\) is a number

\[a\psi = (a\psi_1, a\psi_2, \ldots, a\psi_n)\).

Also

\[\frac{\partial \psi}{\partial x} = \left(\frac{\partial \psi_1}{\partial x}, \frac{\partial \psi_2}{\partial x}, \ldots, \frac{\partial \psi_n}{\partial x}\right).\]

Suppose now we take linear combinations

\[a_{11}\psi_1 + a_{12}\psi_2 + a_{13}\psi_3 \ldots = \psi_1'\]
\[a_{21}\psi_1 + a_{22}\psi_2 + a_{23}\psi_3 \ldots = \psi_2'\]
\[\vdots \quad \quad \quad \quad \quad \quad \quad \quad \vdots \]
\[\vdots \quad \quad \quad \quad \quad \quad \quad \quad \vdots \]

to form a new set

\[\psi' = (\psi_1', \psi_2', \ldots).\]

Then the set

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots \\
a_{21} & a_{22} & a_{23} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is called a matrix.
One writes symbolically
\[ \psi' = \alpha \psi. \]
This is just a shorthand for
\[ \psi'_m = \sum_n \alpha_{mn} \psi_n. \]
Suppose now we consider a second such linear transformation
\[ \psi'' = \sum \beta_{\ell m} \psi'_m \]
or
\[ \psi'' = \beta \psi'. \]
If we define a matrix
\[ (\beta \alpha)_{\ell n} = \sum \beta_{\ell m} \alpha_{mn} \]
then
\[ \psi'' = \beta (\alpha \psi) = (\beta \alpha) \psi = \beta \alpha \psi. \]
Thus we have defined a way of combining two matrices \( \alpha \) and \( \beta \) into a compound \( \alpha \beta \). In general
\[ \alpha \beta \neq \beta \alpha, \]
i.e. matrices, unlike simple numbers, do not always commute.

A particular matrix which must be mentioned is the \underline{unit} matrix \( I \) which has matrix elements
\[ I_{\alpha \beta} = \delta_{\alpha \beta}. \]
However, it is readily seen that in all equations \( I \) can simply be replaced by the number 1; thus, for example,
\[ \psi = I \psi = \psi = I \psi. \]
Thus the notation \( I \) is seldom used.
2. The Weyl equation

We see that the \( \sigma 's \) are to be interpreted as matrices, and the function \( \psi \) is to be a set of functions (called here a spinor). The simplest possibility is that \( \psi \) is a pair \( (\psi_1, \psi_2) \) and the matrices \( \sigma \) have \( 2 \times 2 = 4 \) elements and are such that

\[
\sigma_x \sigma_x = I = 1 = \sigma_y \sigma_y = \sigma_z \sigma_z \\
\sigma_x \sigma_y = - \sigma_y \sigma_x, \text{ etc.}
\]

One suitable set of \( \sigma 's \) is

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

These are known as the Pauli spin matrices.

We rewrite Eq. (4) for particles of zero mass as

\[
\frac{\partial \psi}{\partial t} = \vec{\sigma} \cdot \frac{\partial}{\partial \vec{r}} \psi.
\] (9)

This is the Weyl equation.

An equally good Weyl equation is clearly

\[
\frac{\partial \psi}{\partial t} = - \vec{\sigma} \cdot \frac{\partial}{\partial \vec{r}} \psi.
\] (9')

3. The Dirac equation

To deal with non-zero mass, Dirac added a term in \( \psi \) to the equation. Changing our notation slightly we write

\[
\frac{\partial \psi}{\partial t} = - \alpha_x \frac{\partial \psi}{\partial x} - \alpha_y \frac{\partial \psi}{\partial y} - \alpha_z \frac{\partial \psi}{\partial z} - \text{im} \beta \psi
\]

\[
= - \left( \alpha \cdot \frac{\partial}{\partial \vec{r}} + \text{im} \beta \right) \psi.
\] (10)
Then
\[
\frac{\partial^2 \psi}{\partial t^2} = \left( \alpha \cdot \frac{\partial}{\partial r} + \text{i} \mu \beta \right) \left( \alpha \cdot \frac{\partial}{\partial r} + \text{i} \mu \beta \right) \psi
\]
\[
= \left( \alpha_x^2 \frac{\partial^2}{\partial x^2} + \alpha_y^2 \frac{\partial^2}{\partial y^2} + \alpha_z^2 \frac{\partial^2}{\partial z^2} \right) \psi - m^2 \beta^2 \psi +
\]
\[
+ \left[ \left( \alpha_x \alpha_y + \alpha_y \alpha_x \right) \frac{\partial^2}{\partial x \partial y} + \ldots \right] + \ldots
\]
\[
+ \left\{ \text{i} \mu \beta \alpha_x \frac{\partial}{\partial x} + \ldots \right\}
\]
\[
+ \left\{ \text{i} \mu \alpha_y \beta \frac{\partial}{\partial y} + \ldots \right\} \psi.
\]

Again for the Klein–Gordon equation to be satisfied, i.e.
\[
\frac{\partial^2 \psi}{\partial t^2} = (\nabla^2 - m^2) \psi
\]
we must have
\[
\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1
\]
\[
\alpha_x \alpha_y + \alpha_y \alpha_x = 0, \text{ and so on,}
\]
\[
\alpha_x \beta + \beta \alpha_x = 0, \quad \text{"""
}\]

We need now to find four matrices which anticommute with one another. The simplest are \(4 \times 4\) matrices, and \(\psi\) is then a 4-spinor. Various sets of \(4 \times 4\) matrices can be found which satisfy the conditions; we take

\[
\alpha_x = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
\[
\alpha_y = \begin{pmatrix}
0 & 0 & 0 & -\text{i} \\
0 & 0 & \text{i} & 0 \\
0 & -\text{i} & 0 & 0 \\
\text{i} & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
(11)
\]
\[ \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_z \\ \sigma_z \\ 0 \end{pmatrix} \]

or

\[ \vec{\alpha} = \begin{pmatrix} 0 \\ \sigma \\ \sigma \\ 0 \end{pmatrix} \]

\[ \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \]

(11)

It can be shown that any other set gives merely a different way of expressing the same theory.

Writing out the Dirac equation in full (not normally done) we have

\[ \frac{\partial \psi_1}{\partial t} = -\frac{\partial \psi_4}{\partial x} + i \frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_3}{\partial z} - i \text{m} \psi_1, \]

\[ \frac{\partial \psi_2}{\partial t} = -\frac{\partial \psi_1}{\partial x} - i \frac{\partial \psi_3}{\partial y} + \frac{\partial \psi_4}{\partial z} - i \text{m} \psi_2, \]

\[ \frac{\partial \psi_3}{\partial t} = -\frac{\partial \psi_2}{\partial x} + i \frac{\partial \psi_4}{\partial y} - \frac{\partial \psi_1}{\partial z} - i \text{m} \psi_3, \]

\[ \frac{\partial \psi_4}{\partial t} = -\frac{\partial \psi_3}{\partial x} - i \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial z} + i \text{m} \psi_4. \]

Let us look for solutions of the Dirac equation in the form

\[ \psi_n = U_n e^{i \vec{k} \cdot \vec{r} - i \omega t}. \]

Now \( \psi \) must satisfy the Klein-Gordon equation (7) which gives

\[ \omega^2 = k^2 + m^2 \text{ or } \omega = \pm (k^2 + m^2)^{1/2}. \]
Both signs of $\omega$ occur; so there are solutions with positive and negative energies. Having decided on one of the signs, one can find two independent solutions of the Dirac equation. For example, if $k$ is in the $z$-direction, then

\[
U_{++} = \begin{pmatrix}
1 \\
0 \\
k/(E+m) \\
0
\end{pmatrix}, \quad U_{+-} = \begin{pmatrix}
0 \\
1 \\
0 \\
-k/(E+m)
\end{pmatrix},
\]

\[
U_{-+} = \begin{pmatrix}
-k(|E|+m) \\
0 \\
i \\
0
\end{pmatrix}, \quad U_{--} = \begin{pmatrix}
0 \\
k/(|E|+m) \\
0 \\
1
\end{pmatrix}.
\]

In the $U_{++}$, etc., the first sign refers to the sign of the energy. Thus $U_{++}$ and $U_{+-}$ have $(+)$ne energy; $U_{-+}$ and $U_{--}$ have $(-)$ne energy.

The second symbol occurs because we need more than just $E$ and $p$ to specify the state of the particle. There is an internal degree of freedom, interpreted as the spin. The $(+)$ sign denotes the spin axis along the direction of motion; the $(−)$ sign denotes the spin axis opposite to the direction of motion. Combinations refer to intermediate spin directions.

At first the occurrence of negative energy solutions seems difficult to correlate with observation. Dirac turned the difficulty to remarkable advantage. We are dealing with fermions—a distinct state can be occupied by at most one particle. Thus there is a state of least possible energy, that is, in which all negative energy states are occupied. To make precise the idea of distinct state, the usual device is to require the wave functions to satisfy periodic boundary conditions on the surface of a large cube of side $L$. Then only integral values of

\[
k_x \ L/2\pi, \quad k_y \ L/2\pi, \quad k_z \ L/2\pi
\]
are allowed. At the end of any calculation L must be made to
tend to infinity so that the artificial boundary conditions do not
influence events in any given region of space. We now suppose that
what is normally called vacuum is actually the above state of least
energy, and that when quantities such as energy, momentum, angular
momentum, charge, are measured we really only observe increments in
these quantities as compared with their vacuum values. It is clear
then that observed energies will always be positive -- since the
vacuum has least possible energy. If some disturbance knocks one
of the negative energy particles of the vacuum into a positive
energy state there will be left a hole in the infinite negative
energy sea. Such holes themselves behave very much as particles,
and are called antiparticles. If the unfilled state has negative
energy \(-|E|\) and momentum \(-\vec{k}\), then the corresponding contributions
to the energy and momentum of the whole system (relative to the
vacuum) are \(+|E|\) and \(+\vec{k}\). These then are the energy and momentum
of the antiparticle. Similarly if the unfilled state had spin
angular momentum along (or against) its momentum \(-\vec{k}\), the anti-
particle has spin along (or against) its momentum \(\vec{k}\). If the anti-
particle discussed has charge \(e\) then relative to the vacuum the
hole, and so the antiparticle, has charge \(-e\). The antielectron is
the positron. The antineutrino is a hole in the infinite sea of
negative energy neutrinos.

When a particle is knocked out of a negative energy state
it leaves a hole, and one might think therefore that the number of
particles minus the number of holes (i.e., the number of antiparticles)
is always constant. However, this idea does not work for separate
sorts of particle because of more complicated processes. A similar
idea does appear to work when a number of different sorts of particle
are counted together. Thus muons, electrons, and neutrinos are col-
lectively known as leptons. The negative muons and electrons are
currently thought of as particles, and the positives as antiparticles.
It seems that the number of leptons minus the number of antileptons
is conserved. This is known as lepton conservation.
In this account the antiparticle—the hole—seems to be of quite different nature from the particle. In a more sophisticated field theoretic version this is seen to be illusory; there is really considerable symmetry between particle and antiparticle. Here, however, we avoid discussion requiring field theory, and for this reason also a proper account of lepton conservation is omitted.

In a general account of Dirac theory one would describe other triumphs in addition to the prediction of the positron—for example, the predicted electron magnetic moment and atomic spectra fine structure. But we do not go into these here.

II. INTERACTIONS

Anything which induces particles to change their nature or state of uniform motion is referred to as an interaction. We are interested here in interactions causing processes such as \( \beta \) decay. Consider first, however, the much simpler process of scattering of a particle by a static force centre described by a potential energy \( V(\vec{r}) \), \( \vec{r} \) being the position of the particle. We require the probability per unit time that it is deflected from an initial direction \( \hat{k} \) to a final \( \hat{k}' \), i.e., from an initial state described by

\[
\psi = e^{-i\vec{k} \cdot \vec{r}}
\]

to

\[
\psi' = e^{i\vec{k}' \cdot \vec{r}}.
\]

One knows from elementary quantum mechanics that the transition probability is proportional to the squared modulus of the so-called matrix element for the process, given by

\[
M = \int d\vec{r} \psi^* (\vec{r}) V(\vec{r}) \psi (\vec{r})
\]

to a first approximation in the strength of \( V \). The other factors in the transition probability formula do not depend on \( V \), and we will not need to discuss them.
If, instead of one particle colliding with a fixed scattering centre, we have two particles colliding with one another, the matrix element is

\[
M = \int d\vec{r}_a \ d\vec{r}_b \ \bar{\psi}_b(\vec{r}_b) \ \psi_a(\vec{r}_a) V(\vec{r}_a - \vec{r}_b) \ \bar{\psi}_b(\vec{r}_b) \ \psi_a(\vec{r}_a).
\]

For a very short-range interaction

\[V(\vec{r}_a - \vec{r}_b) = g \ \delta(\vec{r}_a - \vec{r}_b)\]

the formula simplifies to

\[
M = \int d\vec{r}_b \ \bar{\psi}_b(\vec{r}_b) \ \psi_a(\vec{r}_a) \ \bar{\psi}_b(\vec{r}_b) \ \psi_a(\vec{r}).
\]

Consider now the neutron decay

\[N \to P + e^- + \bar{\nu}.\]

We can visualize the process as the collision of the neutron with one of the negative energy neutrinos of the vacuum, the colliding pair being converted into a proton and an electron; this leaves a hole in the sea of negative energy neutrinos—this is then the antineutrino.

That the particles change their nature in the course of the collision is a novelty. Nevertheless, Fermi assumed the matrix element to be of the above type.

\[
M = g \int d\vec{r}_b \ \bar{\psi}_b(\vec{r}_b) \ \psi_a(\vec{r}_a) \ \bar{\psi}_b(\vec{r}_b) \ \psi_a(\vec{r})
\]

if for the moment we suppose the wave functions have only one component and not four. That the extreme short-range form should be chosen is merely because it is the simplest hypothesis that might fit the facts. The obvious generalization for 4-component wave functions is

\[
M = \int d\vec{r} \left[ \sum_{ijkl} \epsilon_{ijkl} \ \phi_i^* \phi_j \ \phi_k^* \phi_l \right] \tag{12a}
\]

3339/NP/Aw
where we take a sum of all possible combinations of components (specified by i, j, k, l) of the plane wave Dirac functions. The quantity in square brackets is referred to as the 'interaction' or 'interaction Lagrangian' or 'interaction Hamiltonian'. The interaction is specified by the 256 constants $g_{ijkl}$.

As has been remarked, from possible reactions others are obtained by replacing antiparticles on one side by particles on the other, and vice versa. The matrix element for

$$N + \nu \rightarrow P + e$$

is obtained simply by using a positive, rather than a negative, energy wave function for $\psi_\nu$ in Eq. (12a). The inverse reaction

$$P + e \rightarrow N + \nu$$

is known from general theory to have the matrix element

$$M = \int dr \left[ \sum g_{ijkl}^* \psi_i^* \psi_j^* N_k^* \psi_i \psi_j P_l \right]$$

with the same g's. This complex conjugacy of direct and inverse processes is known as the 'hermiticity' of the interaction, or the 'unitarity' of the theory, and is related to the conservation of probability.

The interaction is now specified by 256 unknown constants. Fortunately there are symmetry restrictions to reduce this number.

1. Lorentz invariance

The laws of nature do not depend on the orientation or velocity of the laboratory -- they are said to be Lorentz invariant. In ordinary vector analysis one knows that rotational invariance requires the appearance of all three components of vectors on an equal footing. Lorentz invariance, which includes rotational invariance, imposes similar restrictions on, in particular, the combination of spinors that can occur in the weak decay interaction.
Let \( x_1, x_2, x_3, (=x,y,z), \) and \( x_4 (=t) \) be the co-ordinates attached to a space-time point in some inertial frame of reference, and let the same point be denoted in another such reference frame by \( x'_\mu (\mu = 1, ... 4) \). The two sets are connected by a Lorentz transformation

\[
x'_\mu = \sum_\nu a_{\mu\nu} x_\nu
\]

where the coefficients satisfy

\[
\sum_\nu a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}
\]

(scalar of coefficients) \(|a_{\mu\nu}| = 1 \quad a_{44} > 0\).

Here and later we will use the summaion convention according to which summation over repeated indices is always implied without explicit indication. Then these equations read

\[
x'_\mu = a_{\mu\nu} x_\nu, \quad a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda},
\]

\(|a_{\mu\nu}| = 1 \quad a_{44} > 0\).

The final restrictions prevent changes from left to right handed co-ordinate systems, and those in which the sign of time \( t = x_4 / i \) is changed; to stress this, what we call a Lorentz transformation is sometimes called a 'proper isochronous Lorentz transformation'.

Other physical quantities besides the co-ordinates will be measured differently in the two systems. For example, if an electromagnetic field is specified in the old system by potentials \( A_\mu \) (where \( A_1, A_2, A_3 \) are components of the vector potential and \( \varphi = A_4 / i \) is the scalar potential) then it is given in the new system by

\[
A'_\mu = a_{\mu\nu} A_\nu.
\]

A set of quantities like this, which transform in the same way as \( x_\mu \), is known as a 4-vector, or tensor of rank one. From the potentials we form by differentiation the set of \( 4 \times 4 = 16 \) quantities...
\[ F_{\mu \nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}. \]

known as the electromagnetic field tensor (the non-zero components are equal to the components of \( E \) and \( H \), the electric and magnetic fields). For this the transformation law is

\[ F'_{\mu \nu} = a_{\mu \lambda} a_{\nu \rho} F_{\lambda \rho}. \]

A set transforming in this way is known as a **tensor of rank two** -- or simply as a **tensor**. Tensors of higher rank are defined in an obvious way. An important point is that for two tensors of equal rank the sum of products of corresponding components is a **scalar**, i.e. independent of the reference system. Thus

\[ A'_\mu A'_\lambda = A_\mu A_\lambda, \]

\[ F'_{\mu \nu} F'_{\mu \nu} = F_{\mu \nu} F_{\mu \nu}. \]

These scalars are said to be formed by **contraction** of the tensors.

It would now be logical to discuss the transformation law for the spinors \( \psi \) that we have been using. But this is not too simple, and the easiest way to present the required material is in terms of certain bilinear combinations of spinors which form tensors. Then by contraction of these we form scalars which are acceptable as Lorentz invariant interactions. These bilinear combinations are most readily expressed by means of the \( \gamma \) matrices, which we now introduce.

In the Dirac equation

\[ \frac{\partial \psi}{\partial t} = - \alpha_x \frac{\partial \psi}{\partial x} - \alpha_y \frac{\partial \psi}{\partial y} - \alpha_z \frac{\partial \psi}{\partial z} - im \beta \psi, \]

we multiply from the left by \( i\beta \)

\[ 0 = - i\beta \frac{\partial \psi}{\partial t} - i\beta \alpha_x \frac{\partial \psi}{\partial x} - i\beta \alpha_y \frac{\partial \psi}{\partial y} - i\beta \alpha_z \frac{\partial \psi}{\partial z} + m\psi. \]
But \( \mathbf{x} = x_1 \), \( \mathbf{y} = x_2 \), \( \mathbf{z} = x_3 \), \( \mathbf{it} = x_4 \)

\[-i\beta \alpha_x = \gamma_1, \quad -i\beta \alpha_y = \gamma_2, \quad -i\beta \alpha_z = \gamma_3, \quad \beta = \gamma_4.\]

The equation then reads

\[\sum \gamma_\mu \frac{\partial \psi}{\partial x_\mu} + m\psi = 0\]

or with the summation convention

\[(\gamma_\mu \partial_\mu + m)\psi = 0. \tag{14}\]

It is not surprising that the coefficients \( \gamma \) in this form of the equation -- which is very symmetric between space and time -- should be most convenient in expressing Lorentz invariance. As well as \( \gamma_1,2,3,4 \) it is convenient to introduce

\[\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4.\]

It is readily shown that

\[\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} \quad (\mu, \nu = 1, \ldots, 5)\]

and that with the explicit \( \alpha, \beta \) matrices (11)

\[\gamma_1,2,3 = \begin{pmatrix} 0 & -i\sigma_1,2,3 \end{pmatrix}; \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{15}\]

We are now in a position to list (without proof) the transformation properties of bilinear combinations of spinors. The quantity \( \psi^* \beta \psi \) (where the two spinors may be the same or different) is a scalar. The notation here has the following meaning:

\[\psi^* \beta \psi = \sum_{i,j} (\psi^*_i) \beta_{ij} \psi_j.\]

It is convenient to introduce a quantity

\[\bar{\psi} = \psi^* \beta.\]

(meaning the set of quantities

\[\bar{\psi}_j = \sum_i \psi^*_i \beta_{ij}.\])

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Having sufficiently explained the notation we state the following list

\[
\begin{align*}
\bar{\psi} \psi \quad \{ \text{are scalars} \\
\bar{\psi} \gamma_5 \psi \quad \{ \text{are scalars} \\
\bar{\psi} \gamma_{\mu} \psi \quad \{ \text{are vectors} \\
\bar{\psi} \gamma_{\mu} \gamma_5 \psi \quad \{ \text{are vectors} \\
\bar{\psi} \gamma_{\mu} \gamma_{\nu} \psi \quad \{ \text{are tensors} \\
\bar{\psi} \gamma_{\mu} \gamma_{\nu} \gamma_5 \psi \quad \{ \text{are tensors}
\end{align*}
\]

For reasons that will be given shortly the second members of each pair are also called pseudoscalar, pseudo (or axial) vector, and pseudotensor, respectively. Tensors of higher rank can be constructed similarly but are unnecessary because they can be expressed in terms of those already given. (Strictly speaking, the pseudotensor is also redundant in this way, but it is conveniently retained nevertheless.)

By multiplying the scalars and contracting the tensors we can then form the following 12 Lorentz invariant $\beta$-decay interactions:

\[
\left( \begin{array}{c}
\bar{\psi}_P \psi_N \\
\text{or} \quad \bar{\psi}_P \gamma_5 \psi_N \\
\end{array} \right) \left( \begin{array}{c}
\bar{\psi}_e \psi_{\nu} \\
\text{or} \quad \bar{\psi}_e \gamma_5 \psi_{\nu} \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\bar{\psi}_P \gamma_{\mu} \psi_N \\
\text{or} \quad \bar{\psi}_P \gamma_{\mu} \gamma_5 \psi_N \\
\end{array} \right) \left( \begin{array}{c}
\bar{\psi}_e \gamma_{\mu} \psi_{\nu} \\
\text{or} \quad \bar{\psi}_e \gamma_{\mu} \gamma_5 \psi_{\nu} \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\bar{\psi}_P \gamma_{\mu} \gamma_{\nu} \psi_N \\
\text{or} \quad \bar{\psi}_P \gamma_{\mu} \gamma_{\nu} \gamma_5 \psi_N \\
\end{array} \right) \left( \begin{array}{c}
\bar{\psi}_e \gamma_{\mu} \gamma_{\nu} \psi_{\nu} \\
\text{or} \quad \bar{\psi}_e \gamma_{\mu} \gamma_{\nu} \gamma_5 \psi_{\nu} \\
\end{array} \right)
\]

The general Lorentz invariant interaction is an arbitrary linear combination of these. By imposing Lorentz invariance, the original 256 unknown constants have been reduced to 12.
2. Parity

It was for long customary to reduce the possibilities still further by imposing symmetry with respect to inversion in space -- known also as conservation of parity. It might be thought that the inversion of a state \( \psi(\vec{r},t) \) is \( \psi(-\vec{r},t) \). However, if \( \psi(\vec{r},t) \) satisfies the Dirac equation, \( \psi(-\vec{r},t) \) does not -- it actually satisfies

\[
\left( \gamma_4 \frac{\partial}{\partial x_4} - \gamma_1 \frac{\partial}{\partial x_1} - \gamma_2 \frac{\partial}{\partial x_2} - \gamma_3 \frac{\partial}{\partial x_3} + m \right) \psi(-\vec{r},t) = 0.
\]

Multiplying on the left by \( \beta \) gives

\[
\beta \left( \gamma_4 \frac{\partial}{\partial x_4} - \gamma_1 \frac{\partial}{\partial x_1} - \gamma_2 \frac{\partial}{\partial x_2} - \gamma_3 \frac{\partial}{\partial x_3} + m \right) \psi(-\vec{r},t) = 0
\]

or on using the commutation rules

\[
\left( \gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \beta \psi(-\vec{r},t) = 0.
\]

Thus \( \beta \psi(-\vec{r},t) \) does satisfy the Dirac equation and can be called the inversion of the original state. Thus the inversion is given by the replacement

\[
\psi(\vec{r},t) \to \beta \psi(-\vec{r},t)
\]

and similarly

\[
\bar{\psi}(\vec{r},t) \to \bar{\psi}(-\vec{r},t) \beta.
\]

Consider now what happens to the quantities (16) when states are replaced by their inverses. For example

\[
\bar{\psi} \gamma_5 \psi \to \bar{\psi} \gamma_5 \beta \psi = -\bar{\psi} \psi
\]

\[
\bar{\psi} \gamma_5 \psi \to \bar{\psi} \gamma_5 \beta \gamma_5 \psi = \bar{\psi}(-\gamma_5)\psi = -\bar{\psi} \gamma_5 \psi.
\]

It is because of the change of sign here that the second quantity is called a pseudoscalar. In the same way the vectors and pseudovectors, tensors and pseudotensors, transform with different signs under inversion. Suppose now that we require the weak decay interaction to be invariant against inversion (apart, of course, from the change of argument from \( \vec{r} \) to \(-\vec{r}\), unimportant because \( \vec{r} \) is integrated.
over in the matrix element). Then we must discard those forms in (17) where \( y_5 \) is taken in one bracket and not the other—so reducing the number of possibilities by a factor 2.

However, parity conservation is now experimentally disproved and so we must retain all twelve.

III. V-A THEORY

Earlier we arrived at the general Lorentz invariant \( \beta \)-decay interaction. It was a combination of certain basic types and had unpleasantly many arbitrary coefficients. In the past numerous more or less unconvincing hypothetical principles have been proposed to pick out one combination or another as particularly worthy of consideration. Lately several such principles have been advanced which lead to what is known as the V-A combination. This at present appears to account adequately for much of the data. We will now expound the particular argument for this combination due to Feynman and Gell-Mann, approaching it by way of the 2-component neutrino theory of Salam, Landau, and Lee and Yang.

The 2-component neutrino theory consists simply in the representation of the neutrino not by a Dirac 4-component spinor but by a 2-component spinor satisfying the Weyl equation. This is possible because the neutrino has zero mass, and is in a sense the obviously simplest possibility to be considered. It would have been considered sooner were it not that the Weyl equation does not permit inversion (at least in an unsophisticated sense that we do not here elaborate on). With the Dirac equation, corresponding to any solution \( \psi(\hat{r},t) \) we saw that there is a second \( \beta \psi(-\hat{r},t) \). One naturally asks 'Can we not find some \( 2 \times 2 \) matrix, say \( \rho \), such that when \( \varphi(\hat{r},t) \) solves the Weyl equation, \( \rho \varphi(-\hat{r},t) \) does so also'. On trying this we find that \( \rho \) must anticommute with \( \sigma_x \), \( \sigma_y \), and \( \sigma_z \). Were this possible we would have four anticommuting \( 2 \times 2 \) matrices and it would not have been necessary to go to \( 4 \times 4 \) to obtain the Dirac equation.
with finite mass. So no such matrix exists. Thus if the Pauli-Weyl equation is used the theory is not symmetrical against inversion. When parity non-conservation was discovered, this feature was no longer repulsive, but very attractive.

Rather than make a fresh study of the Lorentz invariant combinations of three Dirac with one Weyl spinor, we will show how the imposition of certain restrictions on the interactions involving four Dirac spinors is equivalent to using a Weyl spinor for the neutrino. To do this we write the Dirac equation

$$\frac{\partial \psi}{\partial t} = -\sigma \cdot \frac{\partial}{\partial r} \psi - i \text{m} \beta \psi$$

in a new way. The matrices $\alpha$ and $\beta$ are conveniently expressed in terms of the $2 \times 2$ matrices $\sigma, 1$

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the 4-component $\psi$ is conveniently expressed in terms of two 2-component spinors $\psi^a$ and $\psi^b$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi^a_1 \\ \psi^a_2 \\ \psi^b_1 \\ \psi^b_2 \end{pmatrix} = \begin{pmatrix} \psi^a \\ \psi^b \end{pmatrix}.$$

Then

$$\frac{\partial \psi^a}{\partial t} = -\sigma \cdot \frac{\partial}{\partial r} \psi^b - i \text{m} \psi^a$$

$$\frac{\partial \psi^b}{\partial t} = -\sigma \cdot \frac{\partial}{\partial r} \psi^a + i \text{m} \psi^b.$$

With the notation

$$\varphi = \frac{i}{2} (\psi^a - \psi^b) \quad \chi = \frac{i}{2} (\psi^a + \psi^b)$$

we have

$$\frac{\partial \varphi}{\partial t} = \sigma \cdot \frac{\partial}{\partial r} \varphi - i \text{m} \chi$$

$$\frac{\partial \chi}{\partial t} = -\sigma \cdot \frac{\partial}{\partial r} \chi - i \varphi.$$
Thus in the case \( m = 0 \), the Dirac equation is seen to be equivalent to two independent Weyl equations.

A convenient way of expressing \( \psi \) in terms of \( \varphi \) and \( \chi \) is the following:

\[
\psi = \frac{1}{2} (1 + \gamma_5) \varphi + \frac{1}{2} (1 - \gamma_5) \chi .
\]

Remembering \( \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \) this becomes

\[
\psi = \frac{1}{2} \begin{pmatrix} \varphi_1 - \varphi_3 \\ \varphi_2 - \varphi_4 \\ \varphi_3 - \varphi_1 \\ \varphi_4 - \varphi_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \varphi_1 + \varphi_3 \\ \varphi_2 + \varphi_4 \\ \varphi_3 - \varphi_1 \\ \varphi_4 + \varphi_2 \end{pmatrix}.
\]

Thus \( \varphi \) corresponds to the part of \( \psi \) denoted by \( \frac{1}{2} (1 + \gamma_5) \psi \) and \( \chi \) to \( \frac{1}{2} (1 - \gamma_5) \psi \).

We recall that the general \( \beta \)-decay interaction was a combination of forms

\[
\bar{\psi}_e (1 \text{ or } \gamma_5) \psi_N \bar{\psi}_e (1 \text{ or } \gamma_5) \psi_e
\]

and similar forms involving vectors and tensors. We may equally well take the basic combinations as

\[
\bar{\psi}_p (1 \pm \gamma_5) \psi_N \bar{\psi}_e (1 \pm \gamma_5) \psi_e, \text{ etc.}
\]

Then we see that a simple way to make the theory equivalent to what would have been obtained using a Weyl neutrino from the outset is to permit only those interactions involving \( (1 + \gamma_5) \psi_e \), and not \( (1 - \gamma_5) \psi_e \), so involving only \( \varphi \) and not \( \chi \). The fact that a second Weyl spinor \( \chi \) is still formally present in addition to the one we want does not matter, as a particle which does not interact with anything is unobservable.
We could, in fact, put $\chi = 0$, but it is simpler and fully equivalent just to treat the neutrino uniformly with the other particles, except that only the combination $(1 + \gamma_5)\psi$ is permitted in interactions.

We could, of course, have retained only $(1 - \gamma_5)\psi$ instead of $(1 + \gamma_5)\psi$, that is, the Weyl spinor $\chi$ instead of $\varphi$. Theoretically these possibilities are equally good, and it is only experiment that eventually gives preference to that chosen here.

To discover the meaning of the combination $(1 + \gamma_5)\psi$, let us consider the explicit Dirac spinors

$$\psi_i = U_i e^{ik'r - i\omega t}$$

that we have already had. In the case $m = 0$, $\vec{k} = (0, 0, k)$ these are

$$U_{++} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad U_{+-} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

positive energy

$$U_{--} = \begin{pmatrix} -1 \\ 0 \\ +1 \\ 0 \end{pmatrix}, \quad U_{-+} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

negative energy

$$\frac{1}{2}(1 + \gamma_5)U$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & +1 & 0 \end{pmatrix}$$

positive energy

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

negative energy.
Thus for positive energy particles multiplication by \((1 + \gamma_5)\) picks out spinors representing particles with spin against the direction of motion (left-handed particles), and for negative energy with spin along the direction of motion (right-handed particles), and eliminates the others. Remembering that the antiparticle is a hole in an infinite negative energy background, one sees that the Pauli–Weyl spinor \(\psi\) describes only left-handed neutrinos and right-handed antineutrinos. This once again emphasizes the asymmetry of the theory under reflection.

For the reflection of a left-handed neutrino (thinking of spin in a classical way, \(\nu\) coming out of paper, left-handed: in mirror \(\nu\) right-handed \(\nu\) does not exist) would naturally be right-handed and so does not exist (or, equivalently, is described by \(\chi\) rather than \(\psi\), and so does not interact).

By insisting on the neutrino spinor occurring only in the combination \((1 + \gamma_5)\psi\), we have reduced the number of different interaction forms by a factor of two. It is very tempting to do something similar with the other particles, reducing the number of forms still further. Let us, we might say, take only those interactions involving \(\psi\) rather than \(\chi\) for every particle. On the face of it this is rather unnatural for, when \(m \neq 0\), \(\psi\) and \(\chi\) are not independent but remain coupled by the Dirac equation. It is the merit of Feynman and Gell-Mann to have invented a viewpoint from which what we want to do does seem natural. They observe that although \(\chi\) cannot reasonably be omitted from the theory, it can be expressed in terms of \(\psi\). From the first equation in (18)

\[
\chi = \frac{1}{im} \left( \vec{\sigma} \cdot \frac{\partial}{\partial r} \psi - \frac{\partial \psi}{\partial t} \right) \tag{19}
\]

and then the second equation in (18) becomes

\[
\frac{1}{im} \left( \vec{\sigma} \cdot \frac{\partial}{\partial r} \frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial t^2} \right) = \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial \psi}{\partial t} \left( \vec{\sigma} \cdot \frac{\partial}{\partial r} \right) - im \phi
\]

or

\[
\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m^2 \psi = 0. \tag{K.G. equation}
\]
Thus whenever $\chi$ appears in the theory it can be expressed in terms of $\varphi$ and its derivatives, and so the whole theory can be expressed in terms of a two-component wave function $\varphi$ which satisfies the Klein-Gordon equation. In particular the general $\beta$-decay interaction can be written in terms of four 2-spinors:

$$\psi^\dagger, \psi^\ast, \psi^\ast, \psi$$

(and their complex conjugates) and derivatives of these.

Now suppose that this 2-spinor formalism had been discovered first. Then in setting up possible interactions we would have regarded forms containing derivatives as a stage more complicated than those not containing them, and would have regarded the latter as worthy of prior consideration. Feynman and Gell-Mann suggest that we adopt this viewpoint.

We have the decomposition

$$\psi = \frac{1}{2}(1 + \gamma_5)\psi + \frac{1}{2}(1 - \gamma_5)\psi$$

$$= \begin{pmatrix} \varphi \\ -\varphi \end{pmatrix} + \begin{pmatrix} \chi \\ \chi \end{pmatrix}$$

and analogously

$$\bar{\psi} = \bar{\psi} \frac{1}{2}(1 + \gamma_5) + \bar{\psi} \frac{1}{2}(1 - \gamma_5)$$

$$= \begin{pmatrix} \chi^\ast \\ -\chi^\ast \end{pmatrix} + \begin{pmatrix} \varphi^\ast \\ \varphi^\ast \end{pmatrix}$$

In order to prevent $\chi, \chi^\ast$ (and therefore derivatives of $\varphi, \varphi^\ast$) occurring in the interaction we select those combinations which involve $\frac{1}{2}(1 + \gamma_5)\psi$ and $\bar{\psi} \frac{1}{2}(1 - \gamma_5)$ for each particle and not $\frac{1}{2}(1 + \gamma_5)\psi$ or $\bar{\psi} \frac{1}{2}(1 + \gamma_5)$. The possible forms are then

$$\bar{\psi}^\dagger (1 - \gamma_5) (1 + \gamma_5) \psi^\mu \bar{\psi}^\dagger (1 - \gamma_5) (1 + \gamma_5) \psi^\nu$$

$$\gamma_\mu \gamma_\nu \quad \gamma_\mu \gamma_\nu$$

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The combination arising from the scalars is identically zero since
\[(1 - \gamma_5)(1 + \gamma_5) = 1 - \gamma_5 + \gamma_5 - \gamma_5^2 = 0.\]
Similarly that from the tensors is zero. We are left with the unique form
\[
\bar{\psi}_\mu (1 - \gamma_5) \gamma_\mu (1 + \gamma_5) \psi_\nu (1 - \gamma_5) \gamma_\mu (1 + \gamma_5) \psi_\nu
\]

\[
= \bar{\psi}_\mu (1 + \gamma_5) \gamma_\mu (1 + \gamma_5) \psi_\nu (1 + \gamma_5) \gamma_\mu (1 + \gamma_5) \psi_\nu
\]

\[
= 4 \bar{\psi}_\mu \gamma_\mu (1 + \gamma_5) \psi_\nu (1 + \gamma_5) \psi_\nu.
\]
Introducing a constant $\sqrt{\hbar}$ $G$ to measure the strength of the interaction and letting
\[
a = \frac{1}{2}(1 + \gamma_5)
\]
we have the final unique interaction form
\[
8^{1/2} \sqrt{\hbar} G \bar{\psi}_\mu \gamma_\mu a \psi_\nu \gamma_\mu a \psi_\nu
\]  
(20)
where $G$ may be assumed real by suitably defining $\psi_\nu$. This is built up from the original forms involving vectors and pseudo (or axial) vectors and is referred to as the V-A combination (the minus sign arising from certain historic conventions).

There is left now only one undetermined constant $G$ to be fixed by experiment. For reasons that we will not go into, the $\beta$ decay rate of $^0\text{He}$ is very suitable for this purpose and yields
\[
G = (1.01 \pm 0.01) \times 10^{-5}
\]
in a system of units where the mass of the proton is unity.

Of the many detailed consequences of this interaction which are in agreement with experiment, we will mention only the simplest. It will be recalled that for particles of zero mass the retention of only $\varphi$ and not $\chi$ in the interaction meant the retention of only left-handed neutrinos and right-handed antineutrinos. Even when the mass
is not zero, a similar situation exists for very fast particles -- for
in factors like

\[ \frac{k}{\omega + m} \rightarrow \frac{k}{\omega} \rightarrow 1 \quad \text{for} \quad \omega \gg m \]

the mass is negligible. Thus fast electrons from β decay are fully
polarized in a direction opposite to their motion, and fast positrons
along their motion. This can be demonstrated in scattering experi-
ments with the β rays.

We recall that, including the β decay, we gave three weak
interactions

\[
N \rightarrow P + e + \bar{\nu} \\
\mu \rightarrow \nu + e + \bar{\nu} \\
N \rightarrow P + \mu + \bar{\nu}.
\]

In principle, these might be described by quite different interaction
forms. But if we believe that for all weak interactions the 2-spînîor
description of Dirac particles is the fundamental one, then we are led
to the same unique (V-A) interaction law for all three processes, and
it is tempting to assume also that the three coupling strengths are
identical. Then we have the interactions, writing them with two
particles on each side, rather than with an additional antiparticle
or one side:

\[
\begin{align*}
N + \nu & \rightarrow P + e \\
N + \nu & \rightarrow P + \mu \\
\mu + e & \rightarrow \nu + e
\end{align*}
\]

\[
\begin{align*}
\sqrt{G} \, \bar{\psi} \gamma_{\mu} \gamma_{\nu} a \psi \,, \quad & \sqrt{G} \, \bar{\psi} \gamma_{\mu} a \psi \,, \\
\sqrt{G} \, \bar{\psi} \gamma_{\mu} \gamma_{\nu} a \psi \,, \quad & \sqrt{G} \, \bar{\psi} a \psi \,, \\
\sqrt{G} \, \bar{\psi} \gamma_{\mu} \gamma_{\nu} a \psi \,, \quad & \sqrt{G} \, \bar{\psi} a \psi \,.
\end{align*}
\]

Having determined the constant G from β decay we are in a
position to calculate, for example, the lifetime of the \( \mu \) meson, governed
by the third process above. This was done by Feynman and Gell-Mann
with the result

\[ 2.26 \pm 0.04 \times 10^{-6} \text{ sec.} \]

The experimental value is 2.22 ± 0.02. This is remarkably good agree-
ment.
As well as the interactions above, we have those corresponding to the various inverse reactions:

\[
\begin{align*}
P + e &\rightarrow N + \nu \\
P + \mu &\rightarrow N + \nu \\
\nu + e &\rightarrow \mu + \nu
\end{align*}
\]

\[
\begin{align*}
\sqrt{8} \; G \; \overline{\psi}_N \; \gamma^\mu \; a \; \psi_P \; \overline{\psi}_\nu \; \gamma^\mu \; a \; \psi_e \\
\sqrt{8} \; G \; \overline{\psi}_N \; \gamma^\mu \; a \; \psi_P \; \overline{\psi}_\nu \; \gamma^\mu \; a \; \psi_\mu \\
\sqrt{8} \; G \; \overline{\psi}_\mu \; \gamma^\mu \; a \; \psi_\nu \; \overline{\psi}_\nu \; \gamma^\mu \; a \; \psi_e.
\end{align*}
\]

A very compact way of expressing these interactions is in terms of a 4-vector 'current':

\[
J_\mu = (\overline{\psi}_P \; \gamma^\mu \; a \; \psi_N + \overline{\psi}_\nu \; \gamma^\mu \; a \; \psi_\mu + \overline{\psi}_e \; \gamma^\mu \; a \; \psi_e)
\]

\[
J^*_\mu = -(\overline{\psi}_N \; \gamma^\mu \; a \; \psi_P + \overline{\psi}_\mu \; \gamma^\mu \; a \; \psi_\nu + \overline{\psi}_\nu \; \gamma^\mu \; a \; \psi_e)
\]

where $J^*_\mu$ is in fact the complex conjugate of $J_\mu$. Then all six interactions are contained in the form

\[
- \sqrt{8} \; G \; J_\mu \; J^*_\mu.
\]  \hspace{1cm} (21)

Moreover, this form suggests the possibility of further development in that it could be obtained if the four-fermion interaction were not direct but resulted from the interaction of each with a heavy boson---at present hypothetical. The $J_\mu \; J^*_\mu$ combination also contains three additional interactions representing mutual scattering of any two fermions. Where neutrinos are involved it would be difficult to observe; where only nucleons are involved it would be completely swamped by strong interaction effects which also cause scattering. It must also be borne in mind that the strong interactions of nucleons with $\pi$ mesons, photons, and other particles greatly complicate even the decay processes which are primarily due to weak interactions. They are in general difficult to allow for, and obscure the predictions of the theory. For example, the highly successful prediction of the $\mu$ decay rate (where strong interaction effects are absent) from an ordinary $\beta$ decay rate becomes very surprising indeed. Feynman and Gell-Mann have shown how this particular dilemma can be resolved by adding to
$J_{\mu}$ terms involving the meson wave functions which are precisely such as to cancel the 'renormalization' effects of mesons on the original terms. Apart from these terms added to suppress renormalization effects, the addition of still further terms involving strange particles may lead to a systematic explanation of the weak decay processes of these particles.

* * * *

Glossary for reading Feynman and Gell-Mann

The notation we have used is the most common one, and differs somewhat from that of Feynman and Gell-Mann.

<table>
<thead>
<tr>
<th>What we call</th>
<th>they call</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>$-i \gamma_x$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>$-i \gamma_y$</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>$-i \gamma_z$</td>
</tr>
<tr>
<td>$\gamma_4$</td>
<td>$\gamma_t$</td>
</tr>
<tr>
<td>$\gamma_5$</td>
<td>$i \gamma_5$</td>
</tr>
</tbody>
</table>

They introduce the abbreviation

$$\vec{\gamma} = \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} + \gamma_t \frac{\partial}{\partial t}$$

so that the Dirac equation is written

$$i \vec{\gamma} \psi = m \psi.$$

They also consider the presence of an external electromagnetic field specified by potentials (into the definition of which is absorbed the unit of charge e) $A_x$, $A_y$, $A_z$, $\varphi$. With the abbreviation

$$\vec{A} = \gamma_x A_x + \gamma_y A_y + \gamma_z A_z.$$

The Dirac equation is then

$$(i \vec{\gamma} - \vec{A})\psi = m \psi.$$

The reference is Phys. Rev. 102, 193 (1958).