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GENERALIZED SCHWARZIAN IN SEVERAL VARIABLES AND MÖBIUS INVARIANT DIFFERENTIAL OPERATORS

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ABSTRACT. We introduce certain Möbius invariant differential operators in several variables, which in many ways correctly generalize the classical Schwarzian derivative in one variable. We show that these generalized Schwarzians generate the entire set of Möbius invariant differential operators acting on local biholomorphisms. We then prove the composition formula for generalized Schwarzians from the point of view of connections in vector bundles. We also give the necessary and sufficient condition to prescribe generalized Schwarzians and give the formula for the solution.

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§1 INTRODUCTION AND SUMMARY OF RESULTS

The classical Schwarzian derivative $S[f]$ of a holomorphic function f in a single variable z with nonvanishing derivative on a domain $\Omega \subset \mathbb{C}$ is given by

$$S[f](z) = -\frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)' + \frac{1}{4} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

In the above, we deliberately chose the coefficient of the first term to be $-1/2$, and accordingly the coefficient of the second term is $1/4$. Various formulae come out nicely in this way. Note that if we let $\Delta(z) = d/dz \ln(f')^{-1/2} = -f''/(2f')$, then the Schwarzian derivative can be written as $S[f](z) = \Delta'(z) + \Delta(z)^2$. (See Corollary

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4-3 and its proof.) The major properties of the classical Schwarzian derivative are its Möbius invariance and the characterization of the projective linear maps:

$$\begin{aligned} S\left[\frac{af+b}{cf+d}\right](z) &= S[f](z), & \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), \\ S[f](z) = 0 &\iff f(z) = \frac{az+b}{cz+d}, & \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}). \end{aligned}$$

These properties can be easily checked by direct calculation. In this paper, we construct Möbius invariant differential operators in several variables generalizing the classical Schwarzian derivative $S[\cdot]$, and show that these operators have the right properties corresponding to the above. Furthermore we show that our generalized Schwarzians generate the entire set of Möbius invariant differential operators acting on local biholomorphisms, providing a new proof for one variable case [L]. The key ingredient is the several variable generalization of higher order Schwarzian operators initiated in [T1], where a certain integrality property of the Schwarzian derivative is also discussed. We give necessary and sufficient conditions to prescribe generalized Schwarzians. These conditions are vacuous for the one variable case.

In [T2], our investigation of generalized Schwarzians is continued for locally biholomorphic mappings between complex manifolds with projective structures. There, certain holomorphic flat connections play an important role. We will determine the field of differential invariants of projective structures on complex manifolds. The present paper deals with the local version of our theory.

To describe our generalized Schwarzians, let $\Omega \subset \mathbb{C}^n$ be a connected domain and let $z = (z_1, z_2, \dots, z_n) \in \Omega$ be a point in Ω . Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map from the domain Ω to the complex projective space $\mathbb{CP}^n$. For each multi-index $K \in \mathbb{Z}_+^n$, our generalized Schwarzian $S_K[f] : \Omega \rightarrow \mathbb{C}^{n+1}$ is constructed in terms of the local canonical lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ of f by the formula [Proposition 3-3]:

$$S_K[f](z) = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right)^{-1} \frac{\partial^{[K]} \tilde{f}}{\partial z^K}(z).$$

Here, a lift $\tilde{f}$ is called canonical if $\det(\tilde{f}, \partial \tilde{f} / \partial z_1, \dots, \partial \tilde{f} / \partial z_n) \equiv 1$. For example, the canonical lifts of projective linear maps are linear [Example 2-3]. Canonical lifts of local biholomorphisms always exist locally [Lemma 2-4], and the global existence is guaranteed if the domain Ω is simply connected [Proposition 2-5]. Although $S_K[f]$ is defined in terms of the local canonical lift $\tilde{f}$ of f which is determined up to $(n+1)$ st roots of unity, the generalized Schwarzian $S_K[f]$ is in fact independent of the choice of the local canonical lifts, as the notation suggests [Proposition 3-1]. For more details on canonical lifts, see §2.

Our generalized Schwarzians $S_K[f]$ are Möbius invariant in the sense that for any Möbius (=projective linear) transformation $T \in \mathrm{PSL}_{n+1}(\mathbb{C})$ and local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$, we have

$$S_K[T \circ f] = S_K[f] : \Omega \rightarrow \mathbb{C}^{n+1}.$$

Let $S_K[f](z) = {}^t(S_K^{(0)}[f](z), S_K^{(1)}[f](z), \dots, S_K^{(n)}[f](z)) \in \mathbb{C}^{n+1}$. Then, each component $S_K^{(\ell)}[f] : \Omega \rightarrow \mathbb{C}$ for $0 \leq \ell \leq n$ is Möbius invariant [Proposition 3-3, Corollary 3-4]. We view our Schwarzians $S_K[f]$ as having values in column vectors rather than in row vectors, so that we can multiply matrices from the left.

For a multi-index $K = (k_1, k_2, \dots, k_n)$, let $|K| = k_1 + \dots + k_n$. For $1 \leq i \leq n$, let σ_i be a row vector given by $\sigma_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}_+^n$ with 1 at the i -th entry and 0 at everywhere else. Let $e_j \in \mathbb{C}^{n+1}$ be a column vector with 1 at the j -th entry and 0 at all the other entries. Then from the definition $S_0[f] = e_1$, $S_{\sigma_i}[f] = e_{i+1}$ for $1 \leq i \leq n$. So when $|K| = 1$, we don't obtain anything interesting. When $|K| = 2$, generalized Schwarzians $S_{\sigma_i + \sigma_j}[f]$ denoted by $S_{ij}[f]$ for $1 \leq i, j \leq n$ are called fundamental Schwarzians. These fundamental Schwarzians are indeed the generalization of the classical Schwarzian derivative in the single variable case and they play a fundamental role in our theory. One of the main properties of the fundamental Schwarzians is that they characterize projective linear maps.

Theorem 1 [Theorem 3-10] (Characterization of Projective Linear Maps by Fundamental Schwarzians). *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map from a domain $\Omega \subset \mathbb{C}^n$ to the projective space. Then,*

$$S_{ij}[f](z) = 0, \text{ for all } 1 \leq i, j \leq n \iff f \text{ is projective linear.}$$

For the definition of projective linear maps, see (2-3). We also show that the vanishing of higher Schwarzians imply that the lift $\tilde{f}$ is polynomial [Proposition 3-9]. For the single variable case, $n = 1$, the higher Schwarzians also characterize projective linear maps.

Theorem 2 [Theorem 3-12]. *Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \rightarrow \mathbb{CP}^1$ be a holomorphic map such that $f'(z)$ doesn't vanish on Ω . Then the following statements are equivalent:*

- (1) $S_2[f] = 0$,
- (2) $S_{r+1}[f] = 0$ for some $r \geq 1$,
- (3) the map f is projective linear.

See Example 4-4 for some explicit expressions of the higher Schwarzians in the single variable case. From these expressions as differential operators, it is far from clear that the kernel space of these $\mathbb{C}^2$ -valued higher Schwarzian operators $S_r[\cdot]$ are all the same for the case $n = 1$. The above stronger statement in Theorem 2 is not true for the several variables case. See Example 3-11 for a counter example in the several variables case.

The generalized Schwarzian operator $S_K[\cdot]$ is a homogeneous differential operator in the sense that if we choose a local coordinate (U, φ) on $\mathbb{CP}^n$, then each component of $S_K[f]$ can be expressed as a nonlinear homogeneous differential operator [Definition 4-5] of virtual order $|K|$ or $|K| - 1$ acting on a $\mathbb{C}^n$ -valued function $\varphi \circ f$ [see Proposition 4-7]. For the meaning of differential operators acting on maps to manifolds, see §4 for details.

To give an explicit expression of the fundamental Schwarzians $S_{ij}[f]$ as partial differential operators, we use an inclusion map $\mathbb{C}^n \subset \mathbb{CP}^n$ which identifies $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ with $[1 : w_1 : \dots : w_n] \in \mathbb{CP}^n$ as a local coordinate neighborhood of $\mathbb{CP}^n$.

Proposition 3 [Proposition 4-2, Lemma 4-6] (Fundamental Schwarzians as Differential Operators). *Suppose a locally biholomorphic map $f : \Omega \rightarrow \mathbb{C}^n \subset \mathbb{CP}^n$ from a domain $\Omega \subset \mathbb{C}^n$ is given by $f(z) = {}^t(f_1(z), f_2(z), \dots, f_n(z))$. Then the components of the fundamental Schwarzian $S_{ij}[f] : \Omega \rightarrow \mathbb{C}^{n+1}$ as differential operators are given by*

$$S_{ij}^{(0)}[f](z) = \Delta_{ij}(z) - \Delta_i(z)\Delta_j(z) - \sum_{1 \leq k, \ell \leq n} \Delta_\ell(z)\zeta_{\ell k}(z) \frac{\partial^2 f_k}{\partial z_i \partial z_j}(z),$$

$$S_{ij}^{(\ell)}[f](z) = \sum_{k=1}^n \zeta_{\ell k}(z) \frac{\partial^2 f_k}{\partial z_i \partial z_j}(z) + \delta_{\ell j} \Delta_i(z) + \delta_{\ell i} \Delta_j(z), \quad 1 \leq \ell \leq n.$$

Here $\zeta_{k\ell}$ is the (k, ℓ) matrix component of the inverse matrix $(\partial f_i / \partial z_j)^{-1}$ and

$$\Delta_i(z) = \frac{-1}{n+1} \frac{\partial}{\partial z_i} \log J_f(z), \quad \Delta_{ij}(z) = \frac{\partial \Delta_i(z)}{\partial z_j} = \frac{\partial \Delta_j(z)}{\partial z_i},$$

where $J_f(z)$ is the Jacobian determinant of $f : \Omega \rightarrow \mathbb{C}^n$. The Schwarzian operator $S_{ij}^{(0)}[\cdot]$ is homogeneous of virtual order 2, and $S_{ij}^{(\ell)}[\cdot]$ for $1 \leq \ell \leq n$ is homogeneous of virtual order 1.

When $n = 1$, the only fundamental Schwarzian is $S_2[f]$ and it is given by

$$S_2[f](z) = \begin{pmatrix} S[f](z) \\ 0 \end{pmatrix},$$

where $S[f]$ is the virtual order 2 homogeneous classical Schwarzian derivative of a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ with nonvanishing derivative.

In general, for higher Schwarzian operators $S_K[\cdot]$, its 0-th component $S_K^{(0)}[f]$ is a homogeneous differential operator of virtual order $|K|$, and other components $S_K^{(\ell)}[\cdot]$ for $1 \leq \ell \leq n$ are homogeneous of virtual order $|K| - 1$ [Proposition 4-7].

We show that the higher Schwarzians $S_K[f](z)$ with $|K| \geq 3$ can be expressed as *integral* polynomials in the components $S_{ij}^{(\ell)}[f]$ and their derivatives. To describe these polynomials, let

$$A_i[f](z) = (e_{i+1}, S_{i1}[f](z), S_{i2}[f](z), \dots, S_{in}[f](z)), \quad 1 \leq i \leq n,$$

be an $M_{n+1}(\mathbb{C})$ -valued holomorphic function in $z = (z_1, \dots, z_n) \in \Omega$ consisting of the fundamental Schwarzians and let

$$\mathcal{A}[f](z) = \sum_{i=1}^n A_i[f](z) dz_i$$

be an $M_{n+1}(\mathbb{C})$ -valued differential 1-form on the domain Ω . We also consider differential operators $D_i[f]$ for $1 \leq i \leq n$ acting on $\mathbb{C}^{n+1}$ -valued holomorphic functions (or $M_{n+1}(\mathbb{C})$ -valued holomorphic functions) in $z \in \Omega$ defined by

$$D_i[f] = D_i^z[f] = \frac{\partial}{\partial z_i} + A_i[f](z) \cdot, \quad 1 \leq i \leq n.$$

Here the dot “ $\cdot$ ” in the second term in the right hand side denotes matrix multiplication from the left. We use a superscript z when it is necessary to indicate the variables we are dealing with to avoid confusion. The next theorem describes $S_K[f]$ with $|K| \geq 3$ in terms of $S_{ij}[f]$.

Theorem 4 [Theorem 3-7] (Integrability Conditions and Higher Order Schwarzians). *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be any locally biholomorphic map. The following statements hold.*

- (1) *The first order differential operators $D_i[f]$ mutually commute:*

$$[D_i[f], D_j[f]] = 0, \quad 1 \leq i, j \leq n.$$

Equivalently, the total covariant derivative $D = d + \mathcal{A}[f]$ has vanishing curvature $D^2 = 0$. In other words, as $M_{n+1}(\mathbb{C})$ -valued 2-forms on Ω , we have

$$d\mathcal{A}[f] + \mathcal{A}[f]^2 = 0.$$

- (2) *$\mathcal{A}[f]$ is an $\mathfrak{sl}_{n+1}(\mathbb{C})$ -valued 1-form on Ω :*

$$\text{Tr } \mathcal{A}[f] = 0.$$

- (3) *For each multi-index $K = (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$, $S_K[f]$ is an integral polynomial in the components $S_{ij}^{(\ell)}[f]$ of the fundamental Schwarzians and their derivatives, where $1 \leq i, j \leq n$, $0 \leq \ell \leq n$. More precisely, we have*

$$S_K[f](z) = (D^K[f]p_0)(z),$$

where $p_0 = {}^t(1, 0, \dots, 0) \in \mathbb{C}^{n+1}$ and $D^K[f] = D_1[f]^{k_1} \dots D_n[f]^{k_n}$.

Note that the statement (1) above is vacuous when $n = 1$ because there is only one operator $D_1[f]$ then and it commutes with itself. Also note that the notation $D^K[f]$ in (3) makes sense and well defined since operators $D_i[f]$ commute due to (1). The conditions (1) and (2) further imply the following identity.

Theorem 5 [Theorem 3-8]. *Let $n \geq 2$. With the same notations as above,*

$$(n-1)S_{ij}^{(0)}[f](z) = - \sum_{\ell=1}^n \frac{\partial S_{ij}^{(\ell)}[f]}{\partial z_\ell}(z) + \sum_{k,\ell=1}^n S_{ik}^{(\ell)}[f](z) \cdot S_{j\ell}^{(k)}[f](z).$$

Remark. When $n = 1$, this statement is vacuous, because $S_2^{(1)}[f] = 0$ due to the last statement in Proposition 3 above.

From Proposition 3, we know that the 0-th component of the fundamental Schwarzian $S_{ij}^{(0)}[\cdot]$ is homogeneous of virtual order 2, and other components $S_{ij}^{(\ell)}[\cdot]$ are homogeneous of virtual order 1. Theorem 5 reflects this fact, expressing $S_{ij}^{(0)}[\cdot]$ in terms of first order derivatives and quadratic terms of $S_{ij}^{(\ell)}[\cdot]$ for $1 \leq \ell \leq n$.

The classical Schwarzian derivative $S[f]$ plays a fundamental role when $n = 1$. A similar role is played by the virtual order 1 components of fundamental Schwarzians $S_{ij}^{(\ell)}[f]$, where $1 \leq i, j, \ell \leq n$. In fact, these operators generate the entire set of Möbius invariant differential operators acting on the space of locally biholomorphic maps $f : \Omega \rightarrow \mathbb{CP}^n$, which is the content of the next theorem.

Theorem 6 [Theorem 5-2] (Generating Möbius invariant operators). *Let $I[f] : \Omega \rightarrow \mathbb{C}$ be an arbitrary $\mathbb{C}$ -valued Möbius invariant differential operator acting on locally biholomorphic maps $f : \Omega \rightarrow \mathbb{CP}^n$.*

- (1) *When $n = 1$, the invariant operator $I[f]$ depends only on the derivatives of the classical Schwarzian derivative $S[f]$.*
- (2) *When $n \geq 2$, the invariant operator $I[f]$ depends only on the virtual order 1 fundamental Schwarzians $S_{ij}^{(\ell)}[f]$ and their derivatives, where $1 \leq \ell, i, j \leq n$.*

Part (1) of this result was first proved by M. Lavie [L]. Our method takes care of these two cases at the same time. Although the generalized Schwarzians are invariant when a local biholomorphism f is post-composed by Möbius transformations, they are not invariant in general when f is pre-composed by Möbius transformations. (If they were invariant, they would be constants!) This is just as in the case of the classical Schwarzian derivative $S[f]$ in a single variable. In this case, we have the following formula for composable holomorphic functions f and g with nonvanishing derivatives:

$$S[f \circ g](z) = (S[f] \circ g)(z) \cdot g'(z)^2 + S[g](z).$$

A similar formula holds for our generalized Schwarzians. Although the above formula for the classical Schwarzian derivative looks rather mysterious, its meaning becomes more transparent in the geometric context of covariant derivatives in which we formulate our result in §6.

To describe our result, let $f : \Omega \rightarrow \mathbb{CP}^n$, $g : \Omega' \rightarrow \Omega$ be local biholomorphisms, where Ω, Ω' are domains in $\mathbb{C}^n$. Let $z \in \Omega$, $\zeta \in \Omega'$ be variables in these domains. To maps f and $f \circ g$, we associate total covariant derivatives

$$\begin{aligned} D^z[f] &= \sum_{i=1}^n dz_i \cdot D_i^z[f] = d_z + \mathcal{A}[f](z) \cdot, \\ D^\zeta[f \circ g] &= \sum_{j=1}^n d\zeta_j \cdot D_j^\zeta[f \circ g] = d_\zeta + \mathcal{A}[f \circ g](\zeta) \cdot \end{aligned}$$

acting on sections of trivial bundles over Ω and Ω' with the fibre $M_{n+1}(\mathbb{C})$ or $\mathbb{C}^{n+1}$. Let $W_g(\zeta)$ be an $M_{n+1}(\mathbb{C})$ -valued holomorphic function on Ω' given by

$$W_g(\zeta) = \begin{pmatrix} 1 & & & & \\ -z_1 & 1 & & & \\ -z_2 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -z_n & 0 & 0 & \dots & 1 \end{pmatrix} \left(\tilde{g}(\zeta), \frac{\partial \tilde{g}}{\partial \zeta_1}(\zeta), \frac{\partial \tilde{g}}{\partial \zeta_2}(\zeta), \dots, \frac{\partial \tilde{g}}{\partial \zeta_n}(\zeta) \right),$$

where $z = g(\zeta)$, and $\tilde{g}$ is the canonical lift of g when g is regarded as a map into $\Omega' \subset \mathbb{CP}^n$. Note that the matrix $W_g(\zeta)$ depends only on g , not on f .

Theorem 7 [Theorem 6-1] (The Composition Formula for Fundamental Schwarzians). *With the notations as above, we have*

$$D^\zeta[f \circ g] = W_g(\zeta)^{-1} \cdot g^*(D^z[f]) \cdot W_g(\zeta).$$

In other words, the covariant derivative $D[f \circ g]$ (which contains all information on the generalized Schwarzians for the local biholomorphism $f \circ g$) is given by conjugating the pull back connection $g^*(D^z[f])$ by $W_g(\zeta)$. In the single variable case, the composition formula in Theorem 7 reduces to the above composition formula for the classical Schwarzian derivative [Example 6-4]. When $f = T \in \text{PSL}_{n+1}(\mathbb{C})$ is a Möbius transformation, the above composition formula becomes

$$D^\zeta[T \circ g] = D^\zeta[g]$$

showing the Möbius invariance of the total covariant derivative associated to local biholomorphisms, and hence the Möbius invariance of the fundamental Schwarzians [Example 6-5].

Next, we consider the problem of prescribing the fundamental Schwarzians. Suppose we are given a set of $\mathbb{C}^{n+1}$ -valued holomorphic functions $p_{ij}(z)$ defined on a domain $\Omega \subset \mathbb{C}^n$ for $1 \leq i, j \leq n$. We consider the following system of partial differential equations for an unknown locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$:

$$S_{ij}[f](z) = p_{ij}(z), \quad 1 \leq i, j \leq n.$$

For an arbitrarily given set of $\mathbb{C}^{n+1}$ -valued functions $p_{ij}(z)$, the above system may not have a solution. For the existence of a solution, certain compatibility conditions among the $p_{ij}(z)$'s must be satisfied. To describe these conditions, let

$$\mathcal{A}(z) = \sum_{i=1}^n A_i(z) dz_i, \quad \text{where } A_i = (e_{i+1}, p_{i1}(z), p_{i2}(z), \dots, p_{in}(z))$$

be an $M_{n+1}(\mathbb{C})$ -valued differential 1-form on Ω constructed from the $p_{ij}(z)$'s. We also define first order differential operators D_i out of the $p_{ij}(z)$'s by

$$D_i = \frac{\partial}{\partial z_i} + A_i(z) \cdot, \quad 1 \leq i \leq n.$$

Again, the “ $\cdot$ ” denotes matrix multiplication of $A_i(z)$ from the left.

Theorem 8 [Theorem 7-3] (Prescribing generalized Schwarzians). *Assume that $\Omega \subset \mathbb{C}^n$ be a simply connected domain and let $p_{ij}(z)$ be a given $\mathbb{C}^{n+1}$ -valued holomorphic function defined on the domain Ω for $1 \leq i, j \leq n$. Then a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$ satisfying*

$$S_{ij}[f] = p_{ij}(z), \quad 1 \leq i, j \leq n,$$

exists if and only if $\mathcal{A}$ satisfies

$$d\mathcal{A} + \mathcal{A}^2 = 0, \quad \text{Tr } \mathcal{A} = 0.$$

Equivalently, one can prescribe Schwarzians for given $p_{ij}(z)$'s if and only if

$$[D_i, D_j] = 0, \quad A_i(z) \in \mathfrak{sl}_{n+1}(\mathbb{C}).$$

When either of the above conditions are satisfied and consequently solutions to the Schwarzian equation exist, then any two solutions $f_1, f_2 : \Omega \rightarrow \mathbb{CP}^n$ are related by a Möbius transformation, i.e., $f_1(z) = T \circ f_2(z)$ for some $T \in \text{PSL}_{n+1}(\mathbb{C})$,

For any matrix $T_0 \in \text{SL}_{n+1}(\mathbb{C})$ and any point $z_0 \in \Omega$, there exists a unique solution $f : \Omega \rightarrow \mathbb{CP}^n$ whose canonical lift $\tilde{f}$ is such that

$$\left(\tilde{f}(z_0), \frac{\partial \tilde{f}}{\partial z_1}(z_0), \frac{\partial \tilde{f}}{\partial z_2}(z_0), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z_0) \right) = T_0 \in \text{SL}_{n+1}(\mathbb{C})$$

The Taylor series expansion of the canonical lift $\tilde{f}$ of the solution f around $z = z_0$ is given by the formula

$$\tilde{f}(z) = T_0 \cdot \left\{ \sum_{K \in \mathbb{Z}_+^n} \frac{(z - z_0)^K}{K!} (D^K p_0)(z_0) \right\}.$$

Here $p_0 = {}^t(1, 0, \dots, 0)$ is a constant column vector.

Conversely, the canonical lift $\tilde{f}$ of any solution $f : \Omega \rightarrow \mathbb{CP}^n$ to the Schwarzian equation is of the above form for some $T_0 \in \text{SL}_{n+1}(\mathbb{C})$.

In Example 7-4, we give a concrete application of Theorem 8 in the single variable case. Finally, we discuss the meaning and properties of the power series appearing in the above formula [Proposition 7-5].

§2 CANONICAL LIFTS OF LOCALLY BIHOLOMORPHIC MAPS

Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$ be the canonical projection map. A lift $\tilde{f}$ of $f : \Omega \rightarrow \mathbb{CP}^n$ is a holomorphic map $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ such that $\pi \circ \tilde{f} = f$. Locally, there can be many lifts, but a global lift on Ω may not exist. We will show that if the domain Ω is simply connected, then global lifts always exist.

Lemma 2-1. *Suppose that a lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ of a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$ exists on Ω . Then, as vectors in $\mathbb{C}^{n+1}$,*

$$\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \frac{\partial \tilde{f}}{\partial z_2}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z),$$

are linearly independent for each $z \in \Omega$.

Proof. At any $z_0 \in \Omega$, we show that the above vectors are linearly independent. Let the lift $\tilde{f}$ be given by $\tilde{f}(z) = {}^t(\tilde{f}_0(z), \tilde{f}_1(z), \dots, \tilde{f}_n(z))$. Since $\tilde{f}(z_0) \in \mathbb{C}^{n+1} \setminus \{0\}$, we have $\tilde{f}_i(z_0) \neq 0$ for some $0 \leq i \leq n$. We may assume that $i = 0$. Then, near $z = z_0$, we can write $\pi(\tilde{f}(z)) = [1 : \tilde{f}_1(z)/\tilde{f}_0(z) : \dots : \tilde{f}_n(z)/\tilde{f}_0(z)]$ in homogeneous coordinates. Let $f_j(z) = \tilde{f}_j(z)/\tilde{f}_0(z)$. Since $f : \Omega \rightarrow \mathbb{CP}^n$ is a local biholomorphism, the Jacobian determinant $J_f(z) = \partial(f_1, \dots, f_n)/\partial(z_1, \dots, z_n)$ doesn't vanish near z_0 . We now observe that

$$(2-1) \quad \det \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) = \tilde{f}_0(z)^{n+1} \cdot J_f(z).$$

Since both $\tilde{f}_0(z)$ and $J_f(z)$ don't vanish near $z = z_0$, we see that the vectors in Lemma 2-1 are linearly independent near $z = z_0$. Since $z_0 \in \Omega$ can be arbitrary, we conclude that the above vectors are linearly independent at any point $z \in \Omega$. $\square$

Among all the possible lifts of a local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$, we choose a canonical one.

Definition 2-2 (Canonical Lifts). Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. A canonical lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ of f is a lift of f satisfying

$$(2-2) \quad \det \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \frac{\partial \tilde{f}}{\partial z_2}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) \equiv 1, \quad \text{on } \Omega.$$

Remark. We will consider differential invariants of the map $f : \Omega \rightarrow \mathbb{CP}^n$ via the $\mathbb{C}^{n+1}$ -valued canonical lift. One cannot differentiate a map $f : \Omega \rightarrow \mathbb{CP}^n$ to obtain a map from Ω to $\mathbb{CP}^n$. But differentiation of its $\mathbb{C}^{n+1}$ -valued canonical lift makes sense and is again $\mathbb{C}^{n+1}$ -valued. Since the canonical lift $\tilde{f}$ contains all the information of the original map f , (we can recover f from its canonical lift $\tilde{f}$), we can define differential invariants $S_K[f]$ of the map $f : \Omega \rightarrow \mathbb{CP}^n$ via differentiation of its canonical lift $\tilde{f}$.

Example 2-3. We show that the canonical lift of a projective linear map is a special linear map. Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a projective linear map. In homogeneous coordinates of $\mathbb{CP}^n$, the map f can be represented by

$$(2-3) \quad f(z) = \left[\sum_{j=1}^n a_{1j} z_j + b_1 : \dots : \sum_{j=1}^n a_{nj} z_j + b_n : \sum_{j=1}^n c_j z_j + d \right].$$

Multiplying by a constant if necessary, we may assume that

$$(2-4) \quad \det \begin{pmatrix} b_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n1} & \dots & a_{nn} \\ d & c_1 & \dots & c_n \end{pmatrix} = 1$$

Let a column vector valued linear function $F : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be given by

$$(2-5) \quad F(z) = {}^t \left(\sum_{j=1}^n a_{1j} z_j + b_1, \dots, \sum_{j=1}^n a_{nj} z_j + b_n, \sum_{j=1}^n c_j z_j + d \right).$$

We check that the determinant condition (2-2) for this map $F(z)$ is satisfied:

$$\det \left(F, \frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n} \right) = \begin{pmatrix} \sum_{j=1}^n a_{1j} z_j + b_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n a_{nj} z_j + b_n & a_{n1} & \dots & a_{nn} \\ \sum_{j=1}^n c_j z_j + d & c_1 & \dots & c_n \end{pmatrix} = 1,$$

by (2-4). Hence the linear function $F : \Omega \rightarrow \mathbb{C}^{n+1}$ is the canonical lift $\tilde{f}$ of the projective linear map f . By Lemma 2-4 below, all the canonical lifts are obtained by multiplying by $(n+1)$ st roots of unity.

As for the existence of canonical lifts, we first show that canonical lifts always exist locally.

Lemma 2-4. *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. Then, for each point $z \in \Omega$, there exists an open set $U_z \subset \Omega$ such that a canonical lift $\tilde{f} : U_z \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ of $f|_{U_z}$ exists. Such a canonical lift of f is unique up to a constant multiple of an $(n+1)$ st root of unity.*

Proof. Let $z_0 \in \Omega$ be an arbitrary point in Ω . Let $f(z_0) = [w'_0 : w'_1 : \cdots : w'_n] \in \mathbb{CP}^n$ for some $(w'_0, w'_1, \dots, w'_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. So, $w'_i \neq 0$ for some $1 \leq i \leq n$. Let

$$U_i = \{[w_0 : w_1 : \cdots : w_n] \in \mathbb{CP}^n \mid w_i \neq 0\} \cong \mathbb{C}^n.$$

Let $z_0 \in U_{z_0} \subset \Omega \cap f^{-1}(U_i)$ be a simply connected small open neighborhood of z_0 . By restriction, we have a map $f : U_{z_0} \rightarrow \mathbb{C}^n \cong U_i \subset \mathbb{CP}^n$. Let f be given by $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ as a map into $\mathbb{C}^n$ for $z \in U_{z_0}$. A canonical lift $\tilde{f}$ has the form

$$\tilde{f}(z) = (\lambda(z)f_1(z), \dots, \lambda(z)f_i(z), \lambda(z), \lambda(z)f_{i+1}(z), \dots, \lambda(z)f_n(z))$$

for some nonvanishing holomorphic function $\lambda : U_{z_0} \rightarrow \mathbb{C} \setminus \{0\}$. The determinant condition of the canonical lift $\det(\tilde{f}(z), \partial \tilde{f} / \partial z_1(z), \dots, \partial \tilde{f} / \partial z_n(z)) \equiv 1$ implies that

$$\lambda(z)^{n+1}(z) \cdot J_f(z) \equiv 1, \quad \text{where } J_f(z) = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(z_1, z_2, \dots, z_n)}$$

is the Jacobian determinant which doesn't vanish on $\Omega \cap f^{-1}(U_i)$ because f is a local biholomorphism. Since U_i is simply connected, an $(n+1)$ st root of $J_f(z)^{-1}$ is a well defined holomorphic function on U_{z_0} . Thus, we obtain $(n+1)$ different canonical lifts of f over U_{z_0} , any two of which are related by a constant multiple of an $(n+1)$ st root of unity. $\square$

Next, we discuss the question of global existence of canonical lifts on Ω . For this, we assume that the domain Ω is simply connected.

Proposition 2-5. *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. If $\Omega \subset \mathbb{C}^n$ is a simply connected domain, then there exists a globally defined canonical lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$, unique up to multiplication by $(n+1)$ st roots of unity.*

Proof. By Lemma 2-4, locally there exist $(n+1)$ different canonical lifts; any one of them can be obtained from any other by multiplication of an $(n+1)$ st root of unity. These local lifts form an $(n+1)$ -fold covering space $\tilde{\Omega}$ over Ω , where $\tilde{\Omega} \subset \Omega \times (\mathbb{C}^{n+1} \setminus \{0\})$ consists of pairs $(z, \tilde{f}(z))$ for $z \in \Omega$ and for all local canonical lifts $\tilde{f}$ around z . The projection map $\tilde{\Omega} \rightarrow \Omega$ is induced by the projection map $\Omega \times (\mathbb{C}^{n+1} \setminus \{0\}) \rightarrow \Omega$ onto the first factor. Thus, we have a $(n+1)$ -fold covering space over Ω . This covering space must be trivial since the domain Ω is simply connected by our hypothesis. Hence each sheet of this trivial covering defines a global canonical lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$. Different sheets are related by multiplication by $(n+1)$ st roots of unity. $\square$

We shall simply refer to the canonical lift with the understanding that it is defined up to roots of unity.

As is well known, the automorphism group of $\mathbb{CP}^n$ is $\mathrm{PSL}_{n+1}(\mathbb{C})$. For any automorphism $T : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$, there exist $(n+1)$ special linear transformations $\tilde{T} \in \mathrm{SL}_{n+1}(\mathbb{C})$, determined up to $(n+1)$ st roots of unity, covering T :

$$(2-6) \quad \begin{array}{ccc} \mathbb{C}^{n+1} \setminus \{0\} & \xrightarrow{\tilde{T}} & \mathbb{C}^{n+1} \setminus \{0\} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{CP}^n & \xrightarrow{T} & \mathbb{CP}^n \end{array}$$

From the point of view of the present paper, projective linear maps $T \in \mathrm{PSL}_{n+1}(\mathbb{C})$ of $\mathbb{CP}^n$ are regarded “trivial” which our differential invariants to be constructed below do not see. Our generalized Schwarzians $S_K[f]$, $K \in \mathbb{Z}_+^n$, for local biholomorphisms $f : \Omega \rightarrow \mathbb{CP}^n$ capture the complete information of the “non-projective-linear” part of the map f .

To prepare for the construction of our differential Möbius invariant $S_K[\cdot]$ for each multi-index $K \in \mathbb{Z}_+^n$, let $f_w(z) = f(w+z)$ for $w \in \Omega$, and we look at the linear part of the canonical lift of f_w defined on the shifted domain $\Omega_w = \Omega - w \subset \mathbb{C}^n$. Note that if $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ is the canonical lift of f , then the canonical lift of f_w is given by $\tilde{f}_w(z) = \tilde{f}(w+z)$. The Taylor expansion of $\tilde{f}(z)$ around $z = w \in \Omega$ is the same as the Taylor expansion of $\tilde{f}_w(z)$ around $z = 0 \in \Omega_w$. Its linear part denoted by $L[\tilde{f}_w] : \Omega_w \rightarrow \mathbb{C}^{n+1}$ is given by

$$\begin{aligned} L[\tilde{f}_w](z) &= \tilde{f}_w(0) + \sum_{i=1}^n \frac{\partial \tilde{f}_w}{\partial z_i}(0) \cdot z_i \\ &= \left(\tilde{f}(w), \frac{\partial \tilde{f}(w)}{\partial z_1}, \dots, \frac{\partial \tilde{f}(w)}{\partial z_n} \right) \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix}. \end{aligned}$$

The second equality follows by differentiating $\tilde{f}_w(z) = \tilde{f}(z+w)$ at $z = 0$. The coefficient matrix

$$(2-7) \quad M_{\tilde{f}_w} \equiv \left(\tilde{f}(w), \frac{\partial \tilde{f}(w)}{\partial z_1}, \dots, \frac{\partial \tilde{f}(w)}{\partial z_n} \right) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$$

is called the *matrix of linear approximation* of $\tilde{f}(z)$ at $z = w$. From the definition,

$$(2-8) \quad L[\tilde{f}_w](z) = M_{\tilde{f}_w} \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

Due to the determinant condition on the canonical lift $\tilde{f}(z)$, the matrix of linear approximation $M_{\tilde{f}_w}$ has determinant 1, that is, $M_{\tilde{f}_w} \in \mathrm{SL}_{n+1}(\mathbb{C})$, for any $w \in \Omega$. The transformation of $\mathbb{CP}^n$ induced from $M_{\tilde{f}_w} \in \mathrm{SL}_{n+1}(\mathbb{C})$ in the sense of (2-6) is

denoted by M_{f_w} and is called the *canonical Möbius approximation* of $f(z)$ at $z = w$. Note that

$$(2-9) \quad M_{f_w} = \pi \circ L[\tilde{f}_w] : \Omega_w \rightarrow \mathbb{CP}^n$$

is the “best” projective linear approximation of f at $z = w \in \Omega$. Also note that in the context of the maps into the projective space $\mathbb{CP}^n$, the usual notion of linear approximation of maps is not defined canonically. Instead, the notion of projective linear approximation plays the corresponding role. Observe that M_{f_w} depends only on f as the notation suggests, not on the choice of its canonical lift $\tilde{f}$ through which M_{f_w} was defined. We list several properties of the objects we defined using the canonical lift.

Proposition 2-6. *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map and let $T : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ be any automorphism of $\mathbb{CP}^n$, namely, $T \in \mathrm{PSL}_{n+1}(\mathbb{C})$. For the canonical lift $\tilde{f}$ and $\tilde{T} \in \mathrm{SL}_{n+1}(\mathbb{C})$ covering T , the following statements hold.*

- (i) *The map $\tilde{T} \circ \tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ is the canonical lift of $T \circ f : \Omega \rightarrow \mathbb{CP}^n$.*
- (ii) *The linear approximation of the canonical lift $\tilde{T} \circ \tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1}$ at $w \in \Omega$ is given by*

$$L[(\tilde{T} \circ \tilde{f})_w] = L[\tilde{T} \circ \tilde{f}_w] = \tilde{T} \circ L[\tilde{f}_w] : \Omega_w \rightarrow \mathbb{C}^{n+1} \setminus \{0\}.$$

- (iii) *The matrix of the linear approximation of $\tilde{T} \circ \tilde{f}$ at $w \in \Omega$ is given by*

$$M_{\tilde{T} \circ \tilde{f}_w} = \tilde{T} \cdot M_{\tilde{f}_w} \in \mathrm{SL}_{n+1}(\mathbb{C}),$$

where $\cdot$ denotes matrix multiplication.

- (iv) *The canonical Möbius approximation of $T \circ f : \Omega \rightarrow \mathbb{CP}^n$ at $w \in \Omega$ is*

$$M_{T \circ f_w} = T \circ M_{f_w} : \mathbb{CP}^n \rightarrow \mathbb{CP}^n.$$

Proof. For (i), observe that with respect to the projection map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$, we have $\pi \circ \tilde{T} = T \circ \pi$ and $\pi \circ \tilde{f} = f$. So, $\pi \circ (\tilde{T} \circ \tilde{f}) = T \circ \pi \circ \tilde{f} = T \circ f$. This shows that $\tilde{T} \circ \tilde{f}$ is a lift of $T \circ f$. Furthermore, since $\tilde{T}$ is a constant matrix, we have $\partial/\partial z_i(\tilde{T} \circ \tilde{f}) = \tilde{T} \cdot \partial \tilde{f}/\partial z_i$ for $1 \leq i \leq n$. So,

$$\begin{aligned} & \det \left((\tilde{T} \circ \tilde{f})(z), \frac{\partial}{\partial z_1}(\tilde{T} \circ \tilde{f})(z), \dots, \frac{\partial}{\partial z_n}(\tilde{T} \circ \tilde{f})(z) \right) \\ &= \det \left(\tilde{T} \cdot \left(\tilde{f}, \frac{\partial \tilde{f}}{\partial z_1}, \dots, \frac{\partial \tilde{f}}{\partial z_n} \right) \right) = \det(\tilde{T}) \det \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) \equiv 1. \end{aligned}$$

Thus, $\tilde{T} \circ \tilde{f}$ is the canonical lift of $T \circ f$. This proves (i).

The linear approximation of $\tilde{T} \circ \tilde{f}(z)$ at $z = w \in \Omega$ is the same as the linear approximation of $\tilde{T} \circ \tilde{f}_w(z)$ at $z = 0 \in \Omega_w$, which is given by

$$\begin{aligned} L[\tilde{T} \circ \tilde{f}_w](z) &= (\tilde{T} \circ \tilde{f}_w)(z) + \frac{\partial(\tilde{T} \circ \tilde{f}_w)}{\partial z_1}(0) \cdot z_1 + \dots + \frac{\partial(\tilde{T} \circ \tilde{f}_w)}{\partial z_n}(0) \cdot z_n \\ &= \tilde{T} \cdot \left(\tilde{f}_w(0) + \frac{\partial \tilde{f}_w}{\partial z_1}(0) \cdot z_1 + \dots + \frac{\partial \tilde{f}_w}{\partial z_n}(0) \cdot z_n \right) \\ &= \tilde{T} \cdot L[\tilde{f}_w](z). \end{aligned}$$

This proves the formula in (ii). For (iii), from the calculation in (ii),

$$L[\tilde{T} \circ \tilde{f}_w](z) = \tilde{T} \cdot L[\tilde{f}_w](z) = \tilde{T} \cdot M_{\tilde{f}_w} \cdot {}^t(1, z_1, \dots, z_n).$$

This identity means that $M_{\tilde{T} \circ \tilde{f}_w} = \tilde{T} \cdot M_{\tilde{f}_w} \in \mathrm{SL}_{n+1}(\mathbb{C})$.

(iv) is straightforward. From (2-9) and (ii) above, $M_{T \circ f_w} = \pi \circ L[\tilde{T} \circ \tilde{f}_w] = \pi \circ \tilde{T} \circ L[\tilde{f}_w] = T \circ \pi \circ L[\tilde{f}_w] = T \circ M_{\tilde{f}_w}$. $\square$

§3 GENERALIZED SCHWARZIANs

We define generalized Schwarzian operators in several variables acting on locally biholomorphic maps $f : \Omega \rightarrow \mathbb{CP}^n$ from a domain $\Omega \subset \mathbb{C}^n$ into the projective space. As before, for any point $w \in \Omega$, we let $f_w(z) = f(z + w)$ for $z \in \Omega_w = \Omega - w \subset \mathbb{C}^n$. Let $\tilde{f}_w(z)$ be the $\mathbb{C}^{n+1}$ -valued local canonical lift of $\tilde{f}_w$ near $0 \in \Omega_w$. Let $M_{\tilde{f}_w} : \Omega \rightarrow \mathrm{SL}_{n+1}(\mathbb{C})$ be the matrix of linear approximation of $\tilde{f}_w(z)$ at $z = 0$. Explicitly,

$$M_{\tilde{f}_w} = \left(\tilde{f}(w), \frac{\partial \tilde{f}}{\partial z_1}(w), \dots, \frac{\partial \tilde{f}}{\partial z_n}(w) \right).$$

We then let

$$(3-1) \quad S_w[f](z) = M_{\tilde{f}_w}^{-1} \cdot \tilde{f}_w(z) = \left(\tilde{f}(w), \frac{\partial \tilde{f}}{\partial z_1}(w), \dots, \frac{\partial \tilde{f}}{\partial z_n}(w) \right)^{-1} \cdot \tilde{f}_w(z)$$

be a $\mathbb{C}^{n+1}$ -valued holomorphic function in z defined in a neighborhood of $0 \in \Omega_w$ for each $w \in \Omega$. Note that $M_{\tilde{f}_w}$ is a function in w and depends on the choice of the local canonical lift $\tilde{f}_w(z)$. Although there are $(n+1)$ choices of local canonical lifts $\tilde{f}_w$ near $z = 0$, we observe that the above $S_w[f]$ is independent of the choice of local canonical lifts, and hence depends only on f , which in turn shows that $S_w[f]$ is globally well defined on Ω_w . To see this, note that all the possible local canonical lifts are of the form $\eta \cdot \tilde{f}_w(z)$ for an $(n+1)$ st root of unity η . For this new canonical lift, the corresponding matrix of linear approximation is $M_{\eta \tilde{f}_w} = \eta \cdot M_{\tilde{f}_w}$. So,

$$M_{\eta \tilde{f}_w}^{-1} \cdot \eta \tilde{f}_w(z) = (M_{\tilde{f}_w}^{-1} \cdot \eta^{-1}) \cdot (\eta \cdot \tilde{f}_w(z)) = M_{\tilde{f}_w}^{-1} \cdot \tilde{f}_w(z).$$

This shows that the $\mathbb{C}^{n+1}$ -valued holomorphic function $S_w[f]$ is independent of the choice of local canonical lifts and well defined globally on Ω_w . The most important property of $S_w[f]$ is the Möbius invariance with respect to the action of $\mathrm{PSL}_{n+1}(\mathbb{C})$.

Proposition 3-1. *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. For any $w \in \Omega$, the map $S_w[f] : \Omega_w \rightarrow \mathbb{C}^{n+1}$ given locally by $S_w[f](z) = M_{\tilde{f}_w}^{-1} \cdot \tilde{f}_w(z)$ in terms of the local canonical lift $\tilde{f}_w(z)$ is globally well defined on Ω_w .*

Furthermore, $S_w[f]$ is Möbius invariant in the sense that for any automorphism $T : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$, we have

$$(3-2) \quad S_w[T \circ f] = S_w[f] : \Omega_w \rightarrow \mathbb{C}^{n+1}.$$

Proof. We have already shown that the map $S_w[f]$ is globally well defined on the domain Ω_w . To see the Möbius invariance, let $\tilde{T} \in \mathrm{SL}_{n+1}(\mathbb{C})$ be a lift of $T \in \mathrm{PSL}_{n+1}(\mathbb{C})$. Then, by (i) of Proposition 2-6, as the local canonical lift of $(\widetilde{T \cdot f})_w(z)$ we can take $\tilde{T} \cdot \tilde{f}_w(z)$. For this canonical lift, the matrix of linear approximation is given by $M_{\tilde{T} \cdot \tilde{f}_w} = \tilde{T} \cdot M_{\tilde{f}_w}$ by (iii) of Proposition 2-6. So,

$$\begin{aligned} S_w[T \circ f](z) &= M_{(\widetilde{T \circ f})_w}^{-1} \cdot (\widetilde{T \circ f})_w(z) = (M_{\tilde{f}_w}^{-1} \cdot \tilde{T}^{-1}) \cdot \tilde{T} \cdot \tilde{f}_w(z) = M_{\tilde{f}_w}^{-1} \cdot \tilde{f}_w(z) \\ &= S_w[f](z). \end{aligned}$$

This proves the Möbius invariance of $S_w[f]$. $\square$

We now define the generalized Schwarzian operators acting on local biholomorphisms from a domain in $\mathbb{C}^n$ into projective spaces. These are defined as Taylor coefficients of $S_w[f](z)$.

Definition 3-2 (Generalized Schwarzians). For a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$, let the Taylor expansion of $S_w[f](z)$ around $z = 0$ be given by

$$(3-3) \quad S_w[f](z) = \sum_{K \in \mathbb{Z}_+^n} \frac{z^K}{K!} S_K[f](w).$$

Here, for $K = (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$, $z^K = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}$ and $K! = k_1! k_2! \dots k_n!$. The $\mathbb{C}^{n+1}$ -valued holomorphic function $S_K[f] : \Omega \rightarrow \mathbb{C}^{n+1}$ obtained as the Taylor coefficient of $z^K/K!$ of $S_w[f]$ for $K \in \mathbb{Z}_+^n$ is the K -th *generalized Schwarzian* of f . The generalized Schwarzians $S_K[f]$ with $|K| = 2$ are called the *fundamental Schwarzians*. We call $S_w[f]$ the *generating function* of the generalized Schwarzians of f .

Since the generating function $S_w[f]$ is Möbius invariant, its Taylor coefficients $S_K[f]$ are also Möbius invariant for all $K \in \mathbb{Z}_+^n$ and they are of the form

$$(3-4) \quad S_w[f](z) = \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} + \sum_{|K| \geq 2} \frac{z^K}{K!} \left\{ M_{\tilde{f}_w}^{-1} \cdot \frac{\partial^{|K|} \tilde{f}_w}{\partial z^K}(w) \right\}.$$

This follows from the Taylor expansion of the canonical lift $\tilde{f}_w(z)$ given by

$$\begin{aligned} \tilde{f}_w(z) &= \tilde{f}_w(0) + \sum_{i=1}^n \frac{\partial \tilde{f}_w}{\partial z_i}(0) \cdot z_i + \sum_{|K| \geq 2} \frac{z^K}{K!} \frac{\partial^{|K|} \tilde{f}_w}{\partial z^K}(0) \\ &= \left(\tilde{f}(w), \frac{\partial \tilde{f}}{\partial z_1}(w), \dots, \frac{\partial \tilde{f}}{\partial z_n}(w) \right) \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} + \sum_{|K| \geq 2} \frac{z^K}{K!} \frac{\partial^{|K|} \tilde{f}_w}{\partial z^K}(w) \\ &= M_{\tilde{f}_w} \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} + \sum_{|K| \geq 2} \frac{z^K}{K!} \frac{\partial^{|K|} \tilde{f}_w}{\partial z^K}(w). \end{aligned}$$

From (3-4), we can read off the explicit expression of the generalized Schwarzian operators $S_K[f]$ for local biholomorphism f . By using the previous observation, we obtain the next proposition.

Proposition 3-3 (Möbius invariance of Schwarzians). *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. For any $K \in \mathbb{Z}_+^n$, the K -th generalized Schwarzian $S_K[f]$ is Möbius invariant. Namely, for any $T \in \text{PSL}_{n+1}(\mathbb{C})$ we have*

$$(3-5) \quad S_K[T \circ f] = S_K[f] : \Omega \rightarrow \mathbb{C}^{n+1}.$$

In terms of the local canonical lift, these generalized Schwarzians are given as follows:

$$(3-6) \quad \begin{aligned} S_0[f](w) &= e_1, & S_{\sigma_i}[f](w) &= e_{i+1}, \\ S_K[f](w) &= \left(\tilde{f}(w), \frac{\partial \tilde{f}}{\partial z_1}(w), \dots, \frac{\partial \tilde{f}}{\partial z_n}(w) \right)^{-1} \cdot \frac{\partial^{|K|} \tilde{f}}{\partial z^K}(w), & |K| &\geq 2. \end{aligned}$$

Here $e_j \in \mathbb{C}^{n+1}$ is the column vector with 1 at the j th place and 0 everywhere else.

Let the ℓ -th component of $S_K[f] : \Omega \rightarrow \mathbb{C}^{n+1}$ be denoted by $S_K^{(\ell)}[f]$ for $0 \leq \ell \leq n$. The following corollary is immediate.

Corollary 3-4. *The ℓ -th component $S_K^{(\ell)}[f] : \Omega \rightarrow \mathbb{C}$ of the generalized Schwarzian $S_K[f]$ of a local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$ is Möbius invariant for all $K \in \mathbb{Z}_+^n$ and $0 \leq \ell \leq n$.*

If the local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$ factors through $\mathbb{C}^n \subset \mathbb{CP}^n$ and if we let $f(z) = (f_1(z), f_2(z), \dots, f_n(z)) \in \mathbb{C}^n$, then each component of the generalized Schwarzian $S_K[f]$ depends rationally on the partial derivatives of $f_j(z)$'s. So, $S_K[\cdot]$'s are nonlinear partial differential operators. The explicit expression of the fundamental Schwarzians as differential operators can be found in Proposition 4-2. The expression for the higher Schwarzians as differential operators can be obtained from that for the fundamental Schwarzians using (3-16) in Theorem 3-7 below.

The generalized Schwarzians $S_K[f]$ for $K \in \mathbb{Z}_+^n$ are not differential algebraically independent from each other. It turns out that the higher Schwarzians can be expressed in terms of the fundamental Schwarzians $S_{ij}[f]$. Furthermore, some relations exist among the fundamental Schwarzians. To describe these relations among Schwarzians, let $M(z)$ be the matrix of linear approximations of the canonical lift $\tilde{f}$ at z :

$$(3-7) \quad M(z) = M_{\tilde{f}_z} = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) \in \text{SL}_{n+1}(\mathbb{C}).$$

This $\text{SL}_{n+1}(\mathbb{C})$ -valued holomorphic function $M(z)$ on Ω plays an important role throughout this paper. In terms of $M(z)$, (3-6) can be written as

$$(3-8) \quad \frac{\partial^{|K|} \tilde{f}}{\partial z^K}(z) = M(z) \cdot S_K[f](z).$$

Combining these formulae for $|K| = 2$, we get

$$\frac{\partial M}{\partial z_i} = \left(\frac{\partial \tilde{f}}{\partial z_i}, \frac{\partial \tilde{f}}{\partial z_i \partial z_1}, \dots, \frac{\partial \tilde{f}}{\partial z_i \partial z_n} \right) = M(z) \cdot (e_{i+1}, S_{i1}[f], \dots, S_{in}[f]),$$

for $1 \leq i \leq n$. For convenience, let $A_i[f](z)$ be a $M_{n+1}(\mathbb{C})$ valued holomorphic function on Ω defined by

$$(3-9) \quad A_i[f](z) = (e_{i+1}, S_{i1}[f], \dots, S_{in}[f]), \quad 1 \leq i \leq n.$$

Then the above formula for $\partial M / \partial z_i$ can be concisely written as

$$(3-10) \quad \frac{\partial M}{\partial z_i}(z) = M(z) \cdot A_i[f](z), \quad 1 \leq i \leq n.$$

Remark. We could use this identity (3-10) to *define* the generalized Schwarzians of f . This point of view turns out to be very natural in the geometric framework of vector bundles on manifolds with connections. This point of view will be pursued later when we consider the problem of prescribing the generalized Schwarzians in §7.

To study properties of $A_i[f](z)$, we prepare a lemma on matrices.

Lemma 3-5. *Let $\Lambda : \Omega \rightarrow \text{GL}_{n+1}(\mathbb{C})$ be a matrix valued holomorphic function from a domain $\Omega \subset \mathbb{C}^n$. Then,*

$$\frac{1}{\det \Lambda} \frac{\partial(\det \Lambda)}{\partial z_i} = \text{Tr} \left(\Lambda^{-1} \cdot \frac{\partial \Lambda}{\partial z_i} \right), \quad 1 \leq i \leq n.$$

More concisely, in terms of the exterior derivative d , we have

$$d(\log \det \Lambda) = \text{Tr} (\Lambda^{-1} \cdot d\Lambda).$$

Proof. Let $\Lambda(z) = (\lambda_{ij}(z))$ and let $\tilde{\lambda}_{ij}(z)$ be the (i, j) -cofactor, i.e., $\tilde{\lambda}_{ij}(z) = (-1)^{i+j} \det \Lambda_{ij}$ where Λ_{ij} is an $n \times n$ matrix obtained from Λ by deleting the i -th row and j -th column. Then, it is well known that the inverse matrix Λ^{-1} is given by $\Lambda^{-1} = {}^t(\tilde{\lambda}_{ij}) / \det \Lambda$. So, from the definition of the trace, we have

$$\text{Tr} (\Lambda^{-1} \cdot d\Lambda) = \frac{1}{\det \Lambda} \sum_{i,j=1}^n \tilde{\lambda}_{ij} d\lambda_{ij}.$$

On the other hand, if we let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_i \in \mathbb{C}^{n+1}$ is a column vector, then the exterior derivative of its determinant $\det \Lambda$ is given by

$$d(\det \Lambda) = \sum_{i=1}^n \det(\lambda_1, \dots, \lambda_{i-1}, d\lambda_i, \lambda_{i+1}, \dots, \lambda_n).$$

Expanding the i -th term in the above summation along the i -th column, we get

$$d(\det \Lambda) = \sum_{i=1}^n \sum_{j=1}^n d\lambda_{ij} \cdot \tilde{\lambda}_{ij} = \sum_{i,j=1}^n \tilde{\lambda}_{ij} \cdot d\lambda_{ij}.$$

Comparing the above two calculations, we get our result. $\square$

Now, we go back to our study of the generalized Schwarzians.

Proposition 3-6. *The matrices $A_i[f](z)$ for $1 \leq i \leq n$ consisting of the fundamental Schwarzians have values in the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$, that is,*

$$(3-11) \quad \text{Tr } A_i[f](z) \equiv 0, \quad \text{for } 1 \leq i \leq n.$$

In other words, the components of the fundamental Schwarzians satisfy the following identities:

$$(3-12) \quad \sum_{\ell=1}^n S_{i\ell}^{(\ell)}[f](z) \equiv 0, \quad \text{for } 1 \leq i \leq n.$$

Proof. As before, let $M(z) \in \text{SL}_{n+1}(\mathbb{C})$ be the matrix (3-7) of the linear approximation of the canonical lift $\tilde{f}$ at z . Then, from (3-10) and Lemma 3-5,

$$\text{Tr } A_i[f](z) = \text{Tr} \left(M(z)^{-1} \cdot \frac{\partial M(z)}{\partial z_i} \right) = \frac{1}{\det M(z)} \frac{\partial(\det M(z))}{\partial z_i}.$$

Since $\det M(z) \equiv 1$ by the determinant condition of the canonical lift, all of its partial derivatives vanish identically. Hence $\text{Tr } A_i[f](z) \equiv 0$ for $1 \leq i \leq n$. If we rewrite this identity in terms of the components of the fundamental Schwarzians using the definition of $A_i[f]$ given in (3-9), we get our second identity (3-12). $\square$

Remark. In our present context, it is not necessary to regard $A_i[f]$ as a Lie algebra valued function or as a connection. However, when we consider the composition formula or the problem of prescribing Schwarzians, or in the context of maps between manifolds with projective structures which will be discussed in a future paper, this Lie algebra point of view is crucial.

Given a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$, we introduce first order partial differential operators $D_i = D_i[f]$ for $1 \leq i \leq n$ acting on $\mathbb{C}^{n+1}$ -valued (or $M_{n+1}(\mathbb{C})$ -valued) holomorphic functions on Ω by

$$(3-13) \quad D_i[f] = \frac{\partial}{\partial z_i} + A_i[f](z) \cdot, \quad 1 \leq i \leq n.$$

Here, the dot “ $\cdot$ ” denotes matrix multiplication from the left. These operators should be regarded as partial covariant derivatives in a suitable geometric context. These operators play a crucial role in studying the properties of the generalized Schwarzians. We observe that for any $\mathbb{C}^{n+1}$ -valued (or $M_{n+1}(\mathbb{C})$ -valued) holomorphic function $F(z)$ on Ω , (3-10) implies that

$$\begin{aligned} \frac{\partial}{\partial z_i} (M(z) \cdot F(z)) &= \frac{\partial M}{\partial z_i} \cdot F + M \cdot \frac{\partial F}{\partial z_i} = M \cdot (A_i[f] \cdot F) + M \cdot \frac{\partial F}{\partial z_i} \\ &= M(z) \cdot (D_i[f](F))(z). \end{aligned}$$

Since $M(z)$ is invertible, the above formula shows that the differential operators $\partial/\partial z_i$ and D_i are “ $M(z)$ -equivalent”, making the following diagram commutative for $1 \leq i \leq n$.

$$(3-14) \quad \begin{array}{ccc} \text{Hol}(\Omega, \mathbb{C}^{n+1}) & \xrightarrow[\cong]{M(z) \cdot} & \text{Hol}(\Omega, \mathbb{C}^{n+1}) \\ D_i[f] \downarrow & & \downarrow \frac{\partial}{\partial z_i} \\ \text{Hol}(\Omega, \mathbb{C}^{n+1}) & \xrightarrow[\cong]{M(z) \cdot} & \text{Hol}(\Omega, \mathbb{C}^{n+1}) \end{array}$$

where $\text{Hol}(\Omega, \mathbb{C}^{n+1})$ is the set of $\mathbb{C}^{n+1}$ -valued holomorphic functions defined on Ω .

Theorem 3-7 (Generalized Schwarzians and commuting differential operators). *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map from a domain $\Omega \subset \mathbb{C}^{n+1}$. The associated first order partial differential operators $D_i[f]$ have the following properties.*

(i) *The differential operators $D_i = D_i[f]$ mutually commute:*

$$(3-15) \quad [D_i, D_j] = 0, \quad 1 \leq i, j \leq n.$$

(ii) *All the generalized Schwarzians $S_K[f]$ are obtained from the constant vector $p_0 = {}^t(1, 0, \dots, 0) \in \mathbb{C}^{n+1}$ by applying the partial covariant derivatives $D_i[f]$ successively K times. More precisely,*

$$(3-16) \quad S_K[f](z) = D^K[f] \cdot p_0, \quad K \in \mathbb{Z}_+^n,$$

where $D^K[f] = D_1[f]^{k_1} \cdot D_2[f]^{k_2} \cdots D_n[f]^{k_n}$ if $K = (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$.

Remark. Since the partial differential operators $D_i[f]$ commute due to (i), the notation $D^K[f]$ in (ii) makes sense and depends only on $K \in \mathbb{Z}_+^n$, not on the order in which these operators are applied.

Proof. (i) Since the differential operators $\partial/\partial z_i$ and $D_i[f]$ are $M(z)$ -equivalent in the sense of (3-14), for any $\mathbb{C}^{n+1}$ -valued holomorphic function $F(z) \in \text{Hol}(\Omega, \mathbb{C}^{n+1})$ we have

$$M(z) \cdot ([D_i[f], D_j[f]] \cdot F(z)) = \left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right] (M(z) \cdot F(z)) = 0.$$

The last equality holds because the partial differential operators $\partial/\partial z_i$ and $\partial/\partial z_j$ commute. Since the matrix $M(z)$ is invertible and the function $F(z)$ is arbitrary, as differential operators we have

$$[D_i[f], D_j[f]] = 0.$$

This proves (i). For (ii), from the definition of the generalized Schwarzians (3-8),

$$\frac{\partial^{|K|} \tilde{f}}{\partial z^K} (z) = M(z) \cdot S_K[f](z)$$

for any $K \in \mathbb{Z}_+^n$. Since $\tilde{f}(z)$ is the first column vector of $M(z)$, we can write $\tilde{f}(z) = M(z) \cdot p_0$. The $M(z)$ -equivalence (3-14) between $\partial/\partial z_i$ and $D_i[f]$ gives

$$\frac{\partial^{|K|} \tilde{f}}{\partial z^K} = \frac{\partial^{|K|}}{\partial z^K} (M(z) \cdot p_0) = M(z) \cdot (D^K[f] \cdot p_0).$$

Since $M(z)$ is invertible, equating the right hand sides of the above two identities, we get $S_K[f] = D^K[f] \cdot p_0$ for any multi-index $K \in \mathbb{Z}_+^n$. This completes the proof of Theorem 3-7. $\square$

In terms of matrices $A_i(z)$ in (3-9), the statements in (3-11) and (3-15) are equivalent to

$$(3-17) \quad \begin{cases} \frac{\partial A_i}{\partial z_j} + A_j A_i = \frac{\partial A_j}{\partial z_i} + A_i A_j \\ \text{Tr } A_i = 0, \end{cases} \quad 1 \leq i, j \leq n.$$

For notational convenience, let $p_K(z) = D^K[f] \cdot p_0$ for any multi-index $K \in \mathbb{Z}_+^n$. We also let $p_{\sigma_i} = p_i$ and $p_{\sigma_i + \sigma_j} = p_{ij}$ for $1 \leq i, j \leq n$, where σ_i is a row vector with 1 at the i -th entry and 0 at all the other entries. Note that $p_K(z)$'s are column vector valued functions and we have

$$(3-18) \quad p_i = D_i \cdot p_0 = e_{i+1}, \quad p_{ij} = D_i D_j \cdot p_0 = D_i \cdot p_j = D_j \cdot p_i = S_{ij}[f](z).$$

Equating the corresponding columns of the matrix identity in (3-17), we have

$$(3-19) \quad \begin{cases} p_{ij} = p_{ji} \\ \frac{\partial p_{i\ell}}{\partial z_j} + A_j \cdot p_{i\ell} = \frac{\partial p_{j\ell}}{\partial z_i} + A_i \cdot p_{j\ell}, \end{cases} \quad 1 \leq i, j, \ell \leq n.$$

These identities come from the identity between the first columns or $(\ell + 1)$ st columns of the matrix identity in (3-17). In particular, we see that the generalized Schwarzians $p_{ij} = S_{ij}[f]$ must be symmetric in indices i, j . We deduce one important consequence of (3-17).

Theorem 3-8. *When $n \geq 0$, the 0-th components of the fundamental Schwarzians $p_{ij}^{(0)} = S_{ij}^{(0)}[f]$ depend on the other components. More precisely, the following identity holds.*

$$(3-20) \quad (n-1)p_{ij}^{(0)} = - \sum_{\ell=1}^n \frac{\partial p_{ij}^{(\ell)}}{\partial z_\ell} + \sum_{k,\ell=1}^n p_{ik}^{(\ell)} \cdot p_{j\ell}^{(k)}, \quad n \geq 2.$$

Proof. First note that the commutativity (3-15) of the first order differential operators D_i implies that $D_\ell \cdot p_{ij} = D_\ell D_i D_j \cdot p_0 = D_i D_j \cdot p_\ell$ for $1 \leq i, j, \ell \leq n$. So, we have

$$\begin{aligned} (*) \quad (p_{ij}, D_1 p_{ij}, \dots, D_n p_{ij}) &= (D_i D_j p_0, D_i D_j p_1, \dots, D_i D_j p_n) \\ &= D_i D_j (p_0, p_1, \dots, p_n) = D_i D_j I_{n+1} = D_i A_j. \end{aligned}$$

The third equality holds because $I_{n+1} = (p_0, p_1, \dots, p_n)$ since $p_i = e_{i+1}$ from (3-18). We take the trace of both sides of the above identity. For the right hand side, using the identity $\text{Tr } A_i = 0$ in Proposition 3-6,

$$\begin{aligned} \text{Tr } D_i A_j &= \text{Tr} \left(\frac{\partial A_j}{\partial z_i} + A_i A_j \right) = \frac{\partial}{\partial z_i} (\text{Tr } A_j) + \text{Tr } A_i A_j = \text{Tr } A_i A_j \\ &= 2p_{ij}^{(0)} + \sum_{k,\ell=1}^n p_{ik}^{(\ell)} \cdot p_{j\ell}^{(k)}. \end{aligned}$$

For the trace of the left hand side, since $D_k p_{ij} = \partial p_{ij} / \partial z_k + A_k \cdot p_{ij}$ for $1 \leq k \leq n$,

$$\begin{aligned} \text{Tr}(p_{ij}, D_1 p_{ij}, \dots, D_n p_{ij}) &= \text{Tr}\left(p_{ij}, \frac{\partial p_{ij}}{\partial z_1}, \dots, \frac{\partial p_{ij}}{\partial z_n}\right) + \text{Tr}(0, A_1 \cdot p_{ij}, \dots, A_n \cdot p_{ij}) \\ &= p_{ij}^{(0)} + \sum_{\ell=1}^n \frac{\partial p_{ij}^{(\ell)}}{\partial z_\ell} + \sum_{\ell=1}^n (A_\ell \cdot p_{ij})^{(\ell+1)}, \end{aligned}$$

where $(A_\ell \cdot p_{ij})^{(\ell+1)}$ is the $(\ell+1)$ st component of the column vector $A_\ell \cdot p_{ij}$ and is given by $(A_\ell \cdot p_{ij})^{(\ell+1)} = p_{ij}^{(0)} + \sum_{k=1}^n p_{\ell k}^{(\ell)} \cdot p_{ij}^{(k)}$. Hence the last summation in the above identity is equal to

$$\sum_{\ell=1}^n (A_\ell \cdot p_{ij})^{(\ell+1)} = n p_{ij}^{(0)} + \sum_{\ell,k=1}^n p_{\ell k}^{(\ell)} \cdot p_{ij}^{(k)} = n p_{ij}^{(0)} + \sum_{k=1}^n (\text{Tr } A_k) \cdot p_{ij}^{(k)} = n p_{ij}^{(0)}.$$

The last identity holds because $\text{Tr } A_k = 0$ for $1 \leq k \leq n$. Thus, the trace of the identity (*) yields

$$p_{ij}^{(0)} + \sum_{\ell=1}^n \frac{\partial p_{ij}^{(0)}}{\partial z_\ell} + n p_{ij}^{(0)} = 2 p_{ij}^{(0)} + \sum_{k,\ell=1}^n p_{ik}^{(\ell)} \cdot p_{j\ell}^{(k)}.$$

Collecting similar terms, we get the identity (3-20) in Theorem 3-8. $\square$

The generalized Schwarzians carry various information about the local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$. For example, the vanishing of higher Schwarzians implies the following.

Proposition 3-9 (Vanishing of Schwarzians). *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. Let r be a positive integer. Suppose $S_K[f] \equiv 0$ for all multi-index K with $|K| = r+1$. Then, in terms of the homogeneous coordinate of $\mathbb{CP}^n$, the local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$ has a polynomial representation of the form $f(z) = [f_0(z) : f_1(z) : \dots : f_n(z)]$ where $f_i(z) : \Omega \rightarrow \mathbb{C}$ is a polynomial function of total degree at most r for $0 \leq i \leq n$.*

Proof. Let $\tilde{f} : \Omega' \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be the local canonical lift of f defined on a subdomain $\Omega' \subset \Omega$. By definition,

$$S_K[f](z) = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right)^{-1} \frac{\partial^{r+1} \tilde{f}}{\partial z^K}(z) = M_{\tilde{f}_z}^{-1} \cdot \frac{\partial^{r+1} \tilde{f}}{\partial z^K}(z).$$

Since $M_{\tilde{f}_z} \in \text{SL}_{n+1}(\mathbb{C})$ is invertible, $S_K[f](z) \equiv 0$ implies $\partial^{r+1} \tilde{f} / \partial z^K \equiv 0$ for all K with $|K| = r+1$. This means that each component of the canonical lift $\tilde{f}$ must be a polynomial of total degree at most r . Since $f = \pi \circ \tilde{f}$, f has a polynomial representation of degree at most r in terms of homogeneous coordinate of $\mathbb{CP}^n$. $\square$

In particular, when $r = 1$, the above statement says that if $S_{ij}[f] \equiv 0$ for all $1 \leq i, j \leq n$, then the canonical lift $\tilde{f}$ of f is linear. So, the map f extends to a map from $\mathbb{CP}^n$ and $f : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ itself is projective linear. The converse is also true. That is, the vanishing of all $S_{ij}[f]$ implies that the map f is projective linear, as is demonstrated in the next theorem. This fact demonstrates the basic importance of the fundamental Schwarzians $S_{ij}[f]$.

Theorem 3-10 (Characterization of Möbius transformations). *Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map. Then, we have the following equivalence.*

$$S_{ij}[f] \equiv 0, \quad 1 \leq i, j \leq n \iff \text{the map } f \text{ is projective linear.}$$

Proof. ($\Rightarrow$) This has been proved in Proposition 3-9.

($\Leftarrow$) Let f be a restriction of a projective linear transformation of $\mathbb{CP}^n$ to a domain $\Omega \subset \mathbb{C}^n$. Then, f has a global representation of the form

$$f(z) = [f_0(z) : f_1(z) : \cdots : f_n(z)], \quad z \in \Omega,$$

where $f_i(z)$ for $0 \leq i \leq n$ are all linear functions in z . Let $\bar{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be a global lift of f given by

$$\bar{f}(z) = {}^t(f_0(z), f_1(z), \dots, f_n(z)).$$

We show that the canonical lift $\tilde{f}$ of f differs from the above lift $\bar{f}$ by a multiplicative constant. To see this, let the global canonical lift $\tilde{f}$ of f be of the form

$$\tilde{f}(z) = {}^t(\lambda(z)f_0(z), \lambda(z)f_1(z), \dots, \lambda(z)f_n(z))$$

for some nonvanishing holomorphic function $\lambda(z) : \Omega \rightarrow \mathbb{C}$. The determinant condition of the canonical lift $\tilde{f}$ implies that

$$\lambda(z)^{n+1} \det\left(\bar{f}(z), \frac{\partial \bar{f}}{\partial z_1}(z), \dots, \frac{\partial \bar{f}}{\partial z_n}(z)\right) \equiv 1.$$

Since $\bar{f}$ is linear, $\partial \bar{f} / \partial z_i$ for $1 \leq i \leq n$ is a constant vector and for any point $z_0 \in \Omega$ this vector is equal to $\partial \bar{f} / \partial z_i(z_0)$. Furthermore, the Taylor expansion of $\bar{f}(z)$ around z_0 is a linear combination of $\bar{f}(z_0)$ and $\partial \bar{f} / \partial z_i(z_0)$'s since $\bar{f}(z)$ is linear. Hence

$$\det\left(\bar{f}(z), \frac{\partial \bar{f}}{\partial z_1}(z), \dots, \frac{\partial \bar{f}}{\partial z_n}(z)\right) = \det\left(\bar{f}(z_0), \frac{\partial \bar{f}}{\partial z_1}(z_0), \dots, \frac{\partial \bar{f}}{\partial z_n}(z_0)\right)$$

is a constant and so $\lambda(z)$ must also be a nonzero constant function, determined up to an $(n+1)$ st root of unity. Thus the canonical lift $\tilde{f}$ is of the form

$$\tilde{f}(z) = \lambda \cdot \bar{f}(z) = \lambda \cdot {}^t(f_0(z), f_1(z), \dots, f_n(z)),$$

where the constant $\lambda \in \mathbb{C}^*$ satisfies the following relation for any $z_0 \in \Omega$,

$$\lambda^{n+1} \cdot \det\left(\bar{f}(z_0), \frac{\partial \bar{f}}{\partial z_1}(z_0), \dots, \frac{\partial \bar{f}}{\partial z_n}(z_0)\right) \equiv 1.$$

This shows that the canonical lift $\tilde{f}$ of f is linear and hence all of its second derivative vanishes: $\partial^2 \tilde{f} / \partial z_i \partial z_j \equiv 0$ for $1 \leq i, j \leq n$, which in turn implies that

$$S_{ij}[f](z) = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z)\right)^{-1} \cdot \frac{\partial^2 \tilde{f}}{\partial z_i \partial z_j}(z) \equiv 0.$$

This completes the proof of Theorem 3-10. $\square$

We remark that this result was obtained in [G1] in a different method.

Example 3-11. We remark that the result in Proposition 3-9 is best possible for $n \geq 2$. We give an example to demonstrate this point. Let a $\mathbb{C}^{n+1}$ -valued holomorphic map $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be given by

$$(3-21) \quad \tilde{f}(z) = {}^t(a_0, a_1 z_1, g_1(z_1) + a_2 z_2, g_2(z_1, z_2) + a_3 z_3, \dots, g_{n-1}(z_1, \dots, z_{n-1}) + a_n z_n)$$

for some nonzero constants $a_0, a_1, \dots, a_n \in \mathbb{C} \setminus \{0\}$ such that $\prod_{i=0}^n a_i = 1$. The domain Ω of $\tilde{f}$ can be taken to be a subset of $\mathbb{C}^n$ consisting of those z such that $\tilde{f}(z) \neq 0 \in \mathbb{C}^{n+1}$. In (3-21), $g_1, g_2, \dots, g_{n-1}$ are arbitrary holomorphic functions in specified variables. Then, by construction of $\tilde{f}$, the matrix $(\tilde{f}(z), \partial \tilde{f} / \partial z_1, \dots, \partial \tilde{f} / \partial z_n)$ is lower triangular whose diagonal elements are $a_0, a_1, \dots, a_n$. So its determinant is $\prod a_i = 1$. Hence $\tilde{f}$ is the canonical lift of a local biholomorphism $f = \pi \circ \tilde{f} : \Omega \rightarrow \mathbb{CP}^n$. In particular, if $g_1, g_2, \dots, g_{n-1}$ are polynomials of total degree r in specified variables, then $\partial^{|K|} \tilde{f} / \partial z^K = 0$ for all K with $|K| \geq r + 1$. So, the local biholomorphism $f = \pi \circ \tilde{f}$ satisfies the hypothesis of Proposition 3-9 and the degree of the lift $\tilde{f}$ is precisely r since $n \geq 2$. So, the statement of Proposition 3-9 is best possible.

However, in the single variable case we can make a stronger statement. In fact, the higher Schwarzians also characterize projective linear maps. When $n = 1$, the generalized Schwarzians $S_k[f] : \Omega \rightarrow \mathbb{C}^2$ for $k \geq 0$ are given by

$$(3-22) \quad S_k[f](z) = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z}(z) \right)^{-1} \cdot \frac{\partial^k \tilde{f}}{\partial z^k}(z) \in \mathbb{C}^2.$$

Explicit forms of the higher Schwarzians $S_k[f]$ as differential operators are given in Example 4-4. Although these operators become more and more complicated as k increases, the kernel spaces of these $\mathbb{C}^2$ -valued operators are all the same and consist of the projective linear maps, as shown in the next theorem.

Theorem 3-12. *Let $\Omega \subset \mathbb{C}$ be a 1-dimensional domain and let $f : \Omega \rightarrow \mathbb{CP}^1$ be a locally biholomorphic map. Then, the following statements are equivalent.*

- (i) $S_2[f](z) = 0$.
- (ii) $S_{r+1}[f](z) = 0$ for all $r \geq 1$.
- (iii) $S_{r+1}[f](z) = 0$ for some $r \geq 1$.

When one of the above statements holds, then the map $f : \Omega \rightarrow \mathbb{CP}^1$ is projective linear.

Proof. (i) $\Rightarrow$ (ii). By definition, $S_2[f](z) \equiv 0$ is equivalent to $\partial^2 \tilde{f} / \partial z^2 \equiv 0$. Differentiating several times more, we get $\partial^{r+1} \tilde{f} / \partial z^{r+1} \equiv 0$, which implies that

$$S_{r+1}[f](z) = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z}(z) \right)^{-1} \cdot \frac{\partial^{r+1} \tilde{f}}{\partial z^{r+1}}(z) \equiv 0.$$

This proves (ii). The implication (ii) $\Rightarrow$ (iii) is obvious. Now we assume (iii) and show (i). Since the matrix $(\tilde{f}(z), \partial \tilde{f} / \partial z)$ is invertible for a local biholomorphism f , the vanishing of the generalized Schwarzian $S_{r+1}[f](z)$ implies that $\partial^{r+1} \tilde{f} / \partial z^{r+1} \equiv 0$. This means that both of the two components of the canonical

lift $\tilde{f}$ are polynomials of degree at most r . Let $\tilde{f}(z) = {}^t(p(z), q(z))$, where $p(z), q(z)$ are linearly independent polynomials of degree at most r . The determinant condition of the lift $\tilde{f}$ requires that the polynomials $p(z), q(z)$ must satisfy

$$\det \begin{pmatrix} p(z) & p'(z) \\ q(z) & q'(z) \end{pmatrix} = 1, \quad \text{for all } z \in \Omega.$$

Associated to the lift $\tilde{f}$, we consider another $\mathbb{C}^2$ -valued function $\tilde{g} = {}^t(u(z), v(z))$ defined as follows. If $\deg p(z) \neq \deg q(z)$, then we let $\tilde{g}(z) = \tilde{f}(z)$. When $\deg p(z) = \deg q(z) = m$, let a_m, b_m be the leading coefficients of these polynomials and set $u(z) = p(z), v(z) = q(z) - (b_m/a_m)p(z)$. Then $\deg u(z) \neq \deg v(z)$ and $u(z)$ and $v(z)$ are linearly independent. Now observe that

$$\det(\tilde{g}(z), \partial\tilde{g}/\partial z(z)) = \det \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} = \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \equiv 1,$$

Let $\deg u(z) = d_1 \leq r, \deg v(z) = d_2 \leq r$. Then, by construction of $\tilde{g}$, $d_1 \neq d_2$ and $\max(d_1, d_2) = r$. Let the leading coefficients of $u(z)$ and $v(z)$ be c_1, c_2 . These constants are nonzero since $u(z), v(z)$ are linearly independent. So,

$$\begin{cases} u(z) = c_1 z^{d_1} + (\text{lower order terms}), \\ v(z) = c_2 z^{d_2} + (\text{lower order terms}). \end{cases}$$

To show that $r \leq 1$, we assume that $r \geq 2$ and deduce a contradiction. First observe that

$$\det(\tilde{g}, \partial\tilde{g}/\partial z) = c_1 c_2 (d_2 - d_1) z^{d_1 + d_2 - 1} + (\text{lower order terms}),$$

where $d_1 + d_2 - 1 \geq 1$ since $\max(d_1, d_2) = r \geq 2$. Then the determinant condition on $\tilde{g}$ given by $\det(\tilde{g}, \partial\tilde{g}/\partial z) \equiv 1$ implies that the leading coefficient must vanish, that is, $c_1 c_2 (d_2 - d_1) = 0$. But by our construction of the map $\tilde{g}$, $c_1 \neq 0, c_2 \neq 0, d_1 \neq d_2$. This is a contradiction. Hence we must have $r \leq 1$. Thus, each component of the canonical lift $\tilde{f}$ must be linear and hence $S_2[f] = 0$. In this case, $f : \Omega \rightarrow \mathbb{CP}^1$ is projective linear, or Möbius. This completes the proof of Theorem 3-12. $\square$

§4 GENERALIZED SCHWARZIAN AS DIFFERENTIAL OPERATORS

First we clarify what we mean by a differential operator

$$I[\cdot] : \text{Hol}(\Omega, M) \rightarrow \text{Hol}(\Omega, \mathbb{C})$$

acting on holomorphic maps to a complex manifold M . Here, we assume that the result of the operation $I[f]$ is a $\mathbb{C}$ -valued function on Ω . In general, we can deal with vector valued differential operators (our generalized Schwarzian operators $S_K[\cdot]$ are examples of this), but then each component will have the above form. So without loss of generality, we can consider differential operators of the above form. Let (U, φ) be any local coordinate neighborhood of M , where $U \subset M$ and $\varphi : U \hookrightarrow \mathbb{C}^n$. For any $f \in \text{Hol}(\Omega, M)$, let $f_\varphi = \varphi \circ f$ be a $\mathbb{C}^n$ -valued function

defined on $\Omega \cap f^{-1}(U)$. By definition, $I[\cdot]$ is a differential operator if for any local coordinate neighborhood (U, φ) of M and for any $f \in \text{Hol}(\Omega, M)$, the function $I[f] : \Omega \rightarrow \mathbb{C}$ restricted to $\Omega \cap f^{-1}(U) \subset \Omega$ can be expressed algebraically (or rationally) in terms of the derivatives of components of f_φ with function coefficients which are independent of f . This definition is independent of the choice of local coordinate neighborhoods (U, φ) in the complex manifold M .

Suppose a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$ factors through $\mathbb{C}^n \subset \mathbb{CP}^n$, where $\mathbb{C}^n = \{[w_0 : w_1 : \dots : w_n] \in \mathbb{CP}^n \mid w_0 \neq 0\}$. In this case, we give an explicit expression of the fundamental Schwarzians $S_{ij}[f]$ of f as nonlinear partial differential operators involving at most second order partial derivatives of f . Let a local biholomorphism $f : \Omega \rightarrow \mathbb{C}^n \subset \mathbb{CP}^n$ be given by

$$(4-1) \quad f(z) = {}^t(f_1(z), f_2(z), \dots, f_n(z)) \in \mathbb{C}^n.$$

Since $f(z)$ is a locally biholomorphic map, its Jacobian determinant $J_f(z)$ is finite and never vanishes on Ω . Its canonical lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ is of the form

$$(4-2) \quad \tilde{f}(z) = {}^t(\lambda(z), \lambda(z)f_1(z), \dots, \lambda(z)f_n(z))$$

for some nonvanishing holomorphic function $\lambda : \Omega \rightarrow \mathbb{C}^*$. The determinant condition (2-2) of the canonical lift $\tilde{f}$ implies that

$$(4-3) \quad \lambda(z)^{n+1} \cdot J_f(z) \equiv 1.$$

For notational convenience, let

$$\Delta_i(z) = \frac{-1}{n+1} \frac{\partial}{\partial z_i} \ln J_f(z) = \frac{-1}{n+1} \frac{1}{J_f} \frac{\partial J_f}{\partial z_i}, \quad \Delta_{ij}(z) = \frac{\partial \Delta_i}{\partial z_j}(z) = \frac{\partial \Delta_j}{\partial z_i}(z).$$

The following formulae are immediate by the logarithmic differentiation of (4-3).

Lemma 4-1. *With the above notation, we have*

$$\begin{cases} \frac{\partial \lambda}{\partial z_i}(z) = \lambda(z) \Delta_i(z), & 1 \leq i \leq n \\ \frac{\partial(\lambda f_\ell)}{\partial z_i}(z) = \lambda \left(\frac{\partial f_\ell}{\partial z_i} + \Delta_i f_\ell \right), & 1 \leq i, \ell \leq n. \end{cases}$$

From these formulae, the matrix $M_{\tilde{f}_z} \in \text{SL}_{n+1}(\mathbb{C})$ of the linear approximation of the canonical lift $\tilde{f}$ is given as follows.

$$\begin{aligned} (4-4) \quad M_{\tilde{f}_z} &= \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) \\ &= \lambda \begin{pmatrix} 1 & \Delta_1 & \dots & \Delta_n \\ f_1 & \frac{\partial f_1}{\partial z_1} + \Delta_1 f_1 & \dots & \frac{\partial f_1}{\partial z_n} + \Delta_n f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & \frac{\partial f_n}{\partial z_1} + \Delta_1 f_n & \dots & \frac{\partial f_n}{\partial z_n} + \Delta_n f_n \end{pmatrix} \\ &= \lambda \begin{pmatrix} 1 & & & & \\ f_1 & 1 & & & \\ f_2 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ f_n & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & \Delta_1 & \Delta_2 & \dots & \Delta_n \\ 0 & \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \dots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \dots & \frac{\partial f_n}{\partial z_n} \end{pmatrix} \end{aligned}$$

All the entries of the above matrices are functions in z . Here, the first matrix has nonzero entries only along the main diagonal and along the first column.

We calculate the inverse of the matrix $M_{\tilde{f}_z}$. First observe that

$$(4-5) \quad \begin{pmatrix} 1 & & & & \\ f_1 & 1 & & & \\ f_2 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ f_n & 0 & 0 & \dots & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & \\ -f_1 & 1 & & & \\ -f_2 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -f_n & 0 & 0 & \dots & 1 \end{pmatrix}$$

For the inverse of the second matrix factor of $M_{\tilde{f}_z}$ in (4-4), we can directly check the following identity.

$$(4-6) \quad \begin{pmatrix} 1 & \Delta_1 & \Delta_2 & \dots & \Delta_n \\ 0 & \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \dots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \dots & \frac{\partial f_n}{\partial z_n} \end{pmatrix} \begin{pmatrix} 1 & -\sum_{\ell=1}^n \Delta_\ell \zeta_{\ell 1} & \dots & -\sum_{\ell=1}^n \Delta_\ell \zeta_{\ell n} \\ 0 & \zeta_{11} & \dots & \zeta_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \zeta_{n1} & \dots & \zeta_{nn} \end{pmatrix} = I_{n+1}$$

Here $\zeta = (\zeta_{ij}) = (\partial f_i / \partial z_j)^{-1}$ is the inverse matrix of the Jacobian matrix of the local biholomorphism $f : \Omega \rightarrow \mathbb{C}^n$. So we have the following identities.

$$(4-7) \quad \sum_{i=1}^n \frac{\partial f_k}{\partial z_i} \zeta_{i\ell} = \delta_{k\ell}, \quad 1 \leq k, \ell \leq n.$$

From (4-5) and (4-6), the inverse matrix of $M_{\tilde{f}_z}$ is given by

$$(4-8) \quad M_{\tilde{f}_z}^{-1} = \lambda^{-1} \begin{pmatrix} 1 & -\sum_{\ell=1}^n \Delta_\ell \zeta_{\ell 1} & \dots & -\sum_{\ell=1}^n \Delta_\ell \zeta_{\ell n} \\ 0 & \zeta_{11} & \dots & \zeta_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \zeta_{n1} & \dots & \zeta_{nn} \end{pmatrix} \begin{pmatrix} 1 & & & \\ -f_1 & 1 & & \\ -f_2 & 0 & 1 & \\ \vdots & \vdots & \vdots & \ddots \\ -f_n & 0 & 0 & \dots & 1 \end{pmatrix}$$

where $\lambda(z)^{n+1} \cdot J_f(z) \equiv 1$. To calculate the fundamental Schwarzian $S_{ij}[f]$, we also need to calculate the second order partial derivatives of $\tilde{f}$. A direct calculation using Lemma 4-1 shows that

$$(4-9) \quad \begin{aligned} \frac{\partial^2 \tilde{f}}{\partial z_i \partial z_j}(w) &= \lambda \begin{pmatrix} \Delta_i \Delta_j + \Delta_{ij} \\ \frac{\partial^2 f_1}{\partial z_i \partial z_j} + \Delta_i \frac{\partial f_1}{\partial z_j} + \Delta_j \frac{\partial f_1}{\partial z_i} + (\Delta_i \Delta_j + \Delta_{ij}) f_1 \\ \vdots \\ \frac{\partial^2 f_n}{\partial z_i \partial z_j} + \Delta_i \frac{\partial f_n}{\partial z_j} + \Delta_j \frac{\partial f_n}{\partial z_i} + (\Delta_i \Delta_j + \Delta_{ij}) f_n \end{pmatrix} \\ &= \lambda \begin{pmatrix} 1 & & & \\ f_1 & 1 & & \\ f_2 & 0 & 1 & \\ \vdots & \vdots & \vdots & \ddots \\ f_n & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \Delta_i \Delta_j + \Delta_{ij} \\ \frac{\partial^2 f_1}{\partial z_i \partial z_j} + \Delta_i \frac{\partial f_1}{\partial z_j} + \Delta_j \frac{\partial f_1}{\partial z_i} \\ \vdots \\ \frac{\partial^2 f_n}{\partial z_i \partial z_j} + \Delta_i \frac{\partial f_n}{\partial z_j} + \Delta_j \frac{\partial f_n}{\partial z_i} \end{pmatrix} \end{aligned}$$

Note that the factors at the beginning of the both formulae in (4-4) and (4-9) are the same. Using the calculations of $M_{\tilde{f}_z}^{-1}$ in (4-8) and of $\partial^2 \tilde{f} / \partial z_i \partial z_j$ in (4-9), we see that the fundamental Schwarzian $S_{ij}[f] : \Omega \rightarrow \mathbb{C}^{n+1}$ for any pair of indices $1 \leq i, j \leq n$ is given by

(4-10)

$$\begin{aligned} S_{ij}[f](z) &= M_{\tilde{f}_z}^{-1} \cdot \frac{\partial^2 \tilde{f}}{\partial z_i \partial z_j}(z) \\ &= \begin{pmatrix} 1 & -\sum_{\ell=1}^n \Delta_\ell \zeta_{\ell 1} & \cdots & -\sum_{\ell=1}^n \Delta_\ell \zeta_{\ell n} \\ 0 & \zeta_{11} & \cdots & \zeta_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \zeta_{n1} & \cdots & \zeta_{nn} \end{pmatrix} \begin{pmatrix} \Delta_i \Delta_j + \Delta_{ij} \\ \frac{\partial^2 f_1}{\partial z_i \partial z_j} + \Delta_i \frac{\partial f_1}{\partial z_j} + \Delta_j \frac{\partial f_1}{\partial z_i} \\ \vdots \\ \frac{\partial^2 f_n}{\partial z_i \partial z_j} + \Delta_i \frac{\partial f_n}{\partial z_j} + \Delta_j \frac{\partial f_n}{\partial z_i} \end{pmatrix} \\ &= \begin{pmatrix} \Delta_{ij} - \Delta_i \Delta_j - \sum_{k,\ell=1}^n \Delta_\ell \zeta_{\ell k} \frac{\partial^2 f_k}{\partial z_i \partial z_j} \\ \sum_{k=1}^n \zeta_{1k} \frac{\partial^2 f_k}{\partial z_i \partial z_j} + \Delta_i \delta_{1j} + \Delta_j \delta_{1i} \\ \vdots \\ \sum_{k=1}^n \zeta_{nk} \frac{\partial^2 f_k}{\partial z_i \partial z_j} + \Delta_i \delta_{nj} + \Delta_j \delta_{ni} \end{pmatrix} \end{aligned}$$

In the above calculation, we used a formula

$$\left(\sum_\ell \Delta_\ell \zeta_{\ell 1}, \sum_\ell \Delta_\ell \zeta_{\ell 2}, \dots, \sum_\ell \Delta_\ell \zeta_{\ell n} \right) \begin{pmatrix} \Delta_i \frac{\partial f_1}{\partial z_j} \\ \vdots \\ \Delta_i \frac{\partial f_n}{\partial z_j} \end{pmatrix} = \Delta_i \Delta_j,$$

which follows from (4-7). From these calculations, we have obtained the following result which gives the explicit expression of the fundamental Schwarzian $S_{ij}[f]$ as nonlinear partial differential operator in n complex variables $z_1, z_2, \dots, z_n$.

Proposition 4-2 (Fundamental Schwarzians as Partial Differential Operators). *Let a locally biholomorphic map $f : \Omega \rightarrow \mathbb{C}^n \subset \mathbb{CP}^n$ factoring through $\mathbb{C}^n$ be given by $f(z) = {}^t(f_1(z), \dots, f_n(z))$. Then the ℓ -th component $S_{ij}^{(\ell)}[f] : \Omega \rightarrow \mathbb{C}$, $0 \leq \ell \leq n$ of the fundamental Schwarzian $S_{ij}[f] : \Omega \rightarrow \mathbb{C}^{n+1}$ for $1 \leq i, j \leq n$ is given as follows.*

$$(4-11) \quad \left\{ \begin{aligned} S_{ij}^{(0)}[f](z) &= \Delta_{ij}(z) - \Delta_i(z) \Delta_j(z) - \sum_{k,\ell=1}^n \Delta_\ell(z) \zeta_{\ell k}(z) \frac{\partial^2 f_k}{\partial z_i \partial z_j}(z) \\ &= \Delta_{ij}(z) + \Delta_i(z) \Delta_j(z) - \sum_{\ell=1}^n \Delta_\ell(z) S_{ij}^{(\ell)}[f](z), \\ S_{ij}^{(\ell)}[f](z) &= \sum_{k=1}^n \zeta_{\ell k}(z) \frac{\partial^2 f_k}{\partial z_i \partial z_j}(z) + \Delta_i(z) \delta_{\ell j} + \Delta_j(z) \delta_{\ell i}, \quad 1 \leq \ell \leq n. \end{aligned} \right.$$

Here, if $J_f(z)$ denotes the Jacobian determinant of $f : \Omega \rightarrow \mathbb{C}^n$,

$$\Delta_i(z) = \frac{-1}{n+1} \frac{1}{J_f} \frac{\partial J_f}{\partial z_i}, \quad \Delta_{ij}(z) = \frac{\partial \Delta_i}{\partial z_j}(z), \quad \zeta_{ij}(z) = \left(\frac{\partial f_\star}{\partial z_\star}(z) \right)_{ij}^{-1}.$$

For a single variable case, the classical Schwarzian derivative $S[f]$ for a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ with nonvanishing derivative plays an important role. Recall that $S[f]$ is given by

$$S[f] = -\frac{1}{2} \left(\frac{f''}{f'} \right)' + \frac{1}{4} \left(\frac{f''}{f'} \right)^2.$$

Although $S[f]$ involves the third order derivative of f , its “virtual” order is two. We discuss more about virtual orders for homogeneous differential operators below. We compare our fundamental Schwarzian $S_2[f] : \Omega \rightarrow \mathbb{C}^2$ in the single variable case with the classical Schwarzian derivative $S[f] : \Omega \rightarrow \mathbb{C}$. For this, since $J_f = f'$ when $n = 1$, we observe that

$$\Delta_1 = -\frac{1}{2} \left(\frac{f''}{f'} \right), \quad \Delta_{11} = -\frac{1}{2} \left(\frac{f''}{f'} \right)', \quad \zeta_{11} = \frac{1}{f'}.$$

Thus, from the formula (4-11) of our Schwarzian in Proposition 4-2

$$\begin{aligned} S_2^{(0)}[f](w) &= \Delta_{11} - \Delta_1^2 - \Delta_1 \zeta_{11} \frac{\partial^2 f}{\partial z^2} = -\frac{1}{2} \left\{ \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right\} = S[f], \\ S_2^{(1)}[f](w) &= \zeta_{11} \frac{\partial^2 f}{\partial z^2} + \Delta_1 + \Delta_1 = \left(\frac{f''}{f'} \right) + 2 \left(\frac{-1}{2} \right) \left(\frac{f''}{f'} \right) = 0. \end{aligned}$$

Hence, in the single variable case, the classical Schwarzian derivative and our fundamental Schwarzian are essentially the same and their explicit relationship is given as follows.

Corollary 4-3 (The Fundamental Schwarzian for $n = 1$). *Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \rightarrow \mathbb{C} \subset \mathbb{CP}^1$ be a holomorphic function such that f' doesn't vanish on Ω . Then, the fundamental Schwarzian $S_2[f] : \Omega \rightarrow \mathbb{C}^2$ is given by*

$$(4-12) \quad S_2[f](z) = \begin{pmatrix} S[f](z) \\ 0 \end{pmatrix},$$

where $S[f] = -\frac{1}{2} \left(\frac{f''}{f'} \right)' + \frac{1}{4} \left(\frac{f''}{f'} \right)^2$ is the classical Schwarzian derivative.

Remark. In (4-12), the vanishing of the second component $S_2^{(1)}[f]$ in the single variable case is an immediate consequence of the trace condition (3-12).

Example 4-4 (Higher Schwarzians for $n = 1$). The higher Schwarzians are rather difficult to write down in general even with the help of (3-16). However, in the single variable case, it is relatively easy to explicitly write down higher Schwarzians $S_k[f]$ for a holomorphic function f with nonvanishing derivative. When $n = 1$, the differential operator $D[f]$ associated to f is given by

$$(4-13) \quad D[f] = \frac{d}{dz} + \begin{pmatrix} 0 & S[f](z) \\ 1 & 0 \end{pmatrix}.$$

So, by (3-16), the higher Schwarzians are given by

$$(4-14) \quad S_k[f](z) = D[f]^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad k \geq 0.$$

For example, for small k , the higher Schwarzians are given explicitly as follows.

$$(4-15) \quad \begin{aligned} S_1[f](z) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad S_2[f](z) = \begin{pmatrix} S[f](z) \\ 0 \end{pmatrix}, \\ S_3[f](z) &= \begin{pmatrix} S[f]'(z) \\ S[f](z) \end{pmatrix}, \quad S_4[f](z) = \begin{pmatrix} S[f]''(z) + S[f](z)^2 \\ 2S[f]'(z) \end{pmatrix}, \\ S_5[f](z) &= \begin{pmatrix} S[f]'''(z) + 4S[f]'(z) \cdot S[f](z) \\ 3S[f]''(z) + S[f](z)^2 \end{pmatrix}, \quad \text{etc.} \end{aligned}$$

We now turn to the discussion on the virtual order for homogeneous differential operators. This homogeneity is one feature of the generalized Schwarzian operators. Let $\Omega \subset \mathbb{C}^n$ be a domain and let

$$\text{Hol}(\Omega, \mathbb{C}^N) = \{f : \Omega \rightarrow \mathbb{C}^N \mid f : \text{holomorphic}\}$$

be the set of $\mathbb{C}^N$ -valued holomorphic functions on Ω . To define the notion of homogeneous differential operators, let $\rho_r : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a dilation map given by $\rho_r(z_1, z_2, \dots, z_n) = (rz_1, rz_2, \dots, rz_n)$. First observe that for any $f \in \text{Hol}(\Omega, \mathbb{C}^N)$,

$$\begin{aligned} \frac{\partial^{|K|}}{\partial z^K}(\rho_r^* f)(z) &= \frac{\partial^{|K|}}{\partial z^K} f(rz_1, rz_2, \dots, rz_n) = r^{|K|} \cdot \frac{\partial^{|K|}}{\partial z^K} f(rz_1, rz_2, \dots, rz_n) \\ &= r^{|K|} \cdot \rho_r^* \left(\frac{\partial^{|K|}}{\partial z^K} f \right)(z). \end{aligned}$$

We generalize this observation to define homogeneous differential operators.

Definition 4-5 (Homogeneous Differential Operators). A differential operator $P(\cdot)$ acting on $\text{Hol}(\Omega, \mathbb{C}^N)$ is homogeneous of virtual order $d \in \mathbb{Q}$ if

$$P(\rho_r^* f)(z) = r^d \cdot \rho_r^*(P(f))(z), \quad \text{for any } f \in \text{Hol}(\Omega, \mathbb{C}^N), \quad r \in \mathbb{C}^*.$$

Homogeneous differential operators are nonlinear in general and they depend algebraically on derivatives of components of $f \in \text{Hol}(\Omega, \mathbb{C}^N)$. In this paper, we are interested in those differential operators whose dependence on derivatives of f is rational, i.e., no radicals are present in the expression. For a rational homogeneous differential operator, all the terms in the numerator of the rational expression of the operator involve the same number of differentiations, similarly for terms in the denominator. The difference of these numbers is the virtual order of the homogeneous differential operator. So, for rational homogeneous differential operators, the virtual order is always an integer, $d \in \mathbb{Z}$.

The components of our generalized Schwarzian operators are homogeneous differential operators acting on local biholomorphisms $f : \Omega \rightarrow \mathbb{CP}^n$.

Lemma 4-6 (Homogeneity of $S_{ij}[\cdot]$). *For any $1 \leq \ell \leq n$ and $1 \leq i, j \leq n$, the ℓ -th component of the fundamental Schwarzian operator $S_{ij}^{(\ell)}[\cdot]$ is a homogeneous differential operator of virtual order 1. On the other hand, 0-th component $S_{ij}^{(0)}[\cdot]$ of the fundamental Schwarzian $S_{ij}[\cdot]$ is homogeneous of virtual order 2.*

The classical Schwarzian derivative is homogeneous of virtual order 2.

Proof. Let Λ be the Jacobian matrix of f . Since $J_f = \det \Lambda$ is a sum of monomials each of which is a product of n first derivatives of f , J_f is an n th order homogeneous differential operator. So, the virtual order of Δ_i is $(n+1) - n = 1$. Each component ζ_{mk} of Λ^{-1} is a quotient $\pm \det \Lambda_{mk} / \det \Lambda$ where Λ_{mk} is the matrix obtained from Λ by deleting m th row and k th column. So, $\det \Lambda_{mk}$ is a homogeneous operator of virtual order $(n-1)$. Thus, the virtual order of ζ_{mk} is $(n-1) - n = -1$. Hence each term in $S_{ij}^{(m)}[f]$ has virtual order 1.

Similar order counting shows that $S_{ij}^{(0)}[f]$ has virtual order 2. When $n = 1$, the 0-th component reduces to the classical Schwarzian derivative. So, it is also homogeneous of virtual order 2. $\square$

For higher Schwarzian differential operators $S_K[\cdot]$, a similar statement holds, although their expressions become increasingly complicated and the direct calculation of the virtual order is impractical.

Proposition 4-7. *For any multi-index $K \in \mathbb{Z}_+^n$ with $|K| \geq 2$, each component of the generalized Schwarzian $S_K[\cdot]$ is a homogeneous differential operator such that*

- (i) $S_K^{(0)}[\cdot]$ is of virtual order $|K|$,
- (ii) $S_K^{(\ell)}[\cdot]$ is of virtual order $|K| - 1$ for $1 \leq \ell \leq n$.

Proof. By induction on $|K|$. When $|K| = 2$, the above statement is proved in Lemma 4-6. Now suppose that the above statement holds for $|K| \leq k$ for some $k \geq 2$. From (3-16) in Theorem 3-12, for $1 \leq i \leq n$ we have

$$S_{K+\sigma_i}[f](z) = D_i(S_K[f](z)) = \frac{\partial}{\partial z_i} S_K[f] + (e_{i+1}, S_{i1}[f], \dots, S_{in}[f]) \cdot S_K[f].$$

Here, σ_i is a row vector with 1 at the i th entry and 0 for the remaining entries. Writing out the components of the above column vectors explicitly, we have

$$\begin{aligned} S_{K+\sigma_i}^{(0)}[f](z) &= \frac{\partial S_K^{(0)}[f]}{\partial z_i} + \sum_{j=1}^n S_{ij}^{(0)}[f] \cdot S_K^{(j)}[f], \\ S_{K+\sigma_i}^{(\ell)}[f](z) &= \frac{\partial S_K^{(\ell)}[f]}{\partial z_i} + \sum_{j=1}^n S_{ij}^{(\ell)}[f] \cdot S_K^{(j)}[f], \quad 1 \leq \ell \leq n, \ell \neq i, \\ S_{K+\sigma_i}^{(i)}[f](z) &= \frac{\partial S_K^{(i)}[f]}{\partial z_i} + S_K^{(0)}[f] + \sum_{j=1}^n S_{ij}^{(i)}[f] \cdot S_K^{(j)}[f]. \end{aligned}$$

The inductive hypothesis now shows that for $1 \leq i \leq n$, $S_{K+\sigma_i}^{(0)}[f]$ is homogeneous of virtual order $|K| + 1$, and $S_{K+\sigma_i}^{(\ell)}[f]$ is homogeneous of virtual order $|K|$ for $1 \leq \ell \leq n$, whether $\ell = i$ or not. This completes the inductive step and the proof of Proposition 4-7 is now complete. $\square$

§5 GENERATING MÖBIUS INVARIANT DIFFERENTIAL OPERATORS

Previously in §3 and §4, we showed that the generalized Schwarzian operators $S_K[\cdot]$ for $K \in \mathbb{Z}_+^n$ are Möbius invariant and their components are homogeneous differential operators which can be written as integral polynomials in derivatives of virtual order 1 homogeneous Schwarzian operators $S_{ij}^{(\ell)}[\cdot]$, $1 \leq i, j, \ell \leq n$. More is true. In fact, these virtual order 1 homogeneous Schwarzian operators and their derivatives generate the entire set of Möbius invariant differential operators acting on the space of locally biholomorphic maps from a domain $\Omega \subset \mathbb{C}^n$ to the projective space $\mathbb{CP}^n$.

A differential operator $I[\cdot] : \text{Hol}(\Omega, \mathbb{CP}^n) \rightarrow \text{Hol}(\Omega, \mathbb{C})$ is Möbius invariant if

$$(5-1) \quad I[T \circ f] = I[f],$$

for any $T \in \text{PSL}_{n+1}(\mathbb{C})$ and for any $f \in \text{Hol}(\Omega, \mathbb{CP}^n)$. We shall study Möbius invariant differential operators. We first prepare one lemma.

Lemma 5-1. *Let $I[\cdot]$ be a (rational) differential operator acting on locally biholomorphic maps $f : \Omega \rightarrow \mathbb{CP}^n$. Let $\tilde{f}(z) = (\tilde{f}_0(z), \tilde{f}_1(z), \dots, \tilde{f}_n(z))$ be its canonical lift. Then, $I[f] : \Omega \rightarrow \mathbb{C}$ depends algebraically (or rationally) on derivatives $\partial^{|K|} \tilde{f}_\ell / \partial z^K$, where $K \in \mathbb{Z}_+^n$ and $0 \leq \ell \leq n$. This expression is independent of the choice of the canonical lift $\tilde{f}$.*

Proof. For $1 \leq i \leq n$, let $U_i = \{[\zeta_0 : \zeta_1 : \dots : \zeta_n] \in \mathbb{CP}^n \mid \zeta_i \neq 0\} \subset \mathbb{CP}^n$ be the standard coordinate neighborhood on $\mathbb{CP}^n$. Let $\varphi : U_i \xrightarrow{\cong} \mathbb{C}^n$ be the standard local coordinate. Let the canonical lift $\tilde{f}$ be given by $\tilde{f}(z) = (\tilde{f}_0(z), \tilde{f}_1(z), \dots, \tilde{f}_n(z))$. On the domain $f^{-1}(U_i)$, $I[f]$ depends algebraically (or rationally) on derivatives of $\varphi_i \circ f = (\tilde{f}_0/\tilde{f}_i, \dots, \tilde{f}_{i-1}/\tilde{f}_i, \tilde{f}_{i+1}/\tilde{f}_i, \dots, \tilde{f}_n/\tilde{f}_i)$ by definition of our differential operators. So, $I[f]$ is independent of the choice of the canonical lift of f since it depends on the quotients of the components of $\tilde{f}$: $I[f]$ doesn't change when $\tilde{f}$ is multiplied by an $(n+1)$ st root of unity. Hence $I[f]$ depends algebraically (or rationally) on derivatives of $\tilde{f}_j(z)$'s. This expression in terms of the components of the canonical lift $\tilde{f}$ is independent of the choice of the local coordinate neighborhood U_i used. $\square$

Let the expression of $I[f]$ in terms of the derivatives of the components of the canonical lift be given by

$$(5-2) \quad I[f] = E \left[\frac{\partial^{|K|} \tilde{f}}{\partial z^K} \mid K \in \mathbb{Z}_+^n \right].$$

Here $E[\cdot]$ is an algebraic (or rational) expression in terms of objects inside the bracket $[\cdot]$. Now we state our theorem.

Theorem 5-2 (Generating Möbius invariant differential operators). *Let $I[\cdot]$ be a Möbius invariant (rational) differential operator acting on locally biholomorphic maps $f : \Omega \rightarrow \mathbb{CP}^n$.*

- (i) *When $n \geq 2$, $I[f] : \Omega \rightarrow \mathbb{C}$ depends algebraically (or rationally, respectively) on derivatives of virtual order 1 homogeneous Schwarzians $S_{ij}^{(\ell)}[f]$, $1 \leq i, j \leq n$, where $1 \leq \ell \leq n$.*

(ii) When $n = 1$, $I[f]$ depends only on derivatives of the 0-th component of the Schwarzian $S_2^{(0)}[f]$ which is homogeneous of virtual order 2.

Restricting biholomorphisms f to those which factor through $\mathbb{C}^n \subset \mathbb{CP}^n$, we can obtain equivalent, but more explicit statement in terms of actual explicit differential operators.

Theorem 5-2'. *Suppose the local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$ factors through $\mathbb{C}^n \subset \mathbb{CP}^n$ and let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$. Then, when $n \geq 2$, any Möbius invariant (rational) differential operator $I[f]$ depends algebraically (or rationally, respectively) on the derivatives of*

$$S_{ij}^{(\ell)}[f](z) = \sum_{k=1}^n \zeta_{\ell k} \frac{\partial^2 f_k}{\partial z_i \partial z_j} + \Delta_i \delta_{\ell j} + \Delta_j \delta_{\ell i}, \quad 1 \leq i, j, \ell \leq n,$$

where Δ_i 's and $\zeta_{\ell k}$'s are as before.

When $n = 1$, then $I[f]$ depends only on derivatives of the classical Schwarzian derivative

$$S[f] = -\frac{1}{2} \left(\frac{f''}{f'} \right)' + \frac{1}{4} \left(\frac{f''}{f'} \right)^2.$$

(*Proof of Theorem 5-2*). For $w \in \Omega$, let $f_w(z) = f(w+z)$ for $z \in \Omega_w = \Omega - w \subset \mathbb{C}^n$. If $\tilde{f}(z)$ is the canonical lift of f , then the canonical lift of f_w is given by $\tilde{f}_w(z) = \tilde{f}(w+z)$. Obviously, we have $\partial^{|K|} \tilde{f}_w / \partial z^K(0) = \partial^{|K|} \tilde{f} / \partial z^K(w)$ for all $K \in \mathbb{Z}_+^n$. Hence

$$(*) \quad I[f_w](0) = E \left[\frac{\partial^{|K|} \tilde{f}_w}{\partial z^K}(0) \right] = E \left[\frac{\partial^{|K|} \tilde{f}}{\partial z^K}(w) \right] = I[f](w).$$

When $f : \Omega \rightarrow \mathbb{CP}^n$ is a biholomorphism, then the matrix $M_{\tilde{f}_w}$ of linear approximation of the canonical lift $\tilde{f}_w(z)$ at $z = 0$ is invertible. So, the generating map for the generalized Schwarzians $S_w[f] = M_{\tilde{f}_w}^{-1} \circ \tilde{f}_w : \Omega_w \rightarrow \mathbb{C}^{n+1}$ is defined and is the canonical lift of the map $M_{f_w}^{-1} \circ f_w : \Omega \rightarrow \mathbb{CP}^n$, where $M_{f_w} \in \text{PSL}_{n+1}(\mathbb{C})$. Since our differential operator $I[\cdot]$ is Möbius invariant, by (5-2) we have

$$(**) \quad I[f_w](z) = I[M_{f_w}^{-1} \circ f_w](z) = E \left[\frac{\partial^{|K|} S_w[f]}{\partial z^K}(z) \mid K \in \mathbb{Z}_+^n \right].$$

From the definition (3-3) of the generalized Schwarzian operators and (3-16) in Theorem 3-7, we have

$$\frac{\partial^{|K|} S_w[f]}{\partial z^K}(0) = S_K[f](w) = (D^K[f] \cdot p_0)(w), \quad K \in \mathbb{Z}_+^n.$$

Evaluating (**) at $z = 0$ and using (*), we obtain

$$(***) \quad I[f](w) = E[(D^K[f] \cdot p_0)(w) \mid K \in \mathbb{Z}_+^n].$$

When $n \geq 2$, for any multi-index K , $D^K[f] \cdot p_0$ is an integral polynomial in derivatives of components of fundamental Schwarzians $S_{ij}^{(\ell)}[f]$ with $1 \leq i, j \leq n$ by Theorem 3-7. Furthermore, by Theorem 3-8, 0-th components of $S_{ij}[f]$'s depend on the other components. So, we see that $I[f]$ depends algebraically (or rationally) on derivatives of virtual order 1 homogeneous Schwarzians $S_{ij}^{(\ell)}[f]$, where $1 \leq \ell \leq n$ and $1 \leq i, j \leq n$.

When $n = 1$, $D^K[f]$ depends only on the 0-th component of $S_2[f]$, because the other component $S_2^{(1)}[f]$ identically vanishes by Corollary 4-3. But by (4-12), $S_2^{(0)}[f]$ is exactly the classical Schwarzian derivative which is homogeneous of virtual order 2. Thus (***) becomes $I[f](w) = E[S[f](w)]$ showing that $I[f]$ depends only on derivatives of the classical Schwarzian derivative $S[f]$. This completes the proof of Theorem 5-2. $\square$

We remark that the one dimensional case was proved by M. Lavie in [L] in a different method. But our method above using higher Schwarzians applies to all n .

§6 THE COMPOSITION FORMULA FOR FUNDAMENTAL SCHWARZIAN

Given a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$, our generalized Schwarzians of f are Möbius invariant in the sense that $S_K[T \circ f] = S_K[f]$ for any $T \in \text{PSL}_{n+1}(\mathbb{C})$ and for any $K \in \mathbb{Z}_+^n$. In this section, we study the behaviour of the Schwarzians with respect to the pre-composition by Möbius transformations. We cannot expect invariance in this context. Since $\text{PSL}_{n+1}(\mathbb{C})$ acts transitively on $\mathbb{CP}^n$, for any two points $z_1, z_2 \in \Omega \subset \mathbb{C}^n \subset \mathbb{CP}^n$ there exists $T \in \text{PSL}_{n+1}(\mathbb{C})$ such that $z_1 = T(z_2)$. So, $S_K[f](Tz) = S_K[f](z)$ for all possible T implies that $S_K[f]$ is a constant function on Ω , which is not the case in general.

Let $f : \Omega \rightarrow \mathbb{CP}^n$ be a locally biholomorphic map from a domain Ω in $\mathbb{C}^n$ and let $T : \Omega' \rightarrow \Omega$ be a Möbius transformation from some domain $\Omega' \subset \mathbb{C}^n$. Their composition $h = f \circ T : \Omega' \rightarrow \mathbb{CP}^n$ is another locally biholomorphic map. We compare the generalized Schwarzians of f and h . Since the higher Schwarzians can be described in terms of the fundamental Schwarzians as in (3-16), we only have to describe $S_K[f \circ T]$ with $|K| = 2$ in terms of $S_{K'}[f]$. In fact, we will find that the total covariant derivatives (6-5) below give the most concise description of the relationship.

Actually, we work in a more general context and let $g : \Omega' \rightarrow \Omega$ be an arbitrary locally biholomorphic map, not necessarily a restriction of a projective linear map. Let $z \in \Omega$, $\zeta \in \Omega'$ be variables in these domains so that $z = g(\zeta)$. Then, $h(\zeta) = (f \circ g)(\zeta)$ is again a locally biholomorphic map. Let $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ and $\tilde{h} : \Omega' \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be the canonical lifts of f and h , respectively. Let their matrices of linear approximation be

$$(6-1) \quad \begin{aligned} M(z) &= \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) \in \text{SL}_{n+1}(\mathbb{C}), \\ H(\zeta) &= \left(\tilde{h}(\zeta), \frac{\partial \tilde{h}}{\partial \zeta_1}(\zeta), \dots, \frac{\partial \tilde{h}}{\partial \zeta_n}(\zeta) \right) \in \text{SL}_{n+1}(\mathbb{C}). \end{aligned}$$

Let $A_i[f](z), A_j[f \circ g](\zeta) \in \mathfrak{sl}_{n+1}(\mathbb{C})$ be given by

$$(6-2) \quad \frac{\partial M}{\partial z_i}(z) = M(z) \cdot A_i[f](z), \quad \frac{\partial H}{\partial \zeta_j}(\zeta) = H(\zeta) \cdot A_j[f \circ g](\zeta), \quad 1 \leq i, j \leq n.$$

The appearance of the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ is due to Lemma 3-5. By (3-9) and (3-10), the fundamental Schwarzians of f and $h = f \circ g$ are given by

$$(6-3) \quad \begin{aligned} A_i[f](z) &= (e_{i+1}, S_{i1}[f](z), S_{i2}[f](z), \dots, S_{in}[f](z)), \\ A_j[f \circ g](\zeta) &= (e_{j+1}, S_{j1}[f \circ g](\zeta), S_{j2}[f \circ g](\zeta), \dots, S_{jn}[f \circ g](\zeta)). \end{aligned}$$

As before, we define partial covariant derivatives as follows.

$$(6-4) \quad D_i^z[f] = \frac{\partial}{\partial z_i} + A_i[f](z), \quad D_j^\zeta[f \circ g] = \frac{\partial}{\partial \zeta_j} + A_j[f \circ g](\zeta), \quad 1 \leq i, j \leq n.$$

From these partial covariant derivatives, we form the total covariant derivatives:

$$(6-5) \quad D^z[f] = \sum_{i=1}^n dz_i \cdot D_i^z[f], \quad D^\zeta[f \circ g] = \sum_{j=1}^n d\zeta_j \cdot D_j^\zeta[f \circ g].$$

Let $\tilde{g} : \Omega' \rightarrow \tilde{\Omega} \subset \mathbb{C}^{n+1} \setminus \{0\}$ be the canonical lift of $g : \Omega' \rightarrow \Omega$ and let its matrix of linear approximation at ζ be

$$(6-6) \quad G(\zeta) = \left(\tilde{g}(\zeta), \frac{\partial \tilde{g}}{\partial \zeta_1}(\zeta), \frac{\partial \tilde{g}}{\partial \zeta_2}(\zeta), \dots, \frac{\partial \tilde{g}}{\partial \zeta_n}(\zeta) \right).$$

The relationship between the fundamental Schwarzians $S_{ij}[f \circ g]$'s and $S_{k\ell}[f]$'s is given concisely in terms of their associated total covariant derivatives $D^z[f]$ and $D^\zeta[f \circ g]$. For the statement of our theorem, let a matrix $W_g(\zeta)$ be defined by

$$(6-7) \quad W_g(\zeta) = \begin{pmatrix} 1 & & & & \\ -z_1 & 1 & & & \\ -z_2 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -z_n & 0 & 0 & \dots & 1 \end{pmatrix} \cdot G(\zeta),$$

where in the first matrix factor all the entries above the diagonal is zero, and z_i is the i th component of the map $z = g(\zeta) \in \mathbb{C}^n$.

Theorem 6-1 (Composition Formula for Fundamental Schwarzians). *Let $f : \Omega \rightarrow \mathbb{CP}^n$, $g : \Omega' \rightarrow \Omega$ be local biholomorphisms and let $D^z[f]$ and $D^\zeta[f \circ g]$ be the total covariant derivatives associated to f and $f \circ g$. Then, $D^\zeta[f \circ g]$ is $W_g(\zeta)$ conjugate to the pull back of $D^z[f]$ by g :*

$$(6-8) \quad D^\zeta[f \circ g] = W_g(\zeta)^{-1} \cdot g^*(D^z[f]) \cdot W_g(\zeta).$$

In other words, we have

$$(6-9) \quad A_j[f \circ g](\zeta) = \sum_{i=1}^n W_g(\zeta)^{-1} \cdot A_i[f](g(\zeta)) \cdot W_g(\zeta) \cdot \frac{\partial z_i}{\partial \zeta_j} + W_g(\zeta)^{-1} \cdot \frac{\partial W_g(\zeta)}{\partial \zeta_j}.$$

Remark. The first matrix factor for $W(\zeta)$ in (6-7) looks somewhat unnatural. This factor arises from the fact that our generalized Schwarzians are defined with respect to variables on the base manifolds Ω, Ω' rather than the standard coordinate variables on $E(\xi) - \{\text{zero section}\} \cong \mathbb{C}^{n+1} \setminus \{0\}$, where $E(\xi)$ is the total space of the tautological line bundle ξ over $\mathbb{CP}^n$.

To prove Theorem 6-1, we first calculate the canonical lift $\tilde{h}$ of $h = f \circ g$. This canonical lift is different from the composition $\tilde{f} \circ g : \Omega' \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$, and we need a correction factor.

Lemma 6-2 (Canonical lifts of composed maps). *With the above notation, the local canonical lift $\tilde{h}$ is given by*

$$\tilde{h}(\zeta) = \eta_g(\zeta) \cdot (\tilde{f} \circ g)(\zeta).$$

Here, $\tilde{f}$ is the local canonical lift of f and $\eta_g(\zeta)^{n+1} \cdot J_g(\zeta) \equiv 1$, where $J_g(\zeta)$ is the Jacobian determinant of the map $g : \Omega' \rightarrow \Omega$. If the domain Ω' is simply connected, then the canonical lift $\tilde{h}$ exists globally on Ω' .

Proof. Since the composition $\tilde{f} \circ g : \Omega' \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ is a lift of $h = f \circ g$, the canonical lift of h is of the form $\tilde{h} = \eta_g(\zeta) \cdot (\tilde{f} \circ g)(\zeta)$ for some nowhere vanishing holomorphic function η_g defined on a subdomain of Ω' . The function $\eta_g(\zeta)$ is determined by the determinant condition of the canonical lift:

$$\det\left(\tilde{h}(\zeta), \frac{\partial \tilde{h}}{\partial \zeta_1}(\zeta), \frac{\partial \tilde{h}}{\partial \zeta_2}(\zeta), \dots, \frac{\partial \tilde{h}}{\partial \zeta_n}(\zeta)\right) \equiv 1.$$

Since $\frac{\partial \tilde{h}}{\partial \zeta_j} = \frac{\partial \eta_g}{\partial \zeta_j} \cdot \tilde{f}(g(\zeta)) + \eta_g \cdot \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial z_i} \frac{\partial z_i}{\partial \zeta_j}$ for $1 \leq j \leq n$, the left hand side of the above identity is equal to

$$\begin{aligned} & \eta_g(\zeta)^{n+1} \cdot \det\left(\tilde{f} \circ g, \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial z_i} \frac{\partial z_i}{\partial \zeta_1}, \dots, \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial z_i} \frac{\partial z_i}{\partial \zeta_n}\right) \\ &= \eta_g(\zeta)^{n+1} \cdot \det\left(\tilde{f}(g(\zeta)), \frac{\partial \tilde{f}}{\partial z_1}(g(\zeta)), \dots, \frac{\partial \tilde{f}}{\partial z_n}(g(\zeta))\right) \cdot \det\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{\partial z_1}{\partial \zeta_1} & \dots & \frac{\partial z_1}{\partial \zeta_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial z_n}{\partial \zeta_1} & \dots & \frac{\partial z_n}{\partial \zeta_n} \end{pmatrix} \\ &= \eta_g(\zeta)^{n+1} \cdot J_g(\zeta). \end{aligned}$$

In the second formula, the first determinant is 1 since $\tilde{f}$ is the canonical lift. Hence the determinant condition on the canonical lift $\tilde{h}$ gives $\eta_g(\zeta)^{n+1} \cdot J_g(\zeta) \equiv 1$. So,

locally $\eta_g(\zeta) = J_g(\zeta)^{-\frac{1}{n+1}}$ and $\eta_g(\zeta)$ is determined up to a $(n+1)$ st root of unity. Note that the Jacobian determinant $J_g(\zeta)$ of a local biholomorphism $g : \Omega' \rightarrow \Omega$ never vanishes on Ω' .

If the domain Ω' is simply connected, then one can take global holomorphic $(n+1)$ st roots of unity of $J_g(\zeta)$ on Ω' and $\eta_g(\zeta)$ exists globally on Ω' . $\square$

Since the generalized Schwarzians $S_K[h]$ and $S_{K'}[f]$ are calculated from $H(\zeta)$ and $M(z)$, we clarify their relationship.

Lemma 6-3. *The matrices $M(z)$, $H(\zeta)$ of linear approximations of $\tilde{f}$ and $\tilde{h}$ are related as follows.*

$$H(\zeta) = M(g(\zeta)) \cdot W_g(\zeta),$$

where $M(g(\zeta))$ is $M(z)$ evaluated at $z = g(\zeta)$ and $W_g(\zeta)$ is as in (6-6) and (6-7).

Proof. The logarithmic differentiation of the identity $\eta_g(\zeta)^{n+1} \cdot J_g(\zeta) \equiv 1$ gives

$$(6-10) \quad \frac{1}{\eta_g} \frac{d\eta_g}{d\zeta_j} = -\frac{1}{n+1} \frac{1}{J_g} \frac{\partial J_g}{\partial \zeta_j}, \quad 1 \leq j \leq n,$$

where $J_g(\zeta)$ is the Jacobian determinant of the map g . We denote the above quantity by $\Delta_j(\zeta)$. So, $\partial \eta_g / \partial \zeta_j(\zeta) = \eta_g(\zeta) \cdot \Delta_j(\zeta)$ for $1 \leq j \leq n$. Now, the chain rule applied to the identity $\tilde{h}(\zeta) = \eta_g(\zeta) \cdot \tilde{f}(g(\zeta))$ in Lemma 6-2 gives

$$\frac{\partial \tilde{h}}{\partial \zeta_j} = \eta_g \cdot \Delta_j \cdot \tilde{f}(g(\zeta)) + \eta_g \cdot \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial z_i} \frac{\partial z_i}{\partial \zeta_j}, \quad 1 \leq j \leq n.$$

So, the matrix of the linear approximation $H(\zeta)$ in (6-1) for the canonical lift $\tilde{h}$ is given by

$$(6-11) \quad H(\zeta) = \left(\tilde{f}(z), \frac{\partial \tilde{f}}{\partial z_1}(z), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z) \right) \eta_g(\zeta) \begin{pmatrix} 1 & \Delta_1 & \dots & \Delta_n \\ 0 & \frac{\partial z_1}{\partial \zeta_1} & \dots & \frac{\partial z_1}{\partial \zeta_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial z_n}{\partial \zeta_1} & \dots & \frac{\partial z_n}{\partial \zeta_n} \end{pmatrix}.$$

Here $z = g(\zeta)$ and the first matrix factor is $M(g(\zeta))$. On the other hand, since the canonical lift $\tilde{g}$ is given by

$$\tilde{g}(\zeta) = \eta_g(\zeta) \cdot {}^t(1, z_1(\zeta), z_2(\zeta), \dots, z_n(\zeta))$$

with $\eta_g(\zeta)^{n+1} \cdot J_g(\zeta) \equiv 1$ for $1 \leq j \leq n$, we have

$$\frac{\partial \tilde{g}}{\partial \zeta_j} = \frac{\partial \eta_g}{\partial \zeta_j} \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} + \eta_g(\zeta) \begin{pmatrix} 0 \\ \frac{\partial z_1}{\partial \zeta_j} \\ \vdots \\ \frac{\partial z_n}{\partial \zeta_j} \end{pmatrix} = \eta_g(\zeta) \begin{pmatrix} \Delta_j \\ z_1 \cdot \Delta_j + \frac{\partial z_1}{\partial \zeta_j} \\ \vdots \\ z_n \cdot \Delta_j + \frac{\partial z_n}{\partial \zeta_j} \end{pmatrix}$$

where we used $\partial\eta_g/\partial\zeta_j(\zeta) = \eta_g(\zeta) \cdot \Delta_j(\zeta)$. Hence for the matrix $G(\zeta)$ of linear approximation of the canonical lift $\tilde{g}(\zeta)$, we have
(6-12)

$$G(\zeta) = \left(\tilde{g}, \frac{\partial\tilde{g}}{\partial\zeta_1}, \dots, \frac{\partial\tilde{g}}{\partial\zeta_n} \right) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ z_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_n & 0 & \dots & 1 \end{pmatrix} \eta_g(\zeta) \begin{pmatrix} 1 & \Delta_1 & \dots & \Delta_n \\ 0 & \frac{\partial z_1}{\partial\zeta_1} & \dots & \frac{\partial z_1}{\partial\zeta_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial z_n}{\partial\zeta_1} & \dots & \frac{\partial z_n}{\partial\zeta_n} \end{pmatrix}.$$

Thus, the last two factors in (6-11) are the same as the last two factors in (6-12). From this, Lemma 6-3 follows. $\square$

(*Proof of Theorem (6-1)*). Applying the chain rule to the matrix identity $H(\zeta) = M(z) \cdot W_g(\zeta)$ in Lemma 6-3, we obtain

$$\frac{\partial H}{\partial\zeta_j} = \left(\sum_{i=1}^n \frac{\partial M}{\partial z_i} \frac{\partial z_i}{\partial\zeta_j} \right) \cdot W_g(\zeta) + M(z) \cdot \frac{\partial W_g}{\partial\zeta_j}, \quad 1 \leq j \leq n.$$

Applying $H(\zeta)^{-1} = W_g(\zeta)^{-1} \cdot M(z)^{-1}$ from the left on the above identity and using (6-2), we obtain

$$A_j[f \circ g](\zeta) = W_g(\zeta)^{-1} \cdot \left(\sum_{i=1}^n \frac{\partial z_i}{\partial\zeta_j} A_i[f](g(\zeta)) \right) \cdot W_g(\zeta) + W_g(\zeta)^{-1} \cdot \frac{\partial W_g}{\partial\zeta_j}(\zeta).$$

We can simplify the formula in terms of the total covariant derivatives. Since $D_i^z = \partial/\partial z_i + A_i[f](z)$ for $1 \leq i \leq n$,

$$\begin{aligned} W_g^{-1} \left(\sum_{i=1}^n \frac{\partial z_i}{\partial\zeta_j} D_i^z[f] \right) W_g &= W_g^{-1} \left(\sum_{i=1}^n \frac{\partial z_i}{\partial\zeta_j} \frac{\partial}{\partial z_i} \right) W_g + W_g^{-1} \left(\sum_{i=1}^n \frac{\partial z_i}{\partial\zeta_j} A_i[f] \right) W_g \\ &= \frac{\partial}{\partial\zeta_j} + W_g^{-1} \frac{\partial W_g}{\partial\zeta_j} + W_g^{-1} \left(\sum_{i=1}^n \frac{\partial z_i}{\partial\zeta_j} A_i[f](g(\zeta)) \right) W_g \\ &= \frac{\partial}{\partial\zeta_j} + A_j[f \circ g](\zeta) = D_j^\zeta[f \circ g]. \end{aligned}$$

Here for the second equality, we used the identity $\partial/\partial\zeta_j = \sum_{i=1}^n (\partial z_i/\partial\zeta_j)(\partial/\partial z_i)$ for $1 \leq j \leq n$. In terms of the total covariant derivatives, the above identities for partial covariant derivatives for $1 \leq j \leq n$ imply that

$$\begin{aligned} D^\zeta[f \circ g] &= \sum_{j=1}^n d\zeta_j \cdot D_j^\zeta[f \circ g] = \sum_{j=1}^n W_g(\zeta)^{-1} \left(d\zeta_j \sum_{i=1}^n \frac{\partial z_i}{\partial\zeta_j} D_i^z[f] \right) W_g(\zeta) \\ &= W_g(\zeta)^{-1} g^* \left(\sum_{i=1}^n dz_i \cdot D_i^z[f] \right) W_g(\zeta) = W_g(\zeta)^{-1} \cdot g^*(D^z[f]) \cdot W_g(\zeta). \end{aligned}$$

This completes the proof of Theorem 6-1. $\square$

Example 6-4. The transformation law (or the composition law) for the classical Schwarzian derivative is well known. If f and g are composable holomorphic functions, then we have

$$(6-13) \quad S[f \circ g](\zeta) = S[f](g(\zeta)) \cdot g'(\zeta)^2 + S[g](\zeta).$$

We show that Theorem 6-1 reduces to (6-13) in the single variable case. So, let $f : \Omega \rightarrow \mathbb{C} \subset \mathbb{CP}^1$, $g : \Omega' \rightarrow \Omega$ be holomorphic functions with nonvanishing derivatives, where Ω, Ω' are domains in $\mathbb{C}$. Let $z \in \Omega$, $\zeta \in \Omega'$ be variables in these domains. Then, from (4-12), (6-3), (6-4), (6-5), the total covariant derivatives associated to f and $f \circ g$ are given by

$$D^z = dz \cdot \frac{d}{dz} + \begin{pmatrix} 0 & S[f](z) \\ 1 & 0 \end{pmatrix} dz, \quad D^\zeta = d\zeta \cdot \frac{d}{d\zeta} + \begin{pmatrix} 0 & S[f \circ g](\zeta) \\ 1 & 0 \end{pmatrix} d\zeta.$$

Also the matrix $W_g(\zeta)$ in (6-7) becomes the following in the one variable case.

$$W_g(\zeta) = \eta(\zeta) \cdot \begin{pmatrix} 1 & \Delta(\zeta) \\ 0 & g'(\zeta) \end{pmatrix},$$

in view of (6-12), where $\eta(\zeta)$ and $\Delta(\zeta)$ are given by

$$(6-14) \quad \eta(\zeta)^2 \cdot g'(\zeta) \equiv 1, \quad \Delta(\zeta) = \frac{\eta'(\zeta)}{\eta(\zeta)} = -\frac{1}{2} \frac{g''(\zeta)}{g'(\zeta)}.$$

These identities also imply the following identities which will be used later.

$$(6-15) \quad \eta(\zeta) \cdot \eta'(\zeta) \cdot g'(\zeta) = \Delta(\zeta), \quad \eta(\zeta)^2 \cdot g''(\zeta) = -2\Delta(\zeta).$$

Now, the transformation formula $D^\zeta = W_g(\zeta)^{-1} \cdot D^z \cdot W_g(\zeta)$ in Theorem 6-1 is equivalent to (6-9), which in our one variable context is given by

$$(6-16) \quad \begin{pmatrix} 0 & S[f \circ g](\zeta) \\ 1 & 0 \end{pmatrix} = W_g(\zeta)^{-1} \cdot \begin{pmatrix} 0 & S[f](z) \\ 1 & 0 \end{pmatrix} \cdot W_g(\zeta) + W_g(\zeta)^{-1} \cdot \frac{dW_g}{d\zeta}(\zeta).$$

Since $\eta(\zeta)^2 \cdot g'(\zeta) \equiv 1$, we have $W_g(\zeta)^{-1} = \eta(\zeta) \cdot \begin{pmatrix} g'(\zeta) & -\Delta(\zeta) \\ 0 & 1 \end{pmatrix}$. So, the first term on the right hand side of (6-16) becomes

$$\begin{aligned} W_g(\zeta)^{-1} \begin{pmatrix} 0 & S[f](g(\zeta)) \\ 1 & 0 \end{pmatrix} W_g(\zeta) &= \eta^2 \cdot g' \begin{pmatrix} g' & -\Delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & S[f] \circ g \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \Delta \\ 0 & g' \end{pmatrix} \\ &= \begin{pmatrix} -\Delta(\zeta) & S[f](g(\zeta)) \cdot g'(\zeta)^2 - \Delta(\zeta)^2 \\ 1 & \Delta(\zeta) \end{pmatrix} \end{aligned}$$

As for the second term on the right hand side of (6-16),

$$\begin{aligned} W_g(\zeta)^{-1} \cdot \frac{dW_g}{d\zeta}(\zeta) &= \eta \begin{pmatrix} g' & -\Delta \\ 0 & 1 \end{pmatrix} \left\{ \eta' \begin{pmatrix} 1 & \Delta \\ 0 & g' \end{pmatrix} + \eta \begin{pmatrix} 0 & \Delta' \\ 0 & g'' \end{pmatrix} \right\} = \eta\eta' \begin{pmatrix} g' & 0 \\ 0 & g' \end{pmatrix} \\ &+ \eta^2 \begin{pmatrix} 0 & g'\Delta' - \Delta g'' \\ 0 & g'' \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} + \begin{pmatrix} 0 & \Delta' + 2\Delta^2 \\ 0 & -2\Delta \end{pmatrix} = \begin{pmatrix} \Delta & \Delta' + 2\Delta^2 \\ 0 & -\Delta \end{pmatrix}. \end{aligned}$$

Here, in the third identity, we used (6-15). Combining the previous calculations, we see that (1,2) components of the matrix identity (6-16) gives

$$S[f \circ g](\zeta) = S[f](g(\zeta)) \cdot g'(\zeta)^2 + \Delta'(\zeta) + \Delta(\zeta)^2.$$

Since $\Delta = (-1/2)g''(\zeta)/g'(\zeta)$ by (6-14), we see that $\Delta' + \Delta^2 = S[g](\zeta)$. Now the above formula becomes (6-13).

Example 6-5 (Möbius invariance). We show how the composition formula in Theorem 6-1 gives rise to the Möbius invariance formula when $f = T \in \text{PSL}_{n+1}(\mathbb{C})$. In this case, since $S_{ij}[f] = 0$ for $1 \leq i, j \leq n$, the associated total covariant derivative $D^z[f] = d + \mathcal{A}[f](z)$ is given by

$$D^z[f] = d_z + \begin{pmatrix} 0 & 0 & \dots & 0 \\ dz_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ dz_n & 0 & \dots & 0 \end{pmatrix}.$$

The matrix $W_g(\zeta)$ has two factors. The conjugation of $g^*(D^z[f])$ by the first factor of $W_g(\zeta)$ is given by

$$\begin{pmatrix} 1 & & & \\ z_1 & 1 & & \\ \vdots & \vdots & \ddots & \\ z_n & 0 & \dots & 1 \end{pmatrix} \left\{ d_z + \begin{pmatrix} 0 & 0 & \dots & 0 \\ dz_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ dz_n & 0 & \dots & 0 \end{pmatrix} \right\} \begin{pmatrix} 1 & & & \\ -z_1 & 1 & & \\ \vdots & \vdots & \ddots & \\ -z_n & 0 & \dots & 1 \end{pmatrix} = d_z.$$

Thus, using the definition $dG(\zeta) = G(\zeta) \cdot \mathcal{A}[g](\zeta)$ of the fundamental Schwarzians of g , the right hand side of the composition formula (6-8) becomes

$$\begin{aligned} W_g(\zeta)^{-1} \cdot g^*(D^z[f]) \cdot W_g(\zeta) &= G(\zeta)^{-1} \cdot g^*(d_z) \cdot D(\zeta) = G(\zeta)^{-1} \cdot d_\zeta \cdot G(\zeta) \\ &= d_\zeta + G(\zeta)^{-1} \cdot dG(\zeta) = d_\zeta + \mathcal{A}[g](\zeta) = D^\zeta[g]. \end{aligned}$$

Hence, when $f = T$ is Möbius, the composition formula gives

$$D^\zeta[T \circ g] = D^\zeta[g],$$

implying $\mathcal{A}[T \circ g](\zeta) = \mathcal{A}[g](\zeta)$, which is equivalent to the Möbius invariance of the fundamental Schwarzians $S_{ij}[T \circ g](\zeta) = S_{ij}[g](\zeta)$ for $1 \leq i, j \leq n$ from the definition of $\mathcal{A}$ in (6-3).

§7 PRESCRIBING FUNDAMENTAL SCHWARZIAN

In this section, we consider the problem of prescribing fundamental Schwarzians. This problem is well understood in the classical one variable case, and we start this section with the description of this classical case.

Let $\Omega \subset \mathbb{C}$ be a complex 1-dimensional connected domain and let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic map with $f'(z) \neq 0$ on Ω . It is well known that for any given meromorphic function $p(z)$ on Ω , there exists a meromorphic function $f(z)$ on Ω such that

$$(7-1) \quad S[f] = p(z), \quad \text{where} \quad S[f] = -\frac{1}{2} \left(\frac{f''}{f'} \right)' + \frac{1}{4} \left(\frac{f''}{f'} \right)^2$$

is the classical Möbius invariant Schwarzian derivative of f . Given $p(z)$, this function $f(z)$ is unique up to Möbius transformations, that is, up to the action by the group $\text{PSL}_2(\mathbb{C})$. So, one can *always* prescribe the classical Schwarzian derivative.

To be more precise, given a function $p(z)$, let $g_1(z)$, $g_2(z)$ be two linearly independent solutions to the following *linear* second order equation:

$$(7-2) \quad u'' = p(z) \cdot u.$$

Then a solution to the Schwarzian equation $S[f] = p(z)$ is given by the quotient of these two functions $f(z) = g_1(z)/g_2(z)$. In fact, all the solutions to the Schwarzian equation (7-1) for a given $p(z)$ is of this form. Note that if $h_1(z)$, $h_2(z)$ are another set of two linearly independent solutions to the linear equation (7-2), then two functions $g_1(z)/g_2(z)$ and $h_1(z)/h_2(z)$ are related by a Möbius transformation. So, any solution to (7-1) can be obtained from a particular solution by applying a Möbius transformation. The geometric meaning of this linear equation (7-2) becomes more transparent when we consider the corresponding equations in the general context of several variables.

We now consider the problem of prescribing the Schwarzian in the context of several variables. Let $\Omega \subset \mathbb{C}^n$ be a domain and suppose that we are given a set of $\mathbb{C}^{n+1}$ -valued holomorphic functions $p_{ij}(z)$ defined on Ω for each pair of integers $1 \leq i, j \leq n$. We consider the problem of finding a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$ such that

$$(7-3) \quad S_{ij}[f] = p_{ij}(z), \quad 1 \leq i, j \leq n,$$

where $S_{ij}[f]$ is our fundamental Schwarzian defined in §3. It turns out that a solution to (7-3) does not always exist and certain compatibility (=integrability) conditions have to be satisfied among the given set of $\mathbb{C}^{n+1}$ -valued functions $p_{ij}(z)$.

To describe the system of partial differential equations in (7-3) more explicitly, let's assume that the local biholomorphism $f : \Omega \rightarrow \mathbb{CP}^n$ factors through a subset $\mathbb{C}^n \subset \mathbb{CP}^n$ consisting of those points $[w_0 : w_1 : w_2 : \dots : w_n] \in \mathbb{CP}^n$ such that $w_0 \neq 0$. We let $f(z) = {}^t(f_1(z), \dots, f_n(z))$ as in §4. Then from Proposition 4-2, the system of partial differential equations (7-3) can be written explicitly as follows.

$$(7-4) \quad \begin{cases} \frac{n}{(n+1)^2} \frac{1}{J_f^2} \frac{\partial J_f}{\partial z_i} \frac{\partial J_f}{\partial z_j} - \frac{1}{n+1} \frac{1}{J_f} \frac{\partial^2 J_f}{\partial z_i \partial z_j} + \sum_{\ell, k=1}^n \frac{\zeta_{\ell, k}}{n+1} \frac{1}{J_f} \frac{\partial J_f}{\partial z_\ell} \frac{\partial^2 f_k}{\partial z_i \partial z_j} = p_{ij}^{(0)}(z), \\ \sum_{k=1}^n \zeta_{\ell k} \frac{\partial^2 f_k}{\partial z_i \partial z_j} - \frac{\delta_{\ell j}}{n+1} \frac{1}{J_f} \frac{\partial J_f}{\partial z_i} - \frac{\delta_{\ell i}}{n+1} \frac{1}{J_f} \frac{\partial J_f}{\partial z_j} = p_{ij}^{(\ell)}(z), \quad 1 \leq \ell \leq n. \end{cases}$$

Here, $J_f : \Omega \rightarrow \mathbb{C}^*$ is the Jacobian determinant of the local biholomorphism $f : \Omega \rightarrow \mathbb{C}^n$ and $\zeta_{\ell k}$ is the (ℓ, k) matrix component of the inverse matrix $(\partial f_k / \partial z_j)^{-1}$. The above system of equations for the unknown holomorphic function $f : \Omega \rightarrow \mathbb{C}^n$ is complicated and nonlinear. Geometrically it is more natural to consider the equations satisfied by the canonical lift $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1}$, since the map f should be regarded as a map into the projective space $\mathbb{CP}^n$ rather than as a map into $\mathbb{C}^n$, and the generalized Schwarzians are defined in terms of canonical lifts. Recall that the canonical lift $\tilde{f}$ is determined up to an $(n+1)$ st root of unity from f , and $\tilde{f}$ uniquely determines f . The canonical lift $\tilde{f}$ satisfies the following system of equations for

the unknown $\mathbb{C}^{n+1}$ -valued function $u : \Omega \rightarrow \mathbb{C}^{n+1}$:

$$(7-5) \quad \begin{cases} \frac{\partial^2 u}{\partial z_i \partial z_j} = \left(u, \frac{\partial u}{\partial z_1}, \frac{\partial u}{\partial z_2}, \dots, \frac{\partial u}{\partial z_n} \right) \cdot p_{ij}(z), & 1 \leq i, j \leq n, \\ \det \left(u, \frac{\partial u}{\partial z_1}, \frac{\partial u}{\partial z_2}, \dots, \frac{\partial u}{\partial z_n} \right) \equiv 1. \end{cases}$$

Note that the first set of differential equations are *linear*. The relation between the systems (7-4) and (7-5) is that if $u(z) = {}^t(u_0(z), u_1(z), \dots, u_n(z))$ is a solution to (7-5), then $f(z) = (u_1(z)/u_0(z), \dots, u_n(z)/u_0(z))$ is a solution to (7-4). The above system can be reformulated concisely in terms of matrices. Let

$$(7-6) \quad M(z) = \left(u(z), \frac{\partial u}{\partial z_1}(z), \frac{\partial u}{\partial z_2}(z), \dots, \frac{\partial u}{\partial z_n}(z) \right)$$

be the unknown $M_{n+1}(\mathbb{C})$ -valued function juxtaposing $\mathbb{C}^{n+1}$ -valued function $u(z)$ and its first derivatives. We also let

$$(7-7) \quad A_i(z) = (e_{i+1}, p_{i1}(z), p_{i2}(z), \dots, p_{in}(z)), \quad 1 \leq i \leq n,$$

be another $M_{n+1}(\mathbb{C})$ -valued function on Ω constructed from the given set of $\mathbb{C}^{n+1}$ -valued functions $p_{ij}(z)$. Then the above system (7-5) can be reformulated as

$$(7-8) \quad \begin{cases} \frac{\partial M}{\partial z_i}(z) = M(z) \cdot A_i(z), & 1 \leq i \leq n, \\ \det M(z) \equiv 1. \end{cases}$$

Furthermore, the first set of equations for $1 \leq i \leq n$ can be collected together using the exterior derivative d , and (7-8) is equivalent to

$$(7-9) \quad \begin{cases} dM(z) = M(z) \cdot \mathcal{A}(z), & \text{where } \mathcal{A}(z) = \sum_{i=1}^n A_i(z) dz_i, \\ \det M(z) \equiv 1. \end{cases}$$

Thus, solving the system (7-4) is equivalent to solving (7-9). The latter system is linear and it is easier to deal with. We state the observations we have made so far in the following lemma.

Lemma 7-1. *Given a set of $\mathbb{C}^{n+1}$ -valued functions $p_{ij}(z)$ for $1 \leq i, j \leq n$, there exists a locally biholomorphic map $f : \Omega \rightarrow \mathbb{CP}^n$ satisfying the Schwarzian equation*

$$(*) \quad S_{ij}[f] = p_{ij}(z), \quad 1 \leq i, j \leq n,$$

if and only if there exists a holomorphic solution $M : \Omega \rightarrow M_{n+1}(\mathbb{C})$ to the equations

$$(**) \quad \begin{cases} dM(z) = M(z) \cdot \mathcal{A}(z), \\ \det M(z) \equiv 1, \end{cases}$$

where $\mathcal{A}(z)$ is as in (7-7), (7-9) determined by $p_{ij}(z)$'s.

Given a solution f to $(*)$, let its (local) canonical lift be $\tilde{f} : \Omega \rightarrow \mathbb{C}^{n+1}$. Then a solution to $(**)$ is given by $M(z) = (\tilde{f}(z), \partial\tilde{f}(z)/\partial z_1, \dots, \partial\tilde{f}(z)/\partial z_n(z))$. Conversely, given a solution $M(z)$ to $(**)$, let $\tilde{f}(z)$ be its first column vector. Then, $f(z) = \pi \circ \tilde{f}(z)$ is a solution to $(*)$, where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$.

First we discuss the existence of a solution to the system (7-9). Since $d^2 = 0$ implies $0 = d^2 M = d(M \cdot \mathcal{A}) = dM \cdot \mathcal{A} + M \cdot d\mathcal{A} = M(\mathcal{A}^2 + d\mathcal{A})$ and since $M(z)$ is invertible in view of the determinant condition, we must have

$$(7-10) \quad d\mathcal{A} + \mathcal{A}^2 = 0.$$

When written out in terms of $A_i(z)$'s, (7-10) is equivalent to a set of conditions

$$(7-11) \quad \frac{\partial A_i}{\partial z_j} + A_j A_i = \frac{\partial A_j}{\partial z_i} + A_i A_j, \quad 1 \leq i, j \leq n.$$

This is a highly nontrivial condition on $p_{ij}(z)$'s.

We can express the same condition in terms of differential operators. This formulation is more convenient in order to explicitly construct solutions to (7-3). Let

$$(7-12) \quad D_i = \frac{\partial}{\partial z_i} + A_i(z) \cdot, \quad 1 \leq i \leq n,$$

be the first order differential operators acting on $M_{n+1}(\mathbb{C})$ -valued sections on Ω . Here, $A_i(z) \cdot$ denotes the multiplication of $A_i(z)$ from the left. From (7-8), it is immediate that $\partial/\partial z_i (\partial M/\partial z_j) = M(z) \cdot (D_i A_j)(z)$ for any $1 \leq i, j \leq n$. From the compatibility condition $\partial^2 M/\partial z_i \partial z_j = \partial^2 M/\partial z_j \partial z_i$ for any i, j , we have $D_i A_j(z) = D_j A_i(z)$ for $1 \leq i, j \leq n$. Expanding this condition, we get (7-11). However, it is straightforward to check that this condition, $D_i A_j(z) = D_j A_i(z)$, is also equivalent to the commutativity of the operators D_i 's:

$$(7-13) \quad [D_i, D_j] = 0, \quad 1 \leq i, j \leq n.$$

On the other hand, the determinant condition $\det M(z) = 1$ means that $M(z)$ defines a holomorphic map from the domain Ω to the complex Lie group $SL_{n+1}(\mathbb{C})$. So, $A_i(z) = M(z)^{-1} \cdot (\partial M(z)/\partial z_i)$ is a holomorphic tangent vector to the Lie group $SL_{n+1}(\mathbb{C})$ at the origin, so $A_i(z) \in \mathfrak{sl}_{n+1}(\mathbb{C})$ for $1 \leq i \leq n$. Since the elements in the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ is traceless, we have

$$(7-14) \quad \text{Tr } A_i(z) = 0, \quad 1 \leq i \leq n.$$

This condition is concisely expressed as $\text{Tr } \mathcal{A}(z) = 0$, where $\mathcal{A}(z) = \sum_{i=1}^n A_i(z) dz_i$. Thus, given a set of $\mathbb{C}^{n+1}$ -valued holomorphic functions $p_{ij}(z)$, a solution to the Schwarzian equation (7-3) exists if and only if a solution to the system (7-9) exists, which in turn implies that given $p_{ij}(z)$'s must satisfy the commutativity condition (7-13) and the tracelessness condition (7-14). These are necessary conditions on $p_{ij}(z)$'s for the existence of solutions to the Schwarzian equation (7-3). It turns out that these conditions are actually sufficient.

To show the sufficiency of the conditions (7-13) and (7-14) for the existence of the solutions to the Schwarzian equation (7-3), we reformulate the problem of prescribing the fundamental Schwarzians in the geometric context of connections and vector bundles. The vector bundle E we consider here is a trivial complex $(n+1)^2$ -dimensional vector bundle whose base space is the domain Ω and the fibre is $M_{n+1}(\mathbb{C})$:

$$(7-15) \quad E = \Omega \times M_{n+1}(\mathbb{C}).$$

On E , we introduce a covariant derivative $\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes T^*\Omega)$ by letting

$$(7-16) \quad \nabla = d - \cdot \mathcal{A}(z), \quad \text{where} \quad \mathcal{A} = \sum_{i=1}^n A_i(z) dz_i.$$

Here $\cdot \mathcal{A}$ denotes the multiplication by the matrix valued 1-form $\mathcal{A}$ defined on Ω from the right. In this geometric context, a $M_{n+1}(\mathbb{C})$ -valued function $M(z)$ on Ω can be regarded as a section in the vector bundle E , and $M(z)$ satisfies the equation $dM = M \cdot \mathcal{A}$ if and only if the corresponding section in E is flat with respect to the connection ∇ , i.e., $\nabla M = 0$. Such a flat section in the vector bundle (E, ∇) exists if the curvature of the connection vanishes. For $\nabla = d - \mathcal{A}$, the curvature is given by $\nabla^2 = d\mathcal{A} + \mathcal{A}^2$. So, for the existence of the solution of the Schwarzian equation, it is necessary to have $d\mathcal{A} + \mathcal{A}^2 = 0$. Conversely, if the curvature of the connection in E vanishes, that is, if $d\mathcal{A} + \mathcal{A}^2 = 0$, then there always exist a unique local flat section around any point $z_0 \in \Omega$ passing through an arbitrarily given vector in the fibre E_{z_0} in the vector bundle (E, ∇) . Furthermore if the domain Ω is simply connected, then there is no monodromy and there always exists a unique global parallel section $M(z)$ passing through any vector in E_{z_0} for any $z_0 \in \Omega$.

Now in addition to $d\mathcal{A} + \mathcal{A}^2 = 0$, suppose $\text{Tr } \mathcal{A}(z) = 0$ on a simply connected domain Ω . For any $T_0 \in \text{SL}_{n+1}(\mathbb{C}) \subset M_{n+1}(\mathbb{C})$ and any $z_0 \in \Omega$, let $M(z)$ be the unique global flat section in the bundle (E, ∇) such that $M(z_0) = T_0$. The existence of such a section is guaranteed by the condition $d\mathcal{A} + \mathcal{A}^2 = 0$. Since $dM = M \cdot \mathcal{A}$ for the flat section $M(z)$, Lemma 3-5 implies that

$$\frac{d(\det M(z))}{\det M(z)} = \text{Tr}(M^{-1} \cdot dM) = \text{Tr } \mathcal{A},$$

which is zero by our assumption. So, $\det M(z)$ is a constant function on the domain Ω . Since $\det M(z_0) = 1$ for a point $z_0 \in \Omega$, we conclude that $\det M(z) = 1$ for all $z \in \Omega$ and so $M(z) \in \text{SL}_{n+1}(\mathbb{C})$.

Thus, we have shown that for any constant matrix $T_0 \in \text{SL}_{n+1}(\mathbb{C})$, the initial value problem $dM = M \cdot \mathcal{A}$, $\det M = 1$, $M(z_0) = T_0$ has a unique global solution on the simply connected domain Ω if and only if the matrix valued 1-form $\mathcal{A}$ satisfies $d\mathcal{A} + \mathcal{A}^2 = 0$, $\text{Tr } \mathcal{A} = 0$.

Let $M_1(z)$ and $M_2(z)$ be two matrix valued holomorphic functions satisfying $dM(z) = M(z) \cdot \mathcal{A}$, $\det M(z) = 1$. Suppose at a point $z_0 \in \Omega$, we have $M_1(z_0) = T_0 \cdot M_2(z_0)$ for some constant matrix $T_0 \in \text{SL}_{n+1}(\mathbb{C})$. Then, by the uniqueness of the solutions to our initial value problem, we must have $M_1(z) = T_0 \cdot M_2(z)$ on Ω . Thus, when $\mathcal{A}$ satisfies the conditions $d\mathcal{A} + \mathcal{A}^2 = 0$ and $\text{Tr } \mathcal{A} = 0$, the group

$SL_{n+1}(\mathbb{C})$ acts transitively and effectively on the set of solutions to the system $dM = M \cdot \mathcal{A}$ and $\det M = 1$.

Next, for an arbitrarily given matrix $T_0 \in SL_{n+1}(\mathbb{C})$, we calculate the Taylor series expansion of the solution $M(z)$ around an arbitrary point $z_0 \in \Omega$ with the initial condition $M(z_0) = T_0$ when $\mathcal{A}$ satisfies $d\mathcal{A} + \mathcal{A}^2 = 0$ and $\text{Tr } \mathcal{A} = 0$. Since the condition $d\mathcal{A} + \mathcal{A}^2 = 0$ is equivalent to the commutativity (7-13) of the first order differential operators D_i , the notation

$$(7-17) \quad D^K = D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n}$$

makes sense for any multi-index $K = (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$. Furthermore, the equation $\partial M / \partial z_i = M \cdot A_i$ implies that we have the commutative diagram (3-14). Hence it is straightforward to check that for any multi-index $K \in \mathbb{Z}_+^n$,

$$\frac{\partial^{|K|} M}{\partial z^K}(z) = M(z) \cdot (D^K I_{n+1})(z).$$

Evaluating this identity at $z = z_0$ and noting that $M(z_0) = T_0$ by the initial condition, we have

$$\frac{\partial^{|K|} M}{\partial z^K}(z_0) = T_0 \cdot (D^K I_{n+1})(z_0).$$

This gives the K -th Taylor coefficient at $z = z_0$ of the matrix valued holomorphic function $M(z)$. Hence its Taylor expansion around $z = z_0$ is given by

$$(7-18) \quad M(z) = \sum_{K \in \mathbb{Z}_+^n} \frac{(z - z_0)^K}{K!} \frac{\partial^{|K|} M}{\partial z^K}(z_0) = T_0 \cdot \left\{ \sum_{K \in \mathbb{Z}_+^n} \frac{(z - z_0)^K}{K!} (D^K I_{n+1})(z_0) \right\}.$$

Thus, we have obtained the following proposition on the solutions $M(z)$ to (7-9).

Proposition 7-2. *Let $\Omega \subset \mathbb{C}^n$ be a simply connected domain and suppose we are given holomorphic $\mathbb{C}^{n+1}$ -valued functions $p_{ij}(z)$ on Ω for $1 \leq i, j \leq n$. Then, with $\mathcal{A}$ as in (7-7), (7-9), the system for an unknown $M_{n+1}(\mathbb{C})$ -valued function $M(z)$ on Ω consisting of*

$$(*) \quad dM = M \cdot \mathcal{A}, \quad \det M = 1$$

has a unique solution for an arbitrary initial condition $M(z_0) = T_0 \in SL_{n+1}(\mathbb{C})$ if and only if $\mathcal{A}$ satisfies

$$d\mathcal{A} + \mathcal{A}^2 = 0, \quad \text{Tr } \mathcal{A} = 0.$$

In this case, the Taylor series expansion of $M(z)$ around $z = z_0$ is given by

$$M(z) = T_0 \cdot \left\{ \sum_{K \in \mathbb{Z}_+^n} \frac{(z - z_0)^K}{K!} (D^K I_{n+1})(z_0) \right\}.$$

The group $SL_{n+1}(\mathbb{C})$ acts transitively and effectively on the space of solutions of the above system ().*

Since the canonical lift $\tilde{f}$ of the solution f to the generalized Schwarzian equation (7-3) is given by the first column vector of $M(z)$ by Lemma 7-1, we have $\tilde{f}(z) =$

$M(z)p_0$ where $p_0 = {}^t(1, 0, \dots, 0) \in \mathbb{C}^{n+1}$ is the constant column vector. We then note that $(D^K I_{n+1})p_0 = D^K p_0$. Applying the matrix identity (7-18) to the vector p_0 , we see that the Taylor series expansion of the canonical lift $\tilde{f}(z)$ is given by

$$(7-19) \quad \tilde{f}(z) = T_0 \cdot \left\{ \sum_{K \in \mathbb{Z}_+^n} \frac{(z - z_0)^K}{K!} (D^K p_0)(z_0) \right\}.$$

Restating our results on $M(z)$ in Proposition 7-2 for its first column vector $\tilde{f}(z)$, we have the following theorem.

Theorem 7-3 (Prescribing fundamental Schwarzians). *Let $\Omega \subset \mathbb{C}^n$ be a simply connected domain and let $p_{ij}(z)$ be a given $\mathbb{C}^{n+1}$ -valued holomorphic function defined on the domain Ω for $1 \leq i, j \leq n$. Then, a locally biholomorphic function $f : \Omega \rightarrow \mathbb{CP}^n$ satisfying*

$$S_{ij}[f] = p_{ij}(z), \quad 1 \leq i, j \leq n,$$

exists if and only if

$$d\mathcal{A} + \mathcal{A}^2 = 0, \quad \text{Tr } \mathcal{A} = 0,$$

where $\mathcal{A} = \sum_{i=1}^n A_i(z) dz_i$ and $A_i(z) = (e_{i+1}, p_{i1}(z), p_{i2}(z), \dots, p_{in}(z)) \in M_{n+1}(\mathbb{C})$.

Equivalently, one can prescribe fundamental Schwarzians for given $p_{ij}(z)$'s if and only if

$$[D_i, D_j] = 0, \quad A_i(z) \in \mathfrak{sl}_{n+1}(\mathbb{C}),$$

where $D_i = \partial/\partial z_i + A_i(z)$.

When either of the above conditions are satisfied and consequently solutions to the Schwarzian equation exists, then any two solutions $f_1, f_2 : \Omega \rightarrow \mathbb{CP}^n$ are related by a Möbius transformation i.e., $f_1(z) = T \circ f_2(z)$ for some $T \in \text{PSL}_{n+1}(\mathbb{C})$.

For any matrix $T_0 \in \text{SL}_{n+1}(\mathbb{C})$ and a point $z_0 \in \Omega$, there exists a unique solution $f : \Omega \rightarrow \mathbb{CP}^n$ whose canonical lift $\tilde{f}$ is such that

$$\left(\tilde{f}(z_0), \frac{\partial \tilde{f}}{\partial z_1}(z_0), \frac{\partial \tilde{f}}{\partial z_2}(z_0), \dots, \frac{\partial \tilde{f}}{\partial z_n}(z_0) \right) = T_0 \in \text{SL}_{n+1}(\mathbb{C}).$$

Its Taylor series expansion around $z = z_0$ is given by the formula

$$\tilde{f}(z) = T_0 \cdot \left\{ \sum_{K \in \mathbb{Z}_+^n} \frac{(z - z_0)^K}{K!} (D^K p_0)(z_0) \right\}.$$

Here $p_0 = {}^t(1, 0, \dots, 0)$ is a constant column vector.

Conversely, the canonical lift $\tilde{f}$ of any solution $f : \Omega \rightarrow \mathbb{CP}^n$ to the Schwarzian equation is of the above form for some $T_0 \in \text{SL}_{n+1}(\mathbb{C})$.

Example 7-4. As an illustration, we prescribe the classical Schwarzian using our current method. Let $p(z)$ be a holomorphic function defined on a domain $\Omega \subset \mathbb{C}$. Our problem is to find a solution $f : \Omega \rightarrow \mathbb{C}$ satisfying

$$(7-1) \quad S[f] = -\frac{1}{2} \left(\frac{f''}{f'} \right)' + \frac{1}{4} \left(\frac{f''}{f'} \right)^2 = p(z).$$

As we discussed earlier, the classical way to find a solution to (7-1) is to find the canonical solutions $u_0(z), u_1(z)$ to the linear equation $u'' = p(z) \cdot u$ in (7-2) satisfying the following initial condition at a fixed point $w \in \Omega$:

$$(7-20) \quad \begin{cases} u_0(w) = 1 \\ u_0'(w) = 0, \end{cases} \quad \begin{cases} u_1(w) = 0 \\ u_1'(w) = 1. \end{cases}$$

Then, any solution $f(z)$ to (7-1) is of the form

$$(7-21) \quad f(z) = \frac{au_0(z) + bu_1(z)}{cu_0(z) + du_1(z)}, \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}).$$

Now in our context of the generalized Schwarzians, the equation equivalent to the above (7-1) is $S_2[f] = {}^t(p(z), 0)$ due to (4-12). In this case, $M_2(\mathbb{C})$ -valued 1-form $\mathcal{A} = \begin{pmatrix} 0 & p(z) \\ 1 & 0 \end{pmatrix} dz$ in (7-7), (7-9) trivially satisfies $d\mathcal{A} + \mathcal{A}^2 = 0$, $\text{Tr } \mathcal{A} = 0$. Then, by (7-19) above, the canonical lift $\tilde{f}(z)$ of solutions are of the form

$$\tilde{f}(z) = T_0 \cdot \left\{ \sum_{k \geq 0} \frac{(z-w)^k}{k!} (D^k p_0)(w) \right\},$$

where $T_0 = (\tilde{f}(w), \partial \tilde{f} / \partial z(w)) \in \text{SL}_2(\mathbb{C})$ and the first order differential operator D is given by

$$D = \frac{d}{dz} + \begin{pmatrix} 0 & p(z) \\ 1 & 0 \end{pmatrix}.$$

The object inside of the brace $\{ \}$ is determined by the given function $p(z)$. Let

$$(7-22) \quad \begin{pmatrix} u_0(z) \\ u_1(z) \end{pmatrix} = \sum_{k \geq 0} \frac{(z-w)^k}{k!} (D^k p_0)(w).$$

We can easily calculate the first few $D^k p_0$'s. For example,

$$(7-23) \quad \begin{aligned} (Dp_0)(w) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (D^2 p_0)(w) = \begin{pmatrix} p(w) \\ 0 \end{pmatrix}, \quad (D^3 p_0)(w) = \begin{pmatrix} p'(w) \\ p(w) \end{pmatrix}, \\ (D^4 p_0)(w) &= \begin{pmatrix} p''(w) + p(w)^2 \\ 2p'(w) \end{pmatrix}, \quad (D^5 p_0)(w) = \begin{pmatrix} p'''(w) + 4p(w)p'(w) \\ 3p''(w) + p(w)^2 \end{pmatrix}, \text{ etc.} \end{aligned}$$

From these calculations, we have

$$(7-24) \quad \begin{pmatrix} u_0(z) \\ u_1(z) \end{pmatrix} = \begin{pmatrix} 1 + p(w)z^2/2! + p'(w)z^3/3! + (p''(w) + p(w)^2)z^4/4! + \dots \\ z + p(w)z^3/3! + 2p'(w)z^4/4! + (3p''(w) + p(w)^2)z^5/5! + \dots \end{pmatrix}.$$

So, $u_0(z)$ and $u_1(z)$ satisfy the same initial conditions as (7-20). So, the formula (7-20) actually calculates the canonical solutions to (7-2). According to (7-19), the canonical lift $\tilde{f}$ of any solution to (7-1) is of the form $\tilde{f}(z) = T_0 \cdot {}^t(u_0(z), u_1(z))$, which gives (7-21) if $T_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. What our current method does compared with the classical method is to give an inductive and systematic method (7-19) to calculate the Taylor coefficients (7-23) of the canonical solutions $u_0(z)$, $u_1(z)$, see (7-24).

We study the properties of the power series inside of the brace in (7-18) and in (7-19). So, for any $w \in \Omega$, let

$$(7-25) \quad M_w(z) = \sum_{K \in \mathbb{Z}_+^n} \frac{(z-w)^K}{K!} (D^K I_{n+1})(w) \in \mathrm{SL}_{n+1}(\mathbb{C}).$$

This is a convergent power series for z in a neighborhood of w and from (7-18),

$$(7-26) \quad M(z) = M(w) \cdot M_w(z).$$

The matrix valued holomorphic function $M_w(z)$ can be thought of as the solution to the equation $dM(z) = M(z) \cdot \mathcal{A}(z)$, $\det M(z) = 1$ with the canonical initial condition $M(w) = I_{n+1}$ at $w \in \Omega$. If the initial condition $M(w)$ is a general matrix, then (7-26) gives the solution. From this point of view, it is easy to deduce some properties of $M_w(z)$.

Proposition 7-5. (i) (Transitivity) *For any $w, z, \zeta \in \Omega$ for which power series $M_w(z)$, $M_w(\zeta)$, $M_z(\zeta)$ converge, we have*

$$M_w(\zeta) = M_w(z) \cdot M_z(\zeta).$$

(ii) (Inverse symmetry) *For any $w, z \in \Omega$ for which $M_w(z)$ and $M_z(w)$ converge, we have*

$$M_w(z)^{-1} = M_z(w).$$

Proof. For (i), from (7-26), we have $M(z) = M(w) \cdot M_w(z)$, $M(\zeta) = M(w) \cdot M_w(\zeta)$. On the other hand, we can also write $M(\zeta) = M(z) \cdot M_z(\zeta) = M(w) \cdot M_w(z) \cdot M_z(\zeta)$. Since $M(w)$ is invertible, we have $M_w(\zeta) = M_w(z) \cdot M_z(\zeta)$. For (ii), again from (7-26), $M(z) = M(w) \cdot M_w(z)$ and $M(w) = M(z) \cdot M_z(w)$. From this, we obtain $M_w(z) \cdot M_z(w) = I_{n+1}$. $\square$

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