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Gibbs measures and quasi-periodic solutions for
nonlinear Hamiltonian partial differential equations

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GIBBS MEASURES AND QUASI-PERIODIC SOLUTIONS FOR NONLINEAR HAMILTONIAN PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this exposé is to present a summary of some new developments in the theory of Hamiltonian nonlinear evolution equations, more specifically, nonlinear Schrödinger equations. The themes and methods discussed are closely related to classical mechanics. The first topic is the existence of an invariant measure for the flow. This invariant measure is the (properly normalized) Gibbs measure from statistical mechanics and we establish wellposedness of the equation on its support. Those investigations are closely related to the paper [L-R-S]. Results in this direction are obtained in $1D$ (in the focusing and defocusing case) and in the $2D$ defocusing case.

The second topic concerns the persistency of time periodic and quasi-periodic solutions for Hamiltonian perturbations of linear and integrable equations. We follow a method, the so-called Liapounov-Schmidt decomposition, originating from the works of [C-W1, 2], rather than the KAM procedure (cf. [Kuk1]). The main advantage of this technique is the fact that it overcomes certain limitations of the KAM scheme, which is necessary to deal in particular with the problems in space-dimension $D \geq 2$. This work is a new approach to KAM problems, also in finite dimensional phase space. Persistency results for PDE's are obtained in the time periodic case in arbitrary dimension and for quasi-periodic solutions when $D \leq 2$. The small divisor problems appearing when inverting the linearized operators are related to the works of Frölich and Spencer on the Anderson model.

1. INTRODUCTION

One of the main problems in the theory of nonlinear Hamiltonian evolution equations is the behavior of the solutions for time $t \rightarrow \infty$. We have in mind the general (not integrable) case, with only few conserved quantities at our disposal. Here one may nevertheless obtain certain information on the longtime behavior of the flows, using the Hamiltonian structure of the equation. Recently, there has been progress along these lines in various directions and we do not intend to try to give a complete survey. In this report, we will discuss the following aspects

- (I) Use of methods of statistical mechanics
- (II) Use of methods of dynamical systems

The main theme, regarding the first topic, is to establish the existence of a well defined global flow on the support of the Gibbs measure $e^{-\beta H(\phi)} \prod d\phi$ and show its invariance under this flow. This leads thus to the existence of an invariant measure. This measure lives on functions with poor regularity or fields. Proving

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wellposedness of the Cauchy problem for such data turns out to require a rather delicate analysis of an independent interest. The wellposedness is usually only local in time and a global result is obtained using the Gibbs-measure as a substitute for a conservation law. In this way, global solutions are constructed for data of a regularity considerably below what may be shown by purely PDE methods. We will carry out our discussion here for the nonlinear Schrödinger equation (NLS) in dimensions 1 and 2 and some related equations, completing a line of investigation initiated in the paper of [L-R-S].

The second topic concerns the persistency of periodic or quasi-periodic (in time) solutions of linear or integrable equations under small Hamiltonian perturbation. A natural approach to such questions is to try to adapt the standard KAM technology from classical mechanics. The problem is thus the persistency of finite dimensional tori in an infinite dimensional phase space. In this direction, important contributions are due to S. Kuksin [Kuk1]. His work gives satisfactory results for $1D$ problems with Dirichlet boundary conditions. The shortcoming of the KAM approach is the fact that it seems unable to deal with multiplicities or clusters in the normal frequencies. Those appear in $1D$ under periodic boundary conditions and in dimension ≥ 2 . A different approach, based on the Liapounov-Schmidt decomposition scheme avoids this limitation of the KAM method. It has been elaborated by W. Craig, E. Wayne [C-W1, 2] for $1D$ time-periodic solutions and more recently by the author. The method is more flexible than KAM, depends less on Hamiltonian structure. Research on its applications is still in progress. Presently, we may deal with time periodic problems in any dimension and time quasi-periodic solutions in $1D, 2D$. The second part of the report is devoted to a sketch of the underlying ideas. We first give an overview of the work done by various authors on nonlinear Hamiltonian perturbations of linear equations. The problem for perturbed integrable equations may in principle be treated along the same lines. The use of symplectic transformations in order to introduce appropriate coordinates however complicates considerably some of the analysis used for the linear case. In some cases, it suffices to use these “integrable” coordinates only in a finite dimensional part of the phase space and deal with the remainder of the problem in Fourier coordinates. We illustrate this idea by discussing in some detail the persistency of (small) time periodic solutions of the KdV equation

$$u_t + u_{xxx} + uu_x = 0$$

for perturbations of the form $\varepsilon[f'(u)]_x$. Results on this problem were obtained by S. Kuksin [Kuk2]. The approach followed here is different however and more effective we believe.

2. NONLINEAR SCHRÖDINGER EQUATIONS AND INVARIANT GIBBS MEASURES

We consider the nonlinear Schrödinger equation (NLS)

$$iu_t + \Delta u \pm u|u|^{p-2} = 0 \tag{2.1}$$

with periodic boundary conditions. Thus u is a complex function on $\mathbf{T}^d \times I$ (local) or $\mathbf{T}^d \times \mathbf{R}$ (global). The equation may be rewritten in Hamiltonian format as

$$u_t = i \frac{\partial H}{\partial \bar{u}} \quad (2.2)$$

where $H(\phi) = \frac{1}{2} \int_{\mathbf{T}^d} |\nabla \phi|^2 \mp \frac{1}{p} \int_{\mathbf{T}^d} |\phi|^p$. Both the Hamiltonian $H(\phi)$ and the L^2 -norm $\int |\phi|^2$ are preserved under the flow. The 1D case $p = 4$ is special (1D cubic NLS) because integrable and there are many invariants of motion. This aspect will however play no role in the present discussion. The possible sign choice $\pm$ in (3.1) corresponds to the focusing (resp. defocusing) case. In the focusing case, the Hamiltonian may be unbounded from below and blowup phenomena may occur (for $p \geq 2 + \frac{4}{d}$). The canonical coordinates are $(\operatorname{Re} \phi, \operatorname{Im} \phi)$ or alternatively $(\operatorname{Re} \hat{\phi}, \operatorname{Im} \hat{\phi})$. The formal Gibbs measure on this infinite dimensional phase space is given by

$$d\gamma_\beta = e^{-\beta H(\phi)} \prod_x d\phi(x) = e^{\pm \frac{\beta}{p} \int |\phi|^p} \cdot e^{-\frac{\beta}{2} \int |\nabla \phi|^2} \prod_x d\phi(x) \quad (2.3)$$

($\beta > 0$ is the reciprocal temperature and we may take $\beta = 1$ in this discussion).

From Liouville's theorem, (2.3) defines an invariant measure for the flow of (2.1). Making this statement precise requires to clarify following two issues

- (i) The rigorous construction (normalization) of the measure (2.3)
- (ii) The existence problem for the flow of (2.1) on the support of the measure.

The first issue is well understood in the defocusing case. The case $D = 1$ is trivial, the case $D = 2$, p even integer is based on the Wick-ordering procedure (see [G-J]) and the normalization for $D = 3$, $p = 4$ is due to Jaffe [Ja]. In the focusing case, only the case $D = 1$ is understood [L-R-S] and normalization of the measure is possible for $p \leq 6$, restricting ϕ to an appropriate ball in $L^2(\mathbf{T})$.

The construction of a flow is clearly a PDE issue. The author succeeded in this in the $D = 1$ and $D = 2$, $p = 4$ cases ([B5], [B6]). For $D = 2$, $p = 4$ there is a natural PDE counterpart of the Wick-ordering procedure and equation (2.1) has to be suitably modified (this modification seems physically inessential however). We may summarize the results as follows.

Theorem 2.4. ($D = 1$)

- (i) *In the defocusing case, the measure (2.3) appears as a weighted Wiener measure, the density being given by the first factor. The same statement is true in the focusing case for $p \leq 6$, provided one restricts the measure to an L^2 -ball $[||\phi||_2 \leq B]$. The choice of B is arbitrary for $p < 6$ and B has to be sufficiently small if $p = 6$.*
- (ii) *Assuming the measure exists, the corresponding 1D equation (2.1) is globally wellposed on a K_σ set A of data, $A \subset \cap_{s < \frac{1}{2}} H^s(\mathbf{T})$, carrying the Gibbs measure γ_β . The set A and the Gibbs measure γ_β are invariant under the flow.*

(Recall that the first part of the theorem is due to [L-R-S]).

Remarks.

- (i) In dimension 1, the L^2 -restriction is acceptable, since L^2 is a conserved quantity and a typical ϕ in the support of the Wiener measure is a function in $H^s(\mathbf{T})$, for all $s < \frac{1}{2}$. Instead of restricting to an L^2 -ball, one may alternatively multiply with a weight function with a suitable exponential decay in $\|\phi\|_2$.
- (ii) Let for each $N = 1, 2, \dots$

$$P_N \phi = \phi_N = \sum_{|n| \leq N} \widehat{\phi}(n) e^{i\langle n, x \rangle} \quad (2.5)$$

be the restriction operator to the N first Fourier modes. Finite dimensional versions of the PDE model are obtained considering “truncated” equations

$$\begin{cases} iu_t^N + u_{xx}^N \pm P_N (u^N |u^N|^{p-2}) = 0 \\ u^N(0) = P_N \phi. \end{cases} \quad (2.6)$$

It is proved that for typical ϕ , the solutions u^N of (3.6) converge in the space $C_{H^s(\mathbf{T})}[0, T]$ for all time T and $s < \frac{1}{2}$ to a (strong) solution of

$$\begin{cases} iu_t + u_{xx} \pm P (u |u|^{p-2}) = 0 \\ u(0) = \phi. \end{cases} \quad (2.7)$$

Theorem 2.8. ($D = 2, p = 4$)

- (i) Denote $\tilde{H}_N$ the Wick-ordered Hamiltonians, obtained replacing

$$|\phi_N|^4 \quad \text{by} \quad |\phi_N|^4 - 4a_N |\phi_N|^2 + 2|a_N|^2 \quad \left(a_N = \sum_{|n| \leq N} \frac{1}{|n|^2 + \rho} \sim \log N \right).$$

The corresponding measures $e^{-\beta \tilde{H}_N(\phi)} \prod d\phi$ converge for $N \rightarrow \infty$ to a weighted 2-dimensional Wiener measure which density belongs to all L^p -spaces. Denote $\tilde{\gamma}_\beta$ this “Wick-ordered” Gibbs measure.

- (ii) The measure $\tilde{\gamma}_\beta$ is invariant under the flow of the “Wick-ordered” equation

$$iu_t + \Delta u - \left(u |u|^2 - 2u \int |u|^2 \right) = 0 \quad (2.9)$$

which is welldefined. More precisely, denoting u^N the solutions of

$$\begin{cases} iu_t^N + \Delta u^N - P_N (u^N |u^N|^2 - 2u^N \int |u^N|^2) = 0 \\ u^N(0) = P_N \phi \end{cases} \quad (2.10)$$

the sequence

$$u^N(t) - \sum_{|n| \leq N} \widehat{\phi}(n) e^{i((n,x) + |n|^2 t)} \quad (2.11)$$

converges for typical ϕ in $C_{H^s}(\mathbf{T}^2)[0, T]$ for some $s > 0$, all time T , to

$$u(t) - \sum \widehat{\phi}(n) e^{i((n,x) + |n|^2 t)}. \quad (2.12)$$

Remarks.

- (i) We repeat that the novelty of Theorem 2.8 lies in the second statement on the existence of a flow. The first statement is a classical result.
- (ii) The second terms in (2.11), (2.12) are the solutions to the linear problem

$$\begin{cases} iu_t + \Delta u = 0 \\ u(0) = \phi \end{cases} \quad (2.13)$$

Here a typical ϕ is a distribution, not a function. However the difference (2.12) between solutions of linear and nonlinear equation is an H^s -function for some $s > 0$, which is a rather remarkable fact.

- (iii) The failure in $D = 2$ of typical ϕ to be an L^2 -function makes the [L-R-S] construction for $D = 1$ inadequate to deal with the $D = 2$ focusing case. Some recent work on this issue is due to A. Jaffe, but for cubic nonlinearities in the Hamiltonian only. The problem for $D = 2$, $p = 4$ in the focusing case is open and intimately related to blowup phenomena ($p = 4$ is critical in $2D$). Observe that in the preceding the invariance of the (Wick-ordered) Gibbs-measure implies the “quasi-invariance” of the free measure $\prod d^2\phi$. I. Gelfand* proposed to investigate this fact directly. Such approach would avoid normalization problems. For instance, in $1D$, one may consider the focusing equation $iu_t + u_{xx} + u|u|^{p-2} = 0$ with $p \geq 6$ and prove this quasi-invariance local in time.
- (iv) One may prove an analogue of Theorem 2.8 for $2D$ wave equations, for instance the equation

$$u_{tt} - \Delta u + (u^3 - 3(\int u^2 dx), u) = 0 \quad (2.14)$$

obtained by renormalization of the defocusing cubic NLW

$$u_{tt} - \Delta u + u^3 = 0. \quad (2.15)$$

Observe that the (optimal) PDE result on local wellposedness deals with data $\phi \in H^{1/4}$ (see [L-S]).

*Private communication

The 1D cubic NLSE appears as the limit of the 1D Zakharov model (ZE)

$$\begin{cases} iu_t = -u_{xx} + nu \\ n_{tt} - c^2 n_{xx} = c^2 (|u|^2)_{xx} \end{cases} \quad (2.16)$$

when $c \rightarrow \infty$. The physical meaning of u, n, c are resp. the electrostatic envelope field, the ion density fluctuation field and the ion sound speed. This model is discussed in [L-R-S]. Defining an auxiliary field $V(x, t)$ by

$$\begin{cases} n_t = -c^2 V_x \\ V_t = -n_x - |u|_x^2 \end{cases} \quad (2.17)$$

we may write (2.16) in a Hamiltonian way, where

$$H = \frac{1}{2} \int [|u_x|^2 + \frac{1}{2} (n^2 + c^2 V^2) + n|u|^2] dx \quad (2.18)$$

and $(\operatorname{Re} u, \operatorname{Im} u), (\tilde{n}, \tilde{V})$ with $\tilde{n} = 2^{-1/2}n, \tilde{V} = 2^{-1/2} \int^x V$ as pairs of conjugate variables. Considering the associated Gibbs measure

$$e^{-\beta H} \cdot \chi_{\{\int |u|^2 dx \leq B\}} \prod_x d^2 u(x) d\tilde{n}(x) d\tilde{V}(x) \quad (2.19)$$

one gets the 1D cubic NLS Gibbs measure as marginal distribution of the u -field.

Theorem 2.20. [B3] *The 1D (ZE) is globally wellposed for almost all data $(u_0, \tilde{u}_0, \tilde{V}_0)$ in the support of the Gibbs measure which is invariant under the resulting flow.*

Remarks.

- (i) In the study of invariant Gibbs measures, it suffices to establish local well-posedness of the IVP for typical data in the support of the measure. One may then exploit the invariance of the measure as a conservation law and generate a global flow. For instance, for the 1D NLS $iu_t + u_{xx} \pm u|u|^{p-2} = 0$, there is for $p = 4$ a global wellposedness result for L^2 -data (L^2 is conserved). However, for $p > 4$, we only dispose presently of a local result (in the periodic case) for data ϕ satisfying

$$\begin{cases} \phi \in H^s, s > 0 & (p \leq 6) \\ \phi \in H^s, s > s_*, p = 2 + \frac{4}{1-2s_*} & (p > 6) \end{cases} \quad (2.21)$$

and a global flow is established from the invariant measure considerations.

- (ii) There has been other investigations in 1D on invariant measures, mostly by more probabilistic arguments. In this respect, we mention the works of McKean-Vaninski and in particular McKean [McK] on the 1D cubic NLS. These methods are more general but give less information on the flow.

3. PERSISTENCY OF PERIODIC AND QUASI-PERIODIC SOLUTIONS UNDER PERTURBATION

One of the most exciting recent developments in nonlinear PDE is the use of the classical KAM-type techniques to construct time quasi-periodic solutions of Hamiltonian equations obtained by perturbation of a linear or integrable PDE. This subject is rapidly developing.

We will first consider perturbations of linear equations. We work essentially* in the real analytic category. Important contributions are due to S. Kuksin [Kuk1], using the standard KAM scheme and more precisely infinite dimensional versions of Melnikov's theorem on the persistency of n -dimensional tori in systems with $N > n$ degrees of freedom. His work yields a rather general theory and we mention only some typical examples of applications to $1D$ nonlinear wave or Schrödinger equations

$$w_{tt} = \left(\frac{\partial^2}{\partial x^2} - V(x; a) \right) w - \varepsilon \frac{\partial \varphi}{\partial w} (x, w; a) \quad (3.1)$$

$$-iu_t = -u_{xx} + V(x, a)u + \varepsilon \frac{\partial \varphi}{\partial |u|^2} (x, |u|^2; a) u. \quad (3.2)$$

Here $V(x, a)$ is a real periodic smooth potential, depending on n outer parameters $a = (a_1, \dots, a_n)$. Denote $\{\lambda_j(a)\}$ the Dirichlet spectrum of the Sturm-Liouville operator $-\frac{d^2}{dx^2} + V(x, a)$. Thus $\lambda_j(a) = \pi^2 j^2 + 0(1)$ and we assume following nondegeneracy condition

$$\det \{ \partial \lambda_j(a) / \partial a_k \mid 1 \leq j, k \leq n \} \neq 0 \quad (3.3)$$

(this condition is a substitute for the classical "twist" condition). Denoting $\{\varphi_j\}$ the corresponding eigenfunctions, the $2n$ -dimensional linear space

$$Z^0 = \text{span} \{ \varphi_j, i\varphi_j \mid 1 \leq j \leq n \} \quad (3.4)$$

is invariant under the flow of equation (3.2) for $\varepsilon = 0$ and foliated into invariant n -tori

$$T^n(I) = \left\{ \sum_{j=1}^n (x_j^+ + ix_j^-) \varphi_j \mid (x_j^+)^2 + (x_j^-)^2 = 2I_j, \ j = 1, \dots, n \right\} \quad (3.5)$$

which are filled with quasi-periodic solutions of (3.2) for $\varepsilon = 0$. A typical result from [Kuk1] is that under assumption (3.3), for most parameter values of a there is an invariant torus $\sum_{a,I}^\varepsilon(\mathbf{T}^n)$ nearby the unperturbed torus $\sum_{a,I}^0$ given by (3.5) and filled with quasi-periodic solutions of (3.2). The frequency vector ω_ε of a perturbed solution will be $c\varepsilon$ close to $\omega = (\lambda_1, \dots, \lambda_n)$ of the unperturbed one.

The methods in [K] leave out the case of periodic boundary conditions, because of certain limitations of the KAM method (second Melnikov condition) excluding

*Sometimes Gevrey-type spaces are considered instead

multiplicities in the normal frequencies. A different approach has been recently used by W. Craig and C.E. Wayne [C-W1, 2], based on the Liapounov-Schmidt decomposition and leading to time periodic solutions of perturbed equations under periodic boundary conditions. This method consists in splitting the problem in a (finite dimensional) resonant part (Q -equation) and an infinite dimensional non-resonant part (P -equation). In the PDE-case (contrary to the case of a finite dimensional phase space), small divisor problems appear when solving the P -equation by a Newton iteration method, also in the time periodic case. Writing u in the form

$$u = \sum_{m,k} \hat{u}(m, k) e^{im\lambda t} \varphi_k(x) \quad (3.6)$$

and letting the linearized operator act on the Fourier coefficients $\hat{u}(m, k)$, one gets operators of the form

$$(m\lambda - \lambda_k) + \varepsilon T \quad (3.7)$$

where the first term is diagonal and T is essentially given by Toeplitz operators with exponentially decreasing matrix elements. The main task is then to obtain reasonable bounds on their inverses. The problem is closely related to a line of research around localization in the Anderson model. In this case, the operator T in (3.7) is replaced by $-\Delta$, $\Delta =$ lattice Laplacian, and the first term plays the role of the potential. Of primary importance is the structure of “singular sites”, i.e. the pairs (m, k) such that $|m\lambda - \lambda_k|$ is small. If we think of λ_k as k^2 , it is clear that these sites have separation $\rightarrow \infty$ for $k \rightarrow \infty$. Let us consider the higher dimensional problem for time periodic solutions. We assume the potential $V(x)$ of the form

$$V(x) = V_1(x_1) + \cdots + V_d(x_d) \quad (3.8)$$

in order to avoid certain problems (the dependence of spectrum and eigenfunctions on V) appearing for $d \geq 2$, other than the small divisor issues that is our primary concern here. Alternatively, one may replace in (3.1), (3.2) the term $V.u$ by a Fourier multiplier to introduce a frequency parameter. The eigenfunctions are then simply exponentials. Replacing λ_k by $|k|^2 = k_1^2 + \cdots + k_d^2$, it turns out that the singular sites may still be partitioned into distant clusters, of diameter and mutual distance $\rightarrow \infty$ for $|k| \rightarrow \infty$. This enables one to a large extent to reproduce the arguments from [C-W1, 2] to deal with that case also.

We now pass to the quasi-periodic problem. The singular sites are now the pairs (m, k) satisfying

$$|\langle m, \lambda \rangle - \lambda_k| < 1 \text{ where } m \in \mathbb{Z}^b, b > 1. \quad (3.9)$$

The structure of those is clearly different here and already in case of a finite dimensional phase space (i.e. finitely many k 's), new arguments are needed. There is resemblance with the works of Fröhlich, Spencer, Wittwer [F-S-W] and Surace [Sur] on localization for a quasi-periodic potential. In particular, one relies on a multi-scale analysis. The first step in these investigations is to recover the KAM and Melnikov results using this Liapounov-Schmidt technique. Next, one considers the PDE applications, when the phase space is infinite dimensional. In [B4], we discuss NLS and NLW in 1D obtained by perturbing a linear equation. In [B5],

2D-problems are considered. The presence of spectral clusters of unbounded size leads to certain difficulties we may deal with in $d = 2$ but so far not for larger dimension $d > 2$ (except for time periodic solutions). One may essentially formulate here the main model result as follows.

Theorem 3.10. *(see [B5]) Consider a perturbed Schrödinger equation*

$$iu_t - \Delta u + V(x)u + \varepsilon \frac{\partial H}{\partial \bar{u}}(u, \bar{u}, x) = 0 \quad (3.11)$$

or

$$iu_t - \Delta u + (u * V) + \varepsilon \frac{\partial H}{\partial \bar{u}}(u, \bar{u}, x) = 0 \quad (3.12)$$

where $x \in \prod^2$, $V(x) = V_1(x_1) + V_2(x_2)$ a real periodic potential in (3.11) and $u * V$ a real Fourier multiplier in (3.12). Denote $\{\mu_n\}$ the periodic spectrum of $-\Delta + V$ and $\{\varphi_n\}$ the corresponding eigenfunctions. Fix indices $n_1, n_2, \dots, n_b$ and consider $\lambda_1 = \mu_{n_1}, \dots, \lambda_b = \mu_{n_b}$ as parameters (letting V vary in an appropriate way). We assume following non-resonance property (1^e Melnikov condition, cf. [Kuk1])

$$|\langle m, \lambda \rangle + \mu_n| > N_0^{-c} \quad (3.13)$$

for $|m| \leq N_0$, $|n| \leq N_0$ and $n \notin \{n_1, \dots, n_b\}$.

(The μ_n may have a weak dependence on λ).

Consider the solution to the unperturbed equation $\varepsilon = 0$

$$u_0(x, t) = \sum_{j=1}^b a_j e^{i\lambda_j t} \varphi_{n_j}(x) \quad (3.14)$$

with $\{a_j\}$ fixed. For ε small enough (depending in particular on N_0), there is a (Cantor) set Δ of frequencies λ (depending on $|a_j|$ ($j = 1, \dots, b$)) with $\text{mes}(C\Delta) \rightarrow 0$ for $\varepsilon \rightarrow 0$ in the parameter set, such that if $\lambda \in \Delta$, there is a perturbed frequency

$$|\lambda' - \lambda| < C\varepsilon \quad (3.15)$$

and a perturbed solution

$$u_\varepsilon(x, t) = \sum_n \hat{u}_\varepsilon(n)(t) \varphi_n(x) \quad (3.16)$$

$$\hat{u}_\varepsilon(n)(t) = \sum_m \hat{u}_\varepsilon(n, m) e^{i(\lambda', m)t} \quad (3.17)$$

with

$$\hat{u}_\varepsilon(n_j, e_j) = a_j \quad (e_j = \text{junit vector in } \mathbb{Z}^b) \quad (3.18)$$

and

$$\sum_{(n, m) \notin S} e^{(|m|+|n|)^c} |\hat{u}_\varepsilon(n, m)| < \varepsilon^{1/2} \quad (3.19)$$

with

$$S = \{(n_j, e_j) | j = 1, \dots, b\}.$$

Equations (3.11), (3.12) are parameter dependent, since $V = V(x, \lambda)$. This may sometimes be avoided, relying on amplitude-frequency modulation based on the nonlinear term satisfying an appropriate twist condition (see [C-W1, 2]).

It is our future aim to investigate perturbations of Birkhoff integrable systems, following the same approach.

Written in an appropriate normal form, the unperturbed Hamiltonian may be given the form

$$H(p, q, y) = h(p) + \frac{1}{2} \langle A(p)y, y \rangle + O(|y|^3) \quad (3.20)$$

(cf. [Kuk1], [Kuk2]). Here the phase-space appears as $Z = Z_0 \oplus Y$ and $(p_1, \dots, p_n, q_1, \dots, q_n)$ are action-angle variables for Z_0 . Specifying $p = a$, write $h(p) = h(a) + \langle \frac{\partial h}{\partial p}(a), p - a \rangle + O(|p - a|^2)$. Then the λ -frequency vector is given by $\frac{\partial h}{\partial p}(a)$. Thus the usual twist condition

$$\det(\text{Hess } H) \neq 0 \quad (3.21)$$

corresponds to (3.3). At this stage, the problem appears essentially as perturbation of a linear system. The drawback of this normal form if one tries to apply the Liapounov-Schmidt technique and previously discussed methods, is the fact that the dependence of the perturbation on these new coordinates is quite complicated, although in the PDE one considers for instance a simple local polynomial expression as perturbation. We will next discuss the model of the perturbed KdV equation

$$u_t + u_{xxx} + uu_x + \varepsilon[f'(u)]_x = 0 \quad (3.22)$$

and prove persistency of those periodic (1 gap) solutions. The main idea will be to consider the normal coordinates only in a small (finite dimensional) part of the phase space and solve the remainder (containing the small divisors) using the equation (3.22) directly and standard Fourier coordinates.

§4. CONSTRUCTION OF PERIODIC SOLUTIONS OF KdV -TYPE EQUATIONS

Consider first a perturbation of the usual KdV -equation

$$y_t + y_{xxx} + yy_x + \varepsilon[F'(y)]_x = 0 \quad (4.1)$$

where we assume F a polynomial in y . Our aim is to construct a solution which is periodic in x and λ -periodic in t , of the form

$$y = \sum_{\substack{m, n \\ m \neq 0}} \hat{y}(m, n) e^{i(mx + n\lambda t)} \quad (4.2)$$

with

$$\widehat{y}(-m, -n) = \overline{\widehat{y}(m, n)} \quad (4.3)$$

and, say

$$\sum_{m, n} e^{-(|m|+|n|)^c} |\widehat{y}(m, n)| < \infty \quad (4.4)$$

for some $c > 0$.

This will be achieved by perturbing a periodic (1-gap) solution y_0 of the standard KdV equation

$$y_t + y_{xxx} + yy_x = 0. \quad (4.5)$$

The method used is a variant of the Liapounov-Schmidt decomposition argument used in [C-W1, 2] to construct periodic solutions of nonlinear wave and Schrödinger equations in $1D$ (see also [B1] for the higher dimensional case). It is well known that (5) is integrable and an appropriate fibration of the phase space has for instance been constructed in [Kap], [B-B-G-K]. These “integrable” coordinates $(R_m)_{m \geq 1}$ are related to the Fourier-coefficients $(\widehat{y}(m))_{m \geq 1}$ by a map Φ

$$R = \Phi(\widehat{y}) \quad (4.6)$$

which is real analytic and satisfies (for small data) in particular

$$D\Phi(\widehat{y}) = \mathbb{I} + O(\|\widehat{y}\|). \quad (4.7)$$

If one expresses equation (4.1) in the R -coordinates, the perturbative term becomes however non-local and carrying out the perturbation analysis in this infinite dimensional phase space requires a very detailed understanding of the properties of Φ and its derivatives. Rather than doing that, we will mix both R and Fourier coordinates, considering only the R -coordinates in a low-dimensional part of the phase space. More precisely, take y_0 of size δ and choose $M = M(\delta)$. We will parametrize first the potential y using Fourier coordinates $\widehat{y}(m)|_{m \leq M}$ and solve the remaining coefficients $\widehat{y}(m)|_{m > M}$ from the equation (1), i.e.

$$\left(\lambda \frac{n}{m} - m^2\right) \widehat{y}(m, n) + \widehat{G(y)}(m, n) = 0 \quad (m > M, n \in \mathbb{Z}) \quad (4.8)$$

where

$$G(y) = \frac{1}{2}y^2 + \varepsilon F'(y) \quad (4.9)$$

yielding $\widehat{y} = \widehat{y}_{[P_M \widehat{y}]}$ as a function of $P_M \widehat{y} = \widehat{y}(m)|_{m \leq M}$. The solution of (4.8) is gotten by applying Newton’s method as in the discussion of section 3, ignoring the special structure (4.9) of G at this stage. The linearized operators present singularities and this analysis is much in the spirit of [C-W1, 2] or [B1] (for the time-periodic case) mentioned in section 3.

Next, consider the R -coordinates $R_1, \dots, R_M$. From (4.6), one has

$$P_M R = P_M \Phi(\widehat{y}_{[P_M \widehat{y}]}) \quad (4.10)$$

and, by (4.7), this enables us to express $P_M \hat{y}$ in $P_M R$ using the implicit function theorem. The map Φ transforms equation (4.1) in equations of the form

$$iR_m + h_m(|R|^2)R_m + \varepsilon m(\nabla H_1(R))_m = 0 \quad (4.11)$$

where for small $I = (I_m)_{m \geq 1}$, $h_m(I)$ has the form

$$h_m(I) = h_m^0 + C_m I_m + o(|I|^2) \quad (4.12)$$

with $h_m^0, C_m \sim m^3$ (cf. [B-K], [B-B-E-I-M]).

We will only consider equations (4.11) for $m = 1, \dots, M$. Moreover, in (4.11), replace $R_m|_{m > M}$ by $\Phi_m(\hat{y}|_{[P_M \hat{y}]})|_{m > M}$ and consider $P_M \hat{y}$ as function of $P_M R$ according to (4.10). From this substitution results a system of M equations in $R_1, \dots, R_M$, of the form

$$iR_m + h_m(|P_M R|^2; 0)R_m + \Gamma_m(P_M R) = 0 \quad (m = 1, \dots, M) \quad (4.13)$$

where

$$\Gamma_m(P_M R) = [h_m(|R|^2) - h_m(|P_M R|^2; 0)]R_m + \varepsilon m(\nabla H_1(R))_m \quad (4.14)$$

will be perturbative. Assume

$$\begin{cases} \Phi_1(y_0(t)) = R_{1,1} e^{i h_1(R_{1,1}^2) t} \\ \Phi_m(y_0(t)) = 0 \quad \text{for } m > 1 \end{cases} \quad (4.15)$$

the KdV (unperturbed) solution.

One may then solve $R_m|_{2 \leq m \leq M}$ and $R_{1,n}|_{n \neq 1}$ from (4.13) and the implicit function theorem. The remaining equation for $m = 1$ and the $e^{i\lambda t}$ -coefficient yields the new frequency

$$\lambda = h_1(R_{1,1}^2) + \varepsilon(R_{1,1}) \quad (4.16)$$

(we will show that the number defined this way is real, as a consequence of the Hamiltonian structure). At first stage, restrictions on the frequency λ appear which eventually restrict $R_{1,1}$ to a certain Cantor set (of positive measure). Observe that if $R_{1,1} \sim \delta$, one has $\lambda \sim 1 + \delta^2$ and hence a δ^2 -size parameter set. Thus in solving (4.8), one needs to be careful about the role of δ and $M(\delta)$ will have to be suitably chosen.

Next, we briefly discuss the Newton iteration scheme to solve (4.8). One takes for the starting value $\hat{y}^0 = (\hat{y}|_{m \leq M}, 0)$ and solves essentially

$$\Delta \hat{y} = -T_{y^0}^{-1}[\widehat{G(y^0)}|_{m > M}], \quad (4.17)$$

where T_y is a (suitable restriction of) the linearized operator

$$(\lambda \frac{n}{m} - m^2) + S_{G'(y)}|_{m > M} \quad (4.18)$$

with $\{\lambda \frac{n}{m} - m^2\}$ the diagonal and $S_\phi(\xi, \eta) = \widehat{\phi}(\xi - \eta)$. Observe that $|\widehat{G}(y_0)|_{m > M}$ is of exponential small size.

We will call singular sites the lattice pairs (m, n) such that

$$|\lambda \frac{n}{m} - m^2| < 1. \quad (4.19)$$

Hence, they appear in well-separated islands (for a fixed λ)

$$|\lambda n - m^3| < |m| \quad (4.20)$$

of size $\sim |m|$ and separation $\sim m^2$. Let T_{y^0} be the operator (4.18) for $y = y^0$ with $M < m < CM$. Then $\|T_{y^0}^{-1}\| < K$ provided

$$\text{dist} \left(0, \text{spec} \left(\lambda \frac{n}{m} - m^2 + S_{G'(y_0)}|_{m \sim M} \right) \right) > \frac{1}{K}. \quad (4.21)$$

Consider R_{11} as a parameter with variation range $\sim \delta$. Then λ behaves as $1 + R_{11}^2$ and $\|S_{G'(y)}\| \lesssim \delta$. On singular islands, one has

$$|\lambda \frac{n}{m} - m^2| > \frac{1}{M} |n(1 + R_{11}^2) - m^3| \geq \frac{1}{M} \|R_{11}^2 n\| > \delta^2 M^2 \quad (4.22)$$

provided

$$\delta^2 M^3 \ll 1. \quad (4.23)$$

If M satisfies also

$$\delta M^2 \gg 1 \quad (4.24)$$

one may take for $K = \frac{1}{\delta^2 M^2}$. Define

$$M = M(\delta) \sim \delta^{-1/2} \quad (4.25)$$

so that (4.23) holds. Then

$$\|T_{y^u}^{-1}\| \lesssim \delta^{-1}. \quad (4.26)$$

Performing the Newton iteration, the same bound on the inverse of (4.18) will hold as long as (4.23) holds, i.e. for $\sim \log \frac{1}{\delta}$ steps. After this, one applies first order eigenvalue variation, using the fact that on singular islands $\delta \frac{n}{m} \sim \delta m^2 \gg 1$.

The vector $\widehat{y}_{[P_M \widehat{y}]}|_{m > M}$ constructed this way has the first increment (4.17) as an essential part, which has as main contribution

$$T_{y^u}^{-1} \{(\widehat{y^0})^2\} = T_{y^u}^{-1} \left\{ \begin{array}{c} \sum_{m_1+m_2=M+1} \widehat{y}(m_1)\widehat{y}(m_2) \\ \vdots \\ \sum_{m_1+m_2=2M} \widehat{y}(m_1)\widehat{y}(m_2) \end{array} \right\}. \quad (4.27)$$

Hence, by (4.26), (4.27), it easily follows that

$$\nabla_{[P_M \widehat{y}]} T_{y^u}^{-1} \{(\widehat{y^0})^2\} = 0(1) \quad (4.28)$$

and thus

$$\nabla_{[P_M \widehat{y}]} \widehat{y}_{[P_M \widehat{y}]} = 0(1). \quad (4.29)$$

Next, consider (4.10), which we solve in $P_M \widehat{y}$ by the implicit function theorem. Thus by (4.29)

$$\begin{aligned} \nabla_{[P_M \widehat{y}]} \{P_M \Phi(\widehat{y}_{[P_M \widehat{y}]})\} &= P_M D\Phi(\widehat{y}) \left\{ \begin{array}{c} 1 \quad 0 \\ \vdots \\ 0 \quad 1 \\ 0(1) \end{array} \right\} \\ &= \left(\begin{array}{cc} 1 & 0 \\ & \ddots \\ 0 & 1 \end{array} \right) + P_M (\mathbb{I}K - D\Phi(\widehat{y})) \left\{ \begin{array}{c} 1 \quad 0 \\ \vdots \\ 0 \quad 1 \\ 0(1) \end{array} \right\}. \end{aligned} \quad (4.30)$$

By (4.7), $\mathbb{I} - D\Phi(\widehat{y}) = O(\delta)$ and a precise description of the matrix may be found in [Kap]. As mentioned earlier, we will replace the equations

$$P_M(y_t + y_{xxx} + G(y)_x) = 0 \quad (4.31)$$

by the equations (4.11) or (4.13) for $m \leq M$. This new problem is equivalent, since

$$P_M D\Phi(\hat{y})[(y_t + y_{xxx} + G(y)_x)^\wedge] = 0 \quad (4.32)$$

and (4.7) imply

$$\begin{aligned} P_M[(y_t + y_{xxx} + G(y)_x)^\wedge] &= -P_M(\mathbb{1} - D\Phi(\hat{y}))P_M[(y_t + y_{xxx} + G(y)_x)^\wedge] \\ &= O(\delta)P_M[(y_t + y_{xxx} + G(y)_x)^\wedge] \end{aligned} \quad (4.33)$$

from where also

$$P_M[(y_t + y_{xxx} + G(y)_x)^\wedge] = 0.$$

Next we need to solve (4.13) in $P_M R$

$$iR_m + h_m(|P_M R|^2; 0)R_m + \Gamma_m(P_M R) = 0 \quad (m = 1, \dots, M) \quad (4.34)$$

$$\Gamma_m(P_M R) \equiv [h_m(|R|^2) - h_m(|P_M R|^2; 0)]R_m + \varepsilon m(\nabla H_1(R))_m \quad (4.35)$$

More precisely, we solve in $(R_{m,n})_{\substack{2 \leq m \leq M \\ n \in \mathbb{Z}}}$ and $(R_{1,n})_{n \neq 1}$. The remaining equation for the $e^{i\lambda t}$ -coefficient is used to determine the new frequency λ .

The term (4.35) may be considered perturbative and it remains thus to control the inverse of the linearization of

$$iR_m + h_m(|P_M R|^2; 0)R_m \quad (m = 1, \dots, M) \quad (4.36)$$

so that one may apply again the implicit function theorem.

Write $R_1 = R_{11}e^{i\lambda t} + \sum_{n \neq 1} R_{1n}e^{in\lambda t} = R_{11}e^{i\lambda t} + R'_1$. Recall that R_{11} is fixed, say real. One expects $\|R'_1\|, \|R_m\| (m > 1)$ to be $O(\varepsilon)$.

Writing

$$\begin{aligned} iR_1 + h_1(|P_M R|^2)R_1 &= -\lambda R_{11}e^{i\lambda t} + i(R'_1) \\ &+ h_1(R_{11}^2 + R_{11}e^{i\lambda t}\overline{R'_1} + R_{11}e^{-i\lambda t}R'_1 + |R'_1|^2; |R_m|^2(2 \leq m \leq M))(R_{11}e^{i\lambda t} + R'_1) \end{aligned} \quad (4.37)$$

the essential part of the linearization of (4.36) with respect to $R_{1,n} (n \neq 1)$, $R_{m,n} (m \neq 1, n \in \mathbb{Z})$ is given by

$$\begin{cases} [-n\lambda + h_1(R_{11}^2; 0)]R_{1n} + R_{11}^2 h'_1(R_{11}^2; 0)(\overline{R}_{1,2-n} + R_{1n}) & (n \neq 1) \\ [-n\lambda + h_m(R_{11}^2; 0)]R_{mn} & (m = 2, \dots, M; n). \end{cases} \quad (4.38-4.39)$$

Recall that $\lambda \approx h_1(R_{11}^2; 0)$. To keep the multipliers in (4.39) away from 0 (non-resonance condition) put $I = R_{11}^2$ and consider

$$\frac{\partial}{\partial I}[-nh_1(I; 0) + h_m(I; 0)] \quad (m \neq 1). \quad (4.40)$$

From the analysis in [B-K] (cf. (4.12)), (4.25), one gets

$$\partial_I h_1(I; 0) \sim 1 \quad (4.41)$$

$$\partial_I h_m(I; 0) = 0(m^2 I) \leq 0(M^2 \delta^2) = O(\delta) \quad (4.42)$$

so that one may ensure

$$|-n\lambda + h_m(R_{11}^2; 0)| \geq \delta^2 (m = 2, \dots, M; n \neq 0). \quad (4.43)$$

Rewrite (4.38) as

$$\begin{cases} [-n\lambda + h_1(R_{11}^2) + R_{11}^2 h_1'(R_{11}^2)] R_{1n} + R_{11}^2 h_1'(R_{11}^2) \bar{R}_{1,2-n} \\ [- (2-n)\lambda + h_1(R_{11}^2) + R_{11}^2 h_1'(R_{11}^2)] R_{1,2-n} + R_{11}^2 h_1'(R_{11}^2) \bar{R}_{1,n} \end{cases} \quad (4.44)$$

which inverse is bounded by ($n \neq 1$)

$$|n-1|^{-1} h_1(R_{11}^2; 0)^{-1} < \delta^{-2}. \quad (4.45)$$

Hence, the implicit function theorem applies and yields $R_{1,n} (n \neq 1)$ and $R_{m,n} (m = 2, \dots, M; n)$ as functions of R_{11} and λ , of the form $\delta^{-2} \varepsilon(R_{11}, \lambda)$. The equation for the $e^{i\lambda t}$ -coefficient of R_1 is thus of the form

$$\lambda = h_1(R_{11}^2) + \delta^{-1} \varepsilon(R_{11}, \lambda) \quad (4.46)$$

which again by the implicit function theorem permits to solve

$$\lambda = h_1(R_{11}^2) + \delta^{-1} \varepsilon(R_{11}). \quad (4.47)$$

The fact that λ is real follows from the Hamiltonian structure. We give an argument below. Restrictions on λ appearing especially when solving (4.8) in $P_M \hat{y}$ by the Newton method may be formulated in R_{11} , leading thus to a condition

$$R_{11} \in \Delta \quad (4.48)$$

whose Δ has measure $\sim \delta$ in the parameter set considered.

Remark. Equation (4.46) corresponds to

$$\begin{aligned} \operatorname{Re}[iR_1 + h_1(|P_M R|^2; 0)R_1 + \Gamma_1(P_M R)]^\wedge(1) = \\ -R_{11} \cdot \lambda + \operatorname{Re} \left(\frac{\partial H}{\partial \bar{R}_1} \right)^\wedge(1) = 0. \end{aligned} \quad (4.49)$$

We show that then necessarily also

$$\operatorname{Im} \left(\frac{\partial H}{\partial \bar{R}_1} \right)^\wedge(1) = 0. \quad (4.50)$$

Write

$$\frac{d}{dt}H(R(t), \overline{R(t)}) = Re \left\{ \sum_{m=1}^M \frac{\partial H}{\partial R_m} \dot{R}_m + \sum_{m>M} \frac{\partial H}{\partial R_m} \dot{R}_m \right\}. \quad (4.51)$$

For $m = 2, \dots, M$, one has by construction $i\dot{R}_m = \frac{\partial H}{\partial \overline{R}_m}$, hence $Re \left\{ \sum_{m=2}^M \frac{\partial H}{\partial R_m} \dot{R}_m \right\} = 0$. For $m = 1$, $i\dot{R}_1 = \frac{\partial H}{\partial R_1} - i \left[\text{Im} \left(\frac{\partial H}{\partial R_1} \right)^\wedge (1) \right] e^{i\lambda t}$ and thus $Re \frac{\partial H}{\partial R_1} \dot{R}_1 = Re \left\{ \frac{\partial H}{\partial R_1} \left[\text{Im} \left(\frac{\partial H}{\partial R_1} \right)^\wedge (1) \right] e^{i\lambda t} \right\}$. Since $\frac{\partial H}{\partial R_1} \sim \overline{R}_1 + O(\varepsilon) = R_{11}e^{-i\lambda t} + O(\varepsilon)$, it follows that

$$Re \frac{\partial H}{\partial R_1} \dot{R}_1 \sim (R_{11} + O(\varepsilon)) \text{Im} \left(\frac{\partial H}{\partial \overline{R}_1} \right)^\wedge (1). \quad (4.52)$$

For $m > M$, one satisfies

$$(y' + [y_{xxx} + \frac{1}{2}(y^2)_x + \varepsilon F'(y)_x])^\wedge(m) = 0$$

and hence, since $R = \Phi(y)$, $D\Phi(y) = \mathbb{1} + O(\|y\|)$

$$\begin{aligned} i\dot{R}_m - \frac{\partial H}{\partial \overline{R}_m} \Big|_{m>M} &= D\Phi(y) \left[(y' + (y_{xxx} + \frac{1}{2}(y^2)_x + \varepsilon F'(y)_x))^\wedge \right] \Big|_{m>M} \\ &= (D\Phi(y) - \mathbb{1})[\dots] \Big|_{m>M} \\ \left\| i\dot{R}_m - \frac{\partial H}{\partial \overline{R}_m} \Big|_{m>M} \right\| &\leq O(\delta) \left\| i\dot{R} - \frac{\partial H}{\partial \overline{R}} \right\| \\ \left\| i\dot{R}_m - \frac{\partial H}{\partial \overline{R}_m} \Big|_{m>M} \right\| &\leq O(\delta) \left| \text{Im} \left(\frac{\partial H}{\partial \overline{R}_1} \right)^\wedge (1) \right|. \end{aligned} \quad (4.53)$$

This yields

$$Re \left\{ \sum_{m>M} \frac{\partial H}{\partial R_m} \dot{R}_m \right\} = O(\delta^2) \left| \text{Im} \left(\frac{\partial H}{\partial \overline{R}_1} \right)^\wedge (1) \right|. \quad (4.54)$$

Substitution of (4.52), (4.54) gives for the right side of (4.51)

$$[R_{11} + O(\varepsilon) + O(\delta^2)] \text{Im} \left(\frac{\partial H}{\partial \overline{R}_1} \right)^\wedge (1). \quad (4.55)$$

But since $H(R(t), \overline{R(t)})$ is periodic in t , the left side has no constant term, so (4.50) has to hold.

Consider now a generalized KdV equation of the form

$$u_t + u_{xxx} + [f(u)]_x = 0 \quad (4.56)$$

where $f(u)$ is a polynomial in u . Assume there is a real number c satisfying

$$f''(c) \neq 0 \text{ and } f'''(c) = 0. \quad (4.57)$$

The substitution

$$u = v + c \quad (4.58)$$

transforms (4.56) in

$$v_t + v_{xxx} + [f'(c)v + \frac{f''(c)}{2}v^2 + 0(v^4)]_x = 0. \quad (4.59)$$

Considering a further substitution of the form

$$v(x, t) = w(x + \alpha t, t) \quad (4.60)$$

the linear term in (4.59) may be eliminated too and we get an equation of the form

$$w_t + w_{xxx} + [w^2 + 0(w^4)]_x = 0. \quad (4.61)$$

Letting $w = \delta y$, one gets

$$y_t + y_{xxx} + [\delta y^2 + \delta^3 F(y)]_x = 0. \quad (4.62)$$

The solution of $y_t + y_{xxx} + \delta y^2 = 0$ admits a λ -frequency variation of size δ^2 while $\varepsilon = \delta^3$. This suffices for the preceding analysis. One gets instead of (4.47)

$$\lambda = h_1(R_{11}^2) + 0(R_{11}^3). \quad (4.63)$$

Consequently, one obtains the following

Proposition 4.64. *Consider the equation*

$$u_t + u_{xxx} + [f(u)]_x = 0.$$

where f is a polynomial such that $f'''(c) = 0$, $f''(c) \neq 0$ for some real number c . Then one has solutions of the form

$$u(x, t) = c + K(x + \alpha t, t) \quad (4.65)$$

where $K(x, t)$ is 1-periodic in x and λ -periodic in t .

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