§1. Introduction

Bifurcation and coalescence of black holes are one of the most interesting problems. Before the dynamical behavior of black holes is attacked by the analytic method we need to deepen our knowledge and experience by finding out the static and stationary solutions of multi black holes. In the preceding papers[1,2,3] we have studied the various static solutions of the Einstein equation assuming the axial symmetry. We studied a series of exact solutions to the vacuum Einstein equation which are the multi Weyl solutions including the Schwarzschild solutions[1]. We then showed an infinite series of exact solutions to the Einstein equation with a scalar field[2]. We also gave a multi charged Weyl solution including multi Reissner-Nordström(R-N) solution[3]. More than a decade ago Belinskii and Zakharov[4] showed that the Kerr solution is obtained as the first non-trivial solution in an infinite series of soliton solutions.

These examples indicate that all the static or stationary exact solutions are understood as the soliton solutions and so we can expect particle-like configuration in the space-time for such solutions. If the solutions are the soliton solutions, a descendant series of exact solutions should always follow the exact solution as multi soliton solutions. The physical implication of the multi soliton solutions is multi particle-like configuration located at different positions.

In the context of theories of elementary particles and/or superstrings many people are interested in the Maxwell-Einstein equations with a dilaton field[5,6]. The black hole solution is already given[6]. It is interesting to note that the limit of zero dilaton charge in the solution does not lead to the R-N solution. In this paper we study the multi dilaton black holes showing that our view point that an infinite series of multi soliton solutions should follow an exact solution also applies to the present case.

Since we give an exact multi soliton solution it is possible to discuss the static balance of two dilaton black holes. In the Majumdar-Papapetrou(M-P) solution[7] there are two kinds of force: the attractive gravitational force and the repulsive electric force. They cancel each other and so the static balance
is kept. In the charged dilaton black hole case there are three kinds of force: the attractive gravitational and dilatonic force and repulsive electric force. The mass, dilaton charge and electric charge in our solution are not necessarily tuned to balance each other. Then a question arises. How is the static solution possible? We also discuss this problem in the paper.

In the case of charged Weyl solution we have discussed the condition of the static balance and showed the necessary and sufficient condition for the static balance[3]. We shall also discuss the static balance in the present case and we find that the horizons of both black holes become singular in the limit that the z-axis between two black holes should be Euclidean flat. There are three parameters corresponding to mass, dilaton charge and the electric charge, but only two of them are independent in the solution. The condition that there is no conical singularity along the axis between two black holes reduces the two parameters into one, and it coincides with a cancellation of the three kinds of force due to the mass and charges. This implies that the the z-axis with a conical singularity in the case when we do not impose the Euclidean flatness condition plays the role of "force" besides the dilatonic, the electrical and the gravitational force. When the z-axis is not Euclidean flat or there is no cancellation among force due to mass and charges, one can ask how the static balance is realized? We discuss the role of the conical singularity and give an interpretation of "strut" to the axis.

In the next section we show that we can reduce the dilaton Maxwell-Einstein equations into two sets of vacuum Einstein equations and present the multi soliton solution. In Section 3 we give the definitions of mass, dilaton charge and electric charge in the multi soliton solution. The last section is devoted to a discussion of the equilibrium condition of multi dilaton black holes and gives a physical interpretation to the conical singularity. We use the units $c = G = 1$ throughout this paper.

§2. Structure of the dilaton Maxwell-Einstein equations

In this section we discuss how we get the multi dilaton black hole solution stressing on the structure of the equations. One should note that the solution here is obtained without assuming the conformal-static metric[8]. The assumption may simplify the equations to a great extent but it brings about a loss of generality and might have missed the physically significant aspects. There is no a priori reason to impose such an assumption more than easiness. We should look for solutions as generally as possible and so we simply assume the axial symmetry and static condition.

We consider the Einstein-Maxwell action with a dilaton field:

$$I = \int d^4x\sqrt{-g}[R - 2\partial_\mu\phi\partial^\mu\phi - e^{-2\phi}F_{\mu\nu}F^{\mu\nu}].$$

(2.1)

Here $\phi$ is a dilaton field. The equations of motion are

$$\nabla_\mu(e^{-2\phi}F^{\mu\nu}) = 0,$$

(2.2)

$$\nabla^2\phi + \frac{1}{2}e^{-2\phi}F^2 = 0,$$

(2.3)

$$R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right),$$

(2.4)

where

$$T_{\mu\nu} = T_{\mu\nu}^e + T_{\mu\nu}^d,$$

(2.5)

with

$$T_{\mu\nu}^e = \frac{1}{4\pi}e^{-2\phi}(F_{\mu\rho}F_{\nu}^\rho - \frac{1}{2}g_{\mu\nu}F^2),$$

(2.6)

$$T_{\mu\nu}^d = \frac{1}{4\pi}[
abla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2],$$

(2.7)

$$F_{\mu\nu} = A_{\mu\nu} - A_{\mu\nu},$$

(2.8)

and $T$ is the trace of $T_{\mu\nu}$. We solve the equations under the static and axially symmetric condition. We further assume that the U(1) gauge field has only electric charge and no magnetic charge for simplicity, though it is not difficult to
extend the present solutions to those incorporating the magnetic charge. The metric reads

\[ ds^2 = -f dt^2 + f^{-1}[Q(d\rho^2 + dz^2) + \rho^2 d\phi^2], \tag{2.9} \]

where \( f = f(\rho, z) \) and \( Q = Q(\rho, z) \). The equations of motion become

\[ \nabla^2 (\ln f) = e^{-2\phi} \frac{2}{Q} (\chi_\rho^2 + \chi_z^2) \], \tag{2.10} \]

\[ \nabla^2 \phi = e^{-2\phi} \frac{1}{Q} (\chi_\rho^2 + \chi_z^2) \], \tag{2.11} \]

\[ \nabla^2 \chi = \frac{f}{Q} [\chi_\rho (\ln f)_\rho + \chi_z (\ln f)_z + 2\chi_\rho \phi_\rho + 2\chi_z \phi_z] \], \tag{2.12} \]

\[ (\ln Q)_\rho = \frac{\rho}{2} \left( (\ln f)^2 \phi - (\ln f)^2 \chi_\rho^2 + 2\phi \chi_\rho \phi_\rho + \phi \chi_z \phi_z \right) \], \tag{2.13} \]

\[ (\ln Q)_z = \rho (\ln f)_\rho (\ln f)_z - \frac{4\rho}{f} e^{-2\phi} \chi_\rho \chi_z + 4\rho \phi_\rho \phi_z \], \tag{2.14} \]

where \( \nabla^2 = \frac{1}{f^2} (\delta_\rho^2 + \delta_z^2 + f^{-1} \partial_\rho) \) and \( \chi \) is the zeroth component of the \( U(1) \) gauge field. \( \chi = -A_0 \). When there is no dilaton field, the equations are simply the Maxwell-Einstein equations. When there is no gauge field, the equations are nothing but the Einstein equation coupled with a scalar field. We have already obtained the soliton solutions for both cases. Our strategy is to decouple the entwined fields by introducing new fields. We first introduce \( F \) and \( \tilde{F} \) by

\[ F = f e^{2\phi}, \quad \tilde{F} = f e^{-2\phi}. \tag{2.15} \]

The above equations are rewritten by using new fields as

\[ \nabla^2 (\ln F) = 0, \tag{2.16} \]

\[ \nabla^2 (\ln \tilde{F}) = \frac{4}{Q} \sqrt{\tilde{F}} (\chi_\rho^2 + \chi_z^2), \tag{2.17} \]

\[ \nabla^2 \chi = \frac{1}{Q} \sqrt{\tilde{F} F} (\chi_\rho (\ln F)_\rho + \chi_z (\ln F)_z), \tag{2.18} \]

\[ (\ln Q)_\rho = \frac{\rho}{2} \left( (\ln F)^2 \phi - (\ln F)^2 \chi_\rho^2 + 2\phi \chi_\rho \phi_\rho + \phi \chi_z \phi_z \right) + \frac{\rho}{2} \left( (\ln \tilde{F})^2 \phi - (\ln \tilde{F})^2 \chi_\rho^2 \right), \tag{2.19} \]

\[ (\ln Q)_z = \frac{\rho}{2} (\ln F)_\rho (\ln F)_z - \frac{4\rho}{F} \chi_\rho \chi_z \phi_\rho \phi_z, \tag{2.20} \]

We next introduce \( Q_{NN}, Q_s \) and \( A \) by

\[ Q = \sqrt{Q_{NN} Q_s}, \tag{2.21} \]

\[ \chi = A/\sqrt{2}, \tag{2.22} \]

We can now decompose the equations into two groups. We require that \( Q_s \) should satisfy the first group of equations:

\[ (\ln \tilde{F})_{\rho\rho} + (\ln F)_{zz} + \frac{1}{\rho} (\ln \tilde{F})_\rho = 0, \tag{2.22} \]

\[ (\ln Q_s)_\rho = \frac{\rho}{2} ((\ln F)_{\rho\rho}^2 - (\ln F)_z^2), \tag{2.23} \]

\[ (\ln Q_s)_z = \rho (\ln \tilde{F})_\rho (\ln \tilde{F})_z. \tag{2.24} \]

This is the vacuum Einstein equation for the metric with \( \tilde{F} \) and \( Q_s \), instead of \( f \) and \( Q \), respectively in the metric (2.9). As far as \( Q_s \) satisfies the first group of equations, the second group of equations are

\[ (\ln F)_{\rho\rho} + (\ln F)_{zz} + \frac{1}{\rho} (\ln F)_\rho = \frac{2}{F} (A_{\rho \rho}^2 + A_{zz}^2), \tag{2.25} \]

\[ A_{\rho \rho} + A_{zz} + \frac{1}{\rho} A_\rho = A_\rho (\ln F)_\rho + A_z (\ln F)_z, \tag{2.26} \]

\[ (\ln Q_{NN})_\rho = \frac{\rho}{2} ((\ln F)_{\rho\rho}^2 - (\ln F)_z^2) - \frac{2\rho}{F} (A_{\rho \rho}^2 - A_{zz}^2), \tag{2.27} \]

\[ (\ln Q_{NN})_z = \rho (\ln F)_\rho (\ln F)_z - \frac{4\rho}{F} A_\rho A_z. \tag{2.28} \]

These equations are exactly the Maxwell-Einstein equations for the metric with \( F \) and \( Q_{NN} \) instead of \( f \) and \( Q \), respectively in the metric (2.9) and \( A \) represents the electrostastic potential. We have already given the infinite number of soliton solutions to the two groups and so we can construct the infinite number of solutions by combining the already known solutions and pulling them back to the original fields with a little care about the boundary conditions.

We impose the asymptotic flatness condition on the metric as

\[ f \to 1, \quad \text{as} \quad \sqrt{\rho^2 + z^2} \to \infty, \tag{2.29} \]

and the boundary conditions on the dilaton field and electrostatic field as

\[ \phi \to \phi_0, \quad \chi \to 0, \quad \text{as} \quad \sqrt{\rho^2 + z^2} \to \infty. \tag{2.30} \]
We thus obtain the asymptotic behaviors of \( F = f e^{2\phi_0} \) and \( \tilde{F} = f e^{-2\phi_0} \) as

\[
F \to e^{2\phi_0}, \quad \tilde{F} \to e^{-2\phi_0}, \quad \text{as} \quad \sqrt{\rho^2 + z^2} \to \infty.
\]  

(2.31)

Since this shows the different boundary conditions from the vacuum case we focus on this. We shall illustrate how we can reduce the second group of equations or the Maxwell-Einstein equations to the vacuum Einstein equation paying attention to the boundary conditions. Assuming that \( F \) is the functional of \( A \) field we solve the equations for \( F \) to obtain

\[
F = c_1 - 2c_2 A + A^2,
\]

(2.32)

where \( c_1 \) and \( c_2 \) are constants. By using the asymptotic flatness condition for \( F \), \( c_1 \) is determined to be \( c_1 = e^{2\phi_0} \). This reduces the differential Eqs. (2.25)-(2.28) to

\[
\begin{align*}
A_{\rho \rho} + A_{zz} + \frac{1}{\rho} A_{\rho} &= \frac{2}{F} (A - c_2) \left( A_{\rho}^2 + A_{z}^2 \right), \\
(\ln Q_{HR})_{\rho} &= \frac{2\rho}{F} (c_2^2 - 1) \left( A_{\rho}^2 - A_{z}^2 \right), \\
(\ln Q_{HR})_z &= \frac{4\rho}{F} (c_2^2 - 1) A_{\rho} A_{z}.
\end{align*}
\]

(2.33)

(2.34)

(2.35)

We now introduce a new function \( R(\rho, z) \) by

\[
A = \frac{e e^{2\phi_0}}{R + m},
\]

(2.36)

where \( e \) and \( m \) are constants with \( m = c_2 e^{-2\phi_0} \). We further introduce a new function \( \tilde{f}(\rho, z) \) by

\[
R = \frac{1}{d} \left( \frac{1 + f}{1 - f} \right).
\]

(2.37)

where \( d^2 = m^2 - e^2 \). Then Eq. (2.33) leads to the equation for \( \tilde{f} \):

\[
(\ln \tilde{f})_{\rho \rho} + (\ln \tilde{f})_{zz} + \frac{1}{\rho} (\ln \tilde{f})_{\rho} = 0.
\]

(2.38)

We can also rewrite Eqs. (2.34) and (2.35) by using the function \( \tilde{f} \) as

\[
\begin{align*}
(\ln Q_{HR})_{\rho} &= \frac{\rho}{2} \left( (\ln \tilde{f})_{\rho}^2 - (\ln \tilde{f})_{z}^2 \right), \\
(\ln Q_{HR})_z &= \rho (\ln \tilde{f})_{\rho} (\ln \tilde{f})_{z}.
\end{align*}
\]

(2.39)

(2.40)

These equations are exactly the same as those for the vacuum Einstein equation of the metric (2.9) with \( \tilde{f} \) and \( Q_{HR} \) instead of \( f \) and \( Q \), respectively. We therefore have shown that the Maxwell-Einstein equations with a dilaton field reduces to two groups of vacuum Einstein equations. Since we know the soliton solutions to the vacuum Einstein equation we can construct an infinite number of solutions by combining two infinite series of soliton solutions. Though there is no mathematical reason that the parameters appearing in the pole trajectories of two groups are the same we equate them (i.e., \( \tilde{f} = f \), \( Q_{HR} = Q_s = Q \)) as well as the soliton numbers in order to extract physically interesting aspects. We thus obtain the soliton solutions to the original action:

\[
f = \frac{2d\tilde{f}}{m + d - (m - d)\tilde{f}},
\]

(2.41)

\[
Q = \frac{\rho^2 \prod_{k=1}^{N} (\mu_k - \mu)^{2q_k}}{\prod_{k=1}^{N} (\mu_k^2 + \rho^2)^{1/2} \prod_{k=1}^{N} \mu_k^{(D-2)\eta_k} C(N)},
\]

(2.42)

\[
\phi = \phi_0 + \frac{1}{2} \ln \left[ \frac{m + d - (m - d)\tilde{f}}{d} \right],
\]

(2.43)

\[
\chi = \frac{e e^{2\phi_0} (1 - \tilde{f})}{\sqrt{2[m + d - (m - d)\tilde{f}]}}.
\]

(2.44)

Here

\[
f = \prod_{k=1}^{N} \left( \frac{\mu_k}{\rho} \right)^{q_k},
\]

(2.45)

\[
\mu_k = w_k - \frac{z}{k^2 + 1} \sqrt{(w_k - z)^2 + \rho^2},
\]

(2.46)

where \( w_k (k = 1, 2, \ldots, N) \) are constants. Hereafter we assume that the soliton number \( N \) is an even number in order to guarantee the reality of metric. The positive constants \( q_k \)'s appearing in Eqs. (2.42) and (2.45) are related to the distortion parameters \( \epsilon_i \)'s by \( q_{2i-1} = q_{2i} = \epsilon_i (i = 1, 2, \ldots, N/2) \), and \( D = \sum_{k=1}^{N} q_k \).

The constant \( C(N) \) in Eq. (2.42) is given by

\[
C(N) = \prod_{1<j}^{N/2} [2(w_{2i-1} - w_{2j-1})(w_{2i-1} - w_{2j-1})]^{\epsilon_i \cdot \epsilon_j},
\]

(2.47)
which has been determined so that the solution may satisfy the asymptotic flatness condition.

In order to elucidate the solutions we illustrate the 2-soliton case. We assume that the deformation parameter is 1. Reparameterizing $w_1$ and $w_2$ as

$$w_1 = z_0 - d, \quad w_2 = z_0 + d,$$  \hspace{1cm} (2.48)

and introducing the spherical coordinates ($r$, $\theta$) by

$$\rho = \sqrt{(r - m)^2 - d^2} \sin \theta,$$  \hspace{1cm} (2.49)

$$z - z_0 = (r - m) \cos \theta,$$  \hspace{1cm} (2.50)

we obtain

$$f = -\frac{\mu_1 \mu_2}{\rho^2} = \frac{r - m - d}{r - m + d}.$$  \hspace{1cm} (2.51)

Since the metric and the fields are expressed in terms of $f$ we can express the solution as

$$ds^2 = -\left(1 - 2 \frac{m + d}{2r}\right) dt^2 + \left(1 - 2 \frac{m + d}{2r}\right)^{-1} dr^2 + r(r - m + d) d\Omega^2, \quad (2.52)$$

$$\phi = \phi_0 + \frac{1}{2} \ln \left(1 - \frac{m - d}{r}\right), \quad (2.53)$$

$$\chi = \frac{\epsilon e^{-\phi_0}}{\sqrt{2r}}. \quad (2.54)$$

This shows that the charges defined by $M = (m + d)/2$ and $\Sigma = -(m - d)/2$ are the mass and the dilaton charge, respectively. We define the electric charge by $Q = \epsilon e^{-\phi_0}/\sqrt{2}$. As for the definition of the electric charge we shall discuss it in the next section in detail. Though there are three charges, they are not independent. The condition that $d^2 = m^2 - c^2$ leads to

$$\Sigma = -\frac{(Q e^{\phi_0})^2}{2M}. \quad (2.55)$$

The final form of the 2-soliton solution reads

$$ds^2 = -\left(1 - 2 \frac{M}{r}\right) dt^2 + \left(1 - 2 \frac{M}{r}\right)^{-1} dr^2 + r(r + 2\Sigma) d\Omega^2, \quad (2.56)$$

$$\phi = \phi_0 + \frac{1}{2} \ln \left(1 + \frac{2\Sigma}{r}\right), \quad (2.57)$$

$$\chi = \frac{Q e^{\phi_0}}{r}. \quad (2.58)$$

This is the solution discussed in Ref. [6]. This solution is characterized by a regular horizon lying at $r = 2M$ like the Schwarzschild solution rather than the R-N solution despite it has an electric charge. The area measured at some $r$ goes to zero as $r \to -2\Sigma$ causing this surface to be singular. In order to conceal the singularity in the horizon there should be a relation that

$$M > -\Sigma. \quad (2.59)$$

The solution is the black hole solution as far as this condition is satisfied. In order to discuss the multi-soliton solution we have to clarify what the mass, dilaton and electric charges are. We shall discuss this in the next section.

### 5.3. Definitions of mass and charges of multi solitons

In order to clarify physics of the multi soliton solution we first study the behavior of metric of the 2-soliton solution or the dilaton R-N solution in the vicinity of the horizon. We discuss the behavior by using the ($\rho, \theta$) variables. We examine the behavior of the solution around the $z$-axis in $w_1 < z < w_2$. We let the deformation parameter $\delta$ be free parameters for a while. Since $f$ behaves as $f \sim \rho^{2\delta}$ in the region $w_1 < z < w_2$ at $\rho \sim 0$, we have

$$f \sim \rho^{2\delta}, \quad (3.1)$$

and

$$Q \sim \rho^{2\delta}. \quad (3.2)$$

When $\delta = 1$ we obtain

$$f = -9\rho \sim \rho^2, \quad f^{-1}Q = g_{\rho\rho} = g_{zz} \sim \rho^0. \quad (3.3)$$

If $\delta$ is different from 1, $f^{-1}Q$ approaches 0 or infinite, which means that this region is a naked singularity. Therefore we need to set $\delta = 1$ to get physically interesting solutions and we assume this from now on.

We shall discuss the behavior of metric of the multi soliton case in the regions specified by (i) $z < w_1$, (ii) $w_{2n-1} < z < w_{2n}$ ($n = 1, 2, \ldots, N/2$) and (iii) $z > w_N$.
separately, because the behavior in the three regions differs according as the $z$ variable changes:

(i) $z < w_1; \quad f \sim \rho^0, \quad Q \sim 1$. \hfill (3.4)
(ii) $w_{2n-1} < z < w_{2n} (n = 1, 2, \ldots, N/2); \quad f \sim \rho^0, \quad Q \sim \rho^3$. \hfill (3.5)
(iii) $z > w_N; \quad f \sim \rho^0, \quad Q \sim 1$. \hfill (3.6)

We see from the region (i) and (iii) that $Q = 1$ along the $z$-axis ensures the asymptotic flatness. Moreover, the behavior of $f$ and $Q$ in the region (ii) is the same as the behavior (3.3) of the 2-soliton or the dilaton charged black hole case. This shows that we can interpret this region as the $n$-th dilaton charged black hole which is free from a naked singularity.

We have found that each object in the solution has the same behavior as that of dilaton R-N solution and so the solution can be interpreted as the multiple dilaton R-N solution. However we are still not sure about the charges characterizing the individual objects. We need to define their distinguishable charges. Therefore we shall discuss the definitions of charges.

(1) Mass

We define the total mass by

$$M_{tot} = -\frac{1}{4\pi} \lim_{\infty} R^0 \xi^\nu \sqrt{-g} dV, \hfill (3.7)$$

where $\xi^\nu$ is the time-like killing vector. In the static case this may be simplified as

$$M_{tot} = -\frac{1}{4\pi} \lim_{\infty} R^0 \sqrt{-g} dV. \hfill (3.8)$$

where $dV = dV_0$ is the integration over the infinite volume that is denoted by $\lim_{\infty}$. This definition of mass can be applied to the non-vacuum case as far as the integration is taken over the infinite volume. For example, when this is applied to the R-N solution with a spherical symmetry this gives the mass only when the infinite volume limit is taken. However this is not appropriate to distinguish each mass, because this gives only the total mass. We shall rewrite the definition (3.7) so that we can discuss the multiple solution. If the energy-momentum tensor should satisfy

$$\lim_{\infty} \int \left( T^\mu_\nu - \frac{1}{2} \delta^\mu_\nu T \right) \xi^\nu \sqrt{-g} dV = 0, \hfill (3.9)$$

we can rewrite the mass as

$$M_{tot} = -\frac{1}{4\pi} \lim_{\infty} \int \left[ R^0 \xi^\nu - 8\pi \left( T^\mu_\nu - \frac{1}{2} \delta^\mu_\nu T \right) \right] \xi^\nu \sqrt{-g} dV. \hfill (3.10)$$

We divide the integration region which is the total 3-dimensional volume into a set of region $V_n$ which includes the $n$-th singularity and the remaining $V$. Then, the integration reads

$$M_{tot} = -\frac{1}{4\pi} \sum_n \int_{V_n} \left[ R^0 \xi^\nu - 8\pi \left( T^\mu_\nu - \frac{1}{2} \delta^\mu_\nu T \right) \right] \xi^\nu \sqrt{-g} dV \hfill (3.11)$$

The second term vanishes because the Einstein equation holds. Note that the $n$-th integration in the first term might depend on the volume $V_n$ since the volume includes the $n$-th singularity. Only if we could show that each integration in the first term does not depend on the choice of volume $V_n$, it deserves to be called the $n$-th mass. Only in this case the total mass is given by the sum of each mass:

$$M_{tot} = \sum_n M_n, \hfill (3.12)$$

where $M_n$ is given by

$$M_n = -\frac{1}{4\pi} \int_{V_n} \left[ R^0 \xi^\nu - 8\pi \left( T^\mu_\nu - \frac{1}{2} \delta^\mu_\nu T \right) \right] \xi^\nu \sqrt{-g} dV. \hfill (3.13)$$

Now let us apply this formula to the present case after we show that the $M_n$ does not depend on the volume $V_n$. We note that

$$\lim_{\infty} \int \left( T^\mu_\nu - \frac{1}{2} \delta^\mu_\nu T \right) \xi^\nu \sqrt{-g} dV = -\frac{1}{8\pi} \lim_{\infty} \int e^{2\phi} F^0_0 d\sigma \hfill (3.14)$$

$$= -\frac{1}{8\pi} \int e^{2\phi} F^0_0 d\sigma = 0,$$
where $S_\infty$ is a surface at the spatial infinity and $d\sigma_t$ denotes the surface integration. Therefore we can apply the formula to the dilaton Maxwell-Einstein case. For the convenience of explanation we divide the mass as

$$M_{\text{tot}} = M_s + M_F + M_d,$$

with

$$M_s = -\frac{1}{4\pi} \sum_{n} \int_{V_n} \mathcal{J}^{\mu} \xi_{\nu} \sqrt{-g} \, dV_{\mu}$$  \hspace{1cm} (3.16)

$$M_F = 2 \sum_{n} \int_{V_n} \left( T^{\mu}_{\nu} - \frac{1}{2} g_{\mu \nu} T \right) \xi^\nu \sqrt{-g} \, dV_{\mu}$$  \hspace{1cm} (3.17)

$$M_d = \frac{1}{2\pi} \sum_{n} \int_{V_n} \left( T^{\mu \nu} - \frac{1}{2} g^{\mu \nu} T \right) \xi^\nu \sqrt{-g} \, dV_{\mu}$$  \hspace{1cm} (3.18)

We can show that the integrands can be expressed as

$$\sqrt{-g} \mathcal{J}_a^{\mu} = \left( \sqrt{-g} \, g^{\rho \sigma} \mathcal{F}^{\mu}_{\rho \sigma} \right) \xi_{\nu}$$  \hspace{1cm} (3.19)

$$\sqrt{-g} T_{\mu \nu} = -\frac{1}{8\pi} \left( e^{-2\phi} \mathcal{F}_{\mu \nu} A_0 \right) \xi_{\nu}$$  \hspace{1cm} (3.20)

This shows that $M_d$ vanishes. By using the Gauss theorem we can express the integrals in Eqs. (3.16) and (3.17) as the surface integrals. From now on we specify that the integration volume $V_n$ is a cylinder with $z$-axis as its symmetric axis. We thus obtain

$$M_s = -\frac{1}{8\pi} \sum_{n} \int_{S_n} \left[ \rho (\ln f)_{,\rho} d\sigma_{\rho} + \rho (\ln f)_{,z} d\sigma_z \right]$$  \hspace{1cm} (3.21)

$$M_F = -\frac{1}{8\pi} \sum_{n} \int_{S_n} \rho f^{-1} e^{-2\phi} \left( \chi^{\rho}_{,\rho} d\sigma_{\rho} + \chi^{z}_{,z} d\sigma_z \right)$$  \hspace{1cm} (3.22)

where $S_n$ is the boundary of the cylinder $V_n$: $S_n = \partial V_n$, $d\sigma_{\rho} = d\phi dz$ and $d\sigma_z = d\rho dp$. Since we can express these integrands in $M_s$ and $M_F$ by using $f$, the total mass reads

$$M_{\text{tot}} = M_s + M_F + M_d$$

$$= \frac{1}{16\pi} \sum_{n} \int_{S_n} \left[ \rho (\ln f)_{,\rho} d\sigma_{\rho} + \rho (\ln f)_{,z} d\sigma_z \right].$$  \hspace{1cm} (3.23)

In order to show how the surface integrations are carried out we should consider a cylinder $S_n$ specified by its lower base at $z = z_n$, upper base at $z = z_b$ and the side at $\rho = \rho_n$. (Fig.1). The surface integral on this surface is explicitly calculated as

$$\int_{S_n} \left[ \rho (\ln f)_{,\rho} d\sigma_{\rho} + \rho (\ln f)_{,z} d\sigma_z \right]$$

$$= \int_{z_n}^{z_b} \int_{\rho_n}^{\rho_b} \rho (\ln f)_{,\rho} d\rho \, dz + \int_{\rho_n}^{\rho_b} \rho (\ln f)_{,z} |_{z=z_b} d\rho - \int_{\rho_n}^{\rho_b} \rho (\ln f)_{,z} |_{z=z_n} d\rho$$

$$= 2\pi \left( |w_{2n-1} - z_n| - |w_{2n} - z_b| - |w_{2n-1} - z_b| + |w_{2n} - z_n| \right)$$

$$= 2\pi I(z_n, z_b)$$  \hspace{1cm} (3.24)

Here we note that $I(z_n, z_b)$ does not change its value as far as

$$z_n \leq w_{2n-1} < w_{2n} \leq z_b$$  \hspace{1cm} (3.25)

is satisfied. This means that the integrals in Eq. (3.24) does not depend on the choice of $V_n$ as far as it includes the singularity. Each cylindrical surface $S_n$ is assumed to include a singular region or the mass source in it. Since each mass does not depend on the choice of the cylinder as far as it includes the singularity, we can estimate the $n$-th mass by setting $z_0 = w_{2n-1}$ and $z_b = w_{2n}$. Finally we get

$$M_{\text{tot}} = \sum_{n} M_n,$$

where $M_n$ is given by

$$M_n = \frac{m + d}{8d} I(w_{2n-1}, w_{2n}) = \frac{m + d}{2d} d_n.$$  \hspace{1cm} (3.26)
Here we parameterize \( w_{2n-1} \) and \( w_{2n} \) as
\[
\begin{cases}
  w_{2n-1} = z_n - d_n, \\
  w_{2n} = z_n + d_n.
\end{cases}
\] (3.29)

There is an alternative way of defining mass which is intuitive but might be lacking in strictness. Just as we may obtain the total mass of the system by the asymptotic behavior of \( f(= -g_{00}) \), we can estimate each mass by first taking the asymptotic limit for the solitons except the \( n \)-th soliton and secondly taking the asymptotic limit for the \( n \)-th soliton. The result coincides with the present one.

(2) dilaton charge

As in the mass case we need a definition of each scalar charge when there are multiple scalar sources. We start with the definition of the scalar charge given by the integration at spatial infinity:
\[
\Sigma = -\frac{1}{4\pi} \lim_{\infty} \int \nabla^2 \phi \sqrt{-g} \, dV. \tag{3.30}
\]
This gives the total dilaton charge when applied to the multiple dilaton black holes. This can be written as
\[
\Sigma = -\frac{1}{4\pi} \lim_{\infty} \int \left( \nabla^2 \phi + \frac{1}{2} e^{-2\phi} F^2 \right) \sqrt{-g} \, dV, \tag{3.31}
\]
if the following relation holds:
\[
\lim_{\infty} \int e^{-2\phi} F^2 \sqrt{-g} \, dV = 0. \tag{3.32}
\]
In the present case we can show this by using Eq. (3.20). As in the mass case we divide the volume into two sets of regions: the first consists of a set of 3-volume each of which includes a singularity or dilaton charge source in it and the second is the remaining. In order to find solutions of physical interest we set the positions of the mass and dilaton charge source to be identical. Since the equation of motion for the dilaton field holds in the region where there is no source of dilaton charge the integral in the second region should vanish. Each integration in the first region yields the distinctive dilaton charge only if the condition that the value does not depend on the choice of volume is satisfied as in the definition of distinctive mass. The integral can be rewritten by the cylindrical surface integral as
\[
\Sigma = -\frac{1}{4\pi} \sum_n \oint_{S_n} \left( \nabla^2 \phi - e^{-2\phi} F^{0i} A_i \right) \sqrt{-g} \, d\sigma_i. \tag{3.33}
\]

We can express the integrand in terms of the \( \tilde{f} \) field and we obtain
\[
\Sigma = -\frac{m - d}{4d} \sum_n \oint_{S_n} \left[ \rho(\ln \tilde{f}) \rho \, d\sigma_p + \rho(\ln \tilde{f}) \rho \, d\sigma_q \right]. \tag{3.34}
\]

Since each integration does not depend on the choice of volume \( V_n \) we see that the criterion is satisfied and we may call it the \( n \)-th charge. Finally we obtain
\[
\Sigma = \sum_n \Sigma_n, \tag{3.35}
\]
where the \( n \)-th charge is given by
\[
\Sigma_n = -\frac{m - d}{8d} I(w_{2n-1}, w_{2n}) = -\frac{m - d}{2d} d_n. \tag{3.36}
\]

(3) electric charge

Noting that the Maxwell equation is given by Eq. (2.2):
\[
\nabla_{\mu} \left( e^{-2\phi} F^{\mu\nu} \right) \equiv 4\pi j^\nu = 0, \tag{3.37}
\]
we define the electric charge by
\[
Q = \lim_{\infty} \int j^0 \sqrt{-g} \, dV. \tag{3.38}
\]
Here \( j^\nu \) is a current of the matter field. In order to evaluate the integration, we divide the total volume into two regions as in the case of mass and the dilaton charge. The first region includes the singularities and the second is the remaining. We can discard the integral over the second region by using the equation of motion as in the evaluation of mass and dilaton charge. Then we are left with
\[
Q = \frac{1}{4\pi} \sum_n \int_{V_n} \left( e^{-2\phi} F^{0i} \right) \sqrt{-g} \, dV. \tag{3.39}
\]
We can rewrite this as

\[ Q = -\frac{1}{4\pi} \sum_n \int_{S_n} \rho f^{-1} e^{-2\phi}(\chi_{\rho\phi} d\rho + \chi_{zz} dz). \]  \hspace{1cm} (3.40)

Here we have assumed that the source of the electric charge lies at the same point as the mass and the dilaton charge. Note that the present definition of the individual electric charge does not depend on the volume. Therefore, we can define the individual charge as before. Finally, we obtain

\[ Q = \sum_n Q_n, \]  \hspace{1cm} (3.41)

where each electric charge is given by

\[ Q_n = \frac{e e^{-\phi_0}}{4\sqrt{2}d} |(w_{2n-1}, w_{2n})| = \frac{e e^{-\phi_0}}{\sqrt{2}d} d_n. \]  \hspace{1cm} (3.42)

We have thus obtained the mass, dilaton charge and the electric charge of the n-th dilaton black hole. By using \( a^2 = m^2 - e^2 \) one can express the dilaton charge in terms of the mass and the electric charge:

\[ \Sigma_n = -\frac{(Q_n e^{\phi_0})^2}{2M_n}, \]  \hspace{1cm} (3.43)

of which one body version is given in Eq. (2.55). One can also relate mass and charges with \( d_n \) ’s:

\[ M_n M_m + \Sigma_n \Sigma_m - Q_n Q_m e^{2\phi_0} = d_n d_m. \]  \hspace{1cm} (3.44)

Here we comment on the force between two charged objects in the presence of the dilaton field. The force should be determined by the interaction term between the current and the gauge potential: \( L_I = -j^\mu A_\mu \). We have shown that the gauge potential is given by Eq. (2.58) in the 2-soliton case. We can rewrite this by using the electric charge \( Q_1 \) as

\[ A_\rho = -\frac{Q_1 e^{2\phi_0}}{r}. \]  \hspace{1cm} (3.45)

Then we obtain the force to the matter field with charge \( Q_2 \) in the presence of field \( A_0 \):

\[ F_{\phi\phi} = \frac{Q_1 Q_2 e^{2\phi_0}}{r^2}. \]  \hspace{1cm} (3.46)

One should note that the strength of the electric force is not simply proportional to the multiplication of the charges in the presence of a dilaton field.

\[ \S 4. \text{Condition for static equilibrium} \]

In this section we discuss the condition for static equilibrium of the two dilaton black holes. In the preceding section we have obtained the metric for the multi dilaton black holes aligned along the z-axis. Since these describe the static solution, it is interesting to ask how the balance of power among multi black holes is realized. The mass, the electric charge and the dilaton charge are concerned with this power of balance. Since there is one relation among them, there remain two parameters for each black hole. Despite the existence of two free parameters how can the power of balance be possible? The solution is not a simple superposition of two 2-soliton solutions and the difference from a simple superposition appears in the \( z \)-axis between dilaton black holes.

In order to measure the deviation from the spatially Euclidean metric we evaluate the quantity

\[ P_0^2 = \lim_{\rho \to 0} \left( \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial z} \right) = \lim_{\rho \to 0} Q^{-1}. \]  \hspace{1cm} (4.1)

It is easy to see that \( P_0 = 1 \) for a spatially Euclidean axis. In the 4-soliton solution we consider the region \( w_2 < z < w_3 \) on the \( z \)-axis that is the region between two black holes. In the present solution, \( P_0^2 \) in the region is given by

\[ P_0^2 = \left( \frac{(z_2 - z_1)^2 - (d_2 - d_1)^2}{(z_2 - z_1)^2 - (d_2 + d_1)^2} \right)^2. \]  \hspace{1cm} (4.2)

The requirement that the axis should be spatially Euclidean leads to either

\[ z_2 - z_1 = \sqrt{d_1^2 + d_2^2}. \]  \hspace{1cm} (4.3)
or
\[d_1d_2 = 0.\]  \hspace{1cm} (4.4)

The condition that the black holes should be separated contradicts with Eq. (4.3) and so this should be discarded. The second possibility (4.4) together with the condition \(M_1M_2 \neq 0\) leads to
\[d = 0.\]  \hspace{1cm} (4.5)

Then this guarantees \(Q_n \neq 0\) and \(\Sigma_n \neq 0\) \((n = 1, 2)\). In this case we have the relation that
\[M_n = -\Sigma_n.\]  \hspace{1cm} (4.6)

Eq. (2.59) is a relation for the horizon of a dilaton black hole to conceal the singularity. Similar condition for the \(n\)-th dilaton black hole which is \(M_n > -\Sigma_n\) is not satisfied in this case. This means that the horizon of the \(n\)-th black hole becomes singular. In summary the requirement that there should be no conical singularity between the dilaton black holes inevitably leads to the disappearance of horizons. In this sense we might say that there is no multi dilaton black hole solution, which should be contrasted with the multi charged black hole case or M-P solution. In general conical singularity sits along the \(z\)-axis between black holes as far as we do not impose a constraint between the charges. We shall give a physical interpretation to the conical singularity. We concentrate on the 4-soliton solution, that is, two dilaton black holes. We assume \(d_i > 0\) \((i = 1, 2)\).

By taking the infinite limit of the distance between two black holes, we obtain
\[\rho_0 = \frac{(z_2 - z_1)^2 - (d_2 - d_1)^2}{(z_2 - z_1)^2 - (d_2 + d_1)^2} \to 1 + 4\frac{d_1d_2}{|z_2 - z_1|^2}, \quad \text{as} \quad |z_2 - z_1| \to \infty.\]  \hspace{1cm} (4.7)

By using the relation (3.44), this reads
\[\rho_0 \sim 1 + 4\frac{M_1M_2 + \Sigma_1\Sigma_2 - Q_1Q_2e^{2\rho_0}}{|z_2 - z_1|^2}.\]  \hspace{1cm} (4.8)

On the other hand, a long, thin and straight string with the string tension \(\mu\) yields \(\rho_0 = 1 - 4\mu[9]\). The comparison brings about
\[\mu \sim -\frac{M_1M_2 + \Sigma_1\Sigma_2 - Q_1Q_2e^{2\rho_0}}{|z_2 - z_1|^2}.\]  \hspace{1cm} (4.9)
References

   S. D. Majumdar, Phys. Rev. 72(1947)930;

FIGURE CAPTION

The cylinder that is specified by the lower base at $z = z_a$, upper base at $z = z_b$and side at $\rho = \rho_c$. The surface integral (3.25) calculated on the surface of thiscylinder does not change its value as far as $z_a \leq w_{2n-1} < w_{2n} \leq z_b$ is satisfied.
Fig. 1