Is There an Exponential Bound in Four-Dimensional Simplicial Gravity?

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(Received 16 March 1994)

We have studied a model which has been proposed as a regularization for four-dimensional quantum gravity. The partition function is constructed by performing a weighted sum over all triangulations of the four-sphere. Using numerical simulation we present evidence that the number of such triangulations containing $V$ simplices may grow faster than exponentially with $V$. This property would ensure that the model has no thermodynamic limit.

PACS numbers: 04.60.Nc

In the last few years there has been considerable interest generated in a model for quantum gravity in which the functional integral over metrics (ill defined in the continuum) is replaced by a discrete sum over random triangulations. The initial proposal [1–3] arose as a natural generalization of random surface theories in two dimensions. The results of these numerical studies were encouraging and were confirmed by other groups [4,5]. The most exciting possibility was the observation of a possible phase transition for a critical value of the bare Newton constant. The hope was that a nonperturbative quantum theory for gravity could be recovered in the vicinity of this new fixed point. These observations were rendered more quantitative by the recent work [6] in which a serious finite size scaling study was performed.

The model is defined from the partition function

$$Z = \sum_{T(S^4)} e^{-\kappa_4 N_4 + \kappa_0 N_0}. \quad (1)$$

The sum is restricted to run over all simplicial manifolds (triangulations) with the topology of $S^4$. The first term in the action $N_4$ is just the number of four simplices in the triangulation $T$ and this allows us to identify the corresponding coupling $\kappa_4$ as a bare cosmological constant. The second term depends only on the number of vertices in the triangulation $N_0$ and plays the role of the integrated Ricci scalar—the coupling $\kappa_0$ is then essentially the inverse bare Newton constant.

This correspondence is clear classically from the usual Regge expression for the curvature associated with any triangle $r_{ijk}$ with the extra constraint that the four simplices are all considered equilateral:

$$r_{ijk} = 2\pi - \arccos \left( \frac{1}{4} \right) r_{ij}^{ijk}. \quad (2)$$

Notice that if the volume is bounded, the number of four simplices shared by a given triangle $r_{ij}^{ijk}$ is necessarily also bounded. This automatically ensures that the model is well defined at finite volume—it is a dynamical question as to whether the problems associated with the unboundedness of the continuum action return on taking the large volume limit.

As we have remarked, the analogous model in two dimensions has been studied extensively; see, for example, the review [7]. It seems clear that at least for central charges less than unity, the sum over triangulated graphs correctly mimics the continuum functional integrals over the metric including the conformal anomaly. In four dimensions it is not at all clear that a simple generalization, such as the one described above, is sufficient to explore the space of metrics. However, it constitutes a simple ansatz which may be studied using numerical simulation.

We may rewrite Eq. (1) in the form

$$Z = \sum_{N_4} e^{-\kappa_4 N_4} \Omega (N_4, \kappa_0). \quad (3)$$

The partial sum $\Omega (N_4, \kappa_0)$ counts the number of triangulations (weighted by the Ricci term) with volume $N_4$. The results we discuss here are concerned with the volume dependence of this entropy function $\Omega (N_4, \kappa_0)$. It is helpful at this point to recall the behavior of the equivalent two-dimensional model.

Two-dimensional gravity regulated using dynamical triangulations has a partition analogous to Eq. (1). The number of simplices is now just $N_2$ with corresponding cosmological constant $\kappa_2$. The coupling $\kappa_0$ plays no role in two dimensions as the number of vertices $N_0$ is strictly proportional to the number of two simplices $N_0 = \frac{1}{2} N_2 + \chi$ if the Euler character $\chi$ is kept constant (for example, $S^2$). If we sum over all two-dimensional triangulations with fixed volume we arrive at a quantity $\omega (N_2)$ analogous to $\Omega (N_4,0)$ for the four-dimensional theory.

There are rigorous proofs [8,9] that this quantity is valid.
exponentially bounded:

$$\omega (N_2) \sim e^{\kappa_2 N_2}. \quad (4)$$

This property is crucial for the very existence of the partition function. It implies that for a sufficiently large bare cosmological constant $\kappa_2 > \kappa_2^c$ the partition function will be finite. The thermodynamic limit $N_2 \to \infty$ is then obtained by tuning $\kappa_2$ towards this critical value $\kappa_2^c$. The mean volume $\langle N_2 \rangle$ then behaves as $\langle N_2 \rangle \sim (\kappa_2 - \kappa_2^c)^{-1}$. If the number of triangulations were to increase faster than exponentially, it would be impossible to tune the bare cosmological constant to approach the large volume limit in a regular fashion—the partition function would be dominated by infinite volume triangulations independent of the bare lattice parameters. Constructing a continuum limit would then be impossible.

Thus, it is absolutely essential for the very existence of these higher dimensional models that there be such a bound. Unfortunately, there are no analytic proofs available for dimension greater than two. If the topology is not fixed it can be shown that the number of triangulations increases factorially with volume even in two dimensions [10]. The situation is made worse by the lack of any topological classification of three- and four-dimensional manifolds.

Faced with this we have used numerical simulation to estimate the volume of the triangulation space. While the previous studies [1,2,4,5] have claimed evidence for an exponential bound we believe the issue is of such paramount importance that a very detailed study is required. Indeed, the results we shall present favor a very different scenario.

Method.—For an entropy function that behaves exponentially with volume we have argued that it is possible to choose the coupling $\kappa_4$ to fix the mean volume $\langle N_4 \rangle$. In practice this is a difficult fine tuning problem. Even under the assumption of an exponential bound, the entropy $\Omega (N_4, \kappa_0)$ is of the form

$$\Omega (N_4, \kappa_0) \sim N_4^{\kappa_0} e^{\kappa_2^*(\kappa_0) N_4}. \quad (5)$$

We have included the leading power law correction parametrized by $a(\kappa_0)$. In two dimensions the power $\alpha$ is negative and the partition function is dominated by small or large volumes depending on the sign of $\Delta \kappa_4 = \kappa_4 - \kappa_2^*(\kappa_0)$. Empirically, a similar instability is observed in the numerical simulation of higher-dimensional models.

This problem has been tackled in a variety of ways. We have followed the approach of Agishtein and Migdal [1] and added to the action a small correction term of the form

$$\Delta S = \gamma (N_4 - V)^2. \quad (6)$$

Replacing the sums by integrals and forgetting for the moment any power law corrections it is now simple to obtain a relation between the mean volume $\langle N_4 \rangle$ and the parameters in the action:

$$\langle N_4 \rangle = \frac{1}{2\gamma} \left( (\Delta \kappa_4 + 2\gamma V) \right). \quad (7)$$

Thus tuning $\kappa_4$ to yield an average volume $V$ yields a measurement of the coupling $\kappa_4^*(\kappa_0)$. The auxiliary coupling $\gamma$ merely controls the magnitude of volume fluctuations. We have set $\gamma = 0.005$. The presence of power law (and other subleading) corrections gives $\kappa_4^*(\kappa_0)$ a volume dependence $\kappa_4^*(\kappa_0) = \kappa_4^*(V, \kappa_0)$. The relation Eq. (7) may be rewritten

$$\kappa_4^*(N_4, \kappa_0) = \kappa_4 + 2\gamma (\langle N_4 \rangle - V). \quad (8)$$

In practice we iterate the above relation during the thermalization stage of our simulation and apply it once more at the end of our run to compute our final estimate for $\kappa_4^*$.

In this picture the presence of an exponential bound would be signaled by this critical cosmological constant $\kappa_4^*(V, \kappa_0)$ having a finite limit for large volumes $V$. In contrast $\kappa_4^*(V, \kappa_0)$ would increase logarithmically in a model for which $\Omega (N_4, \kappa_0)$ grew factorially with volume [this just follows from the asymptotic result $(x!^y \sim e^{x \ln x})$].

Notice that it is sufficient to prove an exponential bound for a single value of $\kappa_0$—the following inequality guarantees that there will then be a bound for any other $\kappa_0 > 0$.

$$\Omega (N_4, 0) \leq \Omega (N_4, \kappa_0) \leq \exp (a\kappa_0 N_4) \Omega (N_4, 0). \quad (9)$$

We have used a Monte Carlo (MC) algorithm to sample the triangulation space of the model—the details are given in [11]. Our code is written in such a way as to make the dependence on dimension $d$ trivial—it enters only as an input parameter to the program.

We have simulated systems from size $V = 500$ to $V = 32000$. Typical runs utilized on the order of $4 \times 10^5$ MC sweeps with one sweep corresponding to $V$ trial updates. In addition we performed a series of runs for both the two-dimensional and three-dimensional models. The results of these simulations could then be contrasted with the equivalent four-dimensional data and served as an important test of our code.

Results.—Figure 1 is a plot of the critical cosmological coupling $\kappa_4^*(V, \kappa_0)$ against the logarithm of the volume for the two-, three-, and four-dimensional models at $\kappa_0 = 0$. (To improve clarity we plot $\kappa_5^* - 0.5$, and $\kappa_5^* - 1.0$.) Clearly, the presence of an exponential bound emerges very clearly in the two-dimensional case—$\kappa_5^*(V, 0)$ is statistically consistent with a constant $\kappa_5^*(\infty) = 1.1249(6)$ for volumes $V \geq 2000$.

For three dimensions the situation is rather different.
The finite volume dependence of $\kappa^d_4(V,0)$ is large over the full range of volumes analyzed. However, as the plot reveals there is no strong evidence of a logarithmic component—indeed the best fit we could make to the data corresponds to a convergent power law (the solid line in the figure) $\kappa^d_4 = a + bV^c$. The fit yields $a = 2.01(1)$, $b = -3.2(1)$, and $c = -0.28(1)$ with a $\chi^2$ per degree of freedom 2.0. Thus, our data in three dimensions favor a bound. Indeed these numbers are consistent with the ones quoted in a previous study by Ambjørn and Varsted [12] who give $a = 2.06, b = -3.9$, and $c = -0.32$. Their fit derives from lattice sizes of $V = 14000$ and smaller with lower statistics but it is reassuring to see that we are in pretty good agreement. We are currently extending our dimension three runs to larger lattices to strengthen our confidence in the three-dimensional bound.

The situation in four dimensions appears radically different. Clearly, the data support the hypothesis that there is a logarithmic component to the critical volume coupling $\kappa^d_4(V,0)$. A fit of all the $d = 4$ data to a simple logarithm $\kappa^d_4 = a + b \ln V$ results in a value for $b = 0.0315(3)$ with a $\chi^2$ per degree of freedom 2.7 (solid line shown). While it is possible to fit the large volume data with converging power fits we do not regard this as very reliable. (Such fits will typically involve the determination of three parameters from four data points.) The attraction of the logarithmic fit is that it encompasses a wide range of lattice volumes with only two parameters.

To test this hypothesis further we looked at the situation for nonzero $\kappa_0$. Figure 2 shows a plot of $\kappa^d_4(V,\kappa_0)$ for $\kappa_0 = 0.0$, $\kappa_0 = 0.5$, and $\kappa_0 = 1.0$. The inequality Eq. (9) implies that the coefficient of this logarithm should be universal (independent of $\kappa_0$). The leading effect of a nonzero value for $\kappa_0$ is simply a renormalization of any exponential terms in $\Omega(N_4,0)$. This is confirmed by the data in Fig. 2. Although the curves start out with different gradients their large volume behavior appears to be the same.

However, the plot also makes it clear that the onset of this asymptotic regime is dependent on $\kappa_0$—as $\kappa_0$ increases the curves start off increasingly flat and the logarithm only manifests itself for large volumes.

We found that very long runs were required to thermalize the four-dimensional lattices. The initial configurations were created by employing only the node insertion move which effectively generates lattices corresponding to large values of $\kappa_0$. For the largest volumes we employed, $V = 32000$, we found that subsequent relaxation times were of the order of $10^5$ sweeps. This difficulty of reaching true equilibrium was the main factor in determining the largest volumes we could reach. It is perhaps a practical demonstration of the results reported in [13] in which the algorithmic unrecognizability of four-manifolds is shown to lead to a lack of a reasonable bound on the number of local moves needed to pass from one configuration to another.

Thus our four-dimensional data would favor an entropy $\Omega(V,0)$ function having a leading behavior

$$\Omega(V,0) \sim (V!)^\delta.$$  

(10)

If we fit the $\kappa_0 = 0.0, 0.5$, and 1.0 data for the three largest volumes by straight lines we find consistent estimates for the exponent $\delta$. These are $\delta = 0.027(1)$, 0.026(1), and 0.025(2), respectively. We would then assign our best estimate for $\delta$ as $d_0 = 0.026(5)$.

**Outlook.**—In summary, we have presented results which are consistent with a leading factorial behavior for the entropy of triangulations of the four-sphere $\Omega(V,0)$. Specifically, the number of triangulations of $S^4$ may grow like

$$\Omega(V,0) \sim \exp(aV)(V!)^\delta.$$  

(11)

Furthermore, we estimate the exponent $\delta = 0.026(5)$. Such a rapid growth would render it impossible to take the thermodynamic (large volume) limit—the partition function for any $\kappa_0$ is dominated by large volumes. This in turn would imply that there is no continuum limit for the model.

We have argued that the presence of large finite-volume effects can obscure this behavior for large values of the
inverse bare Newton constant ($\kappa_0$) on lattices that are computationally accessible. It is tempting to speculate that the rather rapid shift of the pseudocritical node coupling reported in [6] is further evidence for the lack of a well-defined continuum limit. The data presented in [6] are not inconsistent with a scenario in which this pseudocritical coupling diverges as the mean volume approaches infinity, leaving the system in an extremely crumpled, degenerate phase.

It is important to notice also that the term added to help fine tune the cosmological constant $\kappa_4$ is now playing a crucial role in defining the partition function. There is now no reason to believe that different methods of doing this are equivalent.

Clearly an extension of this work (with perhaps a more refined method for computing $\kappa_4^2(V,\kappa_0)$) to larger volumes and node couplings would help to confirm these conclusions.

These calculations were supported in part by NSF Grant No. PHY92-00148. S.M.C. acknowledges useful discussions with M. Bauer, T. Morris, and A. Shapere. Some calculations were performed on the Florida State University Cray Y-MP.

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