Bubble wall velocity in a first order electroweak phase transition

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ABSTRACT

We calculate the velocity and thickness of a bubble wall at the electroweak phase transition in the Minimal Standard Model. We model the wall with semiclassical equations of motion and show that friction arises from the deviation of massive particle populations from thermal equilibrium. We treat these with Boltzmann equations in a fluid approximation in the background of the wall. Our analysis improves on the previous work by using the two loop effective potential, accounting for particle transport, and determining the wall thickness dynamically. We find that the wall is significantly thicker than at phase equilibrium, and that the velocity is fairly high, $v_w \approx 0.7c$, and quite weakly dependent on the Higgs mass.

1 INTRODUCTION

There has been a growing interest in the idea that the baryon asymmetry of the Universe may be created at a first order electroweak phase transition. However, ingredients needed to construct the whole picture are still missing. The exact nature of the Higgs mechanism is unknown, and the simplest model, the Minimal Standard Model, apparently does not contain sufficient CP violation for baryogenesis; we must consider extensions, such as the two Higgs model. Even in the Minimal Standard Model we have difficulties computing the finite temperature effective potential (which is needed to determine the strength of the phase transition) and the dynamics of the transition.

Recently there have been advances in calculating the effective potential; the two loop contribution has been evaluated [1, 2], and there has been progress in understanding nonperturbative effects [3, 4, 5]. Both results support the view that the phase transition is first order and strong enough to proceed by bubble nucleation and growth (even when the Higgs mass is moderately large, $m_H \sim m_W$).

To model baryogenesis accurately one also needs to know the profile and velocity of an expanding bubble wall. The wall velocity is friction limited; but determining
the strength of frictive effects involves determining the non-equilibrium populations of massive particles in the vicinity of the wall, which is difficult [6, 7, 8]. The recent discovery that the top quark is very heavy [9] suggests that top quarks may be the dominant source of friction, in which case an approximation which models top quarks with good accuracy should improve our understanding of the wall motion. In this letter we re-analyze the bubble wall’s velocity and its shape, working in the Minimal Standard Model and using the fluid approximation to model the particle populations, which should treat fermions fairly well. The technique also allows us to account for transport and to determine the wall thickness dynamically (taking a specific Ansatz for the wall profile).

2 Equation of Motion for the Higgs vev

We intend to study the dynamics of infrared condensates in the Higgs field $\Phi$. Such condensates should behave semiclassically to a good approximation. From the terms in the Electroweak Lagrangian containing $\Phi$,

$$L = (\mathcal{D}_\mu \Phi)^\dagger \mathcal{D}^\mu \Phi - V(\Phi^\dagger \Phi) - \sum y(\Phi^\dagger \bar{\psi}_R \psi_L + \Phi \bar{\psi}_L \psi_R)$$

(where the sum runs over the quarks and leptons $\psi$, and $y$ denotes the Yukawa coupling), we find the equations of motion for $\phi$ (where $\Phi^\dagger = [0, \phi/\sqrt{2}]$) to be

$$\dot{\phi} + V'(\phi) - \frac{g_w^2}{4} \phi T A^2 + i g_w A^\mu \partial_\mu \phi + \frac{ig}{2} (\partial^\mu A_\mu) \phi + \sum \frac{y}{\sqrt{2}} \bar{\psi} \psi = 0$$

(1)

Here $A$ and $\psi$ are quantum operators. (For simplicity we have set the Weinberg angle $\tan \theta_W = 0$.) It is reasonable to take thermal averages of these operators using WKB wave functions. This is because the wall will be much thicker than the thermal length $T^{-1}$, which characterizes the reciprocal momenta of particles in the plasma. In this approximation we get

$$\dot{\phi} + V'(\phi) + \sum \frac{dm^2}{d\phi} \int \frac{d^3 k}{(2\pi)^3} 2E f(k, x) = 0$$

(2)

where $V$ is the renormalized vacuum potential, $f$ is the phase space population density (in the background of a propagating wall) and the sum includes all massive physical degrees of freedom. Note the condensed notation: $m = y\phi/\sqrt{2}$ for quarks and leptons and $m = g_w \phi/2$ for the gauge fields.

To model the population density $f$ we assume a small departure from the equilibrium population $f_0$ and write $f = f_0 + \delta f$. The vacuum contribution $V'(\phi)$ and the contribution from $f_0$ combine to give the finite temperature effective potential $V_T(\phi)$, Thus we have [10]

$$\dot{\phi} + V_T'(\phi) + \sum \frac{dm^2}{d\phi} \int \frac{d^3 p}{(2\pi)^3} 2E \delta f(p, x) = 0$$

(3)

We see that the frictive force arises due to the departure from thermal equilibrium $\delta f$. We will use this equation, an expression for $V_T$, and equations for $\delta f$ to compute the wall velocity and its shape once it has reached a planar steady state, $\phi = \phi(z + v_w t), \delta f = \delta f(z + v_w t)$. 

2
3 Effective Potential

The high temperature expansion of the one loop effective potential is [7]

$$V_T(\phi) = D(T^2 - T_0^2)\phi^2 - E\phi^3T + \frac{\lambda_T}{4}\phi^4$$  \hfill (4)

with $D = (2m_t^2 + 2m_W^2 + m_Z^2)/(8\pi^2) \approx .167$, $E = (2m_W^2 + m_Z^2)/(4\pi v_0^2) \approx 0.01$, $\lambda_T$ the Higgs self coupling at a scale roughly given by $T$, and $T_0 = [m_t^2 + 3(4m_t^4 - 2m_W^4 - m_Z^4)/8\pi^2v_0^2]/4D$. Recently the authors of [1] and [2] have computed the two loop expression. The most important changes are that $E$ becomes $(4m_W^2 + 2m_Z^2 + 3(1 + \sqrt{3})\lambda_T^2)/(12\pi v_0^2)$ and a qualitatively new term $-B\phi^2T^2\log(\phi/T)$ appears. The result of [2] is $B \approx (1.4g_w^4 - .48g_w^3\sqrt{3} + 5.1g_w^2\lambda - 6.7\lambda^2)/(16\pi^2)$. It may also be important to include nonperturbative effects. Shaposhnikov proposes a term $-(A_f^2T^4/12)\Pi(f)$ [3]. The function $Pit$ describes the contribution of a gauge condensate and it is roughly constant near $\phi = 0$ and falls exponentially as $\exp(-\phi/g^2T)$ for large $\phi$. We add such a term to parametrize our ignorance about the free energy of the symmetric phase; we use $-(A_f^2T^4/12)\sech(32\phi/T)$ because it is simple—the exact form of $Pit$ will have little effect on our calculations—and we take $\lambda_T$ and $A_f$ as unknowns.

What should we use for $T$? Most of space is converted to the broken phase by bubbles which nucleate when the critical bubble free energy reaches $S \approx 100T$ [11]. We have computed $S$ using our form for $V_T$ and the techniques used in [6, 7, 8] and find, for instance, that for $\lambda_T = 0.03$, $A_f = 0$, and $S = 100T$ that $T = 1.00006T_0$. (Note that because of the $B$ term, $T_0$ is no longer the spinodal temperature.)

4 Fluid Equations

Next we must determine $\delta f$, the deviation from equilibrium in the presence of the moving wall. Our starting point is the Boltzmann equation,

$$\partial_t f + \frac{p_z}{E}\partial_z f + \mathbf{p}_z\partial_{p_z} f = -C[f]$$  \hfill (5)

where $C[f]$ represents the scattering integral, $E = (p^2 + m^2)^{1/2}$ is the particle energy, $v_z = p_z/E$ is the velocity (with the broken phase at positive $z$), and $\mathbf{p}_z = -\partial_z E$ is the force on the particle. In a complete description each particle species in the plasma would be described with a Boltzmann equation. We allow ourselves the approximation that all species but top quarks and perhaps $W$ bosons are in equilibrium and we neglect the slight change in the background temperature across the wall. The former approximation is reasonable as the induced deviation from equilibrium goes as $m^2$ and top quarks and $W$ bosons are the heaviest particles. We can correct for the latter approximation by using the work of Enqvist et al. [12] and Heckler [13], who have used hydrodynamic conservation laws to relate wall velocity to entropy production; for simplicity we will not do so here. We also use a single $f$ for tops and antitops of both helicity. In the Minimal Standard Model this is reasonable as there is almost no CP violation, and the difference in transport properties arises only at the subleading level of weak scatterings. In two doublet models we might need to be more careful.

The Boltzmann equations are nonlinear partial integro-differential relations and as such are analytically intractable. To solve them we make the fluid ansatz, assuming $f$ to be of the form
\[
f = \frac{1}{1 + \exp \frac{E - E_\text{eq}/T}{\mu - v}}
\]

where we have written explicitly three types of perturbations: chemical potential \( \mu \), temperature \( \delta T \) and velocity \( v \). This Ansatz is a truncation of an expansion in powers of momentum; it gives a reasonable description of the populations of thermal energy particles when the background varies slowly on the scale of the diffusion length. For top quarks this should be sufficient as the diffusion length is short and the influence of infrared particles is phase space suppressed. For \( W \) bosons, Bose statistics give large infrared particle populations, and the fluid approximation is unreliable unless \( W \) bosons thermalize on time scales short compared to their annihilation rate. We consider the fluid equations for \( W \) bosons as a guide for their importance and concentrate on top quarks. We will furthermore work to linear order in perturbations which are of order \((m/\pi T)^2\), and therefore naturally small.

With a three parameter Ansatz (6) we cannot ask that the full Boltzmann equations be satisfied, but only impose that three moments be satisfied, namely the integrals over \( f \, d^3 p \), \( f \, d^3 p \, E \), and \( f \, d^3 p \, p_+ \). Working in the fluid frame and using \( \partial_t f(z + v_w t) = v_w f' \) we obtain the following equations [14]:

\[
aw \mu' + v_w \frac{\delta T'}{T} + \frac{1}{3} v' + F_1 = -\Gamma \mu \frac{\mu}{T} - \Gamma T_1 \frac{\delta T}{T}
\]

\[
bw \mu' + v_w \frac{\delta T'}{T} + \frac{1}{3} v' + F_2 = -\Gamma \mu \frac{\mu}{T} - \Gamma T_2 \frac{\delta T}{T}
\]

\[
bw \mu' + v_w \frac{\delta T'}{T} + v' + 0 = -\Gamma v v
\]

where \( a = 2\zeta_2/9\zeta_3 \), \( b = 3n_0/4\rho_0 = 3\zeta_3/14\zeta_4 \), \( \zeta \) is the Riemann \( \zeta \)-function: \( \zeta_2 = \pi^2/6 \), \( \zeta_3 \approx 1.202 \), \( \zeta_4 = \pi^4/90 \), \( n_0 = 3\zeta_3 T^3/4\pi^2 \), \( \rho_0 = 21\zeta_4 T^4/8\pi^2 \).

The force terms are:

\[
F_1 = -\frac{v_w \ln 2 (m^2)'}{9\zeta_3} \quad T^2, \quad F_2 = -\frac{v_w \zeta_2 (m^2)'}{42\zeta_4} \quad T^2
\]

They drive the plasma out of equilibrium, while scatterings restore it. Scatterings keep the plasma near equilibrium when the wall pass age time \( L/v_w \gg T^{-1} \).

We have computed the coefficients on the r.h.s. of (7) - (9) including all diagrams which contribute to order \( \alpha_s^2 \log \alpha_s \). We found [15] \( \Gamma_\mu_1 \approx T/26 \), \( \Gamma_{T_1} \approx T/14 \), \( \Gamma_{v_1} \approx T/60 \), \( \Gamma_{T_2} \approx T/16 \), \( \Gamma_{v_2} \approx T/13 \), where we used \( \alpha_s \equiv \alpha_s(m_Z) \approx 0.12 \). For \( W \) bosons the values are about half as large.

The force terms are proportional to \( m^2 \), so in general \( \delta f \propto m^2 \). Note that the friction term in Eq. (3) is proportional to \( m^2 \delta f \propto m^4 \). For bosons the coefficients in Eqs. (7) - (10) differ, and in particular \( F_1 \propto m^2 \log(2/m) \). The integral in the equation of motion involving \( \mu \) also contains a log enhancement; but because \( m_t^4/m_w^4 \) is very large, \( W \) bosons still produce less friction than top quarks.
5 Computing Velocity and Profile

With our Ansatz for $f$ we can rewrite Eq. (3) for top quarks (with $\sum \to 12$) as

$$-(1 - v_w^2)\phi'' + V'_T(\phi) + 12 \frac{dm^2}{d\phi} T \left[ \frac{\mu \ln 2 + \zeta_2 \delta T}{4\pi^2} \right] = 0 \quad (11)$$

This equation and the fluid equations form a system of nonlinear differential equations for the wall profile and velocity. We will attempt to solve them in [15], but here we will content ourselves with an Ansatz for $\phi$,

$$\phi = \frac{\phi_0}{2} \left( 1 + \tanh \frac{z + v_w t}{L} \right) \quad (12)$$

where $\phi_0$ is the value of $\phi$ in the asymmetric phase and $v_w$ the wall velocity and $L$ the wall thickness in the plasma frame are treated as undetermined parameters. This Ansatz is chosen because the static equilibrium wall shape in the one loop approximation is of this form.

Again, because we have restricted the form of $\phi$ we cannot ask that the full equations of motion be satisfied; we can only enforce two moments. The natural choices are the space integral of the equation of motion times $\partial \phi / \partial v_w$ and $\partial \phi / \partial L$. Note that $\partial \phi / \partial v_w = t \phi'$ and $\partial \phi / \partial L = -(x / L) \phi'$, so an equivalent set of conditions is

$$\int [\text{Eq. (11)}] \phi' dz = 0, \quad \int [\text{Eq. (11)}] \frac{x}{L} \phi' dz = 0 \quad (13)$$

These equations have a simple physical interpretation. The first equation is the total pressure on the wall in its rest frame [6]; if it were non-zero the wall would accelerate, changing $v_w$. The second equation is the asymmetry in the pressure between the front and back edges of the wall; if it were nonzero the wall would be compressed or stretched, changing $L$.

The integrals for the first two terms in (11) are

$$\int (\phi + V'_T(\phi)) \phi' = V_T(\phi_0) - V_T(0) \equiv -\Delta V_T \quad (14)$$

$$\int [\phi + V'_T(\phi)] \frac{z}{L} \phi' = \frac{(1 - v_w^2)\phi_0^2}{6L^2} - \frac{1}{2} [\Delta V_T + \Xi] \quad (15)$$

$$\Xi \equiv B\phi_0^2(\zeta_2 - 1)T^2 + \frac{E\phi_0^3 T}{2} - \frac{5\lambda_T \phi_0^4}{24} + \frac{A_f g_w^6 T^4}{12} \left( 2.82 + \frac{1}{2} \ln \frac{\phi_0}{T} \right) \quad (16)$$

Note that the $\phi$ term acts to stretch the wall (increase $L$) while $V_T$ acts to accelerate and compress the wall. The coefficient 2.82 in the last term is the only place where our choice for the function $P$ enters our computation.

We will first get a rough estimate of the wall velocity and thickness by solving Eqs. (7) - (9) and (13) ignoring transport, by which we mean we will ignore the derivative terms on the l.h.s. of the fluid equations. Transport reduces friction because particles tend to flow off the wall, where they contribute less to equations (13). We will also ignore $\delta T$, which turns out to be a good approximation. The expression for $\mu$ now becomes rather simple:

$$\mu = v_w \frac{\ln 2}{9\zeta_3} \frac{g_i}{\Gamma_{\mu_1} T} \phi' \quad (17)$$
Note that $\mu$ does not depend on other decay rates apart from $\Gamma_{\mu}$.

Using $\int (\phi \phi')^2 = \phi_0^4/10L$, $\int x (\phi \phi')^2 = \phi_0^4/24$ and (14) − (16) one can solve for $L_w = L/(1 - v_w^2)^{1/2}$ and $v_w$:

$$\gamma_w v_w = \frac{15\pi^2 \zeta_3}{2\ln^2 2} \Gamma_{\mu l} L_w \frac{\Delta V_T}{m_t^4}$$

(18)

For $m_t = 174$ GeV, $\lambda_T = 0.03$ and $A_f = 0$ these give $L_w \approx 29 T^{-1}$ and $\gamma_w v_w \approx 4.1$, a mildly relativistic and fairly thick wall. Note that $L \approx 20 T^{-1}$ is much thicker than the top quark diffusion constant $D \approx 4 T^{-1}$, so the fluid approximation is in good shape.

Now we solve the problem including transport. First we must find the contributions to (13) involving $\mu$ and $\delta T$. This is most easily accomplished by Fourier analysis. Let us write Eqs. (7) − (9) in a matrix notation,\]

$$A \delta' + \Gamma \delta = F \phi \phi'$$

(19)

where $\delta$ is a column vector of $\mu$, $\delta T$, $v$, $A$ is the matrix of coefficients for the derivative terms, $\Gamma$ is a matrix of the decay constants, and $F$ is a column vector of the coefficients for the force terms. Note that $A$ is velocity dependent, and $F$ is linear in velocity. In Fourier space (19) becomes

$$i k \delta + A^{-1} \Gamma \delta = A^{-1} F \tilde{\phi} \phi'$$

(20)

which may be solved by eigenvalue methods. Denoting the eigenvalues and eigenvectors of $A^{-1} \Gamma$ as $\lambda_i$ and $\xi_i$ and expanding $A^{-1} F = \alpha_i \xi_i$, we find

$$\delta = \sum_i \frac{\alpha_i}{\lambda_i + i k} \xi_i \tilde{\phi} \phi'$$

(21)

$$\tilde{\phi} \phi' = \left( \phi_0^2/2 \right) \left( 1 - \frac{i k L}{2} \right) \left( \frac{k L \pi}{2} \right) \text{csch} \left( \frac{k L \pi}{2} \right)$$

(22)

This gives an explicit expression for $\tilde{\mu}$ and $\delta \delta$.

Finally, we can convert the relevant integrals of the friction term

$$\frac{3y_i^2 T}{\pi^2} \int [\mu(x) \ln 2 + \zeta_2 \delta T(x)] \phi \phi'(x) dx$$

$$\frac{3y_i^2 T}{\pi^2} \int [\mu(x) \ln 2 + \zeta_2 \delta T(x)] x \phi \phi'(x) dx$$

(23)

into k-space integrals using the relations $\int f_1(x) f_2(x) dx = \int \hat{f}_1(-k) \hat{f}_2(k) dk / 2\pi$ and $xf(x) \Rightarrow id \hat{f}(k) / dk$. This yields integrals of form

$$\int \frac{1}{\lambda + i k} \left( 1 + \frac{k^2 L^2}{4} \right) \left( \frac{k L \pi}{2} \text{csch} \frac{k L \pi}{2} \right)^2 \frac{dk}{2\pi}$$

(24)

which may be converted to a rapidly converging infinite sum by residue integration, or performed numerically.

This completes the evaluation of all terms in Eqs. (13). These equations each define a curve in the space of $v_w$ and $L$. The intersection of these curves is a self consistent solution for the wall shape and velocity within the $\text{Ansatz}$ and approximations we have made.
6 Results

We have solved these simultaneous conditions for some representative values of $\lambda_T$ and $A_f$. We find that the friction from $W$ bosons, calculated in the fluid approximation, is about half that from top quarks. Though our techniques are different than those of [8], we get a similar numerical value for the friction from $W$'s. We have included them in our analysis. Using the temperature where the critical bubble action $S = 100T$, we find

\[
\begin{array}{cccccc}
\lambda_T & 0.02 & 0.03 & 0.05 & 0.05 & 0.03 \\
A_f & 0 & 0 & 0 & 0.1 & 0.1 \\
\phi_0/T & 1.06 & 0.78 & 0.57 & 0.78 & 1.03 \\
v_w & 0.84 & 0.68 & 0.66 & 0.96 & \text{no solution} \\
T*L & 43 & 29 & 23 & 9.4 & \text{no solution} \\
v_{no\ str} & 0.33 & 0.39 & 0.48 & 0.68 & 0.54 \\
T*L_{no\ str} & 15.6 & 16.5 & 16.5 & 7.3 & 9.6 \\
\end{array}
\]

The last two columns are the velocity and thickness of the wall when we treat $v_w$ as a free parameter but fix $L$ to the value derived from (15) without the contribution from friction. This value of $L$ approximately equals the thickness at phase equilibrium. We see that the wall is significantly deformed by the friction effects, and that this increases its velocity. Of course, if the deformation is large then we have little reason to believe that our wall shape Ansatz is accurate – one should model the shape more carefully than we have done here. The conclusion that the wall is fast and thick should be reliable, however.

When we include a sizeable value of $A_f$, the parameter describing nonperturbative symmetric phase effects, we find no solution. The two equations (13) turn out to be incompatible; the wall runs away, but maintains finite plasma frame thickness. This result probably comes from neglecting friction from the gauge condensate responsible for $A_f$, which would compress the wall and prevent runaway. To remedy this shortcoming we need a model for the nonequilibrium dynamics of nonperturbative infrared condensates.

The situation in two doublet models may be quite different from what we have found here. In these theories there are several new massive (Higgs) bosons. The ones which do not couple to the top quark have quite long half-lives and sizable diffusion constants, and may be a major source of friction.

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References


[10] Ref. [6] contains an integral form of this expression; it is derived in a careful field theoretic way in [8].


