CERN Lectures

SYMMETRY PROPERTIES
OF PARTICLES AND FIELDS

by

F. Mandl
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PREFACE

This report is based on lectures which I gave at CERN this summer at the invitation of Professor Bernardini. I should like to thank him most warmly for this opportunity to spend some time in the very stimulating atmosphere of CERN. I also want to thank him, Professor Merrison, and everyone else at CERN, for making my stay there so enjoyable.

This report was prepared from my lectures and lecture notes by Drs. P. Astbury and T. Yamagata and I am extremely grateful to them for undertaking this arduous task. No attempt has been made to change the character of the lecture notes into a smooth account such as would appear in print. The only criterion was that the resulting report should be unambiguous. Similarly, a complete bibliography has not been prepared. Apart from the list of review articles, etc., given in the introduction, reference to particular papers has only been made when these would illustrate particular points or in some other way be immediately helpful to the audience.

F. Mandl

29 July 1960.
INTRODUCTION

I would like to describe briefly what I am trying to do in these talks.

I want to discuss the symmetry properties of quantum mechanical systems. These are very important. If you consider a process

\[ A + B \rightarrow C + D \]

state a state b
given by an amplitude

\[ \langle \Phi_b, V \Phi_a \rangle \]

in perturbation theory \( V = \) perturbation causing the transformation
generally

\[ \langle \Phi_b, S \Phi_a \rangle = \langle \Phi_b, V \Phi_a \rangle + \ldots \]

The right-hand side is difficult to evaluate except when perturbation theory is good, i.e. when the interaction is weak. Hence, our first task is to get the maximum information without a dynamical theory to determine the amplitudes, e.g. to find selection rules, etc. This is where symmetries come in.

Only after using these do we attempt to calculate the amplitude, e.g. maximum symmetry arguments and causality have been used in dispersion relations.

Two examples of symmetry relations in dispersion relations are:

a) crossing symmetry in \( \pi-N \) scattering produces a restriction on the amplitude;

b) threshold for inelastic processes in \( \pi-\pi \) scattering

\[ \pi + \pi \not= \text{odd number of } \pi \text{s.} \]

This is forbidden by \( G \) invariance (the combination of charge conjugation and rotation in isospace).

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There are very many applications; you have met  or f them. It is not my object to cover these but rather to indicate the systematic formulation of the ideas. I will demonstrate some applications but mainly as illustrations. Nor do I intend to give everything in nauseating detail; I shall not dot all the i's and cross all the t's. I want to develop the physical ideas.

The details you will find in the many papers dealing with these topics from which these lectures originate. References in the following reviews, etc., deal with similar ideas from different viewpoints:

Corinaldesi, Nuclear Physics 7, 305-372 (1958);
Kemmer, Polkinghorne and Pursey, Rep.Prog.Physics (1959);
Matthews, P.T., Rochester Lectures (1957) [NYO 2097, unpublished report];
Roman, Theory of Elementary Particles, North-Holland (1960);

Lastly, by way of introduction, let me say that I assume you are familiar with the application of perturbation theory to quantum field theory: in particular with the Feynman graphs method of representing the rather involved mathematics of perturbation theory. I will briefly discuss this now to remind you of it, to establish notation, etc.
LECTURE I

PERTURBATION THEORY

Our object is to get an amplitude for a transition. There are several approaches. I shall use one based on perturbation expansion, interpreted in terms of Feynman graphs. There exists a more elegant and abstract alternative, which is due to Lehmann, Symanzik and Zimmermann.

In perturbation theory the state of the system is described in the interaction picture by a state vector $\hat{\psi}(t)$:

- initially, long before scattering, $\hat{\psi}(-\infty) = \hat{\psi}_a$;
- scattering occurs at $t = 0$;
- finally, long after scattering, $\hat{\psi}(\infty) = S\hat{\psi}(-\infty) = S\hat{\psi}_a$;
- hence the amplitude for finding the state $\hat{\psi}_b$ after scattering is given by

$$\left(\hat{\psi}_b, S\hat{\psi}_a\right). \quad (I.1)$$

$S$ is called the scattering matrix: it is found from

$$i\hbar \frac{\partial S(t)}{\partial t} = H_I(t)\hat{\psi}(t) \quad (I.2)$$

where $H_I(t)$ is an interaction Hamiltonian. This means we are considering a composite system which we break into two parts:

$$H = H_A + H_B + H_I \quad (I.3)$$

where $H_A + H_B(= H_0)$.

$H_0$ is the unperturbed Hamiltonian. (In general $\hat{\psi}_a$ and $\hat{\psi}_b$ are taken as eigenstates of $H_0$.)

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Formal integration of Eq. (I.2), by iteration, gives

\[ S = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \int \cdots \int d^{4}x_{1} \cdots d^{4}x_{n} P \{ \mathcal{H}_{I}(x_{1}) \cdots \mathcal{H}_{I}(x_{n}) \} . \]  \hspace{2cm} (I.4)

Here

**Notation**: \( x \) stands for a 4-vector: \( x_{\mu} = (x_{0}, x_{1}, x_{2}, x_{3}) \)

\( n = c = 1 \)

\( d^{4}x = d^{3}x \, dx_{0} \). (This is real.)

Later I will use: \( kx = k_{\mu} x^{\mu} = k^{\mu} x_{\mu} = k_{0} x_{0} \) - scalar product.

Also

Latin indices: 1 to 3 - spatial

Greek indices: 1 to 4.

In Eq. (I.4)

\[ \mathcal{H}_{I}(x) : \text{interaction density:} \]

\[ H_{I}(t) = \int \mathcal{H}_{I}(x) d^{3}x \]

all space at \( x_{0} = t \).

P is the Dyson's chronological product: factors for a given set of values \( x_{10}, x_{20}, \ldots \) of the time arguments to be written so that they appear in chronological order with time running (increasing) from Right to Left: \( \leftarrow \).

One should remember that \( \mathcal{H}_{I} \) is built up out of field operators, which are non-computing quantities, so the order is significant.

Take, as a simple example, neutral ps ps meson theory. Here

\[ \mathcal{H}_{I}(x) = ig \bar{\psi}(x)\gamma_{5}\psi(x)\psi(x) \]  \hspace{2cm} (I.5)

where

\[ \bar{\psi}(x) = \psi^{\dagger}(x)\gamma_{4} \]

and \( \dagger \) indicates hermitian conjugation.
In Eq. (I.5), (i) makes $\mathcal{H}_I$ hermitian (energy density). $\phi$ and $\psi$ and $\psi^\dagger$ are field operators: they contain creation and absorption operators.

**Mesons:** $\phi = \phi^\dagger$ for neutral mesons

$\psi$ is linear in

- $a(k)$ - absorbs meson of momentum $k$
- $a^\dagger(k)$ - creates " " $k$

**Nucleons:**

$\psi$ is linear in:

- $c_r(p)$ - absorption operator for nucleon of momentum $p$ and spin parallel to $\uparrow_p$ ($r = \{\uparrow_1\}$);
- $d_r^\dagger(p)$ - creation operator for antinucleon with same quantum numbers (i.e. momentum and spin).

Similarly:

$\psi^\dagger$ is linear in

- $c_r^\dagger(p)$ - creation operator for nucleons
- $d_r(p)$ - absorption operator for antinucleons.

This completely symmetrical treatment of nucleons and antinucleons results if one uses a Majorana representation for the Dirac $\gamma$-matrices. This is of great advantage, therefore, when studying symmetry properties. I will come back to it later.

After these remarks on the Majorana representation, we go back to the evaluation of $\langle \bar{\Phi}_b, \mathcal{S}_{\Phi_a} \rangle$, from Eq. (I.4):

$$S = \sum_n S_n = \sum \frac{(-i)^n}{n！} \int \cdots \int d^4x_1 \cdots \mathcal{P}\{\mathcal{H}_I(x_1)\} \cdots \ (I.6)$$

We see now that this is very complex. Order has been brought into it by the Dyson, Wick and Feynman point of view. The gist is this: to get a transition $\Phi_a \rightarrow \Phi_b$ we must pick out a term which contains the right absorption and creation operators:

1) to absorb all particles present initially: $\Phi_a \rightarrow \Phi_0 = \text{vac}$;
2) to create all particles present finally: $\Phi_0 \rightarrow \Phi_b$. 

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iii) In addition, there may be present further creation and absorption operators which create and reabsorb virtual particles in intermediate states. It is clear that these must always occur in pairs to create and get rid of a given particle.

By means of the commutation relations and anticommutation relations of the fields, we can arrange them in such an order that:

first, all initial particles are absorbed;
then, a certain number of intermediate particles are created and reabsorbed; and
finally, all final particles are created.

This involves some tricky algebra: signs, factors, etc., for all but the simplest cases. Wick's theorem looks after all this. It is not my purpose to expound this here, and I assume you know it. I will just remind you of the effect.

For a transition \( \phi_a \rightarrow \phi_b \), from a given term in the S-matrix, we pick out operators to absorb initial and create final particles.

The rest we contract over in all possible ways: in pairs to give intermediate particles. The only non-vanishing contractions are (again neutral ps ps meson theory):

\[
\begin{align*}
\Phi(x_2)\Phi(x_1) &= \frac{1}{2} \Delta_p(x_2 - x_1) : \text{meson propagator} \\
\Psi(x_2)\Psi(x_1) &= -\frac{1}{2} S_p(x_2 - x_1) : \text{nucleon propagator}
\end{align*}
\]

These are just vacuum expectation values of time-ordered products, e.g.

\[
\Phi(x_2)\Phi(x_1) = \langle \phi_0, T \{ \Phi(x_2)\Phi(x_1) \} \phi_0 \rangle, \text{ etc.}
\]

\[
\left[ \Delta_p(x) = \frac{-2i}{(2\pi)^4} \int d^4p \frac{e^{ipx}}{p^2 + m^2 - i\epsilon}, S_p(x) = \frac{2}{(2\pi)^4} \int d^4p \frac{e^{ipx}}{\gamma_\mu p^\mu - iM} \right].
\]
We are then led in a natural way to interpret the various possible contracted terms, in terms of Feynman graphs.

The basic unit is

\[ \mathcal{H}_I = \bar{\psi}(x) \gamma_5 \psi(x) \phi(x) : \gamma : \]

\( S_n \) contains \( n \) such basic vertices. After picking out lines for initial and final particles, the other lines must be contracted in pairs always. Thus, for example, \( S_2 \) leads to

\[ \bar{\psi}(2) \gamma_5 \psi(2) \bar{\psi}(2) \gamma_5 \psi(2) \]

\[ \bar{\psi} \psi \bar{\psi} \psi \bar{\psi} \psi \psi \]

and

\[ \bar{\psi} \psi \bar{\psi} \psi \bar{\psi} \psi \bar{\psi} \psi \]

\[ \pi-\mathcal{N} \text{ scattering} \]
\[ \mathcal{N}-\mathcal{N} \text{ scattering} \]
\[ \mathcal{N}-\text{self energy (meson cloud)} \]
\[ \text{etc.} \]
\[ \text{etc.} \]

\[ + \quad \quad \quad + \]

\[ \bar{\psi} \psi \bar{\psi} \psi \bar{\psi} \psi \]

"vacuum polarization"

unobservable, but as we shall see not to be ignored altogether

Here, \( \gamma_5 \) and arguments 1 and 2 have been omitted except for the first term.

"Etc." above stands for many other processes, according to which creation and absorption operators are picked out of the fields
not contracted over, e.g. $\bar{\psi} \, \psi \, \bar{\psi} \, \psi$ also contains, what I might draw as

\[ \text{annihilation in the field of a } \mathcal{N} \text{ (which experiences a recoil).} \]

But this is topologically the same as $\text{[diagram]}$. I have drawn only topologically distinct graphs: they represent a whole class of processes.

This concludes the rather brief résumé of perturbation theory.

* * *
LECTURE II

CLOSED FORM EXPRESSIONS FOR SCATTERING AMPLITUDES

The form \((\mathcal{S}_b, \mathcal{S}_a)\) is not very convenient: one wants or would prefer a closed form, i.e., one not involving an infinite series. There are two ways of getting such a result.

The more elegant and abstract approach is due to Lehmann, Symanzik and Zimmerman\(^1\). However, I will follow Gell-Mann and Low\(^2\). This is a more familiar approach which connects up with Feynman graphs, etc., so one sees what one is after more directly. This method decomposes into two parts. I will illustrate it for a particular case: then the general case will be obvious.

Let us take, as example, the meson propagator. In perturbation theory, this is simply

\[
G_\pi(y-x) = \frac{1}{2} \Delta_\pi(y-x) = \left( \Phi_0, T[\varphi(y)\varphi(x)] \Phi_0 \right) = \begin{array}{c}
\Phi_0 \Phi_0
\end{array} . \quad (\text{II.1})
\]

Here \(T = \) Wick chronological product (in the general case!). This is equal to Dyson chronological product, except for an extra factor \((-1)\) to allow for Fermi-statistics every time two fermion fields are permuted.

Equation (II.1) is the lowest order perturbation theory. Also radiative corrections have to be considered. This leads to a modified propagator

\[
G'_\pi(y-x) = \frac{1}{2} \Delta'_\pi(y-x) = \begin{array}{c}
\Phi_0 \Phi_0
\end{array} ~ \begin{array}{c}
\Phi_0 \Phi_0
\end{array} \quad (\text{II.2})
\]

self-energy effects

\[= \begin{array}{c}
\Phi_0 \Phi_0
\end{array} + \begin{array}{c}
\Phi_0 \Phi_0
\end{array} + \begin{array}{c}
\Phi_0 \Phi_0
\end{array} + \begin{array}{c}
\Phi_0 \Phi_0
\end{array} + \begin{array}{c}
\Phi_0 \Phi_0
\end{array} + \ldots
\]
I will show that:

$$G'_n(y - x) = \frac{\left( \Phi_0, \hat{T}[\varphi(y)\varphi(x)]\Phi_0 \right)}{\left( \Phi_0, S\Phi_0 \right)} \quad (\text{II.3})$$

By T-product in the numerator I mean simply $\varphi(y)$ and $\varphi(x)$ to be inserted in appropriate places in the chronological products which occur in expansion of $S$:

$$\sum_n \left( \Phi_0, \hat{T}[\varphi(y)\varphi(x)S_n]\Phi_0 \right) = \sum_n \left( \frac{-i}{n!} \int \cdots \int dx_n \int dx_1 \cdots dx_1 \left( \Phi_0, \hat{T}[\varphi(x)\varphi(y)K_1(x_1)K_1(x_n)]\right) \right)$$

Graphically:

- $n = 0$:
  $$\begin{array}{c}
  y \\ \hline
  x
  \end{array}$$

- $n = 1$:
  no terms, since $\left( \Phi_0, \varphi(y)\varphi(x)\varphi(x_1)\Phi_0 \right) = 0$
  Similarly $n = 3, 5 \ldots$ give no contribution.

- $n = 2$:
  involves $\left( \Phi_0, \hat{T}[\varphi(x)\varphi(y) (\bar{\varphi}\gamma_5 \varphi)_{x_2} (\bar{\varphi}\gamma_5 \varphi)_{x_1}]\Phi_0 \right)$
  
  $$= \begin{array}{c}
  y_2 \\ \hline
  x_1 \\
  x_2 \\
  \end{array}$$

- $n = 4$:
  involves $\left( \Phi_0, \hat{T}[\varphi(x)\varphi(y) (\bar{\varphi}\gamma_5 \varphi)_{x_4} (\bar{\varphi}\gamma_5 \varphi)_{x_3} (\bar{\varphi}\gamma_5 \varphi)_{x_2} (\bar{\varphi}\gamma_5 \varphi)_{x_1}]\Phi_0 \right)$
  i.e. involves 4 fermion pairs, i.e. propagators

3 boson

To obtain vacuum expectation value, one must contract over all pairs of operators.

$$= \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} + \begin{array}{c}
  y \\ \hline
  x
  \end{array} \quad \text{in \{\}}
  \text{ disconnected vacuum loops} \}.
Next, consider the denominator:

\[(\Phi_0, S\Phi_0) = 1 + \left( \begin{array}{c}
- \\
\end{array} \right) + \left( \begin{array}{c}
- \\
\end{array} \right) + \left( \begin{array}{c}
\mathcal{O} \\
\end{array} \right) + \left( \begin{array}{c}
X \\
\end{array} \right) + \left( \begin{array}{c}
\mathcal{O} \\
\end{array} \right) + \ldots \]

Now \( S \) operating on \( \Phi_0 \) gives a constant times \( \Phi_0 \). This is because \( S \), representing the equation of motion, conserves energy and momentum, i.e. \( \Phi_0 \) is an eigenstate of \( S \):

\[ S \Phi_0 = \lambda \Phi_0 \]

and since \( S \) is unitary

\[ SS^\dagger = 1 \rightarrow \lambda = e^{i\alpha} \quad \text{(II.4)} \]

\[ \therefore (\Phi_0, S\Phi_0) = e^{i\alpha}. \]

Thus we can write the numerator in Eq. (II.3) as

\[ \left[ \begin{array}{c}
- \\
\end{array} + \begin{array}{c}
- \\
\end{array} + \begin{array}{c}
- \\
\end{array} + \ldots \right] \times \left\{ 1 + \begin{array}{c}
\mathcal{O} \\
\ldots \\
\end{array} \right\} \]

connected graphs

\[ = \left[ \begin{array}{c}
- \\
\end{array} + \begin{array}{c}
- \\
\end{array} + \ldots \right] e^{i\alpha}. \]

Whence finally:

\[ \left( \Phi_0, T\{\varphi(y)\varphi(x)S\}\Phi_0 \right) / (\Phi_0, S\Phi_0) = \left[ \begin{array}{c}
- \\
\end{array} + \begin{array}{c}
- \\
\end{array} + \ldots \right] \]

\[ = \begin{array}{c}
\mathcal{O} \\
\end{array} \]

\[ = \frac{1}{2} \Delta_p'(y-x) = G'_\pi(y-x) \quad \text{q.e.d.} \]
Similarly one shows that

\[ G'_N(y-x) = \left( \Phi_0, T[\psi(y)\bar{\psi}(x)S]\Phi_0 \right) / (\Phi_0, S\Phi_0) \]

\[ = \begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\text{y} \quad \text{x} \\
\end{array} \quad \text{(II.5)} \]

\[ = \begin{array}{c}
\bullet \quad \bullet \\
\text{y} \quad \text{x} \\
\end{array} + \begin{array}{c}
\bullet \\
\text{x} \\
\end{array} + \begin{array}{c}
\bullet \\
\text{y} \\
\end{array} \\
\]

\[ = G_N(y-x) = \left( \Phi_0, T[\psi(y)\bar{\psi}(x)]\Phi_0 \right) = -\frac{1}{2} S_N(y-x) \]

and the amplitude for \( \pi - \pi' \) scattering

\[ G(x' x y y') = \left( \Phi_0, T[\psi(x')\bar{\psi}(x)\phi(y)\phi(y')S]\Phi_0 \right) / (\Phi_0, S\Phi_0) \quad \text{(II.6)} \]

\[ = \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{y'} \quad \text{y} \\
\end{array} \]

\[ = \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{x'} \quad \text{x} \\
\end{array} + \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{x'} \quad \text{x} \\
\end{array} + \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{x'} \quad \text{x} \\
\end{array} + \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{x'} \quad \text{x} \\
\end{array} + \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{x'} \quad \text{x} \\
\end{array} \]

Here I have throughout suppressed indices labelling the 4-spinor components, e.g. in Eq. (II.6):

\[ G^{\nu\mu}(x' x y y') = \left( \Phi_0, T[\psi_{\nu}(x')\bar{\psi}_{\mu}(x)\phi(y')\phi(y)]\Phi_0 \right) / (\Phi_0, S\Phi_0) \]

For charged mesons, using isospin, we would similarly have

\[ G_{\rho}^{\nu\mu}(x' x y y') = \left( \Phi_0, T[\psi_{\nu}(x')\bar{\psi}_{\mu}(x)\phi_{\rho}(y')\phi_{\rho}(y)]\Phi_0 \right) / (\Phi_0, S\Phi_0) \]
\( \alpha, \beta = 1, 2, 3 \) and now \( \mu, \nu = 1, \ldots, 8 \), corresponding to spin and isospin of the nucleons.

Introduce the Fourier transform

\[
G(p' k' k) = \int \cdots \int G(x' x y' y) e^{i(px + ky - p'x' - k'y')} d^4x' d^4x d^4y' d^4y
\]

\[
= S_p(p) \delta(p' - p) A_p(k) \delta(k' - k) + S_p(p') A_p(k') \tilde{G}(p'k'k) S_p(p) A_p(k).
\]

Hence amplitude for \( \pi-N' \) scattering is given by

\[
\tilde{u}^{(s)}(p') \tilde{G}_{gsa}(p'k'k) u^{(r)}(p)
\]

[here spinor indices are suppressed]

\[
\tilde{u}^{(s)}_\mu(p') \tilde{G}_{gsa}(p'k'k) u^{(r)}_\mu(p)
\]

The second step is now to write the amplitudes in closed form.

In the interaction picture (abbreviated I.P. hereafter) we had for the development of a state

\[
in \frac{\partial \Phi(t)}{\partial t} = H(t) \Phi(t).
\]  

(II.7)

Let

\[
\Phi(t) = U(tt_0) \Phi(t_0)
\]

(II.8)

i.e. \( U(tt_0) \) is the formal solution of Eq. (II.7), which generates \( \Phi \) at time \( t \) from \( \Phi \) at time \( t_0 \). Then from Eqs. (II.7) and (II.8):

\[
in \frac{\partial U(tt_0)}{\partial t} = H(t) U(tt_0)
\]

(II.9)
U(t_0 t_0) = 1 \tag{II.10}

t_0 is so far arbitrary.

From Eqs. (II.8) - (II.10) one has at once:

(i) from Eq. (II.8)

\[ U(t_3 t_2) U(t_2 t_1) = U(t_3 t_1) \tag{II.11} \]

(this is the group property);

(ii) from Eqs. (II.10) and (II.11)

\[ U(t_2 t_1) = U^{-1}(t_1 t_2). \tag{II.12} \]

(iii) Furthermore, since \( H(t) \) is hermitian, Eq. (II.9) generates a unitary operator:

\[ U^\dagger(t_2 t_1) = U^{-1}(t_2 t_1) = U(t_1 t_2). \tag{II.13} \]

We see from Eq. (II.8) that

\[ S = U(\omega, -\infty). \tag{II.14} \]

Let us now apply the unitary operator \( U(t t_0) \) to states and operators in the following manner. Define:

a) for any operator \( O(t) \) in the I.P. a new operator, denoted by

\[ O_H(t) = U(t_0 t) O(t) U(t t_0) \tag{II.15} \]

and

b) for any state \( \Phi(t) \) in the I.P. a new state

\[ \Phi_H = U(t_0 t) \Phi(t). \tag{II.15^*} \]
We note these special features:

\[(i) \quad (\psi(t), 0(t) \phi(t)) = (\psi_H, 0_H(t) \phi_H)\]

by direct substitution of Eqs. (II.15) and (II.15*), using Eq. (II.13):

\[\left(\psi_H, 0_H(t) \phi_H\right) = \left(\psi(t), U^\dagger(t_0 t) U(t_0 t) 0(t) U(t_0 t) \phi(t)\right)\]

\[= \left(\psi(t), 0(t) \phi(t)\right)\]

This is, of course, what we expect from a unitary transformation. It leaves scalar products invariant.

(ii) From Eqs. (II.8) and (II.15*):

\[\phi_H = U(t_0 t) \phi(t) = U(t_0 t) U(t_0 t) \phi(t) = \phi(t_0)\]

i.e., \(\phi_H\) is really a constant state, independent of \(t\). In fact, the unitary transformation [Eqs. (II.15 - 15*)] has taken us from the I.P. to the H.P. (Heisenberg picture). In the latter, state-vectors are constant: the total time-dependence is carried by the operators. The two pictures agree at time \(t_0\).

In the I.P., which we have used so far, the states have been eigenstates of \(H_0\). \(\phi_\alpha, \phi_\beta \ldots\) have been bare particle states; not eigenstates of the total Hamiltonian \(H\). In fact, these states change with time as a result of the interaction.

Amongst other things, bare particles are transformed into real (dressed) particles. This is connected with the fact that when we use the I.P., we use the adiabatic hypothesis, i.e., the coupling constant (the charge \(e\) in electrodynamics) is switched off adiabatically as \(t \to t_\infty\).
Coupling constant \( g \) as a function of time:

\[ \begin{array}{c}
\infty \rightarrow t \\
\text{bare particles} \\
-\infty \rightarrow t \\
\text{scattering occurs during some time } \tau \ll 2T \\
T \\
\text{bare particles} \\
T \rightarrow +\infty
\end{array} \]

During the interval \( -\infty < t < -T \) (long before the scattering) the interaction has the effect of generating the real particles (including meson clouds, etc.) from the bare particles.

Consider the vacuum. We denote \( \phi_0 \) = bare vacuum, i.e. the eigenstate of zero energy and momentum of \( H_0 \).

By contrast, real vacuum, the lowest energy eigenstate of \( H \), i.e. a steady state, is denoted in h.p. by \( \phi^H_0 \). Then, in i.p., real vacuum at \( t = t_0 \) is

\[ \phi(t_0) = U(t_0 t_0) \phi^H_0 = \phi^H_0. \] (II.16)

This is not a stationary state in the i.p.: in fact it evolved from

\[ \phi(-\infty) = U(-\infty, t_0) \phi^H_0 \] (II.17)

at time \( t = -\infty \). Eq. (II.17) gives

\[ \phi^H_0 = U(t_0, -\infty) \phi(-\infty). \] (II.18)

We interpret \( \phi(-\infty) \) given by Eq. (II.17) as bare vacuum. This is because we have assumed an adiabatic cut-off: the interaction vanishes as time goes to \( \pm \infty \). So at \( t = -\infty \) we have bare vacuum. As time evolves, the interaction is switched on and generates real vacuum out of bare vacuum. At \( t = t_0 \), it has arrived at real vacuum.

Gell-Mann and Low \(^2\) show more properly (not rigorously) that

\[ \phi^H_0 = \frac{c U(t_0, -\infty) \phi_0}{(\phi_0, U(t_0, -\infty) \phi_0)} \] (II.19)
where:
\[ \Phi_c = \text{bare vacuum in I.P.} \]
\[ c = \text{infinite constant to be eliminated eventually.} \]

One can similarly follow the time development of the real vacuum to \( t \to \infty \), as the interaction is switched off. One gets

\[ \Phi_0^H = \frac{c \ U(t_0, \infty) \ \tilde{\Phi}_0}{(\Phi_c, \ U(t_0, \infty) \ \tilde{\Phi}_0)} \quad \text{(II.19a)} \]

Now
\[ (\Phi_0^H, \ \Phi_0^H) = 1 \]

whence directly from Eqs. (II.19) and (II.19a):

\[ |c|^2 \left( \Phi_0, U(\infty, t_0)U(t_0, \infty)\Phi_0 \right) \left/ \left\{ (\Phi_0, U(t_0, \infty)\tilde{\Phi}_0)(\tilde{\Phi}_0, U(t_0, \infty)\Phi_0) \right\} \right. \]

\[ = U(\infty, \infty) = S \ , \]

Therefore

\[ \frac{|c|^2}{\left(\Phi_0, U(t_0, \infty)\tilde{\Phi}_0)(\tilde{\Phi}_0, U(t_0, \infty)\Phi_0) \right)} = \frac{1}{(\Phi_0, S\Phi_0)} \quad \text{(II.20)} \]

Consider now the expression:

\[ \left( \Phi_c^H, T \left[ \begin{array}{c} \Psi_H(y) \\ \Psi_H(x) \end{array} \right] \Phi_c^H \right) \quad \text{(II.21)} \]

(Everything in \( H, P \)) If one takes, say, \( y_0 > x_0 \), then one can omit the \( T \)-product. Using Eqs. (II.15) and (II.19), Eq. (II.19a) to go over to I.P.
\[
\left( \Phi_0, T \left\{ \Phi_H(y) \Phi_H(x) \right\} \Phi_0 \right) = \\
= \left| c \right|^2 (\Phi_0, U(\infty, t_0)U(t_0 y_0)\varphi(y)U(y_0 t_0)U(t_0 x_0)\varphi(x)U(x_0 t_0)U(t_0, -\infty) \Phi_0)
\]

and from Eqs. (II.20) and (II.11), this expression equals:

\[
\frac{1}{(\Phi_0, S \Phi_0)} \left( \Phi_0, U(\infty, y_0)\varphi(y)U(y_0, x_0)\varphi(x)U(x_0, -\infty) \Phi_0 \right).
\]

This I can write

\[
\frac{1}{(\Phi_0, S \Phi_0)} \left( \Phi_0, T \left\{ \varphi(y)\varphi(x)S \right\} \Phi_0 \right) \quad \text{(II.22)}
\]

it being understood that \( T[ \ldots ] \) means insertion of \( \varphi(x) \) and \( \varphi(y) \) in the chronologically correct places in \( S = U(\infty, -\infty) \); this being decomposed by the group property

\[
S = U(\infty, y_0)U(y_0, x_0)U(x_0, -\infty).
\]

But we now note that Eq. (II.22) is also valid for \( x_0 < y_0 \), on account of the \( T - \) product.

Equation (II.22) is just the expression we had previously for the modified meson propagator [Eq. (II.3)]. So:

\[
G_n'(y-x) = \frac{1}{2} \Delta_P'(y-x) = \left( \Phi_0, H \left\{ \varphi_H(y)\varphi_H(x) \right\} \Phi_0 \right) \quad \text{(II.23)}
\]

This is the final result we want in terms of H.P. operators.

Similarly one shows: nucleon propagator:

\[
G_n'(y-x) = \frac{1}{2} S_P'(y-x) = \left( \Phi_0, H \left\{ \varphi_n(y)\varphi_n(x) \right\} \Phi_0 \right), \quad \text{(II.24)}
\]
\[ G_{a\beta}(x'xy'y) = \left( \bar{\psi}_{H'}(y') \psi_{H}(x') \bar{\psi}_{H}(x) \psi_{H}(y) \right) \phi^{H} \phi_{H} \text{.} \] (II.25)

Here \(a, \beta = 1, 2, 3\): the three real fields from which \(\pi^+, (1, 2)\) and \(\pi^0, (3)\) come.

\textbf{Note.} Hereafter I will drop label \(H\) for H.P.: it will be clear from the context which I mean. For the time being, H.P. will always be used.

Eq. (II.25) is the desired form of the scattering amplitude: it involves Heisenberg operators and \textit{real} states.

The analysis of Eq. (II.25), involving four operators, is too complicated although it would probably be most fruitful. Instead, the usual dispersion relations are derived for quantities like

\[ \left( \phi(p'), T \left[ \psi_{a}(y) \phi_{a}(x) \right] \phi(p) \right) \text{.} \] (II.26)

Here \(\phi(p)\) is the Heisenberg state-vector of a \textit{real} nucleon state, etc. The exact expression of the appropriate S-matrix element is, for example, derived in Low\(^3\). He has used exactly the methods shown above: no further physics is necessary. Only a little algebra is required.

[Actually, dispersion relations have not been derived for Eq. (II.26), but for a related quantity: the retarded commutator, to which causality can be applied.]

Expressions of amplitudes, in terms of Heisenberg operators, are well suited to study symmetry properties. A typical example is the \textit{crossing theorem in \(\pi-N\) physics.}

The method of proof I use is due to Feldman and Matthews\(^4\). I shall later on repeatedly apply it to other symmetry questions. Consider Eq. (II.25) in momentum space. For the transition \(p\alpha \rightarrow p' k'\beta'\):
\[ \tilde{G}_{\alpha\beta}(p'k'; pk) = \int d^4x' d^4x' d^4y' d^4y \ e^{i(px'x' + ky'y')} \ \Phi_0, T \left[ \tilde{\psi}(x') \tilde{\overline{\psi}}(x) \varphi_\beta(y') \varphi_\alpha(y) \right] \Phi_0. \] (II.27)

**N.B.** The \( \int^3 \) over the exponentials just pick out the desired absorption and creation operators from the fields.

On the right-hand side one makes these two changes.

(i) Interchange \( \varphi_\beta(y') \leftrightarrow \varphi_\alpha(y) \) inside \( T\{ \ldots \} \); since these are two boson fields, it does not alter the \( T \)-product.

(ii) Rotlabel variables of integration: \( y \leftrightarrow y' \).

Then the total effect is

\[ \tilde{G}_{\alpha\beta}(p'k'; pk) = \int d^4x' \ldots d^4y \ e^{i(px'x' + (-k')y'(-k)y')} \ \Phi_0, T \left[ \tilde{\psi}(x') \tilde{\overline{\psi}}(x) \varphi_\alpha(y') \varphi_\beta(y) \right] \Phi_0 \]

\[ = \tilde{G}_{\alpha\beta}(p', -k; p, -k'). \] (II.28)

This is crossing symmetry: interchange of \( \alpha \leftrightarrow \beta \) and \( k \leftrightarrow -k' \) simply interchanges in- and out- mesons. \( \tilde{G} \) is symmetric with respect to these: thus in perturbation theory graphs always occur in pairs

\[ \tilde{G}_{\alpha\beta}(p'k'; pk) \]

\[ \begin{array}{c}
\text{\( k \)} \\
\text{\( p \)} \\
\text{\( p' \)} \\
\end{array} \]

\[ + \]

\[ \begin{array}{c}
\text{\( k' \)} \\
\text{\( p \)} \\
\text{\( p' \)} \\
\end{array} \]

Note: crossing symmetry is a direct consequence of the boson nature of pions!

It can, of course, be generalized to interchanging an incoming and an outgoing meson in a more general amplitude.

In static no-recoil limit of \( \pi - N \) theory, the amplitude becomes

\[ t_{\alpha\beta}(k', k, \omega), \text{ with } \omega^2 = m^2 + k^2 = m^2 + k'^2. \] Then the crossing symmetry becomes

\[ t_{\alpha\beta}(k', k, \omega) = t_{\alpha\beta}(-k, -k', -\omega). \] (II.29)

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From the above form of the scattering amplitude, one can easily see another result, which recently became of prominence in connection with Mandelstam's dispersion relations. The same analytic expression gives the amplitude for any of the three processes:

1) \( \pi + N^0 \to \pi + N^0 \)
2) \( \pi + \pi \to \bar{N} + N^0 \) or, of course, \( A + B \to C + D \)
3) \( \pi + \bar{N} \to \pi + \bar{N}^0 \)

for any process and other channels.

To see this, we merely write Eq. (II.27) as

\[
\int dx'dx dy' dy \ e^{i(px + p'x' + ky + k'y')} \left( \Phi_0, T\left[ \tilde{\psi}(x') \tilde{\psi}(x) \phi(y') \phi_d(y) \right] \Phi_0 \right)
\]

where now, unlike before, \( k_0 \) may be > or < 0, etc.; then

\( k_0 > 0 \) - picks out the absorption part of \( \phi_d(y) \),

i.e. a meson absorbed;

\( k_0 < 0 \) - picks out the creation part of \( \phi_d(y) \),

i.e. a meson created;

etc., for the other fields. We then get at once:

1) \( k_0 > 0, p_0 > 0 \) - \( p_0 > 0 \) picks out absorption part from \( \psi(x') \),

i.e. \( N^0 \) absorbed;

\( k_0 < 0, p_0 < 0 \) - \( p_0 < 0 \) picks out creation part from \( \psi^+(x) \)

i.e. \( N^0 \) created;

\[ \pi + N^0 \to \pi + N^0. \]

2) \( k_0 > 0, p_0 > 0 \) \( \{ \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) \( \) others < 0

\[ \pi + \bar{N} \to \pi + \bar{N}^0. \]
3) \( k_0 > 0, k_0' > 0 \) \( \pi + \pi \rightarrow \pi + \pi' \).

Crossing, above, really corresponded to this sort of interchange of an initial and a final particle; in this case the particles happen to be identical bosons which gives an additional symmetry.

For later use, one might mention the complete generalization of the above expression. In

\[
\int \cdots \int dy_\ldots e^{-iky} \left( a_0, T \left[ \cdots \varphi(y), \cdots \right] \varphi_0 \right) \cdots
\]

the \( \int dy e^{-iky} \) picks out the term

\[ a(k)e^{iky} \]

from \( \varphi(y) \): "absorbed at y"; i.e., a final emitted particle. Similarly for \( e^{iky} \). Here we are dealing with plane waves. But if we wanted any other state with wave function \( g(y) \) for the final meson, we would simply take

\[
\int dy g^*(y).
\]

This automatically picks out the correct linear combination of absorption operators to create a state with this wave function.

The complete generalization of this is now at once obvious.

For

\[ A_1 + A_2 + \ldots \rightarrow B_1 + B_2 + \ldots \]

wave functions \( f(x_1, x_2, \ldots) \)

\( g(y_1, y_2, \ldots) \).

Amplitude = \( \int dx_1 dx_2 \cdots dy_1 dy_2 \cdots g^*(y_1, \ldots) \times \left( a_0, T \left[ F_{A_1}(x_1), \ldots G_{B_1}(y_1), \ldots \right] \varphi_0 \right) f(x_1, \ldots) \).

(II.30)
where $F_{A_1} (x_1)$ is the field operator containing the absorption operator for particle $A_1$, and $F_{B_1} (y_1)$ is the one containing the creation operator for particle $B_1$, etc.
LECTURE III

SYMMETRIES, INVARIANCE PROPERTIES
AND CONSERVATION LAWS

I now come to what is the basis of the following development:
discussion of the relations of symmetries, invariance and conservation
laws for a physical system.

Here I give a general discussion. In the following lectures
particular applications will be discussed, mainly those which are of
importance in quantum mechanics, rather than in classical theory. This
is partly a question of time. I will, of course, implicitly use other
symmetries, e.g. Lorentz covariance, although I will not discuss it.

Let me first remind you: if $A$ is an operator in H.P.,

$$i \frac{dA}{dt} = [A, H]$$  \hspace{1cm} (III.1)

and if $\frac{dA}{dt} = 0$ ($A =$ const.), then

$$[A, H] = 0.$$  \hspace{1cm} (III.2)

If $A$ is hermitian, it can be interpreted as an observable and then if
Eq. (III.2) is satisfied, $A$ is a constant of the motion.

Now suppose we consider a system. It is invariant under
certain transformations: usually these form a group. [I want to keep
things general, but you might like to think of a system referred both
to a right-hand and a left-hand set of axes, with the two descriptions
identical (inversion invariance)]. The system certainly has the parti-
cular symmetry if the Lagrangian density, $\mathcal{L}$, has it. For everything
else can be derived from $\mathcal{L}$. In particular, the equation of motion and
commutation relations are derived from $\mathcal{L}$.

The invariance with respect to a transformation will enable us
to give two equivalent descriptions for a system. (For example, in the
case of invariance under inversion, I can think of two observers describing
the same system, one using right-hand, the other left-hand axes. They will use different representations for operators and states: however, these must lead to the same physical results.)

In the general case, too, we get two equivalent representations:

\[ A, \phi \] and \[ A', \phi'. \]

These are equivalent, i.e. they have the same equations of motion and commutation relations. One can achieve this if they are related by a similarity transformation:

\[
\begin{align*}
A'(x) &= R A(x) R^{-1} \\
\phi' &= R \phi
\end{align*}
\]  

(III.3)

But we also want the same scalar products (i.e. the same transition amplitudes, cross-sections, etc.), e.g.

\[(\psi, A\phi) = (\psi', A'\phi').\]

From this it follows in the usual way that \( R \) must be unitary:

\[ R^{-1} = R^\dagger. \]  

(III.4)

Now, how do we get conservation laws? If the system is invariant under the transformation \( R \) then, in particular, the Hamiltonian \( H \) is invariant, i.e. the transformed Hamiltonian \( H' \) agrees with the original Hamiltonian:

\[ H' = RHR^\dagger = H \]

whence

\[ [R, H] = 0. \]  

(III.5)

If \( R \) is hermitian, then \( R \) is an observable, which is a constant of the motion.
Now R is unitary, Eq. (III.4); if it is also hermitian, then
\[ R^{-1} = R^\dagger = R \] (III.6)
hence
\[ R^2 = 1 \] (III.7)
i.e. R can only have eigenvalues ±1. Thus if R is hermitian, it defines a discontinuous transformation. This is a serious restriction on the kinds of transformation; nevertheless, it is very important.

When R is not hermitian, we can still define an operator \( \mathcal{W} \)
\[ R = e^{i\mathcal{W}} \] (III.8)
and \( \mathcal{W} \) is now hermitian and so defines an observable which is a constant of the motion (since Eq. (III.5) implies \([\mathcal{W},H] = 0\)).

For continuous transformations, a different approach exists for finding constants of the motion. This is very useful and is based on Noether's theorem. I will here consider only a special case, which is adequate for what I shall want later.

Suppose we derive a field theory from Hamilton's principle for an action integral
\[ I(\Omega) = \int_{\Omega} d^4x \mathcal{L}(\varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x\nu}). \] (III.9)
\( \Omega \) = region of 4-space; \( \varphi^\alpha, \alpha = 1, \ldots N \), a series of fields.

Consider a variation of the fields
\[ \varphi^\alpha \to \varphi^\alpha + \delta \varphi^\alpha \] (III.10)
keeping the region \( \Omega \) fixed. This is the restriction I mentioned. One can also vary \( \Omega \); in fact, this is necessary to get the energy momentum tensor and angular momentum of the field, e.g. spin! But I will not need to do this. A particularly lucid account is given by Bogoliubov and Shirkov ⁵.)
Under Eq. (III.10), Eq. (III.9) becomes

$$\delta I(\Omega) = \int \frac{d^4x}{\Omega} \left\{ \frac{\partial L}{\partial \dot{\phi}^a} \frac{\partial}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha} \frac{\partial L}{\partial \dot{\phi}^a} \right\} \delta \phi^a + \int \frac{d^4x}{\Omega} \left\{ \frac{\partial}{\partial \phi^a} \frac{\partial \phi^a}{\partial x^\alpha} \right\} \delta \phi^a. \quad (III.11)$$

(Here \( \phi^a = \frac{\partial \phi^a}{\partial x^\alpha} \).)

In Eq. (III.11) the first bracket on the right-hand side is zero by the equation of motion. Now if \( L \) and therefore \( I(\Omega) \) are invariant under the transformation (III.10), then \( \delta I = 0 \), i.e. from Eq. (III.11), since \( \Omega \) is arbitrary:

$$\frac{\partial}{\partial x^\alpha} f_{\nu} = 0 \quad (III.12)$$

with

$$f_{\nu} = \frac{\partial L}{\partial \dot{\phi}^a} \delta \phi^a \quad (III.13)$$

Eq. (III.12) is a continuity equation in four dimensions, i.e. the conservation law for \( f_\nu \). (This connection, between symmetry and the conservation law, is Noether's theorem.)

Integrate (III.12) over the region shown, using Gauss' theorem and assuming that the fields vanish sufficiently fast as \( x \to \infty \). To do this, let

$$F_{\nu}(t) = \int d^3x f_\nu(x)$$

$$x_0 = t$$

and \( F_4 = iF_0 \), then

$$\frac{dF_0(x_0)}{dx_0} = \frac{dF_4}{dx_4} - \int d^3x \frac{\partial F_4}{\partial x_4} - \int d^3x \frac{\partial F_k}{\partial x_k} = 0 \quad (x_0 = t, x_0 = t)$$

The "= 0" follows by converting the three-volume integral over all space.
at $x_0 = t$ into a surface integral at $\infty$. Hence

$$F_0(t) = \text{const.} \quad (\text{III.14})$$

This gives the corresponding conserved quantity (e.g. for gauge invariance, the charge).

I would like to make two comments on this result:

i) the treatment given above is quasi-classical; but it is valid also in the quantum case. One must not alter the order of field operators, e.g. $\delta \varphi^a$ must occur at the place of the $\varphi^a$ which was varied.

ii) I would like to mention, without proof, that in the case of continuous transformation there are two methods. Via unitary transformation one obtains $e^{+iW}$, and via Noether's theorem $F_0(t)$. These are, of course, not different constants, but they are different methods for obtaining the same result. One can show

$$F_0 = -W.$$

Before discussing any particular symmetries, I want to mention generally the type of application to which this leads:

i) restrictions on the form of interaction, e.g. on $\mathcal{L}$ or on $S$.
For example, in momentum space, Dalitz and Wolfenstein have derived for $\mathcal{N}$ scattering the invariants with respect to inversion and time reversal, to be obtained from $k, k', \sigma$, and $G_z$.

ii) Selection rules: from $\left(\hat{\varphi}_b, S\hat{\varphi}_a\right)$, e.g. angular momentum, parity, etc.

iii) Related processes: if

$$RS R^* = S$$

then

$$\left(\hat{\varphi}_b, S\hat{\varphi}_a\right) = \left(\hat{\varphi}_b', S\hat{\varphi}_a'ight)$$
where $\tilde{a}' = R^a \tilde{a}$, etc., i.e. the amplitude for the transformation $\tilde{a} \rightarrow \tilde{b}$ is the same as $\tilde{a}' \rightarrow \tilde{b}'$.

Finally, one word of warning. One cannot take all hermitian operators which commute with the Hamiltonian as constants of the motion. They may not commute with each other. For instance, in non-relativistic quantum mechanics, only one component of angular momentum can be quantized. That is, one can only choose a complete set of commuting observables: $A, B, C, \ldots F$. There are cases when mitigating circumstances relax this a little: I shall mention them where they arise.

After this general discussion of invariance and conservation laws, I now shall treat particular symmetries and see what the consequences are.

*  *  *

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LECTURE IV

INVERSION

We now consider inversion, R.H. ↔ L.H. systems:

\[ \mathbf{x} \rightarrow \mathbf{x}' = - \mathbf{x}, \quad x_0 \rightarrow x'_0 = x_0. \]  \hspace{1cm} (IV.1)

Note: we shall always assume rotational invariance. Then inversion = reflection in a plane followed by a rotation. So one often talks about reflection when one is really interested in inversion.

The corresponding conserved quantity to invariance with respect to Eq. (IV.1) is parity. Until three or four years ago, parity conservation was assumed of universal validity. Now it is only accepted for strong and electromagnetic interactions, not for weak ones. I assume that you know the evidence for these statements.

I now want to classify fields according to their transformation properties under inversion. This will lead to certain parity assignments for the corresponding particles. Of course, this whole scheme only makes sense if parity is conserved. If not, it becomes less transparent. I will come back to this.

There is one simple case, the electromagnetic field. Here we know from classical theory what is required for Maxwell's equations to be inversion-invariant, namely:

\[ \mathbf{E} : \text{polar-vector (} \equiv \text{vector)} (\text{like force}) \]

\[ \mathbf{E}'(\mathbf{x}, x_0) = P \mathbf{E}(\mathbf{x}, x_0) P^{-1} = - \mathbf{E}(- \mathbf{x}, x_0) \]  \hspace{1cm} (IV.2a)

and

\[ \mathbf{H} : \text{axial vector (} \equiv \text{pseudo-vector)} (\text{like angular momentum}) \]

\[ \mathbf{H}'(\mathbf{x}, x_0) = P \mathbf{H}(\mathbf{x}, x_0) P^{-1} = + \mathbf{H}(- \mathbf{x}, x_0). \]  \hspace{1cm} (IV.2b)

From

\[ \mathbf{H} = \text{curl} \mathbf{A} \quad \mathbf{E} = - \nabla V + \mathbf{A} \]
the inversion properties of a 4-vector $A_\mu = (A, iv)$ follow:

$$
A'_k(x, xo) = -A_k(-x, xo) \quad \text{and} \quad A_4(x, xo) = -A_4(-x, xo).
$$

(IV.3)

---

**Spin-zero Field**

Spin zero particles are described by a one-component field $\varphi(x)$ which is a scalar under rotations. It satisfies the Klein - Gordon equation

$$(\Box^2 - m^2)\varphi(x) = 0$$

which, together with the commutation relations, can be derived from

$$
L = -\left\{ \sum \frac{\partial \varphi^\dagger}{\partial x^\nu} \frac{\partial \varphi}{\partial x^\nu} + m^2 \varphi^\dagger \varphi \right\}.
$$

(IV.4)

This is for charged mesons. For neutral mesons

$$\varphi^\dagger = \varphi.$$  

(IV.5)

Under inversion, we distinguish two cases:

$$
\varphi'(x) = P \varphi(x) P^{-1} = \xi \varphi(-x, xo)
$$

(IV.6)

with $\xi = \pm 1$ corresponding respectively to [scalar] fields and particles of intrinsic parity $\pm 1$.

That only these two values of $\xi$ arise can be seen by applying inversion twice: this should correspond to the identity transformation, i.e.

$$P^2 \varphi(x) P^{-2} = \varphi(x)$$

but also

$$= \xi^2 \varphi(x)$$

whence

$$\xi^2 = 1, \quad \xi = \pm 1.$$
The invariance of the field equations and the commutation relations follows from that of $\mathcal{L}(x)$. One verifies at once from Eq. (IV.4) that

$$\mathcal{L}'(x) = P\mathcal{L}(x)P^{-1} = \mathcal{L}(-\bar{x},x_0).$$

This is the statement that $\mathcal{L}$ is invariant.

We consider next the Fourier decomposition of $\varphi(x)$ in terms of creation and absorption operators:

$$\varphi(x) = \frac{1}{\sqrt{V}} \sum \frac{1}{\sqrt{2k_0}} \left[ a(k)e^{ikx} + b^\dagger(k)e^{-ikx} \right]$$

(IV.7)

(sum over $k$

$k_0 = \sqrt{m^2 + k^2}$

creates antiparticles' of momentum $k$

absorbs particles of momentum $k$

(One can also identify particles and antiparticles with $\pi^-$ and $\pi^+$, respectively.)

Then at once

$$\varphi'(x) = \frac{1}{\sqrt{V}} \sum \frac{1}{\sqrt{2k_0}} \left[ Pa(k)P^{-1}e^{ikx} + Pb^\dagger(k)P^{-1}e^{-ikx} \right]$$

(IV.8a)

but also

$$\bar{\xi}\varphi(-\bar{x},x_0)$$

$$= \frac{\bar{\xi}}{\sqrt{V}} \sum \frac{1}{\sqrt{2k_0}} \left[ a(-k)e^{-i(kx-k_0x_0)} + b^\dagger(k)e^{-i(-kx-k_0x_0)} \right]$$

$$= \frac{\bar{\xi}}{\sqrt{V}} \sum \frac{1}{\sqrt{2k_0}} \left[ a(-k)e^{ikx} + b^\dagger(-k)e^{-ikx} \right].$$

(IV.8b)

Hence from Eqs. (IV.8 a) and (IV.8b)

$$a'(k) = Pa(k)P^{-1} = \bar{\xi}a(-k)$$

$$b^\dagger'(k) = Pb^\dagger(k)P^{-1} = \bar{\xi}b^\dagger(-k).$$

(IV.9)

Taking adjoints in Eq. (IV.9), or from $\varphi^\dagger$, one gets corresponding relations for $a^\dagger$ and $b$. These are just what one would expect.
In the 'inverted representation' a particle of momentum $\mathbf{k}$ appears as having momentum $-\mathbf{k}$, etc. To see this:

we postulate (convention)

$$\hat{\psi}_0^\dagger = P\hat{\psi}_0 = \hat{\psi}_0 : \text{(the vacuum has even parity) } \quad \text{(IV.10)}$$

then

$$\hat{a}^\dagger(\mathbf{k})\hat{\psi}_0 = Pa^\dagger(\mathbf{k})P^{-1}\hat{\psi}_0 = \xi a^\dagger(-\mathbf{k})\hat{\psi}_0 \quad \text{(IV.11)}$$

etc., i.e. a particle of momentum $\mathbf{k}$ -> particle of momentum $-\mathbf{k}$, with or without change of sign, according as $\xi = \pm 1$, i.e. intrinsic parity $\pm l$.

It is often convenient to decompose $\varphi$, not into plane-wave states but into ones of definite angular momentum, i.e. spherical waves:

$$\varphi(x) = \sum_{\ell \ell m} \frac{1}{\sqrt{2\ell + 1}} \left\{ a(k\ell m)f_\ell(kr)Y_\ell^m(\phi\theta) + b^\dagger(k\ell m)f_\ell^*(kr)Y_\ell^m(\phi\theta) \right\}$$

where $f_\ell(kr)$ are spherical cylinder functions (which kind depends on whether we expand in standing, incoming or outgoing waves). Here $b^\dagger(k\ell m)$ creates an antiparticle of energy $(m^2 + k^2)^{\frac{1}{2}}$ and angular momentum quantum numbers $\ell, m, etc.$

Using the fact that

$$x + x' = -x, \text{ now means } \theta, \phi \rightarrow \pi - \theta, \pi + \phi$$

and

$$Y_\ell^m(\pi - \theta, \phi + \pi) = (-1)^\ell Y_\ell^m(\phi\theta)$$

one proves, quite analogously to Eq. (IV.11) that

$$a'(k\ell m) = P(a(k\ell m)P^{-1} = \xi(-1)^\ell a(k\ell m). \quad \text{(IV.11')}$

Hence for one-particle states

$$P \left[ a^\dagger(k\ell m)\hat{\psi}_0 \right] = \xi(-1)^\ell \left[ a^\dagger(k\ell m)\hat{\psi}_0 \right]$$

i.e. $[a^\dagger(k\ell m)\hat{\psi}_0]$ is a parity eigenstate with eigenvalue $\xi(-1)^\ell$, again just what we expect.
Using this result and assuming parity conservation in $\pi-N^+$ interaction, the intrinsic $\pi^-$ parity is determined in the usual way from

$$\pi^- + D \rightarrow n + n,$$ giving $\xi = -1.$

\[
\begin{align*}
\text{s-wave capture: } & \text{ parity of initial state } = \xi, \\
\text{angular momentum of initial state } & J = 1, \text{ } \therefore \text{ final state is } 3P_1 \text{ with parity } = -1.
\end{align*}
\]

Dirac Field

Next, I want to consider spin $\frac{1}{2}$ particles, whose field $\psi$ satisfies the Dirac equation

$$\left( \gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \psi(x) = 0 \quad \text{(IV.12)}$$

and of course

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} \quad \text{(IV.13)}$$

The $\gamma$'s have to be (at least) $4 \times 4$ matrices and are, of course, not unique. There are roughly two types of representation which are most useful: those which have a simple non-relativistic limit (the Pauli representation), and those which display the symmetry properties of the theory most blatantly. For our purposes, it is obviously the latter kind we want and although this lecture course is not on calculational techniques, I want to deal with this briefly to show how trivial it all is. The current fashion is to avoid a particular representation altogether. This often only complicates the algebra unnecessarily and obscures the symmetries.

We demand two properties of our representation:

\[
\begin{align*}
1) \quad & \gamma_\mu^\dagger = \gamma_\mu \quad \mu = 1, \ldots 4 \quad \text{(IV.14)} \\
2) \quad & \gamma_k^* = \gamma_k \quad k = 1, \ldots 3 \\
& \gamma_4^* = -\gamma_4 \quad \text{ } \therefore \text{ } \{ \gamma_k \}_1^4 \text{ are } 2 \times 2 \text{ matrices.} \quad \text{(IV.15)}
\end{align*}
\]
[Notation: Unless otherwise stated, Latin indices 1 to 3, Greek 1 to 4. Also: $\dagger$ = hermitian adjoint = complex conjugate.transpose,
$*$ = complex conjugate
$\sim$ = transpose.]

Condition 2) singles out the Majorana representations.

This is all one has to remember. Everything else follows trivially.

$\gamma_5$ is defined by

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \quad (IV.16)$$

and, independently of representation, satisfies

$$\gamma_{\mu} \gamma_5 + \gamma_5 \gamma_{\mu} = 2 \delta_{\mu 5}. \quad (IV.13a)$$

From Eqs. (IV.14) and (IV.15)

$$\gamma_5^\dagger = \gamma_5 \quad (IV.14a)$$
$$\gamma_5^* = -\gamma_5 \quad (IV.15a)$$

It follows that

$$\tilde{\gamma}_{\mu} = \gamma_{\mu}^\dagger = \gamma_{\mu}^* \quad \mu = 1, \ldots, 5 \quad (IV.17)$$

(true for any hermitian operator) and in our representation

$$\tilde{\gamma}_k = \gamma_k^* = \gamma_k \quad k = 1, \ldots, 3 \quad (IV.18)$$
$$\tilde{\gamma}_\mu = \gamma_\mu^* = -\gamma_\mu \quad \mu = 4, 5$$

These simple relations between complex conjugate and transpose matrices, etc., make calculations easy, e.g. one verifies at once that

$$(\gamma_k \gamma_\mu)^\sim = \tilde{\gamma}_\mu \tilde{\gamma}_k = \gamma_k \gamma_\mu.$$  

(When $\mu = 4$ this is obvious; when $\mu \neq 4$ there is an extra minus sign which cancels in anticommuting.)
Let us now find the plane-wave solutions of the Dirac equation. For momentum $p$ and energy

$$p_0 = +\sqrt{M^2 + p^2}$$  \hfill (IV.19)

one gets two solutions. These one can choose as spin eigenstates with spin parallel or antiparallel to the momentum $p$:

$$\psi(r)(p) = u(r)(p) e^{ipx} \quad r = \begin{cases} 1, & \text{spin } \uparrow \text{ to } p, \\ 2, & \text{spin } \downarrow \text{ to } -p. \end{cases} \hfill (IV.20)$$

The labelling of the spin states in $u(r)(p)$ is always relative to the argument of the spinor. Thus for $u(r)(-p), u(1)(-p)$ is the state with spin pointing in the direction $-p$, $u(2)(-p)$ is the state with spin pointing in the direction $p$. Thus $r = 1$ is always a particle with right-hand circular polarization; $r = 2$, with left-hand circular polarization.

Equation (IV.20) into the Dirac equation (IV.12) gives

$$(i \gamma^\mu \gamma_5)u(r)(p) = 0$$  \hfill (IV.21)

$$(\gamma^\mu \gamma_5)^\mu.$$  \hfill (IV.22)

The spin matrix $\sigma$ is defined by

$$\sigma_k = i \gamma_4 \gamma_5 \gamma_k, \quad k = 1, 2, 3$$  \hfill (IV.23)

[In general

$$\sigma_{\mu\nu} = -\frac{1}{2}(\gamma_{\mu} \gamma_5 \gamma_{\nu} - \gamma_{\nu} \gamma_5 \gamma_{\mu}).$$

Furthermore, we note that

$$[\gamma_4, \sigma_k] = [\gamma_5, \sigma_k] = 0$$  \hfill (IV.24)

and from Eq. (IV.15)

$$\sigma_k^* = -\sigma_k.$$  \hfill (IV.25)
Hence if
\[
\sigma_{\mathbf{p}} = \frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|}
\]  
(IV.25)

then
\[
\sigma_{\mathbf{p}} u^{(r)}(\mathbf{p}) = \pm u^{(r)}(\mathbf{p}), \quad r = \left\{ \right\}
\]  
(IV.26)

since \( u^{(i)}(\mathbf{p}) \) has spin \( \pm \) to \( \mathbf{p} \), etc.

We take the norm as
\[
u^{(r)}(\mathbf{p}) u^{(r)}(\mathbf{p}) = \bar{\nu}(\mathbf{p}) \gamma_4 u^{(r)}(\mathbf{p}) = (p_0/M). \quad (IV.27)
\]
(This is a relativistically invariant norm.) In Eq. (IV.27) we put
\[
\bar{\nu} = u^{\dagger} \gamma_4
\]  
(IV.28)

and similarly for \( \bar{\nu} \), etc.

This determines the eigensolutions \( u^{(i)}(\mathbf{p}) \), \( u^{(r)}(\mathbf{p}) \) except for phase factors, which I leave open for the moment.

I now want to show how one very easily derives the other plane-wave solutions for a given momentum \( \mathbf{p} \) from \( u^{(i)}(\mathbf{p}) \).

I assert
\[
u^{(r)}(-\mathbf{p}) = \gamma_4 u^{(i)}(\mathbf{p}) \quad (IV.29)
\]

Indeed, premultiplying the Dirac equation [Eq. (IV.21)] by \( \gamma_4 \) and pulling the \( \gamma_4 \) through to the right-hand side

\[
\gamma_4(\imath \mathbf{p} + \mathbf{M}) u^{(i)}(\mathbf{p}) = (-\imath \gamma_4 p_0 + \imath \gamma_4 \mathbf{p_4} + \mathbf{M}) \gamma_4 u^{(i)}(\mathbf{p}) = 0
\]

i.e. \( \gamma_4 u^{(i)}(\mathbf{p}) \) is an eigenfunction with momentum \( (-\mathbf{p}) \).
Similarly on account of Eq. (IV.23)

$$\gamma_4 \sigma_\mathbf{p} \, u^{(1)}(\mathbf{p}) = \sigma_\mathbf{p} \gamma_4 \, u^{(1)}(\mathbf{p})$$

i.e. $\gamma_4 \, u^{(1)}(\mathbf{p})$ has spin $\parallel$ to $+\mathbf{p}$.

Finally, $\gamma_4 \, u^{(1)}(\mathbf{p})$ is correctly normed. Hence Eq. (IV.29) only corresponds to a particular choice of phase factor for $u^{(2)}$, in terms of that for $u^{(1)}$ (which is still left open). Thus we have

$$u^{(1)}(\mathbf{p}) = \gamma_4 \, u^{(1)}(-\mathbf{p}) \quad (IV.29a)$$

where

$$\mathbf{r} = \begin{cases} ^2 & \text{if } r = ^1 \\ ^1 & \text{if } r = ^2 \end{cases} \quad (IV.30)$$

We see $\gamma_4$ is obviously connected with inversion of axes: $\mathbf{p} \rightarrow -\mathbf{p}$, but the spin (an axial vector like $\mathbf{r} \wedge \mathbf{p}$) is not altered. This is what $r = 1 \rightarrow \mathbf{r} = 2$ means in Eq. (IV.29a), e.g. for $r = 1$ both sides of Eq. (IV.29a) state: spin $\parallel$ to $-\mathbf{p}$.

The beauty of the Majorana representation is that it gives the negative energy solutions at once. This is because in going from particles to antiparticles it does not screw up the spins in some arbitrary manner. This is responsible for its great symmetry and ease of interpretation.

This is due to the fact that in the Majorana representation

$$\left( \gamma_\mu \frac{\partial}{\partial x_\mu} + \mathbf{H} \right) \text{ and } \mathbf{H}$$

are real. Hence it follows at once that

$$\left( \gamma_\mu \frac{\partial}{\partial x_\mu} + \mathbf{H} \right) u^{(r)}(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} = 0 \quad (IV.31)$$

or

$$(-i\mathbf{p} + \mathbf{H}) \, u^{(r)}(\mathbf{p}) = 0 \quad (IV.32)$$

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i.e. $u^{(r)*}(\bar{p})$ is an eigenfunction with momentum $-\bar{p}$, energy $-p_0$.

From Eq. (IV.26), viz.

$$\sigma_\bar{p} u^{(r)}(\bar{p}) = \pm u^{(r)}(\bar{p}) \quad r = \begin{cases} 1 \\ 2 \end{cases}$$

since [Eq. (IV. 24)]

$$\sigma^{*}_\bar{p} = - \sigma_\bar{p}$$

$$\sigma_\bar{p} u^{(r)*}(\bar{p}) = \mp u^{(r)*}(\bar{p}) \quad r = \begin{cases} 1 \\ 2 \end{cases}$$

(i.e. $u^{(r)*}(\bar{p})$ has spin $|\bar{p}|^2$ to $\mp \bar{p}$ respectively.

Also the norm of $u^{(r)*}(\bar{p})$ is the same as of $u^{(r)}(\bar{p})$.

Now obviously, there is another way of changing $p_\mu \rightarrow -p_\mu$ in Eq. (IV.21), viz

$$(i\not{\bar{p}} + M)u^{(r)}(\bar{p}) = 0.$$ 

This is what we did to get Eq. (IV. 32). Clearly the operator we now need is $\gamma_5$: this anticommutes with all four $\gamma_\mu$ in the Dirac equation. It is obviously related to inversion ($x \rightarrow -x$) together with time reversal ($t \rightarrow -t$). So consider

$$\gamma_5 u^{(r)}(\bar{p}).$$

Directly from the Dirac equation IV.21

$$(-i\not{\bar{p}} + M)\gamma_5 u^{(r)}(\bar{p}) = 0 \quad (IV.32 \uparrow)$$

and from Eq. (IV.23) (i.e. $[\gamma_5, \sigma_\bar{p}] = 0$)

$$\sigma_\bar{p} \gamma_5 u^{(r)}(\bar{p}) = \pm \gamma_5 u^{(r)}(\bar{p}) \quad (IV.33 \uparrow)$$

and $\gamma_5 u^{(r)}(\bar{p})$ has the same norm as $u^{(r)}(\bar{p})$. 

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Hence, apart from a phase factor, \( u^{(r)*}(\bar{\rho}) \) is the same as \( \gamma_5 u(\bar{\rho}) \), i.e.

\[
 u^{(r)*}(\bar{\rho}) = e^{i\delta} \gamma_5 u(\bar{\rho}) \tag{IV.34}
\]

with \( \delta \) fixed, but so far not determined. Now I can make \( \delta \) anything I like, according to how the phase of \( u^{(1)}(\rho) \) is originally chosen. For if I replace \( u^{(1)} \) by \( u^{(1)}e^{i\alpha} \), then Eq. (IV.34) simply becomes

\[
 u^{(r)*}(\bar{\rho}) = e^{i(\delta + 2\alpha)} \gamma_5 u(\bar{\rho}).
\]

I have taken \( \delta = \pi \) (accidental but convenient) so that finally Eq. (IV.34) becomes

\[
 u^{(r)*}(\bar{\rho}) = -\gamma_5 u(\bar{\rho}). \tag{IV.35}
\]

Finally, although I shall never use it, one particular Majorana representation is:

\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}. \tag{IV.36}
\]

After this digression on the Majorana representation, we go back to our main problem: the discussion of the spinor field transformation under inversion. The Dirac equation (IV.12) can be derived from the Lagrangian

\[
 \mathcal{L}(x) = -\bar{\psi}(x) \left\{ \gamma_\mu \frac{\partial}{\partial x_\mu} + \mathcal{M} \right\} \psi(x). \tag{IV.37}
\]

We now want the invariance of \( \mathcal{L} \) under inversion, i.e.

\[
 \mathcal{L}'(x) = P \mathcal{L}(x) P^{-1} = \mathcal{L}(-x^0, \mathbf{x}_0). \tag{IV.38}
\]

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Try
\[
\psi'(x) = P\psi(x)P^{-1} = \xi\psi(-\bar{x}, x_0) \\
\psi^+(x) = P\psi^+(x)P^{-1} = \xi^*\psi^+(-\bar{x}, x_0)S^+ 
\]
\text{(IV.39)}

where \(\xi\) is a complex number and \(S\) a \(4 \times 4\) matrix. The phase factors \(\xi\) which were introduced in defining the inverted fields [cf. Eqs. (IV.6) and (IV.39)] are, of course, not necessarily the same for different fields. Then directly:
\[
P\mathcal{L}(x)P^{-1} = -\xi^*\psi^+(-\bar{x}, x_0)S^+\left[\gamma_4 \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)\right] \xi\psi(-\bar{x}, x_0)
\]
and this we require =
\[
\mathcal{L}(-\bar{x}, x_0) = -\psi^+(-\bar{x}, x_0) \left[\gamma_4 \left(\gamma_k \frac{\partial}{\partial x_k} + \gamma_4 \frac{\partial}{\partial x_4} + M\right)\right] \psi(-\bar{x}, x_0).
\]
This is achieved if
\[
|\xi|^2 = 1 \quad S^+\gamma_4\gamma_\mu S = \begin{cases} 
-\gamma_4\gamma_\mu & \mu = 1, 2, 3 \\
\gamma_4\gamma_\mu & \mu = 4
\end{cases}.
\]
This is satisfied by
\[
S = \gamma_4.
\]
[This is really obvious from the Dirac equation \((\gamma_\mu \frac{\partial}{\partial x_\mu} + M)\psi = 0\). To change \(x \rightarrow -\bar{x}\), we need an operator which anticommutes with \(\gamma_k\) and commutes with \(\gamma_4\), i.e., \(\gamma_4\). Then
\[
\gamma_4 \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)\psi = 0
\]
and commuting \(\gamma_4\) through to the right, the result follows.]

Hence, the correct final transformation is
\[
\psi'(x) = P\psi(x)P^{-1} = \xi\gamma_4\psi(-\bar{x}, x_0) \\
\psi^+(x) = P\psi^+(x)P^{-1} = \xi^*\psi^+(-\bar{x}, x_0)\gamma_4
\]
\[
|\xi|^2 = 1.
\]
\text{(IV.40)}
We can apply inversion twice:

\[ P^2 \psi(x) P^{-2} = \xi^2 \psi(x). \quad (IV.41) \]

Since this gets us back to the original coordinate system, it must be simply the identity transformation. But it would be wrong to put Eq. (IV.41) equal to \( \psi(x) \). The reason is that the identity transformation for a Dirac 4-spinor is not simply the \( 4 \times 4 \) unit matrix \( I_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Rather it is not unique, but is double-valued and given by \( \pm I_4 \). This is connected with the fact that under a rotation of axes through 360° around any axis a Dirac spinor is multiplied by (−1). However, a rotation through 360° must be the same as no rotation; hence, Dirac spinors are necessarily double-valued, corresponding to a factor (±1). (The way this comes about can already be seen from the non-relativistic limit. Under rotations, Pauli 2-component spinors transform involving the "half-angles" of rotation only.)

It now follows that we can only conclude from Eq. (IV.41) that

\[ \xi^2 = \pm 1 \]

i.e.

\[ \xi = +1, -1, \ i, -i. \quad (IV.42) \]

Equation (IV.42) looks as though there are four types of intrinsic parity. From the above discussion, this is clearly not so. Because of the double-valuedness, the intrinsic parity of a Dirac particle is only determined up to sign: ±1. Thus we get two groups: \( \xi = \pm 1 \) and \( \xi = \pm i \), which are always distinguishable. On the other hand, if one has two different spinor fields, they always have a definite relative parity.

I should next like to look briefly at the Fourier decomposition of the spinor fields:
\[ \psi(x) = \frac{1}{\sqrt{V}} \sum_{p} \left( \frac{M}{p_0} \right)^{\frac{1}{2}} \sum_{r=1}^{2} \left\{ c_r(p) u(r)(p)e^{ipx} + d^+_r(p) u(r)^*(p)e^{-ipx} \right\} \]

and

\[ \psi^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{p} \left( \frac{M}{p_0} \right)^{\frac{1}{2}} \sum_{r=1}^{2} \left\{ c^+_r(p) u(r)^*(p)e^{-ipx} + d_r(p) u(r)(p)e^{ipx} \right\} \]  

Note: These equations are completely symmetric between particles and antiparticles. This is the beauty of the Majorana representation.

In Eq. (IV.43)

\[ p_0 = +\sqrt{\gamma^2 + p^2} \]

and

\[ c_r(p) \text{ absorb particles of momentum } p, \text{ spin } |^1 \text{ to } |^0; \ r = \begin{cases} 1 \\ 2 \end{cases} \]

\[ c^+_r(p) \text{ create } \]

\[ d_r \text{ and } d^+_r \text{ act similarly for antiparticles.} \]

Now as before

\[ \psi'(x) = \frac{1}{\sqrt{V}} \sum_{p} \left( \frac{M}{p_0} \right)^{\frac{1}{2}} \sum_{r=1}^{2} \left\{ p_{0r}(p) \gamma_{r} u(r)(p)e^{ipx} + p_{0r}^+(p) \gamma_{r} u(r)^*(p)e^{-ipx} \right\} \]

\[ = \frac{\xi}{\sqrt{V}} \gamma \psi(\bar{p}, x_0) \]

\[ = \frac{\xi}{\sqrt{V}} \sum_{p} \left( \frac{M}{p_0} \right)^{\frac{1}{2}} \sum_{r=1}^{2} \left\{ c^+_{\bar{r}}(p) \gamma_{\bar{r}} u(r)(p)e^{ipx} + d_{\bar{r}}(p) \gamma_{\bar{r}} u(r)^*(p)e^{-ipx} \right\} \]

(IV.44)

Here I have straight away replaced \( p \) by \( \bar{p} \) as summation variable.

Let

\[ \bar{r} = \begin{cases} 1 & \text{if } r = 1 \\ 2 & \text{if } r = 2 \end{cases} \]
then we had in Eq. (IV.29a) that

$$u^\prime(\vec{r}) = \gamma_4 u^\prime(-\vec{p}) \quad \text{(IV.29a)}$$

Taking the complex conjugate, since in the Majorana representation $\gamma_4$ is purely imaginary,

$$u^\prime(\vec{r})^* = -\gamma_4 u^\prime(-\vec{p}) \quad \text{(IV.45)}$$

Substituting these equations in Eq. (IV.44) and comparing coefficients

$$c_\tau' \vec{p} = P^\tau \vec{p} P^{-1} = \xi c_\tau(-\vec{p})$$

whence

$$c_\tau^\dagger' \vec{p} = P^\tau \vec{p} P^{-1} = \xi^* c_\tau^\dagger(-\vec{p})$$

and similarly

$$d_\tau' \vec{p} = -\xi^* d_\tau(-\vec{p})$$

whence

$$d_\tau^\dagger' \vec{p} = -\xi d_\tau(-\vec{p})$$

These are just the results we expect: under inversion $\vec{p} \to -\vec{p}$ (vector) but the spin stays the same (axial vector), $\vec{r} \to -\vec{r}$.

We also see that the antiparticle has opposite intrinsic parity to the particle. This is for fermions; for bosons they were the same.

Let us briefly consider the one-particle states again with

$$\Psi_0' = P\Psi_0 = \Psi_0$$

$$c_\tau^\dagger' (\vec{p}) \Psi_0' = \xi^* c_\tau^\dagger(-\vec{p}) \Psi_0 \quad d_\tau^\dagger' (\vec{p}) \Psi_0' = -\xi^* d_\tau^\dagger(-\vec{p}) \Psi_0 \quad \text{(IV.47)}$$

Again as expected: $\vec{p} \to -\vec{p}$; $\vec{r} \to +\vec{r}$; parity of particle = - that of antiparticle.
The above basic results, i.e., the transformation properties of fields of various kinds, enable us to apply parity considerations to all questions. I will briefly mention some typical applications.

**Construction of Parity-Conserving Interactions**

Let us consider a Yukawa interaction

\[ \bar{\psi}_A \gamma_4 \psi_B \]

Here \( \gamma_i \) denote the various combinations of \( \gamma \)-matrices which make up Lorentz covariants, i.e., scalar, vector, etc. They are written out in the table below. Now,

\[ P \bar{\psi}_A(x)(\gamma_4 \gamma_i)\psi_B P^{-1} = \bar{\psi}_A \gamma_i \psi_B(-\chi, x_0) \gamma_4 (\gamma_i \gamma_4) \psi_B(-\chi, x_0) \]

\[ = \bar{\psi}_A \gamma_i \psi_B(-\chi, x_0) \gamma_4 \gamma_i \psi_B(-\chi, x_0) \]

where

<table>
<thead>
<tr>
<th>( \gamma_i )</th>
<th>( \gamma_4 \gamma_i \gamma_4 )</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\psi}_A )</td>
<td>( \gamma_4 \gamma_i \psi_B )</td>
<td>scalar</td>
</tr>
<tr>
<td>( \bar{\psi}_A \gamma_i \psi_B )</td>
<td>( \gamma_4 \gamma_i \gamma_4 \psi_B )</td>
<td>vector</td>
</tr>
<tr>
<td>( \sigma_{\mu\nu} = \frac{-i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) )</td>
<td>( \sigma_{\mu\nu} )</td>
<td>tensor</td>
</tr>
<tr>
<td>( \bar{\psi}_A \gamma_i \gamma_5 \psi_B )</td>
<td>( \gamma_4 \gamma_i \gamma_5 \psi_B )</td>
<td>pseudo-vector</td>
</tr>
<tr>
<td>( \bar{\psi}_A \gamma_5 \psi_B )</td>
<td>( -\gamma_4 \gamma_5 \psi_B )</td>
<td>pseudo-scalar</td>
</tr>
</tbody>
</table>

From Eqs. (IV.48) and (IV.49) and the table, we see at once how \( P_4 \) must transform for a true scalar interaction.
If $A = B$ (same particle), which is usually the case, then

$$\xi_A^* \xi_B^* = |\xi_B|^2 = 1,$$

e.g., in electrodynamics

is $\bar{\psi} \gamma_{\mu} A_{\mu} \psi$ scalar.

(We saw that $A_{\mu} \rightarrow \left\{ \begin{array}{ll} A_{\mu}, & \mu \neq 4 \\ A_4, & \mu = 4 \end{array} \right\}$ as required.)

Similarly, for the Yukawa ps(ps) $\pi - \chi'$ interaction

$$\mathcal{L} = -ig_1 \bar{\psi}_n \gamma_{\mu} \psi_\pi \gamma^\dagger \psi_p - ig_1 \bar{\psi}_p \gamma_{\mu} \psi_\pi \gamma^\dagger \psi_n - ig_2 \bar{\psi}_p \gamma_{\mu} \gamma_5 \psi_\pi \gamma^\dagger \psi_n - ig_3 \bar{\psi}_n \gamma_{\mu} \gamma_5 \psi_\pi \gamma^\dagger \psi_n$$

$\pi^+ - \chi'$ $\pi^- - \chi'$ $\pi^0 - p$ $\pi^0 - n$

From either of the last two terms, inversion invariance $\rightarrow$

$$\xi_{\pi^0} = -1.$$ They only involve $|\xi_\pi|^2 = |\xi_p|^2 = 1$, and therefore tell you nothing about $\xi_n$ and $\xi_p$.

The first two terms involve

$$\xi_n^* \xi_p \xi_{\pi^\pm} = -1.$$ This shows what I mean when I say that for spinors only relative parity matters. They always occur bilinearly (essentially because for Lorentz-invariance one needs scalars, vectors, etc.). Hence we can, by convention, take

$$\xi_p = +1.$$ Then, is everything else fixed? Only

$$\xi_n \xi_{\pi^\pm} = -1.$$ We cannot go further without some additional knowledge, e.g. charge-independence, which is most valuable in restricting possibilities.

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Now $\pi^+$ and $\pi^0$ are different charge states of the same particle (field). Hence they have the same parity. The same also applies for the N-P system. Hence

$$\xi_{\pi^+} = \xi_0 \quad \xi_P = \xi_N.$$ 

Of course, from the Yukawa interaction, either implies the other.

The situation becomes much more complex if one has four Dirac fields, as in $\beta$-decay. Parity is not conserved. Hence many more interactions become possible. However, I will not discuss these but only mention a matter of principle.

To what extent is parity defined, if not conserved?

Answer: Parity-violating interactions are very weak. Hence we can imagine them switched off; then parity is valid, i.e. accurate to a very high degree of approximation. That is to say, strong and electromagnetic interactions only must be invariant under inversion; only these may be used to determine the intrinsic parities of particles. On the other hand, there is no way of ascribing parity to a neutrino which takes part only in weak interactions.
LECTURE V

CHARGE CONJUGATION

When discussing parity, I considered the quantum mechanics aspects only. Indeed there is no corresponding conservation law in classical physics, although corresponding symmetry exists, e.g. the distinction between polar and axial vectors: \( \mathbf{E} \) and \( \mathbf{H} \).

Now I want to discuss a symmetry which is quite different. Firstly, it seems to have no analogue in classical physics and, secondly, it is not connected with space-time: or only very subtly as through the spin-statistics connection of relativity and quantum mechanics and through the TCP theorem. This new symmetry is **charge conjugation**.

Roughly speaking, in nature there seems to be a very high degree of symmetry between positive and negative charge. For every particle with certain electromagnetic properties, there exists an antiparticle identical to the particle, except that the electromagnetic properties have been reversed in sign. Electron-positron is the first and best example, but there are many others: \( \pi^+ \) and \( \pi^- \), \( \rho \) and \( \rho^- \), and \( N \) and \( \bar{N} \) (opposite magnetic moments). The definition can be generalized to get away from electromagnetic properties altogether.

The starting point of all this, as I said, was the Dirac theory of the electron. As originally formulated in terms of negative energy states, the positron appeared as a hole in a filled sea of negative energy state electrons. These were unobservable; they produced no observable effects. Hence their infinite charge, energy, etc. (vacuum charge, etc.) had to be subtracted out to get the finite observable effects, which are quite symmetric between electrons and positrons.

One can avoid this devious route and get the same result more directly as follows. Previously [Eq. (IV.37)] I wrote the Dirac Lagrangian

\[
\mathcal{L} = - \psi^+ \gamma_\mu \left( \gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \psi.
\]

(V.1)
Now before quantization $\phi$ and $\phi^\dagger$ are c-numbers; they are not operators. Hence in going to the quantized field theory, the usual ambiguity about order of factor appears. We could equally well write instead of Eq. (V.1)

$$\mathcal{L} = -\psi Y_\mu \left( \gamma_\mu \frac{\partial}{\partial x_\mu} + M \right) \phi^\dagger,$$

or some linear combination of the two.

Note: I am using the Majorana representation always. In this case we saw that $\psi$ and $\phi^\dagger$ are quite symmetric in particles and antiparticles; $\phi \leftrightarrow \phi^\dagger$ interchanges them. Hence

$$\mathcal{L} = -\frac{i}{2} \left\{ \phi^\dagger Y_\mu \left( \gamma_\mu \frac{\partial}{\partial x_\mu} + M \right) \phi + \psi Y_\mu \left( \gamma_\mu \frac{\partial}{\partial x_\mu} + M \right) \phi^\dagger \right\} \quad (V.1b)$$

will be symmetric in particles and antiparticles, as also will be all consequences derived from it.

Now some caution is required at this point. We know that nature is not completely symmetric with respect to interchange of particles and antiparticles, just as parity is not conserved in weak interactions. So the question arises whether it is reasonable to make things too symmetric. The answer is: for the free fields one does want this symmetry. The Lagrangian Eq. (V.1b) is exactly the one which automatically eliminates infinite vacuum charges and energies. For example, Eq. (V.1b) automatically using the usual definition of current in terms of $\mathcal{L}$ (cf. later!) leads to the current density vector

$$-\varepsilon_{\nu}(x) = -\frac{i\varepsilon}{2} \left( \bar{\psi} \gamma_\nu \psi - \psi \gamma_\nu \bar{\psi} \right) = -\frac{i\varepsilon}{2} \left[ \bar{\psi} \gamma_\nu \psi \right]$$

(V.2)

instead of simply $-i\varepsilon \bar{\psi} \gamma_\nu \psi$, (I am taking $\varepsilon > 0$. Then this is the electron current.) Eq. (V.2), you will remember, is just what one gets if one subtracts out the vacuum charges. For the total charge of the system it gives

$$Q = \varepsilon \sum \sum \frac{2}{p} \left[ N^+(p) - N^-(p) \right] \left[ \begin{array}{c} \text{No. of electrons} \\ \text{No. of positrons} \end{array} \right] \text{momentum } p \text{ spin-state } r$$
Hence we shall always use symmetrized free-field Lagrangians. Any deviations from charge conjugation invariance must come from violations in the interactions.

We now want a formalism for describing interchange of particles and antiparticles, which is known as particle-antiparticle conjugation and also as charge-conjugation (although not a good name, as we shall extend it to neutral particles). The procedure is quite analogous to what we used in the case of parity. So I will be brief and omit some intermediate steps.

We introduce a unitary operator $C$ by these properties: ($C$ = the charge conjugation operator or particle-antiparticle operator):

\[ C \xi_c^c = C \bar{\xi}_o = \bar{\xi}_c \]  
\[ C \xi_r^c(p)C^{-1} = \xi_c \bar{d}_r(p) \]
\[ C \bar{d}_r^c(p)C^{-1} = \xi_c^* \bar{c}_r(p) \]

whence, taking adjoints

\[ C \xi_r^c(p)C^{-1} = \xi_c^* \bar{d}_r^c(p) \]
\[ C \bar{d}_r(p)C^{-1} = \xi_c^* \bar{c}_r(p) \]

For the one-particle states this means:

\[ C \xi_r^c(p)\bar{\xi}_o = C \xi_r^c(p)C^{-1} \bar{\xi}_o^c = \xi_c^* \bar{d}_r^c(p)\bar{\xi}_o \]

\[ C \bar{d}_r^c(p)C^{-1} \bar{\xi}_o^c = \xi_c \bar{c}_r^c(p)\bar{\xi}_o \]

i.e. this is just charge-conjugation; for the fields this means

\[ \psi_c^c = C\psi^c C^{-1} = \xi_c^* \psi^{\dagger} \]
\[ \psi_r^{\dagger} = C\psi^{\dagger} C^{-1} = \xi_c \psi \]
Now again performing charge conjugation twice, particles $\rightarrow$ particles, etc., so that

$$C^2 \psi C^{-2} = C \left[ \xi_0 \psi^\dagger \right] C^{-1} = |\xi_0|^2 \psi$$

but also $= \psi$ (no change: identity)

$$\therefore |\xi_0|^2 = 1 \quad \text{(V.7)}$$

This is a restriction on $\xi_0$.

One proceeds in exactly the same way for bosons represented by a non-hermitian field $\varphi$ and its adjoint $\varphi^\dagger$:

$$C a(\mathbf{k}) C^{-1} = \xi_0 b(\mathbf{k}) \quad ,$$
$$C b^\dagger(\mathbf{k}) C^{-1} = \xi_0 a^\dagger(\mathbf{k}) \quad ,$$

etc.,

$$\varphi_0 = C \varphi C^{-1} = \xi_0 \varphi^\dagger \quad , \quad \text{etc.} \quad \text{(V.9)}$$

Of course, $\xi_0$ is different for each field.

For $\mathcal{L}$ to be invariant under charge conjugation, we now also symmetrize it

$$\mathcal{L} = -\frac{1}{2} \left[ \frac{\partial \varphi}{\partial x_{\mu}} , \frac{\partial \varphi^\dagger}{\partial x_{\mu}} \right] + \frac{m^2}{2} \left[ \varphi, \varphi^\dagger \right] \quad \text{.....} \quad \text{(V.9a)}$$

This formulation is independent of charge. It depends wholly on the non-self-adjointness of fields: the expansion involves two kinds of particles, $a$ and $b^\dagger$, or $c$ and $d^\dagger$, etc. Thus it is at once applicable to neutrons.

A special case arises for self-adjoint fields. For example

$$\varphi^\dagger = \varphi \quad \text{(V.10)}$$
In this case
\[ C \phi C^{-1} = \xi_0 \phi \]  \hspace{1cm} (V.11)

and one gets directly [as in the derivation of Eq. (V.7)] that
\[ \xi_0^2 = 1 \, , \, \xi_0 = \pm 1 \quad \text{real} \, . \]  \hspace{1cm} (V.12)

Thus a neutral hermitian field is self-charge-conjugate: particle = antiparticle. An example of such a particle is the \( \pi^0 \) meson. We still have a choice of sign in Eq. (V.11): to get the usual Yukawa \( \pi - K \) interaction charge conjugation invariant, we must take \( \xi_0 = +1 \). Then \( (\psi = \pi^0 \text{ field}) \)
\[ C \psi_3 C^{-1} = \tau_3 \]  \hspace{1cm} (V.13)

For Dirac particles too, one can define a self-adjoint spinor field \( \psi = \psi^\dagger \). This is particularly transparent in the Majorana representation. It is simply \( c_\tau = d_\tau \). This is the Majorana neutrino, for example, it is equal to its charge conjugate particle, the anti-neutrino. One can experimentally distinguish between the two cases. It is found that \( \nu \) is not self-conjugate: there seems to be no Majorana fermions in nature\(^*\).

---

\( ^* \) Evidence:

a) \( n + \nu \rightarrow p^+ + e^- \) is observed; \( n + \bar{\nu} \rightarrow p^+ + e^- \) is not.

b) double \( \beta \) decay is not observed; this is the process
\[ n + n \rightarrow (p^+ + e^- + \bar{\nu}) + n \rightarrow p^+ + e^- + p^+ + e^- : \text{no neutrino is emitted}. \]

Ordinary 'Dirac' \( \beta \) decay has the inverse \( n + \nu \rightarrow p^+ + e^- \) only. It shows up in lifetime measurements. The Majorana neutrinos theory predicts lifetimes of about \( 10^{16} \) years. The experimentally determined lifetimes for double \( \beta \) decay are at least \( 8 \times 10^{16} \) years (this is for decays with 'short' lifetimes) in contradiction to the predictions of the Majorana theory of the neutrino.

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Applications

We now turn to some applications. Looking at interactions, we want to see how the five basic spinor covariants transform under charge conjugation.

Let us take electrodynamics as a typical example to show that after all charge conjugation does concern charge; the interaction [cf. Eq. (V.2)] is

\[-e S_y(x)A_y(x) = -\frac{i e}{2} \left[ \bar{\psi}_y \gamma_y \psi \right] A_y(x). \tag{V.14}\]

Remember: we must always use interactions derived from the symmetrized Lagrangian, Eq. (V.1b), i.e. Eq. (V.14) in this case. Now in the Majorana representation

\[\bar{\gamma}_y \gamma_4 = \gamma_4 \gamma_y. \tag{V.15}\]

Hence

\[\left[ \bar{\psi}, \gamma_y \psi \right] = \bar{\psi} \gamma_y \psi - (\gamma_y \psi) \bar{\psi}\]

\[= \psi^\dagger \gamma_y \gamma_y \psi - \psi \bar{\gamma}_y \gamma_4 \psi^\dagger\]

\[= \psi^\dagger \gamma_4 \gamma_4 \psi - \psi \gamma_4 \gamma_4 \psi^\dagger\]

and under charge conjugation

\[C \left[ \bar{\psi}, \gamma_y \psi \right] C^{-1} = |\xi_0|^2 \left\{ \bar{\psi} \gamma_y \gamma_4 \psi^\dagger - \psi^\dagger \gamma_4 \gamma_y \psi \right\} = - \left[ \bar{\psi}, \gamma_y \psi \right]. \tag{V.16}\]

since

\[|\xi_0|^2 = 1.\]

Thus for Eq. (V.14) to be invariant under charge conjugation, we require

\[C A_y(x) C^{-1} = - A_y(x). \tag{V.17}\]
If we call the $\xi_c$, which has been introduced above, the charge conjugation parity of the field, then Eq. (V.17) means that the photon field has negative charge conjugation parity.

In the same way, one can deal with all other cases. For example

$$C \left[ \Phi_A, \psi_B \right] C^{-1} = \xi_A^* \xi_B \left[ \Phi_B, \psi_A \right].$$

For neutral mesons one then requires for charge conjugation invariance of

$$\left[ \Phi_N, \psi_N \right] \psi_S,$$

since $|\xi_N|^2 = 1$, that

$$\xi_{\pi^*} = +1.$$ (V.18)

For charged mesons one gets similarly

$$\xi_P^* \xi_N \pi^\pm = 1.$$ (V.18a)

Again for charge independence we require $\xi_P = \xi_N$, giving $\xi_P^* \xi_N = 1$, whence $\xi_{\pi^\pm} = 1$, as indeed we also require by charge independence (since $\xi_{\pi^*} = 1$).

We obtain a restriction on permissible parity transformations of spinors, if we demand that the charge conjugate spinor

$$\psi_c = C \psi C^{-1} \quad \text{and} \quad \psi$$

should behave in the same way under inversion. We have

$$\psi_c(x) = \xi_c \psi^\dagger(x) \quad \quad P \psi(x) P^{-1} = \xi_P \psi(-x, x_0)$$

$$\psi^\dagger_c(x) = \xi_c^* \psi(x) \quad \quad \xi_P = \pm 1, \pm i.$$ (V.19)
Hence

\[ P \psi^+_c(x) P^{-1} = \xi_c^* P \psi^+(x) P^{-1} \]

\[ = \xi_c^* \xi_F^* \psi^+(-\Sigma_c, x_c) \gamma_4 \]

\[ = \xi_F^* \gamma_4 \xi_c^* \psi^+(-\Sigma_c, x_c) \]

\[ = -\xi_F^* \gamma_4 \psi_c(-\Sigma_c, x_c) \]  \hspace{1cm} (V.20)

since \[ \gamma_4 = -\gamma_4 \].

Comparing Eqs. (V.19) and (V.20), we see that these transform in the same way, if \( \xi_F = -\xi_F^* \), i.e. \( \xi_F = \pm i \) only. This restriction would rule out spinors of the \( \pm 1 \) class.

In the case of the Fermi-interaction (4 spinor fields) the situation becomes more complicated. But in this case one knows that charge conjugation invariance does not hold, so it is not very fruitful to look for invariant interactions.

That charge conjugation invariance does not hold for weak interactions can best be seen from the \( \mu^+ \) decays. The \( e^+ \) have opposite circular polarization; if charge conjugation invariance held, these would have to be the same, since charge conjugation transforms

\[ (\mu^+ \rightarrow e^+ + \nu + \bar{\nu}) \rightarrow (\mu^- \rightarrow e^- + \bar{\nu} + \nu). \]

[See Eq. (V.5)].

For \( \beta \)-decay the charge conjugation process would involve antinucleons. Hence charge conjugation non-invariance is more difficult to test here. That it can be checked depends on the TCP theorem (v.i.), whence C is equivalent to TP: so one can, after all, check C by checking T and P without going to antiparticles. A theorem of Lee, Yang and Oehme then shows that, neglecting Coulomb effects, certain experiments on \( \beta \)-decay would
not show up P-violation if PT holds. Since P-violation is observed in such experiments and the effect is too big to be explained by Coulomb effects, PT-violation, i.e. C-violation in β-decay, is established.

Furry's Theorem

A different application is Furry's theorem. A Feynman graph with only an odd number of photon lines as external lines vanishes. In perturbation theory one sees this easily enough, e.g.

\[ \text{and} \quad \text{give} \]

opposite contributions: in intermediate states: \( e \leftrightarrow e^+ \). If the first graph is \( a \, e^3 \), the second graph is \( a(-e)^3 \) but otherwise the same. Thus the sum of the two terms vanishes.

However, now we can see this generally. The graph

is represented by

\[
G(k_1, \ldots, k_n) = \int \, d^4x_1 \cdots d^4x_n \, e^{-i(k_1x_1 + \ldots + k_nx_n)} \, \langle \phi_0, T(A_{\lambda_1}(x_1) \cdots A_{\lambda_n}(x_n)) \phi_0 \rangle.
\]

(V.21)

Apply charge conjugation: if the theory is invariant under charge conjugation

\[
CG(k_1, \ldots) C^{-1} = G(k_1, \ldots).
\]

(V.22)

But from Eq. (V.17), viz.

\[
CA_{\lambda} C^{-1} = -A_{\lambda},
\]
also

\[ C G(k_1, \ldots) C^{-1} = (-1)^n G(k_1, \ldots) \]

(V.22a)

Hence

\[ G(k_1, \ldots) = 0 \quad \text{if} \quad n = \text{odd} \]

The same argument can be applied to other processes. For example, C-invariance forbids

\[ n^2 \rightarrow \text{odd number of photons}, \]

since \( C \varphi_\gamma C^{-1} = \varphi_\gamma \) [Eq. (V.13) above].

**Related Processes**

Yet another application follows for a C-invariant theory: in this case the S-matrix must be invariant, i.e.

\[ CSC^{-1} = S \]

and so

\[ (\bar{\Phi}_b, S\bar{\Phi}_a) = (\bar{\Phi}_b', S\bar{\Phi}_a') \]

where \( \Phi_a' = C^{-1}\Phi_a \) is the state charge conjugate to \( \Phi_a \), i.e. the same as \( \Phi_a \), with all particles replaced by their antiparticles. Thus we get the same transition amplitude for related processes. For example the following interactions and scattering are the same

\[ P\bar{N} = \bar{P}N \]

\[ p + p \rightarrow p + n + \pi^+ = \bar{p} + \bar{p} \rightarrow \bar{p} + \bar{n} + \pi^- \quad \text{etc.} \]
Eigenstates of C

If a theory is C-invariant, C commutes with the Hamiltonian. Hence we can look for simultaneous eigenstates of H and C. However, C anti-commutes with the charge operator since it interchanges positive and negative charges

$$CQC^{-1} = -Q.$$ \hfill (V.23)

So Q and C are not compatible: we cannot take both as constant observables. Since we always deal with charge eigenstates, i.e. Q = const., C must suffer. The exceptions are, of course, states for which Q = 0, i.e., zero charge; now Q and C do commute, from Eq. (V.23). This condition is clearly insufficient for an eigenstate of C: C\phi = \lambda \phi; rather \phi must have equal numbers of particles and corresponding antiparticles. For example

$$C\phi (P,\pi^-) = \phi (P,\pi^+) : \text{this can} \neq \lambda \phi (P,\pi^-)$$

$$C\phi (e^+,\bar{\psi}) = \phi (e^-,\bar{\psi}) : \text{this can} = \lambda \phi (e^+,\bar{\psi}) .$$

Since \( C^2 = 1 \), the eigenvalues of C are \( \lambda = \pm 1 \).

We are thus dealing with rather special systems for which we can define a charge conjugation parity, analogous to the usual parity. For C-invariant interactions, this leads to Selection Rules. Typical systems of this kind are \( (e^+,e^-) = \text{positronium}, (P,\bar{P}) = \text{protonium}, (N,\bar{N}), (\pi^+,\pi^-) \) etc. Typical processes are

$$e^+ + e^- \rightarrow \gamma + \gamma + \ldots \quad \text{n photons}$$

$$p + \bar{p} \rightarrow \pi^+ + \pi^- \quad , \text{etc.}$$

Consider, as an example, the positronium selection rules:

$$e^+ + e^- \rightarrow \gamma + \gamma + \ldots \quad (n \text{ photons}) .$$
Apart from factors which are not important here, the amplitude is given by [cf. our general result Eq. (II.30)]

\[
G = \int dx_1 dx_2 dy_1 \ldots dy_n e^{-i(k_1 y_1 + \ldots)} \sum_{\alpha\beta} \left( \langle 0, T \left( \psi_\alpha^\dagger(x_1) \psi_\beta(x_2) \lambda_\lambda(y_1) \ldots \lambda_n(y_n) \right) \rangle \langle 0 \right) \\
\times f(x_1x_2 \alpha\beta)
\]  

(V.24)

where \( f(x_1x_2 \alpha\beta) \) = the wave function (w.f.) of the initial positronium state whose decay we are studying (\( \alpha, \beta \) are the spin components: each \( c^\pm \) is described by a 4-component w.f.).

If charge conjugation invariance holds this = \( C G C^{-1} \)

\[
= (-1)^n \int dx_1 \ldots e^{-i(k_1 y_1 + \ldots)} \sum_{\alpha\beta} \left( \langle 0, T \left( \psi_\alpha^\dagger(x_1) \psi_\beta(x_2) \lambda_\lambda(y_1) \ldots \rangle \langle 0 \right) f(x_1x_2 \alpha\beta)
\]

where \( (-1)^n \) comes from the \( n \) photon operators.

Now exchange \( \psi_\alpha^\dagger(x_1) \leftrightarrow \psi_\beta(x_2) \) [gives (−) sign, since fermions] and \( x_1 \leftrightarrow x_2, \ \alpha \leftrightarrow \beta \)

\[
= (-1)^{n+1} \int dx_1 \ldots e^{-i(k_1 y_1 \ldots)} \sum_{\alpha\beta} \left( \langle 0, T \left( \psi_\alpha(x_1) \psi_\beta^\dagger(x_2) \lambda_\lambda(y_1) \ldots \rangle \langle 0 \right) f(x_2x_1 \beta\alpha).
\]

Now this is exactly the original expression \( G \), except for the factor \( (-1)^{n+1} \) and \( f(x_2x_1 \beta\alpha) \) instead of \( f(x_1x_2 \alpha\beta) \). Hence if

\[
f(x_2x_1 \beta\alpha) = (-1)^\omega f(x_1x_2 \alpha\beta),
\]

i.e. the positronium w.f. is symmetric or antisymmetric with respect to \( c^+ \leftrightarrow c^- \) (i.e. exchange space and spin co-ordinates), then

\[
G = (-1)^{n+1+\omega} G
\]
so \( G \neq 0 \) only if

\[
  n + 1 + \omega = \text{even}.
\]

This is the basic charge conjugation selection rule. We can interpret it as the conservation of charge conjugation parity:

\[
(-1)^n = (-1)^{\omega + 1}
\]

- charge conjugation parity of \( e^+ - e^- \): the 'extra' \((-1)\) comes from Fermi-Dirac statistics.

- charge conjugation parity of \( n \) photon state.

What does this mean for positronium? Under interchange \( e^+ \rightleftharpoons e^- \), how does the wave function \( f(e^+, e^-) \) change. For a state of orbital angular momentum \( \ell \) and spin \( s \) (\( = 0 \) or \( 1 \))

\[
\left\{ \begin{array}{l}
2S + 1 \ f_\ell(e^+ e^-) \\
\end{array} \right. \rightarrow (-1)^{\ell + s + 1} \left\{ \begin{array}{l}
2S + 1 \ f_\ell(e^- e^+) \\
\end{array} \right.
\]

i.e.

\[
\omega = \ell + s + 1
\]

and we get the charge conjugation parity selection rule from Eq. (V.25)

\[
\ell + s + n = \text{even}
\]

i.e.

\[
{}^1S_0 \text{ state} \rightarrow \text{even number of photons } (\gamma + \gamma)
\]

\[
{}^3S_1 \text{ state} \rightarrow \text{odd number of photons } (\gamma + \gamma + \gamma).
\]

One can formulate the above results in a language reminiscent of isobaric spin. Think of \( e^\pm \) as different charge states of the same fermion, specified by an additional quantum number: charge \( q = \pm e \). Then we can
say that the total wave function of positronium is antisymmetric under interchange of $e^+ \leftrightarrow e^-$. This now involves:

\[
\text{interchanging space co-ordinates, gives a factor } (-1)^\ell \\
\quad \text{ " spins } \quad \text{ " } \quad (-1)^{s+1} \\
\quad \text{ " charge } \quad \text{ " } \quad (-1)^\omega .
\]

Therefore in all

\[
(-1)^\ell + s + 1 + \omega = (-1) \quad \text{[Pauli principle]}
\]

\[
\implies (-1)^\omega = (-1)^{\ell + s}
\]

and then, since charge conjugation is a good conserved quantity,

\[
(-1)^{\ell + s} = (-1)^n ,
\]

as before.

This method can readily be applied to processes of this type:

\[
p + \bar{p} \rightarrow \pi^0 + \pi^0 + \ldots + \pi^0 \quad (n \pi^0 \text{ mesons}).
\]

One need not write it down again. The $p\bar{p}$ part behaves exactly like the $e^+e^-$. The difference is that $\pi^0$ has charge conjugation parity $+1$ necessarily. Hence the factor $(-1)^n$ does not occur and we get the selection rule

\[
(-1)^{\ell + s} = 1 .
\]

Hence possible initial states are

\[
^1S_0, \quad ^3P_0, ^1D_2 .
\]

Let us consider the $2\pi^0$ decay

\[
p + \bar{p} \rightarrow \pi^0 + \pi^0
\]
in more detail, looking at the (space) parity and angular momentum selected rules.

Since the final state consists of two identical bosons, their relative orbital angular momentum \( t' \) (equal to the total angular momentum \( J \) of the system) must be even:

\[
J = t' = 0, 2, 4, \ldots
\]

i.e. the \( 2\pi^0 \) system can exist in S, D, G,... states only.

Consider parity conservation next. The parity of the \( 2\pi^0 \) system is \((-1)^{t'} = 1\). The parity of the \( p\bar{p} \) is \((-1)^{t+1}\), \( t \) being the relative orbital angular momentum of the \( p\bar{p} \) system and the extra factor \((-1)\) arising from the opposite intrinsic parities of \( p \) and \( \bar{p} \). For parity conservation we must then have

\[
t + 1 = \text{even},
\]

i.e. only \( P, F, \ldots \) states can decay into \( 2\pi^0 \)'s.

Next consider angular momentum selection rules, taking singlet and triplet \( p\bar{p} \) states in turn. For singlet states \((S = 0)\) we would have \( J = t = \text{odd} \). But we also require, as we saw, \( J = t' = \text{even} \). Hence, only the triplet \((S = 1)\) \( p\bar{p} \) states \( ^3P_0, ^3P_2, \ldots \) can decay into \( 2\pi^0 \)'s. Of the three resultant angular momenta \( J = t - 1, t, t + 1 \), again \( J = t = \text{odd} \) is excluded, leaving as possible decays

\[
\begin{align*}
^3P_0 & \rightarrow 2\pi^0 \\
^3P_2 & \rightarrow S_0 \\
^3P_2 & \rightarrow D_2
\end{align*}
\]

etc.

We note, in particular, that the ground state \( ^1S_0 \) of the \( p\bar{p} \) system can not decay into \( 2\pi^0 \)'s.
Comparing the above selection rules for parity and angular momentum with those for charge conjugation, one sees that for the $p\bar{p} \rightarrow 2\pi^0$ decay, charge conjugation invariances does not give further additional restrictions: any transition forbidden by charge conjugation invariance is already forbidden by parity and angular momentum selection rules. The same is true for the decay into two charged mesons

$$p\bar{p} \rightarrow \pi^+ + \pi^- .$$

However, for decays into 3 or more mesons, charge conjugation invariance does give new selection rules, additional to those obtained from parity and angular momentum.

We see from the above - in fact from the $\pi^0$ having charge conjugation parity (+1) - that if a particular process is allowed as far as charge conjugation is concerned, then the same process is still allowed if we add an arbitrary number of $\pi^0$ mesons. Thus the charge conjugation selection rules for

$$p + \bar{p} \rightarrow \pi^+ + \pi^- + \pi^0$$

are the same as those for

$$p + \bar{p} \rightarrow \pi^+ + \pi^- .$$

Thus, although these selection rules do not give any new information for reaction (B) they are of interest for reaction (A).

The amplitude for reaction (B) can be written (we omit everything except the relevant factors referring to the final state pions):

$$\int d\gamma_1 d\gamma_2 \ldots g^*(\gamma_1 \gamma_n) \left( \Phi_c, T\left[ \ldots \varphi(y_1) \varphi(y_2) \ldots \right] \Phi_0 \right) f(\ldots) .$$

Under charge conjugation this goes over into

$$\int d\gamma_1 d\gamma_2 \ldots g^*(\gamma_2 y_1) \left( \Phi_c, T\left[ \ldots \varphi(y_1) \varphi(y_2) \ldots \right] \Phi_0 \right) f(\ldots) .$$
In order to obtain the last line I have, apart from performing charge conjugation, interchanged the meson field operators inside the $T$-product and relabelled the integration variables $y_1 \leftrightarrow y_2$. Now if the two mesons are in a state of orbital angular momentum $\ell'$, then

$$g(y_2 y_1) = (-1)^{\ell'} g(y_1 y_2),$$

i.e., the charge conjugation parity of the $\pi^+\pi^-$ system in this state is $(-1)^{\ell'}$ and we get instead of Eq. (V.27) the selection rule

$$(-1)^{\ell + s} = (-1)^{\ell'}, \quad \text{(V.27a)}$$

The parity selection rule for reaction (B) is, as before

$$(-1)^{\ell + 1} = (-1)^{\ell'},$$

i.e.,

$$\ell + \ell' = \text{odd}.$$

We see, for triplet $\pi^-\pi^-$ states ($S = 1$), the parity selection rule identical with Eq. (V.27a). On the other hand $S = 0$ is in any case forbidden by parity together with angular momentum conservation. The latter implies, for singlet $\pi^-\pi^-$ states ($S = 0$), that

$$J = \ell = \ell',$$

whence $\ell + \ell' = \text{even}$, inconsistently with parity conservation. This establishes our original assertion that charge conjugation gives no new selection rules for the 2-pion decays. We thus get the following transitions:

$$\begin{align*}
\bar{p}p & \rightarrow \pi^+ + \pi^- \\
^{3}S_1 & \rightarrow P_1 \\
^{3}P_0 & \rightarrow S_0 \\
^{3}P_2 & \rightarrow D_2 \\
^{1}D_1 & \rightarrow P_1 \\
\text{etc.}
\end{align*}$$
To illustrate that additional selection rules do arise for 3-pion decays consider reaction (A) above, from the ground state $1S_0$ of the $p\bar{p}$ system to the state $(P_p)_0$. Here the $\pi^+\pi^-$ mesons are in an angular momentum P-state ($\epsilon'=1$) and the $\pi^0$ has relative angular momentum 1 (p-state) about the centre of mass of the ($\pi^+\pi^-$) system. One easily sees that both initial and final states have parity (-1) and total angular momentum 0. However, from Eq. (27a) which is directly applicable to this reaction, the $1S_0$ $p\bar{p}$ state has charge conjugation parity (+1), whereas the $(P_p)_0$ state of the 3-pion system has charge conjugation parity (-1), thus forbidding this transition.

* * * *
LECTURE VI

TIME-REVERSAL

I now want to discuss time-reversal, essentially $t \rightarrow -t$. This leads to difficulties: it is different from $P$ and $C$ and cannot be represented by a unitary transformation. The result is that we do not get any conservation law out of it. It is nevertheless very useful and has many applications.

To illustrate the difficulty roughly: classically, if the motion of a system is described by $p(t), q(t)$, then one can define a time-reversed motion for which

$$p'(t) = -p(-t), \quad q'(t) = q(-t);$$

the system moves backwards in time. This is possible if the Lagrangian does not depend on $t$ explicitly and involves only even powers of $p$.

In quantum mechanics the Heisenberg picture, $q(t)$ and $p(t)$, are operators for which the usual commutation relations hold.

$$\left[ q_r(t), p_s(t) \right] = i\hbar \delta_{rs} .$$

But for the time-reversed motion it then follows that

$$\left[ q'_r(-t), -p'_s(-t) \right] = i\hbar \delta_{rs}$$

i.e.

$$\left[ q'_r(-t), p'_s(-t) \right] = -i\hbar \delta_{rs} ,$$

i.e. the commutation relations are not invariant: so the transition cannot be unitary. This difficulty in single-particle quantum mechanics persists in quantum field theory.
There are now several different approaches possible. They are all formal tricks to get around the non-unitary nature of time-reversal. I will follow Schwinger's method and that of Pauli; other formulations are given by Luders and Bell.

The method gives transformation laws for states and operators; but the transformation is not linear, and we therefore have a novel formulation. It suffices to define:

1) time-reversed operators for the absorption and creation operators $a(k),...,a^\dagger(p),..., $ are

$$ a(k) \to a_T(k) = T a(k) T^{-1} .$$  \hspace{1cm} (VI.1)

Then the transformation of the fields which are linear in these are also defined (and vice-versa):

$$ \varphi(x) \to \varphi_T(x) = T \varphi(x) T^{-1} .$$  \hspace{1cm} (VI.1a)

ii) For products of operators we define

$$ (AB)_T = B_T A_T = T(AB) T^{-1}$$  \hspace{1cm} (VI.2)

i.e. the order is to be reversed. It is this reversal of the order of factors which expresses the non-linear nature of the time-reversal operation.

iii) The vacuum transforms according to

$$ T|\phi_o \rangle = <\phi_o|, $$

$$ <\phi_o| T^{-1} = |\phi_o> $$  \hspace{1cm} (VI.3)

Any other state $|\psi >$ can be built up by applying creation operators $a^\dagger,...,d^\dagger,...$ to the vacuum $|\phi_o >$; say:

$$ |\psi > = a^\dagger...d^\dagger |\phi_o > .$$
If $a_T^\dagger$ is the time-reversed operator obtained from $a_T^\dagger$, etc.,
then the time-reversed state is

$$< \Phi_T | = T|\Phi > = < \Phi_0 | a_T^\dagger \ldots a_T^\dagger ,$$

i.e., the time-reversed operators are applied in the reversed order to the
vacuum written as a bra-vector.

**NOTE:** conditions (ii) and (iii) allow for the antilinear nature
of time-reversal; e.g. Eq. (VI.2) obviously just restores the correct order
in the commutation relations, etc.

Eq. (VI.3) explicitly separates bra- and ket-vectors: it will,
of course, correspond eventually to the interchange of initial states (kets)
and final states (bras). As long as we only consider vacuum expectation
values, $< \Phi_0 | \ldots | \Phi_0 >$, it follows from Eq. (VI.3) that we can forget about
the transformation of the vacuum states altogether.

All this is an abstract scheme so far. What can we do with it?
Again, consider specific fields.

**Scalar/pseudo-scalar mesons**

Consider charged mesons $\varphi(x)$. Define the time-reversed field
$\varphi_T(x) = T\varphi(x)T^{-1}$ by

$$\varphi_T(x) = T\varphi(x)T^{-1} = \xi_T^\dagger \varphi(x,-x_0) .$$

Hence taking adjoints

$$\varphi_T^\dagger(x) = \xi_T^* \varphi(x,-x_0) .$$

Carrying out two successive transformations, one gets

$$|\xi_T|^2 = 1 .$$
We again expand the fields in Eq. (VI.4) *):

$$\varphi_T(x) = \sum \left\{ a^T(k) e^{i k x} + b^T_k e^{-i k x} \right\}$$  \hspace{1cm} (VI.5)$$

$$\xi_T \varphi^+(x, -x_0) = \xi_T \sum \left\{ a^+(k) e^{-i(k x + k_0 x_0)} + b(k) e^{i(k x + k_0 x_0)} \right\}$$

$$= \xi_T \sum \left\{ a^+(-k) e^{i k x} + b(-k) e^{-i k x} \right\}.$$  \hspace{1cm} (VI.5a)

Comparing coefficients of $e^{+i k x}$ in Eqs. (VI.5) and (VI.5a):

$$a^T_k = \xi_T a^+(-k) \quad \text{and} \quad a^+T_k = \xi_T^* a(-k)$$

$$b^T_k = \xi_T b(-k) \quad \text{and} \quad b^+T_k = \xi_T^* b^+(-k).$$  \hspace{1cm} (VI.6)

This is what we expect. $k$ is a momentum, so under time-reversal it becomes $-k$. Consider single particle states [cf. Eq. (VI.3a)]:

$$\begin{align*}
T\ a^+(k) |\phi_\circ > &= <\phi_\circ | a^+(k) = \xi_T^* a(-k) |\phi_\circ >.
\end{align*}$$  \hspace{1cm} (VI.7)

As expected: initial $\pi^-$, momentum $k \rightarrow$ final $\pi^-$, momentum $(-k)$; etc.

Now consider the Lagrangian

$$\mathcal{L}(x) = \frac{i}{2} \left[ \left( \frac{\partial \varphi(x)}{\partial x^\mu} , \frac{\partial \varphi^+(x)}{\partial x^\mu} \right) + m^2 \left[ \varphi(x), \varphi^+(x) \right] \right].$$  \hspace{1cm} (VI.8)

*) I am hereafter omitting tiresome factors $\frac{1}{\sqrt{V}} \frac{1}{\sqrt{2k_0}}$, etc., if you like, they are incorporated in $a, b, ...$
Then one verifies directly,

\[ T\mathcal{L}(x)T^{-1} = \mathcal{L}(\bar{x}, -x_0), \quad (VI.9) \]

i.e., invariant under time-reversal. Hence equations of motion and commutation relations are also invariant under time-reversal. This justifies the above procedure: it makes it useful. Eq. (VI.9) means that a system with the Lagrangian \( T\mathcal{L}(x)T^{-1} \) behaves in the same way as a system with the original Lagrangian in which the sense of time is reversed.

\**Fermions**

The treatment of these is rather more complicated. We proceed by analogy with spin 0. Define

\[ x' = (\bar{x}, -x_0) \]

\[ \psi_T(x) = T\psi(x)T^{-1} = \xi_T \gamma_4 \gamma_5 \psi^\dagger(x') = \xi_T \psi^\dagger(x') \gamma_5 \gamma_4 \]

\[ = -\xi_T \bar{\psi}(x') \gamma_5 = \xi_T \gamma_5 \bar{\psi}(x') \]

This is the spinor transformation we expect for time-reversal:

\[ \gamma_5 : x_\mu \to -x_\mu, \quad \gamma_4 : x_\mu \to -x_\mu, \quad \text{so} \quad \gamma_4 \gamma_5 : (\bar{x}, x_0) \to (\bar{x}, -x_0) \]

\[ \begin{bmatrix} \text{e.g.} & \gamma_4 \gamma_5 \psi^\dagger = \psi^\dagger (\gamma_4 \gamma_5)^\sim = \psi^\dagger \gamma_5 \gamma_4 \gamma_5 \gamma_4 = \psi^\dagger (\gamma_4)(-\gamma_4) = \psi^\dagger \gamma_5 \gamma_4 \end{bmatrix} \]

Then

\[ \psi_T^\dagger = \xi_T^* \psi(x') \gamma_5 \gamma_4 = \xi_T^* \gamma_4 \gamma_5 \psi(x') \]

\[ \bar{\psi}_T = \psi_T^\dagger \gamma_4 = \xi_T^* \psi(x') \gamma_5 = -\xi_T^* \gamma_5 \psi(x') \]

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Expanding, say, $\psi_T(x) = \xi_T^\dagger\gamma_4\gamma_5\psi^\dagger(x')$ one gets by the same method as before

$$\psi_T(x) = \sum \left\{ c_{rT}(\mathbf{p})u(\mathbf{r})(\mathbf{p})e^{ipx} + d_{rT}(\mathbf{p})u(\mathbf{r})^*(\mathbf{p})e^{-ipx} \right\}$$

$$\xi_T^\dagger\gamma_4\gamma_5\psi^\dagger(x') = \xi_T^\dagger\sum \left\{ c_{r}^\dagger(-\mathbf{p})\gamma_4\gamma_5 u(\mathbf{r})^*(-\mathbf{p})e^{ipx} + d_{r}^\dagger(-\mathbf{p})\gamma_4\gamma_5 u(\mathbf{r})(-\mathbf{p})e^{-ipx} \right\}.$$  

(VI.11)

Now from Eqs. (IV.29a) and (IV.35):

$$\gamma_4\gamma_5 u(\mathbf{r})(-\mathbf{p}) = -\gamma_5 u(\mathbf{r})(\mathbf{p}) = u(\mathbf{r})^*(\mathbf{p})$$

$$\gamma_4\gamma_5 u(\mathbf{r})^*(-\mathbf{p}) = u(\mathbf{r})^*(\mathbf{p})$$

$$\gamma_4\gamma_5 u(\mathbf{r})(-\mathbf{p}) = u(\mathbf{r})(\mathbf{p})$$

(VI.12)

[This follows just by complex conjugation]

Hence comparing Eqs. (VI.11):

$$c_{rT}(\mathbf{p}) = \xi_T^\dagger c_{r}^\dagger(-\mathbf{p})$$

$$c_{rT}(\mathbf{p}) = \xi_T c_{r}(-\mathbf{p})$$

and then

$$d_{rT}(\mathbf{p}) = \xi_T d_{r}(-\mathbf{p})$$

$$d_{rT}(\mathbf{p}) = \xi_T^\dagger d_{r}^\dagger(-\mathbf{p})$$

(VI.13)

Again, as expected; e.g., the one-particle states:

$$T c_{r}^\dagger(\mathbf{p}|\phi_0 > = \xi_T^\dagger < \phi_0 | c_{r}(-\mathbf{p})$$

$$T d_{r}^\dagger(\mathbf{p}|\phi_r > = \xi_T < \phi_0 | d_{r}(-\mathbf{p})$$

(VI.14)
This result is as expected:

an initial particle of momentum $\mathbf{p}$, spin $\|\pm\mathbf{p}\|$ for $r = \{1\}$

$\rightarrow$ final " " " $-\mathbf{p}$, spin $\|\pm\mathbf{p}\|$ for $r = \{1\}$

i.e. particle $\rightarrow$ particle

$\mathbf{p} \rightarrow -\mathbf{p}$ : reversed under time-reversal

spin $\rightarrow -$ spin " " " $=$ angular momentum: $- r \times p$.

Now one can show by direct transformation that the Dirac Lagrangian is invariant under time-reversal, i.e.

$$\mathcal{L}(x) = -\frac{i}{2} \left\{ \bar{\psi}(x) \gamma_{\mu} \left( \frac{\partial}{\partial x_{\mu}} + M \right) \psi(x) + \bar{\psi}(x) \gamma_{\mu} \left( \frac{\partial}{\partial x_{\mu}} - M \right) \psi(x) \right\}$$

(VI.14a)

transforms as

$$T \mathcal{L}(x) T^{-1} = \mathcal{L}(x'),$$

(VI.15)

provided

$$|\xi_T|^2 = 1$$

(VI.16)

which we demand. The proof is not very instructive other than as an exorcise in handling $\gamma$-s. Only note two points.

1) Each half itself invariant, as expected: same as for $P$. They transform into each other under charge conjugation or combined operations which include charge conjugation.

ii) In proving Eq. (VI.16) one carries out a differentiation by parts. This leads to a term which is a 4-dimensional divergence.
\[ \frac{\partial}{\partial x^\mu} \left[ \phi(x') \gamma_\mu \phi(x') \right] \]  

One can omit this, as a divergence does not affect the field equations, etc.

The invariance of \( \mathcal{L} \) again makes \( T \) a useful symmetry transformation. The time-reversed system satisfies the same dynamical equations as the original system.

So far we have dealt only with free Lagrangians. However, if an interaction is also \( T \)-invariant, the same is true for the total equation and then one gets the usual types of relations for inverse processes. The basic equation is Reciprocity (*) :

\[ < \phi' | S | \phi > = < \phi_T | S_T | \phi_T' > . \]  

(VI.17)

To prove Eq. (VI.17), we have by our definition of the time-reversal operation that

\[ < \phi' | S | \phi > = < \phi_T | S_T | \phi_T' > . \]  

(VI.17a)

where \( < \phi_T | \) is the time-reversed state of \( | \phi > \), etc., and

\[ S_T = T S T^{-1} . \]  

(VI.18)

*) Eq. (VI.17) relates the amplitudes for inverse reactions :

\[ A + B + \ldots \leftrightarrow C + D + \ldots ; \]

initial state \( i \)  
final state \( f \)

\[ < p', s', \ldots | S, p, s, \ldots > = \pm < -p', -s', \ldots | S | -p, -s, \ldots > . \]

Here \( p, \ldots \) and \( s, \ldots \) are the momenta and spins in the state \( i \), and \( p', \ldots \) and \( s', \ldots \) have similar meanings for the state \( f \). The \( \pm \) sign comes from the phase factors in defining the time-reversed states. These phase factors will not be important except when one is interested in interference effects.

In writing \( \pm s \), etc., for the spin quantum numbers, I have reverted from the notation of Lecture IV [where spin was expressed as right-hand \( (r=1) \) and left-hand \( (r=2) \) polarization] to the more usual notation \( s = \pm 1 \), etc., referring to a definite axis, which may be the appropriate momentum \( \alpha \), if a non-relativistic approximation is justified, a fixed direction in space.
Equation (VI.17a) follows merely by writing $|\Phi> = c^T_T \ c^T_S \ ... \ d^T_t \ ... |\Phi_o>$; etc., inverting the order of all operators and replacing them by their time-reversed operators [cf. Eq. (VI.3a)].

Equations (VI.17a) and (VI.18) are merely transformations from one representation to the time-reversed representation. No physics has gone into this. We shall now show that if the system is invariant under time-reversal then

$$S_T = S.$$  

$S$ was defined by

$$S = U(\infty, -\infty),$$

where $U(t_2, t_1)$ is the solution of

$$ih \frac{\partial U(t_2, t_1)}{\partial t_2} = H(t_2)U(t_2, t_1) \quad (VI.19)$$

with

$$U(t_2, t_1) = 1 \text{ for } t_2 = t_1.$$  

Here $H(t_2)$ is the interaction Hamiltonian in the interaction picture. [We derived Eq. (VI.19) when we discussed the transition from H.P. \to I.P.].

Put Eq. (VI.19) in a more symmetrical form \(^{9}\). Take adjoint, use $U^T(t_2, t_1) = U(t_1, t_2)$ and $t_1 \leftrightarrow t_2$:

$$-ih \frac{\partial U(t_2, t_1)}{\partial t_1} = U(t_2, t_1)H(t_1). \quad (VI.19a)$$

Hence, adding Eqs. (VI.19) and (VI.19a) with $t_2 = t$, $t_1 = -t$

$$ih \frac{\partial U(t, -t)}{\partial t} = H(t)U(t, -t) + U(t, -t)H(-t). \quad (VI.20)$$

Apply time-reversal: $U(t, -t) \to U_T(t, -t)$, $H(t) \to H_T(t)$ and invert order of operators

$$ih \frac{\partial U_T(t, -t)}{\partial t} = U_T(t, -t)H_T(t) + H_T(-t)U_T(t, -t). \quad (VI.21)$$

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Now time-reversal invariance means

\[ H_T(t) = H(-t) \quad (VI.22) \]

So if time-reversal invariance holds, Eq. (VI.21) \( \rightarrow \)

\[ \frac{\partial U_T(t,-t)}{\partial t} = U_T(t,-t)H(-t) + H(t)U_T(t,-t) \quad (VI.23) \]

Compare Eqs. (VI.20) and (VI.23): they are identical equations for \( U \) and \( U_T \). Hence, for a \( T \)-invariant theory,

\[ U_T(t,-t) = U(t,-t) \]

and in particular

\[ S_T = S \]

This completes the proof of the Reciprocity theorem [Eq. (VI.17)].

As was stated earlier (in the footnote to page 71) the Reciprocity theorem can be written:

\[ \langle p's'...|S|p's... \rangle = \pm \langle -p-s...|S|-p'-s'... \rangle, \]

where the left-hand side is the element of the \( S \)-matrix from a state with particles of momenta \( p,... \) and spins \( s,... \) to a state with particles of momenta \( p',..., \) and spins \( s',..., \), and similarly for the right-hand side. The \( \pm \) sign depends on the phase factors in defining the time-reversed states.

The Reciprocity theorem is not the same as the detailed balancing theorem which states

\[ \left| \langle p's'...|S|p's... \rangle \right|^2 = \left| \langle p's...|S|p's'... \rangle \right|^2. \]

This detailed balancing theorem does not hold rigorously but only in perturbation theory, if the hermitian interaction \( H_{int} \) causing the
reaction is weak enough for the approximation

\[ S = H_{\text{int}} \]

to be valid. In this case, detailed balancing follows at once from

\[ \langle \mathbf{E}' \mathbf{s}' \ldots | H_{\text{int}} | \mathbf{E} \mathbf{s} \ldots \rangle = \langle \mathbf{E} \mathbf{s} \ldots | H_{\text{int}} | \mathbf{E}' \mathbf{s}' \ldots \rangle^*. \]

However, rigorous theorems similar to detailed balancing can be derived from reciprocity.

i) If the system is invariant under space-inversion, the reciprocity theorem gives

\[ \langle \mathbf{E}' \mathbf{s}' \ldots | S | \mathbf{E} \mathbf{s} \ldots \rangle = \pm \langle \mathbf{E} \mathbf{s} \ldots | S | \mathbf{E}' \mathbf{s}' \ldots \rangle. \]

Hence if the spins are not observed

\[ \sum_{\text{spins}} | \langle \mathbf{E}' \mathbf{s}' \ldots | S | \mathbf{E} \mathbf{s} \ldots \rangle |^2 = \sum_{\text{spins}} | \langle \mathbf{E} \mathbf{s} \ldots | S | \mathbf{E}' \mathbf{s}' \ldots \rangle |^2 \]

which is the same as detailed balancing except for the spin summations.

ii) A different form of detailed balancing theorem (also rigorously true) results if we analyse into terms of angular momentum states assuming invariance under 3-dimensional rotations.

Suppose, instead of the plane waves used above, we analyse the reaction in terms of states of definite angular momentum \( J, M \) and other variables \( \lambda_1, \lambda_2 \ldots \) to make a complete set. If the theory is T-invariant, then

\[ \langle \lambda_1 \lambda_2 \ldots J M | S | \lambda_1 \lambda_2 \ldots J M \rangle = \pm \langle \lambda_1 \lambda_2 \ldots J - M | S | \lambda_1' \lambda_2' \ldots J - M \rangle. \]

If \( S \) is also invariant under rotations, this matrix element is also

\[ = \pm \langle \lambda_1 \lambda_2 \ldots J M | S | \lambda_1' \lambda_2' \ldots J M \rangle. \]
This gives a detailed balancing theorem in terms of angular momentum states for the inverse reactions in the channels specified by

\[ \lambda_1 \lambda_2 \cdots \longleftrightarrow \lambda_1' \lambda_2' \cdots \]

Next I want to mention (rather than treat exhaustively) that we are now in a position to discuss time-reversal invariance of interactions, from the time-reversal properties of the fields.

We have seen that [Eqs. (VI.10)]

\[ \psi_T(x) = \xi_T \overline{\psi}(x') \tilde{\gamma}_S ; \quad \overline{\psi}_T(x) = \xi_T^* \tilde{\overline{\gamma}}_S \psi(x') \]

Then consider the 5 basic spinor covariants \( O_1 = 1, i\gamma_\mu, \ldots \)

\[ T \left[ \overline{\psi}, O_1 \psi \right] T^{-1} = T \left[ \overline{\psi} O_1 \psi - \overline{\psi} \overline{O}_1 \overline{\psi} \right] T^{-1} \]

\[ = |\xi_T|^2 \left( \overline{\psi} \tilde{\gamma}_S \overline{O}_1 \gamma_S \psi - \overline{\psi} \gamma_S O_1 \gamma_S \psi \right) . \]

Now

\[ (\tilde{\gamma}_S \overline{O}_1 \gamma_S) = (\gamma_S O_1 \gamma_S) = c_1 \overline{c}_1 \text{, say.} \]

\[ (\gamma_S O_1 \gamma_S) = c_1 \overline{c}_1 \]

\[ \left[ \overline{\psi}, O_1 \psi \right] \rightarrow c_1 \left[ \overline{\psi}, O_1 \psi \right] \]

One finds directly and easily, for example

\[ O_1 = 1 \quad c_1 = 1 \]

\[ O_1 = i\gamma_\mu \quad (\gamma_S i\gamma_\mu \gamma_S)^* = -i\gamma_\mu = \begin{cases} -i\gamma_\mu & \mu = 1, 2, 3 \\ +i\gamma_\mu & \mu = 4 \end{cases} \]
\[ \begin{aligned}
  \sigma_{\mu
u} \rightarrow 
  \begin{cases}
    c_i = -1, & \text{if } \mu \text{ and } \nu \text{ both } \neq 4 \\
    +1, & \text{if either } = 4.
  \end{cases}
\end{aligned} \]

Similarly one finds

This last result, incidentally also shows how the electromagnetic field transforms:

\[ E = 4k \text{ components of second rank tensor, so } E \rightarrow +E \]

\[ H = kE \rightarrow -H. \]

Or, of course, from

\[ E \sim \frac{\partial A}{\partial t} : \frac{\partial}{\partial t} \text{ space part of } A_\mu \rightarrow +1 \]

\[ H = \text{curl } A : \frac{\partial}{\partial x_k} ( ) \rightarrow -1 \]

or directly from Maxwell's equations.
TCP THEOREM

We are now ready to state and prove a most remarkable theorem: the TCP theorem.

Call the transformation TCP: P, then C, then T; in this order. (Any other order would do; it would lead to minor differences of sign etc., so we must be consistent and we adhere to PCT.)

The TCP theorem states: a theory which is invariant under proper Lorentz transforms, and has the correct spin statistics relation, is automatically invariant under (TCP), i.e. demanding TCP invariances does NOT imply further restrictions but is automatically satisfied. This is different from previous cases where, for example, invariance with respect to P implied special symmetry properties for the system.

There are many ways of deriving the TCP theorem. It is originally due to Lüders. We will use the Feldman - Matthews approach.

If the theorem holds for a theory of the above kind, then all Green's functions must be invariant under TCP. I will prove it for, say,

$$\pi^- + P \rightarrow \pi^- + P,$$

which is typical, and will show the general procedure.

From the individual transformations under T, C and P we have for the combined transformation:

$$\varphi(y) \xrightarrow{P} \xi_P \varphi(-y,y') \xrightarrow{C} \xi_P \xi_C \varphi^*(-y,y') \xrightarrow{T} \xi \varphi(-y), \quad \xi = \xi_P \xi_C \xi_T^*$$

and

$$\varphi^+(y) \xrightarrow{TCP} \xi^* \varphi^+(-y).$$

(VII.1)
Similarly

\[ \psi(x) \xrightarrow{\xi_1^T} \gamma_4(-x, x_0) \xrightarrow{C} \xi_2^T \xi_4^T \gamma_4 \psi((-x, x_0)) \xrightarrow{T} \xi' \gamma_5 \psi(-x), \quad \xi' = \xi_2^T \xi_4^T \xi_1^T \]

and

\[ \bar{\psi}(x) \xrightarrow{TCP} - \xi' \bar{\psi}(-x) \gamma_5 \]

The amplitude for the process

\[ \pi^- + p \rightarrow \pi^- + p \]

is

\[ A_{RS}(p_1 p_2 k_1 k_2) = \int dx_1 \ldots dy_2 \ e^{-i(k_1 y_1 - k_2 y_2 + p_1 x_1 - p_2 x_2)} \]

\[ \times \sum_{\alpha, \beta} \bar{u}_d^{(r)}(p_1) \left( \bar{\phi}_0, T \left[ \bar{\psi}_\beta(x_2) \psi_\alpha(x_1) \bar{\psi}(y_2) \phi(y_1) \right] \phi_0 \right) u_\beta(s)(p_2) \]

\[ = \sum_{\alpha \beta} \bar{u}_d^{(r)}(p_1) C_{d \beta}(p_1 p_2 k_1 k_2) u_\beta(s)(p_2). \]  

(VII)

\[ \psi_\alpha(x_1) \text{ absorbs } p \text{ at } x_1; \]

\[ \bar{\psi}_\beta(x_2) \text{ creates } p \text{ at } x_2; \]

\[ \phi(y_1) \text{ absorbs } \pi^- \text{ at } y_1, \text{ etc.} \]

Under TCP this transforms into (remember as in time-reversal the order of the operators must be reversed):

\[ \sum_{\alpha \beta} \bar{u}_d^{(r)}(p_1) \int dx_1 \ldots dy_2 \ e^{-i(k_1 y_1 - k_2 y_2 - p_1 x_1 + p_2 x_2)} \times \]

\[ \times \left( \bar{\phi}_0, T \left[ \phi(-y_1) \phi(-y_2) \left[ \gamma_5 \psi_\alpha(-x_1) \right] \left[ \psi_\beta(-x_2) \gamma_5 \right] \right] \phi_0 \right) u_\beta(s)(p_2). \]

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Now interchange operators in $\mathcal{T}\{\ldots\}$ back to their original order: $\psi^\dagger \psi \phi^\dagger \phi$. For the bosons this does nothing. For the fermions it gives a factor $(-1)$ which cancels the one marked ($\cdot$). Also it replaces $x_i \rightarrow -x_i, y_i \rightarrow -y_i$. Then the last expression (i.e. the amplitude transformed by TCP) becomes

$$(\text{TCP})(\text{Ampl.})(\text{TCP})^{-1} =$$

$$= \sum_{\alpha\beta} \overline{u}_a^{(p_1)}(p_1) Y_5 \alpha^\lambda \int dx_1 \ldots dx_2 \ e^{+i(k_1 y_1 \ldots - p_2 x_2)} \left\{ \psi_0, \mathcal{T}\left[ \phi^\dagger (x_2) \phi_\lambda (x_1) \phi^\dagger (y_2) \phi(y_1) \right] \right\} \psi_0 \right] \frac{1}{r_s} \left[ u_\beta^{(s)} (p_2) \right].$$

(VII.4)

Now for the amplitude Eq. (VII.3) to be Lorentz-covariant, $G_{\alpha\beta}(p_1 p_2 k_1 k_2)$ may depend on the 4-vectors $p_1, \ldots, k_2$ : either through scalar products, like $p_1 \cdot k_1, p_1 \cdot p_2, \ldots$. For these the minus signs in the arguments of $G$, Eq. (VII.4), are not significant. Or $G$ depends on the 4-vectors through quantities like $\not{\psi}_1$, etc. Now, since for any 4-vector $k_\mu$

$$y_5 \not{\psi} = - \not{y},$$

it follows that

$$\left\{ y_5 \ G(-p_1, -p_2, -k_1, -k_2) y_5 \right\} = G(p_1 p_2 k_1 k_2).$$

(VII.5)

Comparing Eqs. (VII.3) and (VII.4) one sees that, on account of Eq. (VII.5), the transformed amplitude [Eq. (VII.4)] is identical to the original amplitude [Eq. (VII.3)]: $A_{\alpha \beta} (p_1, \ldots) \xrightarrow{\text{TCP}} A_{\alpha \beta} (p_1, \ldots)$. This completes the proof of the TCP theorem. The generalizations to other cases are now pretty obvious.
Comments

i) Firstly, some remarks about the validity of the TCP theorem. Different proofs do not attain the same degree of generality.

The above proof depends on the validity of the expressions for amplitudes, e.g. Eq. (VII.3), in terms of Heisenberg operators. Without going into complete details, the essential condition is the assumption of microcausality, i.e. the vanishing of commutators (anticommutators) for boson (formion) fields at points separated by space-like distances.

A more general rigorous proof has been given by Jost. Jost replaces the requirement of microcausality by a weaker condition, which demands the vanishing of certain vacuum expectation values of products of field operators at points bearing certain space-like relations to each other.

ii) TCP invariance does not mean TCP = 1. This is obvious from the above considerations; it is a question of invariance in form of $L$, etc., under certain transformations.

iii) The TCP theorem then states a restriction which is implied for any Lorentz-invariant theory with the right spin statistics connection. This leaves open several possibilities.

a) T, C, P separately invariant: strong interactions, and electro-magnetic.

b) If not invariant under one of T, C, P, then non-invariant under one or both the others.

c) If invariant under the product of two of T, C, P, then also invariant under the third separately.

For weak interactions one knows P and C are not invariant separately, but almost certainly (PC) and T are invariant.
The TCP theorem allows many very useful applications. Without
giving derivations let me remind you of a few of those:

a) Final-state theorem of Lee, Gohm and Yang\textsuperscript{11}; if in the
final state of a decay process there are no electromagnetic or strong
interactions present, then asymmetries due to parity violation in the
decay can only be observed if C is also violated. This is to first order
in the weak interaction.

This theorem implies at once that C cannot hold for
\[
\pi \rightarrow \mu + \nu \\
\mu \rightarrow e + \nu + \bar{\nu}
\]
since P does not hold. Furthermore in \(\beta\)-decay
\[
n \rightarrow p + e^- + \bar{\nu}
\]
C cannot be conserved because of the observed P-violation. Because of
the electromagnetic final state interaction, one cannot conclude this
quite so readily. However, one can estimate the asymmetries which should
be observed due to P-violation resulting from the rather weak Coulomb
effects. The observed asymmetries are much larger.

b) The lifetime and mass of a particle are the same as those
of its antiparticle. This theorem was first proved by Lee, Gohm and Yang\textsuperscript{11}
using perturbation theory. A proof not dependent on perturbation theory
has been given by Zumino and by Roman\textsuperscript{12}.

The important point about this theorem is, of course, that it
does not only hold if charge conjugation invariance holds (this is of
course clearly a sufficient condition). In fact, TCP invariance alone
suffices.

The above theorems can, of course, also be applied to weak
decays involving hyperons, e.g. \(\Lambda \rightarrow p + \pi^-\), etc.

***
LECTURE VIII

ISOSPAC

So far we have treated rather extensively discrete transformations. We now deal more briefly - or rather more selectively - with continuous transformations.

Amongst continuous transformation, proper Lorentz transformations and subgroups, three-dimensional rotations are very important. In particular, the spin properties of elementary particles come out automatically from the tensorial or spinorial character of the fields. But I will not discuss these important aspects as I assume you know them. In any case, I have always used Lorentz-invariance, etc.

I want, instead, to talk about isospace. I also assume that you know the general basic ideas, so I shall sum up briefly and then discuss certain special features.

The essence of the concept is to treat certain sets of particles as different states of the same field, e.g. p and n. Correspondingly, one combines $\psi_p$ and $\psi_n$ into

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}.$$  \hspace{1cm} (VIII.1)

One can think of this as an eight-component spinor, but this is not the most profitable way. It is better treated as two components, each in itself an ordinary four-component field.

We can now introduce Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$ \hspace{1cm} (VIII.2)

$$\tau_\pm = \frac{1}{\sqrt{2}} (\tau_1 \pm i \tau_2)$$ \hspace{1cm} (VIII.3)
which transform the proton and neutron fields amongst themselves

\[
\tau_-(\begin{pmatrix} \psi \\ 0 \end{pmatrix}) = \sqrt{2} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \quad p \rightarrow n
\]

\[
\tau_+\left(\begin{pmatrix} 0 \\ \psi \end{pmatrix}\right) = \sqrt{2} \begin{pmatrix} \psi \\ 0 \end{pmatrix} \quad n \rightarrow p
\]

\[
\tau_3\left(\begin{pmatrix} \psi \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \quad p \rightarrow p
\]

\[
\tau_3\left(\begin{pmatrix} 0 \\ \psi \end{pmatrix}\right) = -\begin{pmatrix} 0 \\ \psi \end{pmatrix} \quad n \rightarrow n.
\]

The Yukawa $\pi$-$N$ interaction, which I discussed some lectures ago (Lecture IV) can then be written:

\[
\mathcal{H} = ig_1 \frac{1}{\sqrt{2}} \left\{ \bar{\psi} Y_3 \begin{pmatrix} \phi^+ \\ 0 \end{pmatrix} + \bar{\psi} Y_3 \begin{pmatrix} \phi^- \\ 0 \end{pmatrix} + ig_2 \bar{\psi} Y_3 \begin{pmatrix} 1 + T_3 \overline{\psi} \psi \end{pmatrix} + ig_3 \bar{\psi} Y_3 \begin{pmatrix} 1 - T_3 \overline{\psi} \psi \end{pmatrix} \right\}.
\]

\[
\begin{array}{ccc}
\text{p} & \rightarrow & \text{n} \\
\text{absorb} & \pi^- & \rightarrow \pi^0 \\
\text{create} & \pi^+ & \rightarrow p
\end{array}
\]

\[
\begin{array}{ccc}
\text{p} & \rightarrow & \text{p} \\
\text{n} & \rightarrow & \text{n}
\end{array}
\]

(VIII.4)

As is well known, a special choice of coupling constants now leads to charge independence (C.I.), viz.,

\[
\frac{1}{\sqrt{2}} g_1 = g_2 = -g_3 = g.
\]

Then decomposing the charged fields $\phi$, $\phi^\dagger$ with two hermitian fields:

\[
\phi = (\phi_1 + i\phi_2)/\sqrt{2}, \text{ etc.,}
\]

(of course $\phi_3 = \phi_3 : \pi^0$)

equation (VIII.4) becomes

\[
\mathcal{H} = ig \bar{\psi} Y_3 \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} \psi.
\]

(VIII.5)
Here I have introduced $\mathbf{I} = (I_1, I_2, I_3)$, $\mathbf{\varrho} = (\varrho_1, \varrho_2, \varrho_3)$ as the three components of a vector in a new abstract space of three dimensions: isobaric space = isospace. C.I. is expressed by the fact that Eq. (VIII.5) is a scalar in isospace: $\mathbf{I} \cdot \mathbf{\varrho}$ is a scalar product. Invariance under rotations in isospace is the general expression for charge independence, e.g. the most general charge independence $\mathcal{N}-\mathcal{N}$ interaction is of the form

$$V_1 + \mathbf{I}_1 \cdot \mathbf{I}_2 V_2$$

(of course, $V_1$ and $V_2$ are spin-dependent). Just as $\mathbf{\varrho}$ is now thought of as a vector in isospace so, of course, $\psi = (\varphi_{\pi}, \varphi_n)$ is a two-component spinor in isospace. It transforms under rotations in isospace just like an ordinary spinor does under ordinary rotations. The parallelism is complete! One refers to $\mathbf{\varrho}$ as an isovector, to $\psi$ as an isospinor. A field has now two transformation properties in space and in isospace, e.g. the pion field is a pseudo-scalar isovector; the nucleon field is a spinor isospinor, etc.

Interactions which are invariant under rotations in isospace then lead to isospin selection rules, e.g. for

$$\pi + \mathcal{N} \to \pi + \pi + \mathcal{N}$$

since $\pi$ has isospin $\frac{1}{2}$, the $\mathcal{N}^0$ isospin $\frac{1}{2}$. The initial state can have a resultant $T = \frac{1}{2}$ or $\frac{3}{2}$. What $T$-values can one obtain for a $(\pi \pi \mathcal{N})$ system?

$$\pi + \pi \rightarrow T = 0, 1, 2$$

and combined with the $\mathcal{N}(T = \frac{1}{2})$:

$$T = \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}.$$   

However, if $T$ is conserved, the $T = \frac{1}{2}$ part of a $\pi-\mathcal{N}$ state can only go into $T = \frac{1}{2}$, etc. In particular, the $T = \frac{3}{2}$ states cannot be excited at all in this reaction.

Electromagnetic interactions obviously violate charge independence. They essentially contain the charge operator $\frac{1}{2}(1 + \tau_3)$:
= 1 for \( p \), = 0 for \( n \). C.I. holds for strong interactions, in so far as electromagnetic interactions are comparatively weak, i.e. if they can be neglected. It is worth mentioning that even with electromagnetic interactions, one gets isospin selection rules. This is because a non-C.I. interaction will usually be of the form

\[ \lambda + \mu \tau_3. \]

The first term is an isoscalar, the second an isovector. Thus for a transformation \( i \to f \)

\[ <f|\lambda + \mu \tau_3|i> = <f|\lambda|i> + <f|\mu \tau_3|i> \]

the first term can only effect transformations with \( T_f = T_i \),
the second " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " " 
Similarly, evaluating $T_3$ from the expansion of the fields one finds

$$T_3 = \frac{i}{2} (N_p - N_n - N_x + N_y).$$  \hspace{1cm} (VIII.6b)

From Eqs. (VIII.6 a-b) one gets

$$Q = \frac{i}{2} N + T_3 = (N_p - N_n)$$  \hspace{1cm} (VIII.6a)

as one wants it.

From Eq. (VIII.6b) we see that if a given particle has isospin 3-component $T_3$, its antiparticle has 3-component $(-T_3)$.

If we do have charge independence we can also consider any other rotation in isospace. A particularly interesting one is $T_2(\pi)$, i.e. through $\pi$ around the 2-axis:

Thus

$$(\varphi_1, \varphi_2, \varphi_3) \xrightarrow{T_2(\pi)} (-\varphi_1, \varphi_2, -\varphi_3).$$  \hspace{1cm} (VIII.7)

The interesting transformation is to combine this with $C$

$$\varphi \leftrightarrow \varphi^\dagger$$

$$(\varphi_1, \varphi_2, \varphi_3) \xrightarrow{C} (\varphi_1, -\varphi_2, \varphi_3).$$  \hspace{1cm} (VIII.8)

Hence under the operation

$$G = C \cdot T_2(\pi)$$

$$(\varphi_1, \varphi_2, \varphi_3) \xrightarrow{G} (-\varphi_1, -\varphi_2, -\varphi_3).$$  \hspace{1cm} (VIII.9)
Thus, the $\pi$-meson has negative G-parity. (G was first introduced by Lee and Yang.) This leads to interesting selection rules for interactions invariant under G. (This is always the case if C and C.I. hold.)

Firstly we get the generalization of Furry's theorem: any graph

with an odd number of meson lines coming out is identically zero. The proof is as for Furry's theorem using G-invariance instead of C-invariance.

To find how

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

transforms under G, we need its transformation property with respect to $T_2(\pi)$. Under a rotation through $\vartheta$ about the $j^{th}$ axis, $\psi$ transforms into

$$e^{i\tau_j \vartheta / 2} \psi = \sum_{n=0}^{\infty} \frac{(i \tau_j \vartheta / 2)^n}{n!} \psi .$$

Since $(i \tau_j)^2 = -1$, this

$$= \left\{ \sum_{n=0}^{\infty} (-1)^n \frac{(\vartheta / 2)^{2n}}{(2n)!} + i \tau_j \sum_{n=0}^{\infty} (-1)^n \frac{(\vartheta / 2)^{2n+1}}{(2n+1)!} \right\} \psi$$

$$= \left( \cos \frac{\vartheta}{2} + i \tau_j \sin \frac{\vartheta}{2} \right) \psi .$$
In our case \( \vartheta = \pi, r_1 = r_2 \) and \( i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), so

\[
\begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \xrightarrow{T_2(\pi)} \begin{pmatrix} \phi_p \\ -\phi_n \end{pmatrix} \xrightarrow{C} \begin{pmatrix} \phi^+_p \\ -\phi^+_n \end{pmatrix}
\]

(VIII.10)

i.e. under \( G = CT_2(\pi) \):

\[
\begin{align*}
p &\to \bar{n} \\
n &\to -\bar{p}
\end{align*}
\]

(minus indicates that there is a phase involved). (VIII.11)

It is now possible to apply G-parity arguments to obtain selection rules. [This has been done by Amati and Vitale\(^{13}\), and Lee and Yang\(^{14}\).] But as in the case of \( C \), \( G \) does not always form a good quantum number, so some care is needed. Consider, for example, the decay of the \( pp \) system, studied earlier (Lecture V).

We see from Eqs. (VIII.10) and (VIII.11) that the \( pp \) system is not an eigenstate of \( G \). But we can write it as a superposition of two eigenstates:

\[
\phi_{pp} = \frac{1}{\sqrt{2}} (\phi_{pp} - \phi_{nn}) + \frac{1}{\sqrt{2}} (\phi_{pp} + \phi_{nn})
\]

which are eigenstates with eigenvalues \( \pm 1 \). Since the pion has G-parity \(-1\), it follows that

(i) only half the decays can be with an odd number of \( \pi \)'s; phase-space arguments then overwhelmingly favour \( 3\pi \)-decay.

(ii) In the other half of decays, if pions are involved, there must always be an even number. Since we saw earlier that decay into \( 2\pi \) is strictly forbidden, some other mode such as \( 4\pi \)-decay, or a non-pionic decay such as \( K\bar{K} \), must occur in half the cases.

* * *

9253/SC/kw
LEcTUrE IX

STraNgE PaRTiClES

So far I have not talked about strange particles, etc. This must appear rather scandalous to you, but actually it is not quite so bad. The invariance principles and methods we have been discussing are at once applicable; only the situation becomes very complex. Because of our ignorance in this field, it is often not easy to decide between alternatives. Also the number of new particles means that detailed schemes for describing their symmetries become very complex, admitting considerable extension of the ideas we discussed. This is so, particularly for isospace which, like ordinary space, has acquired a fourth dimension as well as reflection properties. Time will not allow me to discuss these ideas and I will have to select very fiercely what to talk about. The outstanding idea dominating strange particle physics is, of course, that of strangeness. It was introduced independently by Gell-Mann and Nishijima to explain the large production cross-sections for strange particles (associated production) and the long lifetimes (weak decay modes). I will start with gauge invariance, a very important topic which I have not yet mentioned; it connects up naturally with strangeness.

When discussing isospace, we characterized charge conservation as invariance under rotation about the 3-axis in isospace. For a rotation through $\alpha$, the meson field transforms thus:

$$\begin{align*}
\varphi_1 &\rightarrow \varphi_1' = \varphi_1 \cos \alpha - \varphi_2 \sin \alpha \\
\varphi_2 &\rightarrow \varphi_2' = \varphi_1 \sin \alpha + \varphi_2 \cos \alpha \\
\varphi_3 &\rightarrow \varphi_3' = \varphi_3,
\end{align*}$$

(IX.1a)

In terms of the charged fields

$$\left\{ \begin{array}{c}
\varphi \\
\varphi^* \\
\varphi^* \\
\end{array} \right\} = \frac{1}{\sqrt{2}} \left( \varphi_1 + i\varphi_2 \right).$$

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The first two equations give
\[
\begin{align*}
\phi & \rightarrow \phi' = \phi \, e^{i\alpha} \\
\bar{\phi} & \rightarrow \bar{\phi}' = \bar{\phi} \, e^{-i\alpha}
\end{align*}
\]  \hspace{1cm} \text{(IX.1b)}

A transformation of this kind is called a gauge transformation. Charge conservation now means invariance of the theory, i.e. of \( \mathcal{L} \), under gauge transformation. This is ensured if \( \mathcal{L} \) only involves bilinear terms like \( \phi \bar{\phi} \), etc., but not terms like \( \phi^2 \). All our Lagrangians have been of this type.

It is straightforward to apply Noether's theorem (see Lecture III) to a particular gauge-invariant Lagrangian to obtain the conserved current \( S_\mu (x) \). For an infinitesimal gauge transformation
\[
\delta \phi = \phi' - \phi = i \alpha \phi, \quad \delta \bar{\phi} = -i \alpha \bar{\phi}
\]  \hspace{1cm} \text{(IX.2)}

the current
\[
S_\mu = \left\{ \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi + \frac{\partial \mathcal{L}}{\partial \bar{\phi}_{,\mu}} \bar{\phi} \right\}
\]  \hspace{1cm} \text{(IX.3)}

(where \( \phi_{,\mu} = \partial \phi / \partial x_\mu \), etc.) satisfies, by Noether's theorem,
\[
\frac{\partial S_\mu (x)}{\partial x_\mu} = 0 .
\]  \hspace{1cm} \text{(IX.4)}

One easily verifies that, for example, for the Lagrangian for charged mesons [Eq. (V.9a)], equation (IX.3) gives the charge-current density. In particular, one finds that
\[
Q = e \int [S_\mu (x)/i] \, d^3 x = e \sum_k \{ N_+ (k) - N_- (k) \}
\]  \hspace{1cm} \text{(IX.5)}

\( N_\pm (k) \) being the occupation number operators for \( \pi^\pm \) of momentum \( k \). Hence interpretation of \( Q \) as charge, and \( S_\mu (x) \) as current, etc., is obvious.
This method can be applied generally, also to other particles, e.g. protons. Again the Dirac Lagrangian was bilinear in $\bar{\psi}$ and $\psi^\dagger$ ensuring conservation of charge. For a system of different particles we then require invariance of $\mathcal{L}$ if all charged particle fields are jointly subjected to the same gauge transformation, i.e. if $F$ is a field operator which emits positively charged particles (like $\phi$ in our convention) then

$$F \rightarrow F' = F e^{i\alpha}$$

and if $G$ emits negatively charged particles, then of course

$$G \rightarrow G' = G e^{-i\alpha}.$$ 

In this way charge conservation is ensured. For example, in the Yukawa interaction:

$$\bar{\psi}_p \gamma^5 \psi_n \phi$$

\(\bar{\psi}_p\) creates protons, under gauge transformation, \(x e^{i\alpha}\)

\(\psi_n\) involves only neutrals, so it is unaffected by the gauge transformation.

In this way one can construct charge-conserving theories by constructing gauge-invariant interactions, etc. This is known as the charge-gauge, because of its connection with charge conservation.

One important generalization must be mentioned. The above gauge transformations are not sufficiently general to cope with the interactions of charged particles with the electromagnetic field. In this case it is well known that the potentials, which enter the interactions [e.g. in quantum electrodynamics]

$$\text{ie} \bar{\psi} \gamma_\mu A_\mu \psi$$

are arbitrary to the extent of a transformation

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{\partial \Lambda(x)}{\partial x^\mu},$$ (IX.6a)

with $\Box^2 \Lambda = 0$. In order that the theory be invariant under this
transformation (IX.6a) too, one must demand that the fields be at the
same time transformed according to

\[ \psi \rightarrow \psi e^{i \Lambda(x)}, \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i \Lambda(x)}. \]  

(IX.6b)

We must thus admit these coupled gauge transformations [Eqs. (IX.6a) and
(IX.6b)] to ensure invariance under electromagnetic gauge transformations.
These coupled gauge transformations are known as gauge transformation of
the second kind.

We have obtained charge conservation by demanding invariance
under gauge transformation applied to all fields representing charged
particles. But the fun only starts. Independently of the charge-gauge
transformations we can apply different independent gauge transformations
to arbitrarily selected groups of non-hermitian fields. Each such gauge
transformation will lead to a conservation law, if the original \( S \) is in-
variant under it for that group of particles, and there is clearly no
reason why these sets should not overlap.

It was pointed out by d'Espagnat and Prentki that in this way
one can understand the Gell-Mann - Nishijima scheme. We divide the
strongly interacting particles into four groups.

1. **Charged particles**

   For these a charge-gauge group of transformations is defined
as above and we demand invariance under these transformations to get
charge conservation.

2. **\( \pi^0 \)-meson**

   This is represented by a hermitian field, so we leave it alone.

3. **Baryons**

   \( \Sigma^0, \Xi^- \Sigma^+, \Sigma^0 \Xi^- \Lambda^0, \text{p n, and their antiparticles.} \) (These are
   the heavy spin \( \frac{1}{2} \) fermions.) We demand invariance under the simultaneous
   baryon gauge transformations

   \[ \varphi \rightarrow \varphi e^{i \beta}, \quad \varphi^\dagger \rightarrow \varphi^\dagger e^{-i \beta}. \]  

   (IX.7)
4. Hypercharge gauge

Similarly we demand invariance under a gauge transformation applied to all isofermions. (The reason for this nomenclature will only become clear later.) For p, n, K⁺, K⁰ and their antiparticles

\[ \varphi \rightarrow \varphi e^{iy}, \quad \varphi^+ \rightarrow \varphi^+ e^{-iy}; \]  

(IX.8a)

and for Ξ⁰, Ξ⁻ and their antiparticles

\[ \varphi \rightarrow \varphi e^{-iy}, \quad \varphi^+ \rightarrow \varphi^+ e^{iy}. \]  

(IX.8b)

With these invariance conditions, the strong interactions are strangeness conserving and vice-versa. Each gauge group leads to a conserved quantity. For the baryons this is the baryon number \( N \) defined by

\[ N = \text{number of baryons} - \text{number of antibaryons}. \]  

(IX.9)

This is a natural generalization of what I called the nucleon number when dealing with p and n only. Schwinger calls this the nucleonic charge.

It is clear that baryon conservation demands that the Lagrangian should be bilinear in baryons and antibaryons, as explained above.

In the same way the conserved quantity hypercharge \( Y \) is defined by

\[ Y = \text{number of isofermions} - \text{number of anti-isofermions}. \]  

(IX.10)

Here p, n, K⁺ and K⁰ are isofermions, Ξ⁰ and Ξ⁻ are anti-isofermions. (They have the opposite transformation property under hypercharge gauge transformations.) The antiparticles are of the 'opposite' nature: i.e.,

\( \bar{p}, \bar{n}, K⁻ \text{ and } \bar{K}⁰ \) are anti-isofermions

\( \bar{Ξ}⁰, \bar{Ξ}⁻ \) are isofermions.
Again $Y$-conservation is ensured if $\mathcal{L}$ is appropriately bilinear. Strangeness $S$ is defined by

$$S = Y - N$$  \hspace{1cm} (IX.11)

and is conserved if $Y$ and $N$ are conserved. This is the case for strong interactions. For weak interactions the baryon number $N$ (and, of course, charge $Q$) are conserved but not the hypercharge $Y$. In fact, for weak interactions the selection rule

$$|\Delta S| = |\Delta Y| = 1$$

seems very well established.

The surprising thing is that the above formulation is without reference to isospace. We have, however, talked about "isofermions" in anticipation of a connection. We now show how this connection is made.

To do this, we combine the isofermions into Pauli isospinors, i.e. two-component quantities transforming like spinors in isospace

$$\mathcal{N} = \begin{pmatrix} P \\ N \end{pmatrix}, \quad K = \begin{pmatrix} K^+ \\ K^- \end{pmatrix}, \quad \Xi = \begin{pmatrix} \Xi^+ \\ \Xi^- \end{pmatrix}.$$  

In each case the upper component corresponds to $\tau_3 = +1$, the lower to $\tau_3 = -1$. The nucleon case has been treated in Lecture VIII. Amongst other things we calculated $T_3$ for a system of nucleons [Eq. VIII.6b)]. Correspondingly we now find for a system of $\mathcal{N}$, $K$ and $\Xi$ and their antiparticles that

$$T_3 = \sum_{i=\mathcal{N},K,\Xi} \frac{i}{2} \left( N^i_\alpha - N^i_\beta - N^i_\bar{\alpha} + N^i_\bar{\beta} \right).$$  \hspace{1cm} (IX.12)

Here $\alpha, \beta$ denotes the particles corresponding to $\tau_3 = \pm 1$, and $\bar{\alpha}, \bar{\beta}$ the
corresponding antiparticles. Thus the \( i = \mathcal{M} \) term in Eq. (IX.12) can be written as
\[
\frac{1}{2}(N_p - N_{\bar{p}} - N_n + N_{\bar{n}})
\]
i.e. for
\[
p \quad \bar{p} \quad n \quad \bar{n}
\]
\[
T_3 = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2}.
\]

The hypercharge \( Y \) (\( = \) isofermion number) was defined in Eq. (IX.10) as the number of isofermions \((p, n, K^+, K^0, \Xi^-, \Xi^0)\) minus the number of anti-isofermions \((\bar{p}, \ldots, \Xi^0)\). In our present notation
\[
Y = \sum_{i=N,K} \left( N^i_\alpha - N^i_\bar{\alpha} + N^i_\rho - N^i_{\bar{\rho}} \right) - \left( N^i_{\Xi^0} - N^i_{\Xi^0} + N^i_{\Xi^+} - N^i_{\Xi^-} \right).
\]

(IX.13)

From Eqs. (IX.12) and (IX.13), one sees that \( Y \) and \( T_3 \) are related to the charge of a system through
\[
Q = T_3 + \frac{1}{2} Y.
\]

(IX.14)

This is the generalization of Eq. (VIII.6c) for nucleons. From our definition of strangeness, Eq. (IX.11)
\[
S = Y - N = 2(Q - T_3) - N.
\]

(IX.15)

This is the original definition of strangeness. We see that if we have conservation of charge \( (Q) \) and of baryon number \( (N) \), then conservation of strangeness \( (S) \) follows from conservation of \( T_3 \), i.e., from invariance of rotations about the 3-axis in isospace.

So far charge independence has not been assumed. In particular, therefore, we can discuss electromagnetic (EM) interactions. From Eq. (IX.15) it follows since charge and baryon number are conserved that \( T_3 \)-conservation implies conservation of \( S \).

The 'usual' EM interactions are obtained by replacing in \( L \),
\[
\frac{\partial L}{\partial x^\mu}
\]

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\[ \left( \frac{\partial}{\partial x_\mu} - i e Q A_\mu \right) \varphi, \text{ etc.} \]

where \( eQ \) is the charge associated with the \( \varphi \)-field. This EM interaction, of course, leaves \( T_3 \) invariant; hence also \( S \). We shall restrict ourselves to these EM interactions (principle of minimal EM interactions) as this excludes EM interactions which would permit fast EM decays of strange particles, such as

\[ \Sigma^+ \rightarrow p + \gamma \]

This could, for example, occur through a term

\[ \mu \bar{\psi}_p G_{\alpha\beta} \rho_{\alpha\beta} \psi_{\Sigma^+} \]

(here \( G_{\alpha\beta} \) is the EM field tensor) which satisfies all invariance requirements but is not invariant under rotations about the \( \beta \)-axis in isospace.

The exceptional EM decay which does occur is, of course,

\[ \Sigma^0 \rightarrow \Lambda^0 + \gamma \]

This occurs via the mechanism

\[ \Sigma^0 \xrightarrow{\text{strong}} \Sigma^+ + \pi^+ \xrightarrow{\text{EM}} \Sigma^+ + \pi^+ + \gamma \xrightarrow{\text{strong}} \Lambda^0 + \gamma \]

and has a very short lifetime (experimentally \( \ll 10^{-11} \) sec, theoretically \( 10^{-19} \) sec).

The classification of particles in terms of isospace acquires its real justification when we assume that charge independence does hold for all strong interactions, i.e. that strong interactions are rotation invariant in isospace. This is a severe restriction on the form of the interaction. We must now also classify the strange particles other than isofermions with respect to definite transformation properties in isospace. These residual particles, \( \Sigma^{2\circ} \) and \( \Lambda^0 \), are taken as isovectors and isoscalars respectively. We have previously classified the \( \pi \) meson as an isovector.

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I briefly summarize these properties of the elementary particles:

<table>
<thead>
<tr>
<th>Name</th>
<th>T</th>
<th>Y</th>
<th>Q = Y / 2 + T_3</th>
<th>N</th>
<th>S = Y - N</th>
<th>Transformation in isospace</th>
<th>Transformation in space</th>
</tr>
</thead>
<tbody>
<tr>
<td>π^+</td>
<td>1</td>
<td>0</td>
<td>T_3</td>
<td>0</td>
<td>0</td>
<td>isovector</td>
<td>ps., scalar</td>
</tr>
<tr>
<td>K^+</td>
<td>½</td>
<td>1</td>
<td>½ + T_3</td>
<td>0</td>
<td>1</td>
<td>isospinor</td>
<td></td>
</tr>
<tr>
<td>hn</td>
<td>½</td>
<td>1</td>
<td>½ + T_3</td>
<td>1</td>
<td>0</td>
<td>spinor</td>
<td></td>
</tr>
<tr>
<td>Λ</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>isoscalar</td>
<td></td>
</tr>
<tr>
<td>Σ^0</td>
<td>½</td>
<td>0</td>
<td>T_3</td>
<td>1</td>
<td>-1</td>
<td>isovector</td>
<td></td>
</tr>
<tr>
<td>Σ^-</td>
<td>½</td>
<td>-1</td>
<td>-½ + T_3</td>
<td>1</td>
<td>-2</td>
<td>isospinor</td>
<td></td>
</tr>
</tbody>
</table>

Denoting corresponding quantities for antiparticles by an affix $A$, one has

$$
\begin{align*}
\zeta^A &= -Q \\
N^A &= -N \\
\chi^A &= -Y
\end{align*}
$$

These also follow from gauge invariance with respect to charge, baryon and hypercharge. From the defining equations then

$$
\begin{align*}
T_3^A &= -T_3 \\
S^A &= -S
\end{align*}
$$

The above ideas attain a particularly lucid form if one extends the permissible operations in isospace to include inversion. This was done by d’Espagnat and Frenki.
It can be shown that under invariance in isospaces, a state $\Phi$ transforms into

$$\Phi' = I \Phi = e^{-i \frac{\pi}{2} Y} \Phi$$

where $I$ is the operator of inversion in isospaces and $Y$ is the hypercharge of the state $\Phi$. We also know that under a rotation through $\pi$ about the $3$-axis $\Phi$ transforms into

$$\Phi' = e^{-i \pi T_3} \Phi$$

(cf. Roman, "Theory of Elementary Particles", p. 437), where $T_3$ is the three-component of isospin in the state $\Phi$. Thus carrying out these transformations in succession

$$\Phi \xrightarrow{I} e^{-i \frac{\pi}{2} Y} \Phi \xrightarrow{T_3(\pi)} e^{-i \pi(T_3 + \frac{1}{2} Y)} \Phi.$$

Thus invariance under both these transformations implies that

$$T_3 + \frac{1}{2} Y$$

is a constant. This is just what we previously defined as the charge $Q$ of the system [Eq. (IX.14)]. Thus invariance under inversion ensures the correct linear relation between $Q$ and $T_3$. Previously it was necessary to introduce

$$Q = T_3 \quad \text{for } \pi \text{'s}$$

$$Q = \frac{1}{2} + T_3 \text{ for } p, n$$

etc., in a rather ad hoc manner. Now it follows automatically.

One can try and obtain the most general expression for strong interactions subject to certain invariance requirements.
If one demands:

1) charge conservation
2) charge independence
3) baryon conservation, and
4) hypercharge conservation (or equivalently invariance under inversion in isospace),

and restricts oneself to Yukawa-type interactions not involving derivative couplings, one is led to the famous d'Espagnat–Prentki interaction containing eight arbitrary coupling constants \( g_1 \ldots g_8 \). It is of the form

\[
\mathcal{H} = i \sum_{\nu,\bar{\nu}} g_{\nu} \gamma_5 \bar{\nu} \gamma \nu + \sum_{i=2}^{A} g_i(BB') + \sum_{i=5}^{8} g_i(BK'). \quad \text{(IX.16)}
\]

Here \( B, B' \) denote baryons, and \( K \) denotes any \( K \)-mesons.

The \( \pi^-N^0 \) coupling constant one knows pretty well

\[
\frac{g_{\pi}}{\sqrt{2}} \sim 15.
\]

Experiments also indicate that the \( K-B \) interactions are much weaker than the \( \pi-B \). For example, experiments on photoproduction of \( \pi^+ \) and \( K^+ \)

\[
\gamma + p \rightarrow n + \pi^+ \\
\gamma + p \rightarrow \Lambda + K^+
\]

suggest

\[
\frac{g_{K}^2}{g_{\pi}^2} \sim \frac{1}{15} \text{ to } \frac{1}{5}
\]

where

\[
g_{\pi} = g_3 \sim g_1, \quad i = 2 \ldots 4 \\
g_{K} \sim g_1, \quad i = 5 \ldots 8.
\]
Unfortunately one's ignorance at this stage, both theoretical and experimental, becomes rather profound. Many schemes, introducing further symmetries of various kinds, have been proposed which enjoy varying degrees of popularity. It is not my purpose to try to discuss these. I only want to mention one idea.

The above assignment of coupling constants divides the strong interactions into two groups:

very strong - involving \( \pi \)'s
medium - involving \( K \)'s.

One may then postulate that

\[
\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_\pi \\
\varepsilon_5 = \cdots = \varepsilon_9 = \varepsilon_K.
\]

(IX.17)

(IX.18)

There is, of course, little experimental evidence for either of these conjectures. There are some theoretical doubts at least about the second one [Eq. (IX.18)].

Equation (IX.17), known as global symmetry, is due to Gell-Mann and Schwinger. If we define

\[
N_1 = \begin{pmatrix} P \\ n \end{pmatrix}, \quad N_2 = \begin{pmatrix} \Sigma^+ \\ (\Lambda^0 + \Sigma^0) / \sqrt{2} \end{pmatrix}, \quad N_3 = \begin{pmatrix} \Lambda^0 + \Sigma^0 \end{pmatrix}, \quad N_4 = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}
\]

then the \( \pi-B \) interactions in Eq. (IX.16) (i.e. first four terms) can be written as

\[
ig \sum_{i=1}^{4} \bar{N}_i \gamma_5 T_i N_i
\]

(IX.19)

displaying a rather nice-looking symmetry. The difficulty of knowing whether it has anything to do with nature is, of course, partly due to
the fact that global symmetry is spoiled by the presence of medium strong and weaker interactions, and the medium strong interactions are, after all, pretty strong; so large experimental deviations from global symmetry may not disprove this scheme. They may just mean that we are not behaving reasonably in switching off the medium strong interactions.

* * *

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LECTURE X

INVERSION IN ISO SPACE

We mentioned above, without going into details, that hypercharge conservation can be interpreted as due to invariance under inversion in isospace. I now want to establish this convention more properly.

Introducing inversion as well as rotations in isospace, all quantities become double-valued in the usual way. For example, we now have isoscalars for which

\[ s \rightarrow \mathbb{I} s \mathbb{I}^{-1} = s' = \pm s \]  \hspace{1cm} (X.1)

and isopseudoscalars for which

\[ p \rightarrow \mathbb{I} p \mathbb{I}^{-1} = p' = -p \]  \hspace{1cm} (X.2)

e tc.

How does an isospinor \( \psi \) transform? We want the transformation

\[ \psi \rightarrow \mathbb{I} \psi \mathbb{I}^{-1} = \psi' = A \psi \]  \hspace{1cm} (X.3)

To find \( A \) we consider the "vector"

\[ \psi^\dagger \mathbb{I} \psi \]

which under inversion transforms into

\[ \psi^\dagger \mathbb{I}^{-1} A \mathbb{I} A \psi = \pm \psi^\dagger \mathbb{I} \psi , \]  \hspace{1cm} (*)

\[ \hspace{1cm} (X.4) \]

\[ (*) \quad \text{One easily proves that} \quad A \text{ is unitary, i.e.} \quad A^\dagger = A^{-1}. \text{ From} \]

\[ A^\dagger \mathbb{I} A = \pm \mathbb{I} \]

\[ \text{taking the inverse, since} \quad \mathbb{I}^{-1} = \mathbb{I}, \]

\[ A^{-1} \mathbb{I} A^\dagger = \pm \mathbb{I}, \]

whence

\[ (A A^\dagger) \mathbb{I} (A A^\dagger) = \mathbb{I}. \]

But \( S \mathbb{I} S = \mathbb{I} \) implies \( S = \pm \mathbb{I} \). To see this we write

\[ S = a + \sum b_m \tau_m \]

Then by direct calculation using only \([\tau_m, \tau_n]_+ = 2\delta_{mn}\), we find that \( S \mathbb{I} S = \mathbb{I} \) has \( a = \pm 1 \), \( b_m = 0 \) as the only solutions. Thus \( S = \pm \mathbb{I} \). But in our case \( S = A A^\dagger \) is positive definite, so we must have \( A A^\dagger = + \mathbb{I} \).
the \( \pm \) sign depending on whether \( \psi \uparrow \psi \) is an axial or polar vector. But there is no \( 2 \times 2 \) matrix which anticommutes with \( r_1, r_2 \text{ and } r_3 \). (This follows from the fact that we can always write

\[
A = a I + b_i r_i,
\]

together with the anticommutation laws for the \( r_i \)'s.) Hence we can only have the \( + \) sign in Eq. (X.4). This is, of course, satisfied if

\[
A = \lambda I
\]

and this is the only solution. To find \( \lambda \) we apply inversion twice. Then

\[
A^2 = \lambda^2 I = \pm I. \tag{X.5}
\]

The \( \pm \) sign here is due to the fact that for spinors the identity is represented by \( \pm I \); this is because under a rotation through \( 2\pi \) about any axis a spinor changes sign. Hence

\[
A = \pm 1, \pm i.
\]

Now we know that

\[
\psi' = \pm \psi
\]

is a rotation, and is thus never equivalent to an inversion. Hence for an inversion we can only have

\[
\psi' = \pm i \psi.
\]

Because of the inherently double-valued nature of spinors we cannot separate these two cases. However, we can distinguish the relative parity of two spinors. So we get two parity classes, iso-spinors of the first kind:

\[
\xi' = I \xi I^{-1} = \pm i \xi, \tag{X.6}
\]

and of the second kind

\[
\eta' = I \eta I^{-1} = \mp i \eta. \tag{X.6a}
\]

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It is now easy to construct covariants in isospace out of spinors and \( r \)-matrices. Eqs. (X.6) and (X.6a) give the transformation properties under inversion. Under rotation through \( \Theta \) about the \( i \)th axis one has

\[
\xi \to e^{i\frac{1}{2} \Theta_{1}^i} \xi \quad \eta \to e^{i\frac{1}{2} \Theta_{1}^i} \eta \quad (X.7)
\]

whence

\[
\xi^\dagger \to \xi^\dagger e^{-i\frac{1}{2} \Theta_{1}^i} \quad \eta^\dagger \to \eta^\dagger e^{-i\frac{1}{2} \Theta_{1}^i} \quad (X.7a)
\]

and taking complex conjugates

\[
\tilde{\xi} \to \tilde{\xi} e^{i\frac{1}{2} \Theta_{1}^i} \quad \tilde{\eta} \to \tilde{\eta} e^{i\frac{1}{2} \Theta_{1}^i} \quad (X.7b)
\]

From Eqs. (X.7), (X.7a) and (X.7b) one easily constructs all possible covariants. For example, \( \xi^\dagger \xi \) is a scalar:

under inversion : \( \to (\mp i \xi^\dagger)(\pm i \xi) = \xi^\dagger \xi \)

under rotation : \( \to (\xi^\dagger e^{-i\frac{1}{2} \Theta_{1}^i})(e^{i\frac{1}{2} \Theta_{1}^i} \xi) = \xi^\dagger \xi \).

But we can also get new types of covariants by combining first- and second-kind spinors. For example

\[
\tilde{\xi} \tau_2 \eta \quad \text{is also a scalar :}
\]

under inversion : \( \to (\pm i \tilde{\xi}) \tau_2 (\mp i \eta) = \tilde{\xi} \tau_2 \eta \)

under rotations : \( \to \tilde{\xi} e^{i\frac{1}{2} \Theta_{1}^i} \tau_2 e^{i\frac{1}{2} \Theta_{1}^i} \eta \).

Now for \( i = 2 \) \( \tau_1^* = -\tau_2 \) and \( \tau_2 \) commutes with \( e^{i\frac{1}{2} \Theta_{1}^i} \), so one gets simply \( \tilde{\xi} \tau_2 \eta \).

For \( i = 1 \) or \( 3 \) \( \tau_1^* = \tau_1 \), but now
\[ \tau_2 \cdot e^{i \frac{1}{2} \Theta_1} = \tau_2 \cdot \left( \cos \frac{1}{2} \Theta + i \tau_1 \sin \frac{1}{2} \Theta \right) \]

\[ = \left( \cos \frac{1}{2} \Theta - i \tau_1 \sin \frac{1}{2} \Theta \right) \tau_2 = e^{-i \frac{1}{2} \Theta_1} \tau_2 , \]

so one again ends up with \( \tilde{\xi} \tau_2 \eta \).

In this way one gets 12 possible covariants, listed in the table below:

<table>
<thead>
<tr>
<th>Isoscaler</th>
<th>( \xi^\dagger \xi )</th>
<th>( \eta^\dagger \eta )</th>
<th>( \tilde{\xi} \tau_2 \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isopseudoscalar</td>
<td>( \tilde{\xi} \tau_2 \xi )</td>
<td>( \tilde{\eta} \tau_2 \eta )</td>
<td>( \xi^\dagger \eta )</td>
</tr>
<tr>
<td>Isovector</td>
<td>( \tilde{\xi} \tau_2 \eta \xi )</td>
<td>( \tilde{\eta} \tau_2 \eta \eta )</td>
<td>( \xi^\dagger \eta \eta )</td>
</tr>
<tr>
<td>Isopseudovector</td>
<td>( \xi\eta^\dagger \xi )</td>
<td>( \eta\xi^\dagger \eta )</td>
<td>( \eta \tilde{\xi} \eta \eta )</td>
</tr>
</tbody>
</table>

(X.8)

It is not difficult to construct the operator I explicitly. Writing it as

\[ I = e^{-i \frac{1}{2} \Theta Y} \]

we look for the Hermitian operator \( Y \).

For an isoscalar, we have from Eq. (X.1) that

\[ Y = 0 \]

(X.10)

and similarly \( Y \) is zero for an isopseudovector defined to transform under inversion like

\[ I \alpha I^{-1} = \alpha . \]

(X.11)

The only other case I shall consider is that of isospinors.

If

\[ \xi = \begin{pmatrix} \psi_x \\ \psi_\theta \end{pmatrix} \]

(X.12)
is an isospinor of the first kind, it is not difficult to show that

\[ Y = \left( N_\alpha - N_\bar{\alpha} + N_\beta - N_\bar{\beta} \right) \]  
(X.13)

and if

\[ \eta = \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix} \]  
(X.12a)

is an isospinor of the second kind, then

\[ Y = -\left( N_\alpha - N_\bar{\alpha} + N_\beta - N_\bar{\beta} \right). \]  
(X.13a)

Here \( N_\alpha, N_\beta \) are the occupation number operators for particles of kind \( \alpha \) and \( \beta \); \( N_\bar{\alpha}, N_\bar{\beta} \) similarly for antiparticles. Eqs. (X.13) and (X.13a) are seen to be identical with Eq. (IX.13). Thus \( Y \) is again the hyperchange, if one considers nucleons and \( K \)-mesons as isospinors of the first kind and \( \Xi \)'s as isospinors of the second kind.

To verify Eq. (X.13) we write more explicitly

\[ Y = \sum_\mathcal{P} \left[ N_\alpha(\mathcal{P}) - N_\bar{\alpha}(\mathcal{P}) + N_\beta(\mathcal{P}) - N_\bar{\beta}(\mathcal{P}) \right] \]

where \( N_\alpha(\mathcal{P}) \) are now occupation number operators for particles of kind \( \alpha \), and \( \mathcal{P} \) labels their momentum and any other quantum numbers that may be required, e.g. spin.

Since \( \xi \) is linear in absorption and creation operators, it follows that

\[ I \xi I^{-1} = i \xi, \]

provided

\[ I c_\alpha(\mathcal{P}) I^{-1} = ic_\alpha(\mathcal{P}), \]

\( c_\alpha(\mathcal{P}) \) being the absorption operator for a particle of kind \( (\alpha, \mathcal{P}) \).
However, the last equation follows directly for any state \( \Phi(n_\alpha(P), \ldots) \), \( n_\alpha(P) \) being the number of particles of kind \( \alpha \) in this state.

From Eq. (X.9)

\[
I \Phi_\alpha(P) I^{-1} \Phi(n_\alpha(P), \ldots)
\]

\[
= I \Phi_\alpha(P) \exp \left[ \frac{i\pi}{2} \sum \left[ n_\alpha(P) - \ldots \right] \right] \Phi(n_\alpha(P), \ldots)
\]

\[
= \exp \left[ \frac{i\pi}{2} \sum \left[ n_\alpha(P) - \ldots \right] \right] I \sqrt{n_\alpha(P)} \Phi(n_\alpha(P) - 1, \ldots)
\]

\[
= \exp \left[ \frac{i\pi}{2} \sum \left[ n_\alpha(P) - \ldots \right] \right] \exp \left[ -\frac{i\pi}{2} \sum \left[ n_\alpha(P) - 1 - \ldots \right] \right] \sqrt{n_\alpha(P)} \Phi(n_\alpha(P) - 1, \ldots)
\]

\[
= e^{i\pi/2} \alpha_\alpha(P) \Phi(n_\alpha(P), \ldots)
\]

\[
= i \alpha_\alpha(P) \Phi(n_\alpha(P), \ldots)
\]

which completes the proof. For Eq. (X.13a) one proceeds similarly.

Combining the above results we consider a state \( \Phi \) which contains isoscalar, isospinovector and isospinor particles. We find that under inversion

\[
\Phi \rightarrow \Phi' = I \Phi = e^{-i\pi Y} \Phi
\]

(X.14)

with

\[
Y = \sum_{\text{first kind}} \left( N_\alpha - N_{\bar{\alpha}} + N_\beta - N_{\bar{\beta}} \right) - \sum_{\text{second kind}} \left( N_\alpha - N_{\bar{\alpha}} + N_\beta - N_{\bar{\beta}} \right).
\]

(X.14a)

For rotations through \( \Theta \) about the \( i \)th axis, one similarly has

\[
\Phi \rightarrow \Phi' = R_i(\Theta) \Phi = e^{-iT_2 \Theta} \Phi
\]

(X.15)
with [cf. Eq. (IX.12)]

\[ T_3 = \sum_{1st \text{ and } 2nd \text{ kind}} \frac{1}{2} \left( N_u - N_p - N_d + N_s \right) + \sum_{\text{pseudo-vector}} \left( N_+ - N_- \right), \quad (X.15a) \]

where \( N_+ \) are the occupation number operators for the \( T_3 = \pm 1 \) components of the isospin vector particles.

If we now consider a rotation through \( \pi \) about the \( 3 \)-axis following an inversion, then we see that

\[ \phi \rightarrow e^{-i\pi T_3} e^{-i\frac{\pi}{2} Y} \phi = e^{-i\pi (T_3 + \frac{1}{2} Y)} \phi. \quad (X.16) \]

This operation is the same as reflection in the \( 1,2 \) plane.

Demanding invariance under both these transformations, implies that

\[ (T_3 + \frac{1}{2} Y) \]

is conserved, and this quantity we previously interpreted as the charge of the system. Thus invariance under rotation and inversion in isospace, together with the definition of charge by

\[ Q = (T_3 + \frac{1}{2} Y) \quad (X.17) \]

automatically leads to charge conservation. These are the statements I made in Lecture IX. That Eq. (X.17) is the correct definition can, of course, be checked for any particle from Eqs. (X.14a) and (X.15a). By convention one takes the proton as an isospinor of the first kind. Then one indeed finds for a state containing one proton

\[ T_3 = + \frac{1}{2}, \quad Y = 1, \quad Q = 1 \]

and for one neutron

\[ T_3 = - \frac{1}{2}, \quad Y = 1, \quad Q = 0 \quad \text{etc.} \]
From Eq. (X.17) the value of $Y$ follows for any particle, if we know $Q$ and $T_3$. For example, for a $\Lambda^0$ particle, occurring alone, $T = 0$, $T_3 = 0$, $Q = 0$ and hence $Y = 0$, so the $\Lambda^0$ particle is an isoscalar. The $\Xi$ particles occur in pairs, $T = \frac{1}{2}$, $T_3 = \pm \frac{1}{2}$. However, in order to get $Q = -1$ we must have $Y = -1$ for the $\Xi^-$, i.e., the $\Xi$ is an isofermion of the second kind. In this way we get back to the classification of particles given above, except that we now have to distinguish between isoscalar, isopseudoscalar, etc. One obtains

<table>
<thead>
<tr>
<th>$T$</th>
<th>$Y$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0$</td>
<td>1 0</td>
<td>pseudoscalar, isopseudovector (*)</td>
</tr>
<tr>
<td>$K^0$</td>
<td>$\frac{1}{2}$ 1</td>
<td>pseudoscalar, isospinor of 1st kind</td>
</tr>
<tr>
<td>$\rho_n$</td>
<td>$\frac{1}{2}$ 1</td>
<td>spinor, isospinor of 1st kind</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>0 0</td>
<td>spinor, isoscalar</td>
</tr>
<tr>
<td>$\Xi^0$</td>
<td>1 0</td>
<td>spinor, isopseudovector (*)</td>
</tr>
<tr>
<td>$\Xi^-_0$</td>
<td>$\frac{1}{2}$-1</td>
<td>spinor, isospinor of 2nd kind</td>
</tr>
</tbody>
</table>

\*) That these are isopseudovectors follows from the fact that we want to get $Q = 1, 0, -1$ from Eq. (X.17), viz.,

$$ Q = T_3 + \frac{Y}{2} $$

Now $Y = 0$ for isopseudovectors, giving the correct assignment of charge, but for isovectors $Y = \pm 2$ (we have not shown this) and this would give the wrong charge.

Alternatively one can, of course, see this from the interactions, if known. For example, from the $\pi - \Sigma$ case

$$ \bar{\psi} \gamma_5 \Sigma \psi $$

it follows, since $\bar{\psi} \gamma_5 \Sigma \psi$ is an isopseudovector, that $\Sigma$ must be an isopseudovector.

53/56/nc
REFERENCES

4) Feldman and Matthews, Phys.Rev. 102, 1421 (1956).
6) Schwinger, Phys.Rev. 82, 914 (1951).

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