Quantum group symmetry of the Quantum Hall effect 
on the non-flat surfaces

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Abstract

After showing that the magnetic translation operators are not the symmetries of the QHE on non-flat surfaces, we show that there exist another set of operators which leads to the quantum group symmetries for some of these surfaces. As a first example we show that the $su(2)$ symmetry of the QHE on sphere leads to $su_q(2)$ algebra in the equator. We explain this result by a contraction of $su(2)$. Secondly, with the help of the symmetry operators of QHE on the Pioncare upper half plane, we will show that the ground state wave functions form a representation of the $su_q(2)$ algebra.
1 Introduction

After the discovery of the quantum Hall effect (QHE)\textsuperscript{[1]} and the fractional quantum Hall effect (FQHE)\textsuperscript{[2]}, Laughlin introduced his interacting electrons model and showed that the incompressible quantum fluid can explain the appearance of the plateaus in FQHE in filling factor $\nu = \frac{1}{m}$, $m$ an odd integer\textsuperscript{[3]}. In recent years there was many efforts to explain this feature of incompressibility by the symmetries of the quantum mechanics of the two dimensional planar motion of a nonrelativistic particle in a uniform magnetic field. Recently I.I.Kogan\textsuperscript{[4]} and H.T.Sato\textsuperscript{[5]}, by using the magnetic translation operator showed that there exists a quantum group symmetry in this problem. They found that the following combination of the magnetic translation operator, $T_a = \exp(a.(\nabla + iA))$, where $a$ is a constant vector and $A$ is electromagnetic potential, could represent the $su_q(2)$ algebra:

$$J_\pm = \frac{1}{q - q^{-1}}(\alpha_\pm T_\pm \pm + \beta_\pm T_{\pm \pm}) \quad q^{2J_0} = T_{\pm \pm}$$

where $q = \exp(i\frac{B}{2}.(a \times b))$ and $\alpha_+ \beta_- = \beta_+ \alpha_- = -1$. Let $W_{n,\bar{n}} = T_a$ where $a_i = \epsilon_{ij} n_j$. It can be shown that $W_{n,\bar{n}}$ satisfies the Fairlie-Fletcher-Zachos (FFZ) trigonometric algebra\textsuperscript{[6]}. This algebra in the weak field limit ($B \to 0$) leads to $w_\infty$ algebra, the algebra of the area-preserving diffeomorphism\textsuperscript{[7]}. Therefore this $su_q(2)$ symmetry indicates the incompressibility feature of the FQHE. The same symmetry was also found for the topological torus\textsuperscript{[8]}.

In this paper we will study the same quantum group symmetry for non flat surfaces. In section 2 we will show that the magnetic traslation operators are the symmetries of the Hamiltonian only when the metric is flat. Therefore in the case of non flat surfaces we first must look for the symmetry operators and after that try to find any quantum group stucture of it. In section 3 we will begin with the first non-trivial surface, the sphere.
The problem of the motion of the electrons in the presence of the magnetic monopole and when the electrons are restricted to move on a sphere, was first considered by Haldane. He formulated the problem such that the symmetry algebra of the Hamiltonian is $su(2)$ algebra, with the generators which are represented by a special combination of the rotation and gauge transformation. We will consider the group elements of this algebra with non constant parameters. That is the set of maps from $S^2$ to $SU(2)$. By studying its multiplication law we will recover the FFZ algebra for a special region of the sphere. We will try to explain this appearance of $su_q(2)$ from $su(2)$ by a special contraction of $su(2)$.

It is well known that the group of automorphisms of any genus $g \geq 2$ compact Riemann surface is discrete. So to look for any quantum group symmetry we have to consider non-compact surfaces. For this purpose we consider the Poincare upper half plane in section 3, and we will show that the $su_q(2)$ algebra is the symmetry of this surface.

After the writing of this paper was nearly finished, we was aware about the recent preprint [10] in which the quantum group symmetry on the sphere had been discussed.

2 Symmetry properties of the magnetic translation operator

Consider a particle on a Riemann surface interacting with a monopole field. That is the integral of the field strength out of the surface is different from zero. The natural definition of the constant magnetic field is $[11]$

$$F_{\mu\nu} = B\sqrt{g}e_{\mu\nu} ,$$
and the Hamiltonian of the electron is given by

\[ H = \frac{1}{2m} \frac{1}{\sqrt{g}} (\partial_{\mu} - iA_{\mu}) \sqrt{g} g^{\mu\nu} (\partial_{\nu} - iA_{\nu}) = \frac{1}{2m} \nabla_{\mu}^2 + \frac{B}{2m}, \]

where \( \nabla_{\mu} = \partial_{\mu} - iA_{\mu}. \) In Ref. (11) this Hamiltonian was solved by choosing some special metrics. Now consider the magnetic translation operator \( T_\xi = e^{\xi_{\mu} D_\mu} \) which acts on scalars. Here \( \xi = \xi_{\mu} \partial_{\mu} \) is a vector field and \( D_{\mu} = \partial_{\mu} + iA_{\mu}. \) The operators \( T_\xi \) is a symmetry operator only when

\[ [D_{\mu}, \nabla_{\nu}] = -\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = 0 \]

In other words we must be able to choose the symmetric gauge. But solving the eqs. (1) and (2) give the following condition

\[ \partial_{\mu} (B \sqrt{g}) = 0 \] (3)

As \( B \) is constant, this equation shows that the symmetric gauge is possible only when the surface is flat. Therefore \( T_\xi \) does not commute with the Hamiltonian when the surface is not flat.

### 3 Electron on sphere

Consider an electron which is restricted to move on a sphere with radius \( R, \) in the presence of the magnetic monopole at the center of the sphere. The flux of the magnetic field \( B \) is quantized by Dirac quantization condition \( B = \frac{\hbar S}{4\pi R^2}, \) where \( S \) is half integer. The single-particle Hamiltonian is [9]

\[ H = \frac{\Lambda^2}{2mR^2}, \]

where \( \Lambda = r \times [-i\hbar \vec{\nabla} + eA], \) \( A \) satisfies \( \vec{\nabla} \times A = B\Omega (\Omega = \frac{R}{\hbar}) \) and \( \Lambda.\Omega = 0. \) By using gauge freedom, the electromagnetic potential can be taken

\[ A = -\frac{\hbar S}{eR} \cot \theta \hat{\phi} \]

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The eigenvalues of $\Lambda^2$ are $(\ell(\ell + 1) - S^2)\hbar^2$ and the first Landau Level is obtained for $\ell = s$ and is equal to $\hbar w_s/2 \left( w_s = eB/m \right)$. It can be shown that the $\Lambda_i$'s satisfy the following relations

$$[\Lambda_i, \Lambda_j] = i\hbar e_{ijk}(\Lambda_k - \hbar S\Omega_k) \tag{6}$$

Let $L = \Lambda + \hbar s\Omega$, then

$$[L_i, L_j] = i\hbar e_{ijk}L_k \tag{7}$$

The operators $L_i$ are the generators of the symmetries of the Hamiltonian: $[H, L_i] = 0$.

Now consider the group elements of this algebra

$$R_\xi = e^{\frac{i}{\hbar} L_i \frac{\xi_i}{2}} \quad \text{and} \quad R_\eta = e^{\frac{i}{\hbar} L_i \frac{\eta_i}{2}} \tag{8}$$

where $\xi = \xi \hat{\theta}$ and $\eta = \eta \hat{\phi}$. The product of these operators is

$$R_\xi R_\eta = \exp\left\{ \frac{i}{\hbar R}(\xi + \eta) \cdot \Lambda - \frac{1}{2R^2\hbar^2}[\xi \cdot \Lambda, \eta \cdot \Lambda] + \frac{i}{3\hbar R} [\xi \cdot \Lambda, M] + \ldots \right\}, \tag{9}$$

where $M$ denotes the second term of the exponent. A simple calculation shows that

$$[\xi \cdot \Lambda, \eta \cdot \Lambda] = -i\hbar\xi\eta(\hbar S + \cot g\theta\Lambda_\theta),$$

where $\Lambda_\theta = \hbar(S\cot g\theta + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi})$. Now the equality $\xi \cdot \Lambda = \xi \Lambda_\theta$ implies that $[\xi \cdot \Lambda, M] = 0$.

Therefore

$$R_\xi R_\eta = \exp\left\{ \frac{i}{\hbar R}(\xi + \eta) \cdot \Lambda + \frac{i\xi\eta}{2R^2 \sin^2 \theta}(S + \cos \theta \frac{\partial}{\partial \varphi}) \right\}. \tag{10}$$

If we restrict ourselves to the region $\theta = \frac{\pi}{2}$, we find that

$$R_\xi R_\eta = e^{\frac{i\xi B\hbar}{2R}} R_{\xi + \eta} \tag{10}$$

In this way we recover the magnetic translation algebra and therefore the $su_q(2)$ algebra with $q = \exp\left( \frac{i\xi}{2R} B \cdot (\xi \times \eta) \right)$. This result shows that one can generate the $su_q(2)$ algebra by the contraction of the $SU(2)$ group elements. This result has origin in the fact that the
contraction of $su(2)$ algebra leads to Heisenberg algebra. Let $H, X^{\pm}$ be the generators of $su(2)$:

$$[H, X^{\pm}] = \pm 2X^{\pm}, \quad [X^{\pm}, X^{-}] = H.$$ 

Assume $H'$ and $P^{\pm}$ in $H = H' + 1/e^2$ and $P^{\pm} = \epsilon X^{\pm}$. Then

$$[H', P^{\pm}] = \pm 2P^{\pm}, \quad [P^{+}, P^{-}] = \epsilon^2 H + 1.$$ 

At the limit $\epsilon \to 0$ we have

$$[H', P^{\pm}] = \pm 2P^{\pm}, \quad [P^{+}, P^{-}] = 1,$$

which is the Heisenberg algebra. Now the magnetic translation operator in the plane has the following expression in terms of $b = 2p_z - i\frac{B}{2} \zeta$ and $b^\dagger = 2p_z + i\frac{B}{2} \zeta$ [4]

$$T_a = W_{\alpha, \bar{\alpha}} = \exp \left( \frac{1}{2} (n b^\dagger - \bar{n} b) \right)$$

where the commutation relation of the operators $b$ and $b^\dagger$ is $[b, b^\dagger] = 2B$. Therefore $T_a$ are the exponential of the Heisenberg algebra. But on the sphere $R_\zeta$ are the exponential of the $su(2)$ algebra. So the above contraction of $su(2)$ to Heisenberg algebra shed some light on the reduction of the algebra of $R_\zeta$ to the $su_q(2)$ algebra.

4 Electron on the Poincare upper half plane

In this section we consider the Poincare upper half plane $H = \{ z = x + iy, y > 0 \}$, with the following metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$ 

For a covariant constant magnetic field $B$, a particular gauge choice leads to

$$A_z = A_\bar{z} = \frac{B}{2y}.$$
In this gauge the Hamiltonian (1) reduces to (for simplicity we take \( m = 2 \))

\[
H = -y^2 \partial \bar{\partial} + \frac{iB}{2} y(\partial + \bar{\partial}) + \frac{B^2}{4},
\]

(11)

and the ground states with energy \( B/4 \) are given by the solutions of the equation

\[
\nabla \psi_0 = (\bar{\partial} + \frac{B}{2iy}) \psi_0 = 0,
\]

(12)

which are \( \psi_0(z, \bar{z}) = y^B \psi_0(z) \).

It can be easily checked that there are two operators which commutes with the Hamiltonian (11):

\[
L_1 = \partial + \bar{\partial} = \partial_x, \quad L_2 = z\partial + \bar{z}\bar{\partial} = x\partial_x + y\partial_y.
\]

(13)

\( L_1 \) and \( L_2 \) are the generators of a subalgebra of \( sl(2, R) \). Now let \( b = L_1 \) and \( b^\dagger = L_1^{-1} L_2 \) where \([b, b^\dagger] = 1\). Then we can choose the ground state wave functions to be the eigenfunctions of \( b^\dagger \). By a direct calculation it can be shown that

\[
b^\dagger \psi_0(\lambda|z, \bar{z}) = \lambda \psi_0(\lambda|z, \bar{z}),
\]

where

\[
\psi_0(\lambda|z, \bar{z}) = y^{B(\lambda - z)^{-B}}.
\]

(14)

Then if we consider the symmetry operator \( T_\xi = e^{\xi \partial + \bar{\xi} \bar{\partial}} \) it can be shown that:

\[
T_\xi \psi_0(\lambda|z, \bar{z}) = e^{\xi_2 \lambda - \frac{1}{2} \xi_2 \bar{\xi}_2} \psi_0(\lambda - \xi_1|z, \bar{z}).
\]

(15)

Now it can be verified that the generators of \( su_q(2) \) are

\[
J_+ = \frac{T_\xi - T_{\bar{\xi}}}{q - q^{-1}}, \quad J_- = \frac{T_{-\xi} - T_{-\bar{\xi}}}{q - q^{-1}}
\]

\[
q^{2J_0} = T_{\xi - \bar{\xi}} \quad \text{where} \quad q = \exp\left(\frac{1}{2} \xi \times \bar{\eta}\right),
\]

(16)

and the ground states wavefunctions are a representation of this algebra.
\[ J_+ \psi_0(\lambda|z, \bar{z}) = [1/2 - \lambda/\xi_1]_q \psi_0(\lambda - \xi_1|z, \bar{z}) \]

\[ J_- \psi_0(\lambda|z, \bar{z}) = [1/2 + \lambda/\xi_1]_q \psi_0(\lambda + \xi_1|z, \bar{z}) \]

\[ q^{\pm \lambda_0} \psi_0(\lambda|z, \bar{z}) = q^{\pm \lambda/\xi_1} \psi_0(\lambda|z, \bar{z}), \]

where the quantum symbol \([x]_q\) is defined by

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \]

In this way we showed the quantum group symmetry of QHE on the Poincare upper half plane.

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**References**


[8] H. T. Sato, Quantum group symmetry and quantum Hall wavefunctions on a torus, Osaka University preprint, OS-GF-39-93


