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MICROSCOPIC GENERALIZATION OF THE STANDARD INTERACTING BOSON MODEL HAMILTONIAN IN THE RESTRICTED DYNAMICS APPROACH

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ABSTRACT

The evaluation of the microscopic Generalized Interacting Boson Model (GIBM) Hamiltonian, deduced from the general microscopic nuclear Hamiltonian via collective O_{A-1} invariant microscopic Hamiltonian of the general Restricted Dynamics Model (RDM) in the case of central multipole and multipole-Gauss type effective NN -potential is briefly discussed. The GIBM version, which includes all 6th order terms in the expansion of the collective part of NN -potential, has been obtained. This GIBM Hamiltonian contains additional terms in comparison with the standard (sd-boson) Interacting Boson Model (IBM). The microscopic expressions for the standard IBM Hamiltonian parameters in terms of the employed effective NN -potential parameters have been obtained as well.

1. Introduction

Many of the nuclear collective models have been developed from macroscopic (phenomenological) considerations, and later on attempts were made to provide them with a microscopic background. It was pointed out in [1, 2], that most of these models can be regarded as particular cases of a general approach to the collectivity in nucleus - the Restricted Dynamics Approach (RDA) [1].

The essence of RDA, if described in a very simplified form, is the restriction of the Hilbert space of the nuclear microscopic wave functions to some definite subspaces, invariant under the action of the generators of selected groups as well as the action of the one- and two body nuclear operators. In this way it was established that the Hilbert space of the nuclear collective motion consists of the subspaces of the group O_{A-1} irreducible representations (starting with the "minimum" one allowed by Pauli principle), spanned by the particle degrees of freedom of the $A-1$ relative Jacobi vectors of a A nucleon system [1, 2, 3, 4, 5, 6].

The relationships between the macroscopical and microscopical collective models

in RDA have been studied in [7, 8], where a nuclear model with the same mathematical structure as the standard (sd-boson) Interacting Boson Model (IBM) [9], though with different physical background, has been obtained. This, so called Generalized Interacting Boson Model (GIBM), allows one to give a microscopical interpretation to the parameters of the standard IBM Hamiltonian. Moreover, the microscopical GIBM Hamiltonian opens the perspective of inclusion of new terms, which do not appear in the standard IBM, but are inherent only to GIBM.

In this research, continuing the investigations started in [7, 8, 10], we adopt the proposals of [8] and obtain the explicit expressions for the GIBM Hamiltonian terms, related to the first extension of the standard IBM Hamiltonian, namely, the 6th order terms of a collective potential energy expansion (including the 3rd order terms with respect to U_6 generators) as well as the kinetic energy operator term of GIBM.

In the first three sections of this paper we shall consider the evaluation of the microscopic GIBM version from general microscopic nuclear Hamiltonian in RDA. Then we shall describe the presentation of the standard IBM Hamiltonian in general form and its microscopic version - the Microscopic Interacting Boson Model (MIBM) Hamiltonian -, evaluated via GIBM. In the last section we shall consider the extension of the GIBM Hamiltonian beyond the standard IBM structure. There one can find also the microscopical expressions for the standard IBM parameters. In conclusion we shall summarize the obtained results and discuss briefly the future prospects for the microscopic GIBM studies.

2. The General Restricted Dynamics Model

We start from the general translationally invariant nuclear Hamiltonian for the system of A nucleons, interacting by two body central forces

$$H^{micro} = T^{micro} + V^{micro} = -\frac{\hbar^2}{2m} \frac{1}{A} \sum_{i < j=2}^A (\vec{\nabla}_i - \vec{\nabla}_j)^2 + \sum_{i < j=2}^A \left(V_W(r_{ij}) + P_{ij}^{(r)} \cdot V_M(r_{ij}) + P_{ij}^{(s)} \cdot V_B(r_{ij}) + P_{ij}^{(t)} \cdot V_H(r_{ij}) \right), \quad (1)$$

where for the sake of simplicity we have neglected the difference between protons and neutrons in the operator T^{micro} . The subscripts W, M, B and H in Eq.(1) refer to Wigner, Majorana, Bartlett and Heisenberg interactions, correspondingly. The $P_{ij}^{(-)}$ are the projection operators of spatial, spin and isospin states of pairs of nucleons and $r_{ij} = |\vec{r}_i - \vec{r}_j|$.

From the Hamiltonian (Eq.(1)) we will project its collective part, which, according to RDA [1, 2], is its O_{A-1} invariant part. The projection will be carried out in three steps:

- 1) Instead of the interparticle position vectors $|\vec{r}_i - \vec{r}_j|$, we shall use the $A-1$

translationally invariant Jacobi vectors [11]

$$\vec{\rho}_i = \frac{1}{\sqrt{i(i+1)}} \left(\sum_{j=1}^i \vec{r}_j - i\vec{r}_{i+1} \right); \quad i = 1, 2, \dots, A-1 \quad (2)$$

2) Next we shall express these vectors in terms of Dzublik-Zickendraht variables [12, 13]

$$\rho_i^{(s)} = \sum_{k=1}^3 \rho^{(k)} D_{k\alpha}^{(1)}(\Theta, \Phi, \Psi) D_{A-1+k, \alpha}^{(1, A-1)}(\theta_1, \theta_2, \dots, \theta_{3A-9}; \sigma) \quad (3)$$

$s = 1, 2, 3; \quad i = 1, 2, \dots, A-1.$

Here $D^{(13)}$ and $D^{(1, A-1)}$ are Wigner matrices of the groups O_3^+ and O_{A-1} , correspondingly. These matrices are associated with the defining representations of these groups. $\rho^{(k)}$ are three radial variables, Θ, Φ, Ψ are three Euler angles, $\theta_1, \theta_2, \dots, \theta_{3A-9}$ is a set of internal angles, and σ denotes two discrete parameters referring to the O_{A-1} group [1, 2]. From all the $3(A-1)$ variables, introduced in Eq.(3), only six - $\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \Theta, \Phi, \Psi$ (and only these) are scalars with respect to the symmetric group $S_A^{(r)}$, acting on the space variables $\vec{r}_i$ [14]. Therefore, they are the collective microscopic variables.

3) Finally, the resulting collective part of the Hamiltonian (Eq.(1)), which depends only on the six collective variables, one can obtain if one integrates it over the internal $(3A-9)$ variables $\theta_1, \theta_2, \dots, \theta_{3A-9}$, i.e.

$$\begin{aligned} H_{coll} &= H_{coll}(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \Theta, \Phi, \Psi, \sigma_{ij}, \tau_{ij}) \\ &= \int H^{micr}(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \Theta, \Phi, \Psi; \theta_1, \theta_2, \dots, \theta_{3A-9}; \sigma_{ij}, \tau_{ij}) dV_{\theta_1} dV_{\theta_2} \dots dV_{\theta_{3A-9}} = \\ &= T_{coll}^{micr} + V_{coll}^{micr}. \end{aligned} \quad (4)$$

In order not to worry about the physical dimensions, we shall work with dimensionless variables. In such a case the radial variables $\rho^{(k)}$ in Eq.(4) are given in units $\rho^{(k)}/r_\psi$ and energy in units $\hbar\omega_{osc, coll}/\sqrt{A}$, with oscillator scaling parameter (see §15 in [11])

$$r_\psi^{-2} = \nu_{osc, coll} = m\omega_{osc, coll}/\hbar\sqrt{A} \quad [\text{fm}^{-2}]. \quad (5)$$

Analogously one can introduce a more simple RDA model - the Strictly Restricted Dynamics Model (SRDM) [1, 2, 11]. The collective part of SRDM Hamiltonian is U_{A-1} invariant.

3. Derivation of the Microscopic Generalized Bohr-Mottelson Model(GBMM) Hamiltonian

The problem, how to obtain from Eq.(4) an explicit expression for the collective part V_{coll}^{micr} of the general microscopic central potential V^{micr} , is still open. At present it is solved only for central Wigner potentials of the form

$$V_{mW}^{micr} = V_0 \sum_{i < j=2}^A \sum_{t=0,1,2,\dots} b_t r_{ij}^{-2t}, \quad (6)$$

which, despite its simple form, encompasses quite a large class of interactions. Among others the potential Eq.(6) includes also the superposition of multipole Gauss type interactions:

$$V_{mGW}^{micr} = V_0 \sum_{i < j=2}^A (r_{ij})^{2a} e^{-\alpha(r_{ij})^2}, \quad (7)$$

where $a = 0, 1, 2, \dots$ and α are the potential parameters. In the case of potential Eq.(7) the coefficients b_t of Eq.(6) have the form

$$b_t = \frac{(-\alpha)^{t-a}}{(t-a)!}, \quad (8)$$

where $t = a, a+1, a+2, \dots$

In [15, 7] the following result for the collective potential energy term V_{coll}^{micr} with the potential (Eq.(6)) was obtained

$$\begin{aligned} V_{mWcoll}^{micr}(A, \rho^{(1)}, \rho^{(2)}, \rho^{(3)}) &= \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2,\dots} b_t \frac{(A-3)!!(2t)!!}{(A+2t-3)!!} \\ &\times \sum_{k_1+k_2+k_3=t} \frac{(2k_1-1)!!(2k_2-1)!!(2k_3-1)!!}{k_1!k_2!k_3!} (\rho^{(1)})^{2k_1} (\rho^{(2)})^{2k_2} (\rho^{(3)})^{2k_3}. \end{aligned} \quad (9)$$

In order to associate this collective potential energy term with the one of the Bohr-Mottelson model [16], first of all we replace $(\rho^{(1)})^2$, $(\rho^{(2)})^2$ and $(\rho^{(3)})^2$ in Eq.(9) by three new variables given in [17, 7, 8, 11]:

$$\begin{aligned} \rho^2 &= (\rho^{(1)})^2 + (\rho^{(2)})^2 + (\rho^{(3)})^2, \\ \beta \sin \gamma &= \frac{1}{\sqrt{2}} \left[(\rho^{(1)})^2 - (\rho^{(2)})^2 \right], \\ \beta \cos \gamma &= \frac{1}{\sqrt{6}} \left[(\rho^{(1)})^2 + (\rho^{(2)})^2 - 2(\rho^{(3)})^2 \right], \end{aligned} \quad (10)$$

obtaining

$$\begin{aligned} V_{mWcoll}^{micr}(A, \rho, \beta, \gamma) &= \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2,\dots} b_t \frac{(A-3)!!(2t+1)!!}{(A+2t-3)!!} \left(\frac{2t}{3} \right)^t \times \\ &\times \sum_{k=0}^t \frac{t!}{k!(t-k)!} \left(-\frac{\sqrt{6}\beta}{\rho^2} \right)^k B_k(\cos 3\gamma), \end{aligned} \quad (11)$$

where [18]

$$B_k(\cos 3\gamma) = \frac{k!}{(2k+1)!!} \sum_{j=\Delta, \Delta+2, \dots}^{j_{max}} \frac{(k-j-1)!!}{[(k-3j)/2]! j!} \left(\frac{3}{2}\right)^{(k-3j)/2} (\cos 3\gamma)^j \quad (12)$$

with $\Delta = 0(1)$, if k is even(odd), and $j_{max} = \Delta + 2 \{ [k/6] + [(k+4)/6] - [(k+5)/6] \}$, where $[x]$ denotes the integer part of x .

Combining Eq.(11) with Eq.(12), one obtains the final expression for the collective potential energy term, expressed in terms of $\rho^2, \beta^2, \beta^3 \cos 3\gamma$ (which are O_3^+ scalars, as we shall see below):

$$V_{mWcoll}^{micr}(A, \rho^2, \beta^2, \beta^3 \cos 3\gamma) = \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2, \dots} b_t \frac{(A-3)!!}{(1+2t-3)!!} \times \sum_{i+2l+3m=t} C_t(i, l, m) (\rho^2)^i (\beta^2)^l (\beta^3 \cos 3\gamma)^m, \quad (13)$$

where $C_t(i, l, m)$ are some definite coefficients.

The collective part of the central Wigner interaction (Eq.(6)), given in the form of Eq.(13), is the microscopic generalization of the potential energy term of Bohr-Mottelson type phenomenological models in the sense of RDA. Eq.(13) describes the Generalized Bohr-Mottelson model (GBMM) potential energy term in 6 dimensions, where one can obtain the phenomenological collective (5-dimensional Bohr-Mottelson, or Frankfurt) model, if one "freezes" the variable ρ^2 .

As yet we have considered only the potential energy term V_{coll}^{micr} of the microscopic collective Hamiltonian Eq.(4). The evaluation of its kinetic energy term T_{coll}^{micr} from the general microscopic expression of T^{micr} (see Eq.(1)), is still an open problem. Hence, we postulate that the GBMM kinetic energy term - the kinetic counterpart of Eq.(13), is the usual kinetic energy operator in 6-dimensional space. If one introduces six new spherical coordinates $q^{[lm]} (l=0,2)$, the detailed description of which will be given below, then one can define this kinetic energy operator in the canonical way

$$T_{coll}^{micr} = \frac{C_k}{2} \sum_{l=0,2} \sum_{m=-l}^l (-1)^{l-m} \frac{\partial}{\partial q^{[lm]}} \frac{\partial}{\partial q^{[lm]}} \quad (14)$$

where C_k is the kinetic energy constant.

The new collective variables $q^{[lm]} (l=0,2)$, which were used in GBMM kinetic energy term Eq.(14), one can employ also in order to obtain from GBMM a microscopic generalization of standard IBM - GIBM. In such a case one must replace the six collective variables $\rho^2, \beta, \gamma, \Theta, \Phi, \Psi$ by the O_3^+ irreducible bilinear forms of the relative Jacobi vectors, summed on the particle index, namely:

$$q^{[lm]} = \sum_{j=1}^{A-1} \left(\sum_{m_1 m_2} (-1)^{m_1-m_2} \frac{1}{\rho_j^{m_1}} \rho_j^{m_2} \right); \quad l=0,2; \quad -l \leq m \leq l, \quad (15)$$

which turn out to be the only functions of the collective variables one needs.

If one takes Eq.(3), in which the Cartesian index s is replaced by the spherical normalized index $m=0, \pm 1$, using standard relations

$$\rho_j^{\pm 1} = \mp \frac{1}{\sqrt{2}} (\rho_j^{(1)} + i\rho_j^{(2)}); \quad \rho_j^0 = \rho_j^{(3)}, \quad (16)$$

and also the explicit expression of the $D^{(1s)}$ matrices, taking into account the orthogonality of the $D^{(1s-1)}$ matrices, then one obtains from Eq.(15) the following results

$$q^{[00]} = -\frac{1}{\sqrt{3}} \rho^2, \quad (17)$$

$$q^{[2m]} = -\beta \cos \gamma D_{0m}^{(2)}(\Theta, \Phi, \Psi) + \frac{\beta}{\sqrt{2}} \sin \gamma (D_{2m}^{(2)}(\Theta, \Phi, \Psi) + D_{-2m}^{(2)}(\Theta, \Phi, \Psi)), \quad (18)$$

where $D_{lm}^{(2)}$ are usual O_3^+ Wigner matrices and ρ, β and γ are given by Eq.(10).

Using Eqs.(17) and (18) one can relate $\rho^2, \beta^2, \beta^3 \cos 3\gamma$ to O_3^+ scalars as follows:

$$\begin{aligned} \rho^2 &= -\sqrt{3} q^{[00]}, \\ \beta^2 &= \sqrt{5} [q^{[2]}, q^{[2]}]_0^0, \\ \beta^3 \cos 3\gamma &= \sqrt{\frac{35}{2}} [[q^{[2]}, q^{[2]}]^2, q^{[2]}]_0^0, \end{aligned} \quad (19)$$

where

$$[q^{[l]}, q^{[l']}]_{m''}^{l''} = \sum_{mm'} C_{mm'l''}^{l \ l' \ l''} q^{[lm]} q^{[l'm']}. \quad (20)$$

Therefore we see, that the set of six new collective variables $q^{[lm]}$ is equivalent to the set of previous collective variables $\rho^2, \beta, \gamma, \Theta, \Phi, \Psi$, and the GBMM Hamiltonian [7] can be presented as

$$H^{GBMM} = T^{GBMM}(\rho^2, \beta, \gamma, \Theta, \Phi, \Psi) + V^{GBMM}(\rho^2, \beta, \gamma). \quad (21)$$

The kinetic energy expression in Eq.(21) one can obtain from Eq.(14), using Eqs.(17) and (18):

$$\begin{aligned} T^{GBMM}(\rho^2, \beta, \gamma, \Theta, \Phi, \Psi) &= \\ &= -\frac{3}{2} C_k \frac{\partial^2}{\partial (\rho^2)^2} - 2C_k \left[\sum_{j=1,2,3} \frac{(\dot{I}_j^2)^2}{I_j^2(\beta, \gamma)} + \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right] = \\ &= -\frac{3}{2} C_k \frac{\partial^2}{\partial (\rho^2)^2} + T^{BM}(\beta, \gamma, \Theta, \Phi, \Psi). \end{aligned} \quad (22)$$

where $\hat{L}'_p$ are the components of the orbital momentum operator in the internal reference frame of the nucleus, $J'_p(\beta, \gamma)$ is the momentum of inertia

$$J'_p(\beta, \gamma) = 4\beta^2 \sin^2 \left(\gamma - p \frac{2}{3} \pi \right); \quad p = 1, 2, 3 \quad (23)$$

and $T^{BM}(\dots)$ denotes the usual (5-dimensional) Bohr Mottelson model kinetic energy operator [17, 7, 16].

The potential energy operator in GBMM Hamiltonian Eq.(21) is the same one as in the General Restricted Dynamics Model Hamiltonian (see Eq.(13))

$$V^{GBMM}(\rho^2, \beta^2, \gamma) \equiv V_{mW_{coll}}^{inter}(A, \rho^2, \beta^2, \beta^3 \cos 3\gamma). \quad (24)$$

4. Derivation of the Microscopic Generalized Interacting Boson Model (GIBM) Hamiltonian

In order to derivate from the GBMM Hamiltonian the microscopic GIBM Hamiltonian let us introduce six creation (η^{lm}) and six annihilation (η^{+lm}) operators as usual:

$$\eta^{lm} = \frac{1}{\sqrt{2}} \left(q^{l|m} + \frac{\partial}{\partial q^{l|m}} \right); \quad \eta^{+lm} = \frac{1}{\sqrt{2}} \left(q^{l|m} + \frac{\partial}{\partial q^{l|m}} \right). \quad (25)$$

By taking second order expressions in η^{lm} and η^{+lm} one can realize the algebra of the group $Sp(6, R)$, whose compact unitary subgroup U_6 generators are

$$C_{l_1 m_1}^{+l_2 m_2} = \eta_{l_1 m_1} \eta^{+l_2 m_2}. \quad (26)$$

The contravariant and covariant indices in Eq.(26) are related by

$$\eta_{lm} = (-1)^{l-m} \eta^{l-m}. \quad (27)$$

Taking linear combinations of U_6 generators (Eq.(26)) one constructs all the generators of the three U_6 subgroup chains, ending with O_3^+ [19]:

$$\begin{aligned} U_6 &\supset U_5 \supset O_5 \supset O_3^+ \\ U_6 &\supset O_6 \supset O_5 \supset O_3^+ \\ U_6 &\supset SU_3 \supset O_3^+ \end{aligned} \quad (28)$$

If one inverts Eq.(25), one gets

$$q^{l|m} = \frac{1}{\sqrt{2}} (\eta^{+lm} + \eta^{lm}); \quad \frac{\partial}{\partial q^{l|m}} = \frac{1}{\sqrt{2}} (\eta^{+lm} - \eta^{lm}), \quad (29)$$

which allows one to express, using Eq.(19), the invariants $\rho^2, \beta^2, \beta^3 \cos 3\gamma$ in terms of creation and annihilation operators as

$$\begin{aligned} \rho^2 &= -\sqrt{\frac{3}{2}} (\eta^{00} + \eta^{+00}), \\ \beta^2 &= \frac{1}{2} \sum_m (\eta_{2m} + \eta_{2m}^+) (\eta^{2m} + \eta^{+2m}), \\ \beta^3 \cos 3\gamma &= -\frac{\sqrt{35}}{4} \left[(\eta_2, \eta_2, \eta_2)_0^0 + (\eta_2^+, \eta_2^+, \eta_2^+)_0^0 + 3(\eta_2, \eta_2, \eta_2^+)_0^0 + 3(\eta_2^+, \eta_2^+, \eta_2)_0^0 \right], \end{aligned} \quad (30)$$

where

$$(\eta_2, \eta_2, \eta_2)_0^0 = [(\eta_2, \eta_2)]^2 \cdot \eta_2]_0^0 = \frac{1}{\sqrt{5}} \sum_{m_1 m_2 m_3} C_{m_1 m_2 m_3}^{2 \ 2 \ 2} \eta_{2m_1} \eta_{2m_2} \eta^{+2m_3}, \quad (31)$$

and similarly for the other terms of the $\beta^3 \cos 3\gamma$ expression in Eq.(30)

Now let us rewrite the GIBM Hamiltonian Eq.(21) in terms of creation annihilation operators Eq.(25)

$$H^{GBMM}(\eta^{lm}, \eta^{+l'm'}) = T^{GBMM}(\eta^{lm}, \eta^{+l'm'}) + V^{GBMM}(\eta^{lm}, \eta^{+l'm'}), \quad (32)$$

where

1) the kinetic energy operator is obtained by means of the canonic expression (Eq.(14)), where the replacements given by Eq.(29) are made:

$$T^{GBMM}(\eta^{lm}, \eta^{+l'm'}) = \frac{C_k}{1} \sum_{lm} (1 + 2\eta^{lm} \eta_{lm}^+) = \frac{C_k}{4} \sum_{lm} (\eta^{lm} \eta_{lm} + \eta^{+lm} \eta_{lm}^+); \quad (33)$$

2) for the potential energy operator, from Eq.(13) one obtains

$$\begin{aligned} V^{GBMM}(\eta^{lm}, \eta^{+l'm'}) &= \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2,\dots} b_t R_t < \\ &\times \sum_{l+2t+3m=0} C_t(l, t, m) \left[(\rho^2(\dots))^t (\beta^2(\dots))^t (\beta^3 \cos 3\gamma(\dots))^m \right], \end{aligned} \quad (34)$$

where

$$R_t = \frac{(A-3)!!}{(A+2t-3)!!} \quad (35)$$

and $\rho^2(\dots), \beta^2(\dots), \beta^3 \cos 3\gamma(\dots)$ denote the corresponding expressions given by Eq.(30).

Now one splits the microscopic GBMM Hamiltonian (Eq.32) into two parts, the first of which containing only U_6 generators and constant terms, i.e.,

$$H^{GBMM}(\eta^{lm}, \eta^{+l'm'}) = H^{GIBM}(\eta^{lm}, \eta^{+l'm'}) + H'^{GBMM}(\eta^{lm}, \eta^{+l'm'}). \quad (36)$$

The GIBM Hamiltonian H^{GIBM} will then be equal to $H^{GIBM}(\eta^{lm}, \eta^{+l'm'})$, that is, the U_6 -restricted part of the GBMM Hamiltonian H^{GBMM} . Again one can write it as

$$H^{GIBM}(\eta_{lm}, \eta^{l'm'}) = T^{GIBM}(\eta_{lm}, \eta^{l'm'}) + V^{GIBM}(\eta_{lm}, \eta^{+l'm'}). \quad (37)$$

Its kinetic part can easily be written as the U_6 -restriction of the kinetic operator (Eq.(33)):

$$T^{GIBM} = \frac{C_k}{4} \sum_{lm} (1 + 2\eta^{lm} \eta_{lm}^+) = \frac{C_k}{2} (\hat{N} + 3). \quad (38)$$

Here $\hat{N}$ is the operator of the total number of U_6 quanta (see Eq.(44) below).

Concerning the potential energy operator Eq.(34), then, taking into account Eq.(30), one can easily see, that from the terms $\rho^2(\dots)$, $\beta^2(\dots)$, $\beta^3 \cos 3\gamma(\dots)$ only those with $i + 2l + 3m = t = \text{even}$ have U_6 restricted terms.

Finally, from Eq.(37), taking into account Eq.(38) and the $t = 0, 2, 4, 6$ terms of the potential energy expansion Eq.(34), one can write the GIBM Hamiltonian in the following form:

$$\begin{aligned} H^{GIBM} = & \frac{1}{2} C_k (\hat{N} + 3) + C_p \left[b_0 R_0 + b_2 R_2 \left(\frac{20}{3} F_2(2, 0, 0) + 8 F_2(0, 1, 0) \right) + \right. \\ & + b_4 R_4 \left(-\frac{560}{3} F_4(4, 0, 0) + 576 F_4(0, 2, 0) + 1344 F_4(2, 1, 0) \right. \\ & - 256\sqrt{6} F_4(1, 0, 1) \left. \right) + b_6 R_6 \left(-\frac{320320}{27} F_6(6, 0, 0) + \frac{640640}{3} F_6(1, 1, 0) \right. \\ & - \frac{732160\sqrt{6}}{9} F_6(3, 0, 1) + 549120 F_6(2, 2, 0) - 133120\sqrt{6} F_6(1, 1, 1) + \\ & \left. \left. + 115200 F_6(0, 3, 0) + 20480 F_6(0, 0, 2) \right) \right], \end{aligned} \quad (39)$$

where we have used the notation

$$F_t(i, l, m) = \left[\left(\rho^2(\dots) \right)^i \left(\beta^2(\dots) \right)^l \left(\beta^3 \cos 3\gamma(\dots) \right)^m \right]_{U_6}; \quad t = i + 2l + 3m, \quad (40)$$

for the U_6 restriction of the corresponding monomials of O_3^+ -scalars. The coefficients R_t in Eq.(39) are given by Eq.(35). The parameters b_t describe the features of the selected NN -potential of form Eq.(6). In the case of multipole-Gauss type potential Eq.(7) b_t are given by Eq.(8).

Besides the kinetic energy constant C_k , in Eq.(39) there is one more constant - the potential energy constant C_p . For the Wigner type central NN interaction it assumes simple form

$$C_p = N_0 V_0 = \frac{A(A-1)}{2} V_0 [MeV]. \quad (41)$$

One can easily obtain the C_p expression also in more general case, when the central NN -interaction contains Wigner, Majorana, Bartlett and Heisenberg forces of equal potential depth and radial form [8], given by expressions Eq.(6), or Eq.(7):

$$C_p = [N_0 c_W + \Lambda(f) c_M + \Lambda(S) c_B + \Lambda(T) c_H] V_0 [MeV], \quad (42)$$

where $\Lambda(f), \Lambda(S), \Lambda(T)$ are the eigenvalues of the symmetric group class operators, consisting of orbital, spin and isospin exchange operators [20, 21], and c_W, c_M, c_B, c_H are the exchange constants of the central interaction, normalized with the condition [21] $c_W + c_M + c_B + c_H = 1$.

Now we are ready for the more detailed analysis of GIBM Hamiltonian (Eq.(39)) and for its comparison with the standard IBM Hamiltonian.

5. Standard IBM Hamiltonian and its Microscopical Derivation

In order to study the relationship between the microscopical GIBM and the standard IBM Hamiltonian, let us first consider the presentation of the latter.

The standard IBM Hamiltonian, accounting for s and d bosons, one can write using two one-boson operators - $\hat{N}, \hat{n}_d$ (with $\hat{N} = \hat{n}_0 + \hat{n}_d$) and seven two-boson interaction operators - $\hat{A}_0, \hat{A}_2, \hat{A}_4, \hat{D}, \hat{E}, \hat{B}, \hat{C}$ (see [9, 19, 8]):

$$\begin{aligned} H^{IBM} = & \epsilon_0 + \epsilon_s (\hat{N} - \hat{n}_d) + \epsilon_d \hat{n}_d + \frac{1}{2} \epsilon_0 \hat{A}_0 + \frac{\sqrt{5}}{2} \epsilon_2 \hat{A}_2 + \frac{3}{2} \epsilon_4 \hat{A}_4 + \\ & + \frac{1}{2} v_0 \hat{D} + \frac{1}{\sqrt{2}} v_2 \hat{E} + u_2 \hat{B} + \frac{1}{2} u_0 \hat{C}, \end{aligned} \quad (43)$$

where $\epsilon_0, \epsilon_s, \epsilon_d, \epsilon_0, \epsilon_2, \epsilon_4, v_0, v_2, u_2, u_0$ are model parameters.

If one presents the two-boson interaction operators in Eq.(43), using first and second order Casimir operators (expressed via U_6 generators Eq.(26)) of the three U_6 subgroup chains (Eq.(28)):

$$C_1(U_6) = \sum_{lm} C_{lm}^{lm} = \sum_{lm} \eta_{lm} \eta^{+lm} = \eta_{00} \eta^{+00} + \sum_m \eta_{2m} \eta^{+2m} = \hat{n}_0 + \hat{n}_d = \hat{N} \quad (44)$$

$$\begin{aligned} C_2(U_6) &= \sum_{\substack{l_1 m_1 \\ l_2 m_2}} C_{l_1 m_1 l_2 m_2}^{l_1 m_1 l_2 m_2} = \sum_{\substack{l_1 m_1 \\ l_2 m_2}} \eta_{l_1 m_1} \eta^{+l_2 m_2} \eta_{l_2 m_2} \eta^{+l_1 m_1} = \\ &= \hat{N}(\hat{N} + 5); \end{aligned} \quad (45)$$

$$C_3(U_6) = \sum_m C_{2m 2m}^{2m 2m} = \sum_m \eta_{2m} \eta^{+2m} = \hat{n}_d; \quad (46)$$

$$C_2(U_5) = \sum_{m_1 m_2} C_{2m_1 2m_2}^{2m_1 2m_2} = \sum_{m_1 m_2} \eta_{2m_1} \eta^{+2m_2} \eta_{2m_2} \eta^{+2m_1} = \hat{n}_d(\hat{n}_d + 4); \quad (47)$$

$$\begin{aligned} C_2(O_6) &= \sum_{\substack{l_1 m_1 \\ l_2 m_2}} \Lambda_{l_1 m_1 l_2 m_2} \Lambda_{l_2 m_2 l_1 m_1} = \\ &= \sum_{\substack{l_1 m_1 \\ l_2 m_2}} (C_{l_1 m_1 l_2 m_2}^{l_1 m_1 l_2 m_2} - C_{l_2 m_2 l_1 m_1}^{l_1 m_1 l_2 m_2}) (C_{l_2 m_2 l_1 m_1}^{l_1 m_1 l_2 m_2} - C_{l_1 m_1 l_2 m_2}^{l_1 m_1 l_2 m_2}); \end{aligned} \quad (48)$$

$$C_2(O_5) = \sum_{m_1 m_2} \Lambda_{2m_1}^{2m_2} \Lambda_{2m_2}^{2m_1} = \sum_{m_1 m_2} (C_{2m_1}^{2m_2} + C_{2m_2}^{2m_1}) (C_{2m_2}^{2m_1} + C_{2m_1}^{2m_2}); \quad (19)$$

$$C_2(O_3^+) = \hat{L}^2 = \sum_m (-1)^m L_m L_{-m}; \quad (20)$$

$$C_2(SU_3) = \hat{Q}^2 + \frac{1}{2} \hat{L}^2 = \sum_m (-1)^m Q_m Q_{-m} + \frac{1}{2} \sum_m (-1)^m L_m L_{-m} \quad (21)$$

with

$$L_m = \sqrt{\frac{5}{2}} \sum_{m' m''} C_{m' m'' m}^{2 2 1} \Lambda_{2m, 2m'} = \sqrt{10} (\eta, \eta^+)_{-m}^1; \quad (22)$$

$$Q_m = \sqrt{\frac{7}{3}} \sum_{m_1 m_2} C_{m_1 m_2 m}^{2 2 2} (C_{2m_1, 2m_2} + \frac{2}{\sqrt{3}} (C_{00, 2m} + C_{2m, 00})) = \sqrt{\frac{7}{3}} (\eta, \eta^+)_{-m}^2 + \frac{2}{\sqrt{3}} (\eta_{00} \eta_{2m}^+ + \eta_{2m} \eta_{00}^+), \quad (23)$$

then one obtains (see [19]):

$$\hat{A}_0 = \frac{1}{5} \left\{ C_2(U_5) - \frac{1}{2} C_2(O_5) - \hat{n}_d \right\}; \quad (24)$$

$$\hat{A}_2 = \frac{1}{7\sqrt{5}} \left\{ 2C_2(U_5) + C_2(O_5) - \hat{L}^2 - 12\hat{n}_d \right\}; \quad (25)$$

$$\hat{A}_4 = \frac{1}{210} \left\{ 36C_2(U_5) + 3C_2(O_5) + 10\hat{L}^2 - 216\hat{n}_d \right\}; \quad (26)$$

$$\hat{B} = \frac{1}{\sqrt{5}} \left\{ -C_2(U_5) + \hat{N} \cdot \hat{n}_d + 4\hat{n}_d \right\}; \quad (27)$$

$$\hat{C} = C_2(U_6) + C_2(U_5) - 2\hat{N} \cdot \hat{n}_d - 6\hat{N} - 3\hat{n}_d; \quad (28)$$

$$\hat{D} = \frac{1}{2\sqrt{5}} \left\{ -C_2(O_6) + C_2(O_5) - 4C_2(U_5) + 4\hat{N} \cdot \hat{n}_d + 10\hat{N} + 8\hat{n}_d \right\}; \quad (29)$$

$$\hat{E} = \frac{1}{4\sqrt{35}} \left\{ 2C_2(O_6) + 11C_2(U_5) - C_2(O_5) + 3C_2(SU_3) - 2\hat{L}^2 - 16\hat{N} \cdot \hat{n}_d - 40\hat{N} - 34\hat{n}_d \right\}, \quad (30)$$

where (see [19] and Eq.(51)) $\hat{Q}^2 = C_2(SU_3) - \hat{L}^2/2$. Here we have used Casimir invariants $C_2(O_k) = 2C_2^M(O_k)$ ($k = 3, 5, 6$), where $C_2^M(O_k)$ are the ones used in paper [19].

In our future analysis we shall use the equivalent form of the standard IBM Hamiltonian (Eq.(43)), written directly via Casimir invariants (Eqs.(44)-(51)) (see Eq.(2.4) in [8]):

$$H^{IBM} = e_0 + e_1 \hat{N} + e_2 C_2(U_6) + e_3 \hat{n}_d + e_4 C_2(U_5) + e_5 \hat{N} \cdot \hat{n}_d + \beta C_2(O_5) + 2\gamma \hat{L}^2 + \delta C_2(SU_3) + \eta C_2(O_6) \quad (31)$$

The relationships between the parameters of both IBM Hamiltonian forms Eq.(43) and Eq.(61) are given by expressions (2.6) and (2.7) of [8].

The next, most difficult, step in the study of connections between GIBM and IBM Hamiltonians Eq.(39) and (61) is to obtain, in terms of Casimir invariants of the U_6 subgroup chains Eq.(28) (Eqs.(44)-(51)), the explicit expressions for the U_6 restricted monomials $F_t(i, l, m)$ (see Eq.(40)) of O_3^+ scalars, entering in the potential energy terms of Eq.(39). The calculation methods, examples and results of $F_t(i, l, m)$ evaluation for $t = 2, 4, 6$ values are given in Appendix.

If one inserts the obtained $F_t(i, l, m)$ expressions for $t = 0, 2, 4$ values (see formulae (A.6),(A.11),(A.17),(A.22),(A.30),(A.35)) into the GIBM Hamiltonian (Eq.(39)), one can see that all terms, which enter in the resulting expression, are the ones already present in standard IBM Hamiltonian (Eq.(61)). Such comparison for the potential energy part of GIBM has been done in [8].

Taking into account, besides the $t = 0, 2, 4$ potential energy terms, also the kinetic energy term, one can obtain from Eq.(39) the microscopic version of IBM Hamiltonian:

$$H^{MIBM} = T^{MIBM} + V_{(t=0)}^{MIBM} + V_{(t=2)}^{MIBM} + V_{(t=4)}^{MIBM} = \frac{C_k}{2} (\hat{N} + 3) + C_p \left\{ b_0 R_0 + b_2 R_2 (20\hat{N} - 12\hat{n}_d + 30) + b_4 R_4 (11340 - 720\hat{N} + 2520C_2(U_6) - 4536\hat{n}_d - 648C_2(U_5) - 1296\hat{N}\hat{n}_d + 216C_2(O_5) - 288\hat{L}^2 + 432C_2(SU_3) - 216C_2(O_6)) \right\} \quad (32)$$

Comparing the standard IBM Hamiltonian Eq.(61) with this H^{MIBM} expression, one can express the H^{MIBM} parameters $e_0, e_1, e_2, \dots$ via b_t ($t = 0, 2, 4$) values (see formulae (6.5) in [8] and also Eq.(64) below), connected with the NN -interaction potential (see Eq.(8)), and in this way to get their microscopic meaning.

6. Microscopic Extension of the Standard IBM Hamiltonian

The first extension of the microscopic IBM version (Eq.(62)) beyond the structure of the standard IBM Hamiltonian (Eq.(61)) is related with the inclusion of the $t = 6$ term in the GIBM Hamiltonian (Eq.(39)). Using formulae (A.15),(A.29),(A.31),(A.32),(A.37)-(A.39) for the monomials $F_6(i, l, m)$ and disclosing all of them in terms of Casimir invariants, entering in Eq.(61), the GIBM Hamiltonian Eq.(39) can be presented in the form

$$H^{GIBM} = H^{MIBM} + H^{GIBM}. \quad (33)$$

In this decomposition the first part H^{MIBM} denotes the microscopic IBM Hamiltonian, containing all additional terms (with respect to the H^{MIBM} given by Eq.(62)), which include $b_6 R_6$ and have the structure of Casimir invariants used in Eq.(61). Other terms arising from $b_6 R_6$ in Eq.(39) (e.g. containing such Casimir invariants as

$\hat{N}^3, \hat{n}_d^3, \dots$, or their combinations, which do not enter in Eq.(61)), are included in the part H'^{GIBM} of Eq.(63).

In order to make this decomposition unambiguous one first takes a basis of the zero-, first- and second-order O_3^+ scalars that includes the ones present in Eq.(61) plus two more ones that we choose as $[\hat{n}_d, C_2(O_6)]$ and $[\hat{n}_d, C_2(SU_3)]$. Then one expresses all the terms in Eq.(39) in terms of these basis elements replacing every product of two non-commuting operators by the semisum of their commutator plus anticommutator.

If one compares the coefficients at corresponding Casimir invariants of Eq.(61) and H'^{MIBM} , one can derive the following microscopical expressions for the parameters of the standard IBM Hamiltonian Eq.(61):

$$\begin{aligned} c_0 &= \frac{3}{2} C_k + C_p(b_0 R_0 + 30b_2 R_2 + 11340b_4 R_4 + 15575160b_6 R_6); \\ c_1 &= \frac{1}{2} C_k + 20 C_p(b_2 R_2 + 36b_4 R_4 + 3213700b_6 R_6); \\ c_2 &= 360 C_p(7b_4 R_4 + 32930b_6 R_6); \\ \varepsilon &= -12 C_p(b_2 R_2 + 378b_4 R_4 + 3874240b_6 R_6); \\ \alpha &= 8 C_p(-81b_4 R_4 + 1320550b_6 R_6); \\ \beta &= 24 C_p(9b_4 R_4 + 39880b_6 R_6); \\ \delta &= 48 C_p(9b_4 R_4 + 2630b_6 R_6); \\ \varepsilon_1 &= 16 C_p(-81b_4 R_4 + 5428925b_6 R_6); \\ \gamma &= 16 C_p(-9b_4 R_4 + 11390b_6 R_6); \\ \eta &= -8 C_p(27b_4 R_4 + 282400b_6 R_6). \end{aligned} \quad (64)$$

These results generalize the microscopic IBM Hamiltonian parameter calculations presented in [8]. Indeed, omitting in Eqs.(64), for all parameters, the contributions from the $b_6 R_6$ term, assuming the potential energy constant $C_p = 1$ (when for Wigner interaction Eqs.(6) and (7) the potential depth V_0 is included in b_t term (compare Eq.(8) with (2.12) of [8]) and N_0 - in R_t term (compare Eq.(35) with (5.9) of [8])) and omitting the kinetic energy, one obtains for parameters c_0, c_1 the corresponding expressions (6.5) of [8].

7. Conclusions

Our studies of microscopic generalization of the Interacting Boson Model in the Restricted Dynamics Approach and microscopic evaluation of standard IBM parameters continues the investigations started in [8, 10]. In comparison with earlier studies [7, 8], we took into account the kinetic energy term of the microscopic GIBM Hamiltonian and included the 6th order ($t = 6$) terms of the potential energy expansion for multipole Gauss type collective NN potential. It allows us to obtain an extended version of the microscopic GIBM Hamiltonian, containing the first terms which do not enter into the standard IBM Hamiltonian.

The extraction of the collective degrees of freedom in Restricted Dynamics Approach allows one to introduce in GIBM the spin-isospin variables. Therefore, it is possible to take into account explicitly all exchange components of the central effective NN potential of above mentioned type. In this paper it was done in the case of equal radial dependence for all exchange forces, introducing the generalized potential energy constant C_p (see Eq.(42)), but one can easily obtain the corresponding expressions also for exchange force components with different radial shapes.

Our investigations can aid in further studies of the microscopically derived GIBM versions. First of all, now there is the possibility of numerical estimations in the frameworks of obtained results Eqs.(64), using the results of calculations based on the Restricted Dynamics Approach. Such results are available for some selected groups of even-even nuclei [21, 22]. The procedure for microscopical evaluation (in the sense of RDA) of the total U_6 boson number N and other IBM quantum numbers is known [7, 17]. We plan to perform such studies in future.

The solution of the extended microscopical GIBM Hamiltonian version, where 6th order terms of the potential energy expansion are included, indicates another direction for future investigations. The calculation of this GIBM Hamiltonian matrix elements for some selected U_6 subgroup chain of Eq.(28) can clarify the question about the suitability of the models of such type for the description of nuclear structure.

In future, the inclusion of $t = 8, 10, \dots$ order terms in GIBM Hamiltonian expansion also can be of interest. In order to evaluate such terms, one must calculate the O_3^+ scalars of third and higher degrees in U_6 generators with the subsequent expression of those O_3^+ scalars via Casimir invariants of U_6 subgroup chains Eq.(28). Even the calculation of all O_3^+ scalars of the third degree in U_6 generators is a complicated computational problem. For this task one needs highly efficient computer codes in order to evaluate the transformation brackets between three chains of Eq.(28), since the corresponding expressions (see Appendix) involve deeply the Casimir invariants of all their subgroups.

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Appendix A. Derivation of the U_6 -Restricted Monomials $F_t(i, l, m)$

All quantities

$$F_t^0(i, l, m) = \left[\left(\rho^2(\eta^{00}, \eta^{+00}) \right)^l \left(\beta^2(\eta^{2m}, \eta^{+2m}) \right)^l \left(\beta^3 \cos 3\gamma(\eta^{2m}, \eta^{+2m}) \right)^m \right], \quad (\text{A.1})$$

whose U_6 restricted parts, denoted as $F_t(i, l, m)$, appear in the GIBM Hamiltonian (Eq.(39)), one can decompose in two parts. The first, containing only U_6 generators (Eq.(26)), will be the (A.1) restriction to the U_6 group while the second contains all the remaining terms.

Let us denote the U_6 restricted part of (A.1) as "a" and the second part as "b". So we have

$$F_t^0(i, l, m) = a(i, l, m) + b(i, l, m) \quad (\text{A.2})$$

with

$$a(i, l, m) = F_t^0(i, l, m)|_{U_6} \equiv F_t(i, l, m). \quad (\text{A.3})$$

When one multiplies two decompositions of (A.2) type, one gets

$$F_t^0(i, l, m) \cdot F_{t'}^{0'}(i', l', m') = (a + b) \cdot (a' + b') = a \cdot a' + (a \cdot b' + b \cdot a') + b \cdot b'. \quad (\text{A.4})$$

It is clear that the term $a \cdot a'$ contains only U_6 generators (Eq.(26)) and therefore belongs to the U_6 -restricted part. The term $(a \cdot b' + b \cdot a')$ can not have a pure U_6 part and the term $b \cdot b'$ may have both a pure U_6 part and a remaining part. Reasoning along these lines, one obtains the decomposition of a complicated expression breaking it into small pieces and then composing again the part of these pieces together. This technique allows one to obtain in a systematic way the U_6 restriction for any $F_t^0(i, l, m)$ quantity in terms of O_3^+ scalars, expressed via U_6 generators.

The next, most difficult, step is to express the U_6 restricted parts (A.3) in terms of Casimir invariants Eqs.(44)-(51) of the U_6 subgroup chains Eq.(28). For $t = i + 2l + 3m = 0, 1, 2$ one can use the expressions of O_3^+ scalars of first and second degree in U_6 generators, expressed in terms of Casimir invariants of U_6 subgroups, given in [19, 8]. For $t = 3$ one uses the products of the first degree scalars with the second degree scalars, as well as the commutators of two second degree scalars.

As a first example let us apply the above described technique to the evaluation of monomials $F_t^0(2n, 0, 0)$, containing the U_6 restricted terms of type $(\rho^1)^n$.

From Eq.(30) one has

$$\begin{aligned} \rho^1 &= \left[-\sqrt{3/2} (\eta^{+00} + \eta_{00}) \right]^2 = \\ &= \frac{3}{2} \left[(\eta^{+00} \cdot \eta_{00} + \eta_{00} \cdot \eta^{+00}) + (\eta^{+00})^2 + (\eta_{00})^2 \right] = a_1 + b_1 \end{aligned} \quad (\text{A.5})$$

with

$$a_1 = a(2, 0, 0) = \rho_{|U_6}^1 = F_2(2, 0, 0) = \frac{3}{2} (\eta^{+00} \cdot \eta_{00} + \eta_{00} \cdot \eta^{+00}) =$$

$$= \frac{3}{2} (2\hat{n}_0 + 1) = 3(\hat{N} - \hat{n}_d) + \frac{3}{2}; \quad (\text{A.6})$$

$$b_1 = b(2, 0, 0) = \frac{3}{2} \left[(\eta^{+00})^2 + (\eta_{00})^2 \right]. \quad (\text{A.7})$$

From (A.5), (A.6) and (A.7) one has

$$\rho^8 = (a_1 + b_1)^2 = a_1^2 + b_1^2 + (a_1 \cdot b_1 + b_1 \cdot a_1). \quad (\text{A.8})$$

Since

$$\begin{aligned} b_1^2 &= \frac{9}{4} \left[(\eta^{+00})^2 \cdot (\eta_{00})^2 + (\eta_{00})^2 \cdot (\eta^{+00})^2 \right] + \frac{9}{4} \left[(\eta^{+00})^4 + (\eta_{00})^4 \right] = \\ &= \frac{9}{4} (2\hat{n}_0^2 + 2\hat{n}_0 + 2) + \frac{9}{4} \left[(\eta^{+00})^4 + (\eta_{00})^4 \right], \end{aligned} \quad (\text{A.9})$$

one obtains

$$\rho^8 = a_8 + b_8 \quad (\text{A.10})$$

with

$$\begin{aligned} a_8 &= a(4, 0, 0) = \rho_{|U_6}^8 = F_4(4, 0, 0) = \frac{27}{4} (2\hat{n}_0^2 + 2\hat{n}_0 + 1) = \frac{27}{4} \left\{ 2C_2(U_6) - 8\hat{N} + \right. \\ &\quad \left. + 2C_2(U_5) - 4\hat{N} - \hat{n}_d - 10\hat{n}_d + 1 \right\}; \end{aligned} \quad (\text{A.11})$$

$$b_8 = b(4, 0, 0) = \frac{9}{4} \left[(\eta^{+00})^4 + (\eta_{00})^4 + (4\hat{n}_0 + 6)(\eta^{+00})^2 + (4\hat{n}_0 - 2)(\eta_{00})^2 \right]. \quad (\text{A.12})$$

In this way, using the decomposition of $(\rho^1)^{n-1}$, ρ^1 and the relationships

$$(\eta^{+00})^n \cdot (\eta_{00})^n = (\hat{n}_0 + 1)(\hat{n}_0 + 2) \dots (\hat{n}_0 + n), \quad (\text{A.13})$$

$$(\eta_{00})^n \cdot (\eta^{+00})^n = \hat{n}_0(\hat{n}_0 - 1) \dots (\hat{n}_0 - (n - 1)), \quad (\text{A.14})$$

one obtains the decomposition of all $(\rho^1)^n$ type terms. The last of them, entering in GIBM Hamiltonian Eq.(39), is

$$\begin{aligned} \rho_{|U_6}^{12} &= F_6(6, 0, 0) = \frac{135}{8} (4\hat{n}_0^3 + 6\hat{n}_0^2 + 8\hat{n}_0 + 3) = \frac{135}{8} \left\{ C_2(U_6)(\hat{N} - 3\hat{n}_d) + \right. \\ &\quad \left. + C_2(U_5)(3\hat{N} - \hat{n}_d) + 11C_2(U_6) + 22C_2(U_5) + 78\hat{N} - 96\hat{n}_d + 3 \right\}. \end{aligned} \quad (\text{A.15})$$

As a next example let us consider the calculation of monomials $F_t^0(0, l, 0)$, containing the U_6 restricted terms of type $(\beta^2)^n$.

From Eqs.(30) one has

$$\beta^2 = \frac{1}{2} \sum_m (\eta_{2m} + \eta_{2m}^\dagger) \cdot (\eta^{2m} + \eta^{+2m}) = a_2 + b_2 \quad (\text{A.16})$$

with

$$\begin{aligned} a_2 &= a(0, 1, 0) = \beta_{U_6}^2 = F_2(0, 1, 0) = \frac{1}{2} \sum_m (\eta_{2m} \eta^{+2m} + \eta_{2m}^\dagger \eta^{+2m}) = \\ &= \frac{1}{2} (2\hat{n}_d + 5), \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} b_2 &= b(0, 1, 0) = \frac{1}{2} \sum_m (\eta_{2m} \eta^{2m} + \eta_{2m}^\dagger \eta^{+2m}) = \\ &= \frac{\sqrt{5}}{2} \left[(\eta_2, \eta_2)_0^0 + (\eta_2^\dagger, \eta_2^\dagger)_0^0 \right], \end{aligned} \quad (\text{A.18})$$

where we have used the following notation for the coupling of O_4^+ tensors

$$(t_k, t'_l)_0^0 = \frac{1}{\sqrt{2k+1}} \sum_m t_{km} \cdot t'^{+lm}. \quad (\text{A.19})$$

Using (A.16), (A.17) and (A.18) one has

$$\beta^4 = (a_2 + b_2)^2 = a_2^2 + b_2^2 + a_2 b_2 + b_2 a_2 = a_4 + b_4. \quad (\text{A.20})$$

Since

$$\begin{aligned} b_2^2 &= \frac{5}{4} \left\{ \left[(\eta_2, \eta_2)_0^0 \cdot (\eta_2^\dagger, \eta_2^\dagger)_0^0 + (\eta_2^\dagger, \eta_2^\dagger)_0^0 \cdot (\eta_2, \eta_2)_0^0 \right] + \right. \\ &+ \left[\left((\eta_2, \eta_2)_0^0 \right)^2 + \left((\eta_2^\dagger, \eta_2^\dagger)_0^0 \right)^2 \right] \Big\} = \frac{1}{4} \left\{ 2C_2(U_5) + C_2(O_5) + 2\hat{n}_d + 10 + \right. \\ &+ \left. 5 \left[\left((\eta_2, \eta_2)_0^0 \right)^2 + \left((\eta_2^\dagger, \eta_2^\dagger)_0^0 \right)^2 \right] \right\}. \end{aligned} \quad (\text{A.21})$$

then one obtains

$$\begin{aligned} a_4 &= a(0, 2, 0) = \beta_{U_6}^4 = F_4(0, 2, 0) = \frac{1}{4} (6\hat{n}_d^2 + 30\hat{n}_d + 35 + C_2(O_5)) = \\ &= \frac{1}{4} \{ 6C_2(U_5) + C_2(O_5) + 6\hat{n}_d + 35 \}; \end{aligned} \quad (\text{A.22})$$

$$\begin{aligned} b_4 &= b(0, 2, 0) = \frac{1}{4} \left\{ 5 \left((\eta_2, \eta_2)_0^0 \right)^2 + 5 \left((\eta_2^\dagger, \eta_2^\dagger)_0^0 \right)^2 + \right. \\ &+ \left. \sqrt{5} (\eta_2, \eta_2)_0^0 \cdot (4\hat{n}_d + 14) + \sqrt{5} (\eta_2^\dagger, \eta_2^\dagger)_0^0 \cdot (4\hat{n}_d + 6) \right\} \end{aligned} \quad (\text{A.23})$$

In such a way, using the decompositions of $(\beta^2)^{n-1}$ and β^2 and the relationships

$$(\eta_2, \eta_2)_0^0 \cdot (\eta_2^\dagger, \eta_2^\dagger)_0^0 = \frac{1}{5} \left[C_2(U_5) + \frac{1}{2} C_2(O_5) + \hat{n}_d \right]; \quad (\text{A.24})$$

$$(\eta_2^\dagger, \eta_2^\dagger)_0^0 \cdot (\eta_2, \eta_2)_0^0 = \frac{1}{5} \left[C_2(U_5) + \frac{1}{2} C_2(O_5) + 3\hat{n}_d + 10 \right]; \quad (\text{A.25})$$

$$\left[(\eta_2, \eta_2)_0^0, C_2(U_5) \right] = -(\eta_2, \eta_2)_0^0 \cdot (4\hat{n}_d + 12) = -(4\hat{n}_d + 4)(\eta_2, \eta_2)_0^0; \quad (\text{A.26})$$

$$\left[(\eta_2^\dagger, \eta_2^\dagger)_0^0, C_2(U_5) \right] = (\eta_2^\dagger, \eta_2^\dagger)_0^0 \cdot (4\hat{n}_d + 4) = (4\hat{n}_d + 12)(\eta_2^\dagger, \eta_2^\dagger)_0^0; \quad (\text{A.27})$$

$$\left[(\eta_2, \eta_2)_0^0, C_2(O_5) \right] = \left[(\eta_2^\dagger, \eta_2^\dagger)_0^0, C_2(O_5) \right] = 0, \quad (\text{A.28})$$

one can obtain the decompositions of all $(\beta^2)^n$ type terms. The last of them, entering in Eq.(39), is

$$\begin{aligned} \beta_{U_6}^6 &= F_6(0, 3, 0) = \frac{1}{8} \left\{ (20\hat{n}_d^3 + 150\hat{n}_d^2 + 376\hat{n}_d + 315) + C_2(O_5) (6\hat{n}_d + 15) \right\} = \\ &= \frac{1}{8} \{ 20C_2(U_5)\hat{n}_d + 6C_2(O_5)\hat{n}_d + 70C_2(U_5) + 15C_2(O_5) + 96\hat{n}_d + 315 \}. \end{aligned} \quad (\text{A.29})$$

Expressions of the $F^0(i, l, 0)$ type one can decompose by combining the above given decompositions of $(\rho^1)^n$ and $(\beta^2)^n$. The results for terms of this type, which enter in Eq.(39), are as follows:

$$\begin{aligned} (\rho^1 \beta^2)_{U_6} &= F_3(2, 1, 0) = \frac{3}{8} \{ 12\hat{n}_0 \cdot \hat{n}_d + 30\hat{n}_0 + 6\hat{n}_d + 10 + C_2(O_5) + C_2(O_6) \} = \\ &= \frac{3}{8} \{ -C_2(O_6) + 12C_2(U_5) + C_2(O_5) + 12\hat{N} \cdot \hat{n}_d + 30\hat{N} + 24\hat{n}_d \}, \end{aligned} \quad (\text{A.30})$$

$$\begin{aligned} (\rho^8 \beta^2)_{U_6} &= F_6(4, 1, 0) = \frac{9\sqrt{5}}{4} \{ (2\hat{n}_0 + 1) \hat{D} - [\hat{n}_0, \hat{D}] \} + \frac{27}{8} (2\hat{n}_0^2 + 2\hat{n}_0 + 1) \cdot \\ &\cdot (2\hat{n}_d + 5) = \frac{9}{8} \{ \{ \hat{n}_d, C_2(O_6) \} + 20C_2(U_5)\hat{n}_d - 2C_2(O_6)\hat{N} - 32C_2(U_5)\hat{N} - \\ &+ 20C_2(U_5)\hat{n}_d - 6C_2(O_5)\hat{N} - 2C_2(O_5)\hat{n}_d + 30C_2(U_5) + C_2(O_6) - \\ &- 30C_2(U_5) + C_2(O_5) - 20\hat{N} \cdot \hat{n}_d - 240\hat{N} + 198\hat{n}_d \}, \end{aligned} \quad (\text{A.31})$$

$$\begin{aligned} (\rho^4 \beta^4)_{U_6} &= F_6(2, 2, 0) = \frac{3}{8} \{ \sqrt{5} (\hat{D}(4\hat{n}_d + 10) - 2[\hat{n}_0, \hat{D}]) + (2\hat{n}_0 + 1) \cdot \\ &\cdot (6\hat{n}_d^2 + 30\hat{n}_d + 35 + C_2(O_5)) \} = \frac{3}{8} \{ -\{ \hat{n}_d, C_2(O_6) \} + \\ &+ 20C_2(U_5)\hat{N} - 20C_2(U_5)\hat{n}_d - 2C_2(O_5)\hat{N} + 4C_2(O_5)\hat{n}_d - \\ &- 5C_2(O_6) - 10C_2(U_5) + 10C_2(U_5) + 4C_2(O_5) + 20\hat{N} \cdot \hat{n}_d + \\ &+ 120\hat{N} + 40\hat{n}_d + 35 \}, \end{aligned} \quad (\text{A.32})$$

where $[\hat{A}, \hat{B}]$ denotes the commutator and $\{\hat{A}, \hat{B}\}$ the anticommutator.

For the group of most complicated monomials $F_i^n(i, l, m)$, containing the invariant $\beta^3 \cos 3\gamma$, one uses (from Eqs.(30) and (31)):

$$\beta^3 \cos 3\gamma = \frac{\sqrt{35}}{4} \left[(\eta_2, \eta_2, \eta_2)_0^0 + (\eta_2^\dagger, \eta_2^\dagger, \eta_2^\dagger)_0^0 + 3(\eta_2, \eta_2, \eta_2^\dagger)_0^0 + 3(\eta_2, \eta_2^\dagger, \eta_2^\dagger)_0^0 \right]. \quad (\text{A.33})$$

Because of the properties of Clebsch-Gordan coefficients and the commutation rules between the operators η^{2m} and η^{+2m} , the order of operators, appearing in the O_3^+ scalar $\beta^3 \cos 3\gamma$ is irrelevant.

Using (A.30) and (A.2), one obtains

$$F_4^0(1, 0, 1) = \rho^2 \beta^3 \cos 3\gamma = a + b \quad (\text{A.31})$$

with

$$\begin{aligned} a &= (\rho^2 \beta^3 \cos 3\gamma)_{|\mu_k} = F_4(1, 0, 1) = \\ &= -\frac{3\sqrt{105}}{4\sqrt{2}} \left[\eta^{+00}(\eta_2, \eta_2, \eta_2^+) \Big|_0^0 + \eta_{00}(\eta_2, \eta_2^+, \eta_2^+) \Big|_0^0 \right] - \frac{3\sqrt{105}}{4\sqrt{2}} \hat{E}, \quad (\text{A.35}) \end{aligned}$$

$$\begin{aligned} b &= -\frac{\sqrt{105}}{4\sqrt{2}} \left\{ (\eta^{+00} + \eta_{00}) \left[(\eta_2, \eta_2, \eta_2) \Big|_0^0 + (\eta_2^+, \eta_2^+, \eta_2^+) \Big|_0^0 \right] + \right. \\ &\quad \left. + 3 \left[\eta^{+00}(\eta_2, \eta_2^+, \eta_2^+) \Big|_0^0 + \eta_{00}(\eta_2, \eta_2, \eta_2^+) \Big|_0^0 \right] \right\}. \quad (\text{A.36}) \end{aligned}$$

The quantity $\hat{E}$ is given by Eq.(60) and (6.5g) in [19].

The U_6 restriction of the third order expressions in the U_6 generators, containing $\beta^3 \cos 3\gamma$, can not be done in a systematic way. They were obtained as linear combinations of commutators of lower order scalars. The results for the terms of this type, which enter in Eq.(39), are

$$\begin{aligned} (\rho^6 \beta^3 \cos 3\gamma)_{|\mu_k} &= F_6(3, 0, 1) = \frac{3\sqrt{105}}{32\sqrt{2}} \left\{ \frac{\sqrt{5}}{2} [\hat{E}, [\hat{n}_0, \hat{D}]] + \sqrt{5} [[\hat{n}_0, \hat{E}], \hat{D}] \right. \\ &\quad \left. - 18(2\hat{n}_0 + 1)\hat{E} + 18[\hat{n}_0, \hat{E}] \right\} = \frac{3\sqrt{3}}{512\sqrt{2}} \left\{ 3[C_2(SU_3), [\hat{n}_d, C_2(O_6)]] \right. \\ &\quad - 6[C_2(O_6), [\hat{n}_d, C_2(SU_3)]] + 6[C_2(O_5), [\hat{n}_d, C_2(SU_3)]] + 2[C_2(O_6), [\hat{n}_d, C_2(O_6)]] \\ &\quad - 184\{\hat{n}_d, C_2(O_6)\} + 408\{\hat{n}_d, C_2(SU_3)\} + 4736C_2(U_6)\hat{n}_d + 164C_2(O_6)\hat{N} - \\ &\quad - 8976C_2(U_5)\hat{N} + 4256C_2(U_5)\hat{n}_d + 280C_2(O_5)\hat{N} - 384C_2(O_5)\hat{n}_d + 560\hat{L}^2\hat{N} - \\ &\quad - 840C_2(SU_3)\hat{N} - 544\hat{L}^2\hat{n}_d + 11840C_2(U_6) - 12032C_2(U_5) - 120C_2(O_5) + \\ &\quad + 472C_2(O_6) - 480C_2(SU_3) + 256\hat{L}^2 - 61480\hat{N} \cdot \hat{n}_d - \\ &\quad \left. - 53680\hat{N} + 45552\hat{n}_d \right\}, \quad (\text{A.37}) \\ (\rho^2 \beta^5 \cos 3\gamma)_{|\mu_k} &= F_6(1, 1, 1) = -\frac{\sqrt{105}}{8\sqrt{2}} \left\{ \frac{5\sqrt{5}}{8} ([\hat{E}, [\hat{n}_0, \hat{D}]] - 2[[\hat{n}_0, \hat{E}], \hat{D}]) - \right. \\ &\quad \left. - \frac{5}{2} [[\hat{n}_0, \hat{E}], \hat{A}_0] + (10\hat{n}_0 + 6\hat{n}_d + 19)\hat{E} - 2[\hat{n}_0, \hat{E}] \right\} = \\ &= \frac{\sqrt{3}}{512\sqrt{2}} \left\{ 9[C_2(SU_3), [\hat{n}_d, C_2(O_6)]] + 18[C_2(O_6), [\hat{n}_d, C_2(SU_3)]] \right. \\ &\quad \left. - 6[C_2(O_5), [\hat{n}_d, C_2(SU_3)]] - 56\{\hat{n}_d, C_2(O_6)\} + 96\{\hat{n}_d, C_2(SU_3)\} + 128C_2(U_6)\hat{n}_d + \right. \end{aligned}$$

$$\begin{aligned} &+ 176C_2(O_6)\hat{N} - 2576C_2(U_5)\hat{N} + 2240C_2(U_5)\hat{n}_d + 140C_2(O_5)\hat{N} - 400C_2(O_5)\hat{n}_d \\ &- 80\hat{L}^2\hat{N} - 128\hat{L}^2\hat{n}_d + 120C_2(SU_3)\hat{N} + 320C_2(U_6) + 248C_2(O_6) + 2296C_2(U_5) + \\ &+ 320C_2(O_5) + 840C_2(SU_3) - 560\hat{L}^2 - 3960\hat{N} \cdot \hat{n}_d - 12560\hat{N} + 3648\hat{n}_d \Big\}, \quad (\text{A.38}) \end{aligned}$$

$$\begin{aligned} (\beta^6 (\cos 3\gamma)^2)_{|\mu_k} &= F_6(0, 0, 2) = \frac{1}{128} \left\{ 280 \left\{ (\eta_2, \eta_2, \eta_2) \Big|_0^0, (\eta_2^+, \eta_2^+, \eta_2^+) \Big|_0^0 \right\} + \right. \\ &+ 81[C_2(SU_3), [\hat{n}_d, C_2(SU_3)]] - 27[C_2(O_5), [\hat{n}_d, C_2(SU_3)]] \\ &+ 54[C_2(O_6), [\hat{n}_d, C_2(SU_3)]] + 54[C_2(SU_3), [\hat{n}_d, C_2(O_6)]] - 324\{\hat{n}_d, C_2(O_6)\} + \\ &+ 378\{\hat{n}_d, C_2(SU_3)\} + 2304C_2(U_6)\hat{n}_d + 288C_2(O_6)\hat{N} - 5184C_2(U_5)\hat{N} + \\ &+ 2664C_2(U_5)\hat{n}_d + 468C_2(O_5)\hat{n}_d - 432C_2(SU_3)\hat{N} + 720\hat{L}^2\hat{N} - 936\hat{L}^2\hat{n}_d + \\ &+ 5760C_2(U_6) - 756C_2(O_6) - 6228C_2(U_5) + 1170C_2(O_5) + 594C_2(SU_3) + \\ &\left. + 1836\hat{L}^2 - 21600\hat{N} \cdot \hat{n}_d - 30960\hat{N} + 20196\hat{n}_d \right\} \quad (\text{A.39}) \end{aligned}$$

where

$$\begin{aligned} \left\{ (\eta_2, \eta_2, \eta_2) \Big|_0^0, (\eta_2^+, \eta_2^+, \eta_2^+) \Big|_0^0 \right\} &= (35/(18(3\hat{n}_0^2 + 3\hat{n}_0 + 2))) \cdot \\ &\cdot \left\{ \frac{5}{64} \left[\left\{ [\hat{E}, [\hat{n}_0, \hat{D}]] + 2[[\hat{n}_0, \hat{E}], \hat{D}] \right\}, \left\{ 2[\hat{E}, \hat{D}] + [[\hat{n}_0, \hat{E}], [\hat{n}_0, \hat{D}]] \right\} \right] + \right. \\ &\left. + \frac{6}{5}(3\sqrt{5}\hat{A}_2 + 6\hat{n}_d + 10)\hat{n}_0(\hat{n}_0 - 1)(\hat{n}_0 - 2) \right\} \quad (\text{A.40}) \end{aligned}$$

contains only third order terms.

Let us note, that our results for all monomials $F_t(i, l, m)$ with $l = 2, 1$ and for $F_6(6, 0, 0)$ coincide with those presented in [8].

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