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LIFETIMES OF CHAOTIC ATTRACTORS IN A MULTIDIMENSIONAL LASER SYSTEM

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ABSTRACT

We study the lifetimes of chaotic attractors at crises in a multidimensional laser system. This system describes the CO_2 laser with modulated losses and is known as the four-level model. The critical exponents which are related to the lifetimes of the attractors are estimated in terms of the corresponding eigenvalues and the measured characteristic lifetimes in the model. The critical exponents in this model and those of its center manifold version are in good agreement. We conjecture that generically in the four-level model the critical exponents are close to $\frac{1}{2}$ at crises. In addition, we compare predictions of a simpler and popular model known as the two-level model with those of the above mentioned models.

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I. INTRODUCTION

Several studies in the last few years were devoted to the theoretical, numerical and experimental analysis of the power-law scaling for the characteristic times near crises [1–4]. Crises are sudden changes that occur in a strange attractor as a control parameter is changed [1]. There are three different types of crises, classified as attractor merging, attractor widening and attractor destruction. The latter applies to the sudden appearance of a new attractor while the first two are characterized by intermittencies between different regions of the phase space [1]. First, in this article we study the critical exponents in a multidimensional dynamical system known as the four-level model (4-LM) which describes the dynamics of the CO_2 laser with modulated losses [5–7]. We underline that as regards the physics the critical exponents are related to the mean lifetime of chaotic attractors in the vicinity of crises. Second, in this work we find that a popular model for the CO_2 laser, known as the two-level model (2-LM) has unrealistic eigenvalues for its unstable periodic orbits when compared to those extracted from the 4-LM. The lifetimes of chaotic attractors as well as the eigenvalues of the unstable periodic orbits can be measured experimentally [2,3].

A general theory to determine the critical exponents η for homoclinic and heteroclinic crises in three dimensional flows or two-dimensional invertible maps was given in reference [1]. However, there is no corresponding theory for multidimensional dynamical systems [1,8]. In the present work we determine the critical exponents by studying the relation between the power-law scaling for the characteristic times, the eigenvalues of the unstable orbits in the 4-LM and the corresponding eigenvalues of the flow obtained upon application of the center manifold method to the original 4-LM equations. The center manifold equations (CME) give solutions similar to those of the 4-LM for a broad range of parameters and were derived in reference [7]. A relevant difference is that the 4-LM reduces to a five-dimensional invertible map while the CME reduces to a two-dimensional invertible map. Therefore, for the CME we may apply the existing theory [1].

A critical exponent η close to $\frac{1}{2}$ is a characteristic of a highly dissipative system [1]. It has

been shown that for a certain range of parameters, the 4-LM may be described qualitatively by an effective one-dimensional map [6]. This suggests that there the critical exponents $\eta \approx \frac{1}{2}$. However, the 4-LM shows properties [6,7] such as the coexistence of attractors that do not exist in one-dimensional maps. Therefore, in order to determine consistently the critical exponents in the 4-LM, the following points need to be considered. (a) the relation between the eigenvalues of the unstable orbits, the critical exponents η and the dynamics at the crises in the 4-LM (we consider the three different kind of crises mentioned above in the 4-LM) : (b) the extent to which the CME are useful to describe crises in the 4-LM. To our knowledge there is no theory that relates the critical exponents of a given model and those of its center manifold version at crises [8,9]. We show in this article that both the 4-LM and the CME support the conjecture that the existence of critical exponents $\eta \approx \frac{1}{2}$ is a generic property at crises.

On the other hand, saddle-node bifurcations play a relevant role in governing the behavior of dynamical systems [8,9], which in our case describe laser systems. Indeed, saddle points can create and destroy attractors [10,11] or determine the linking properties of chaotic attractors [12]. In this paper, the eigenvalues that we have obtained for the CME are compared with those of the 2-LM and no qualitative agreement is found. We underline that using embedding techniques it is possible to obtain the eigenvalues of the relevant saddle points from experimental data [8].

In the next section we describe the 4-LM, the CME and the 2-LM. In section three we study the critical exponents in the 4-LM and the CME. Further, we find the typical eigenvalues predicted by the 2-LM and compare them with those of the previous models. In section four we give the conclusions.

II. THE MODEL FOR THE CO_2 LASER

The 4-LM for the CO_2 laser with modulated losses is based upon the usual field-matter equations for two-level atoms in a resonant cavity [13]. However, the lasing transitions

take place between rotational levels which belong to different vibrational bands. These lasing rotational levels are in turn coupled nonradiatively with the other rotational levels of the same vibrational band. The nonlasing rotational levels are assumed to have global populations [5,14]. When the adiabatic elimination of the polarization is carried out, the 4-LM is described by the following equations [5,6]:

$$\begin{aligned} \frac{dI}{dt} &= -k(t)I + I(N_2 - N_1) \\ \frac{dN_2}{dt} &= -I(N_2 - N_1) - \gamma_2 N_2 - \gamma_R N_2 + \gamma'_R M_2 + \gamma_2 Q \\ \frac{dN_1}{dt} &= I(N_2 - N_1) - \gamma_1 N_1 - \gamma_R N_1 + \gamma'_R M_1 \\ \frac{dM_2}{dt} &= -\gamma_2 M_2 + \gamma_R N_2 - \gamma'_R M_2 + Z\gamma_2 Q \\ \frac{dM_1}{dt} &= -\gamma_1 M_1 + \gamma_R N_1 - \gamma'_R M_1 \end{aligned} \quad (1)$$

Here I is the intensity of the electric field, $N_2(M_2)$ and $N_1(M_1)$ are the upper and lower resonant (nonresonant) rotational levels. In eq.(1) the populations N_1 , N_2 , M_1 , M_2 , the intensity I and the pump Q have been renormalized with respect to the coupling constant. Z is the effective number of rotational levels in each vibrational band. γ'_R is the rotational relaxation rate for the transitions $M_2 \rightarrow N_2$ and $M_1 \rightarrow N_1$. γ_R is the rotational relaxation rates for the inverse transitions. $\gamma_R/\gamma'_R = Z$ [5]. The vibrational relaxation rates in N_1 and N_2 are denoted respectively by γ_1 and γ_2 . The same relaxation rates hold for M_1 and M_2 respectively. $k(t) = k_0(1 + m \cos(\omega t + \varphi))$ is the damping rate for the intensity. m is the modulation amplitude, ω is the modulation frequency and φ is the initial phase. The numerical values of the parameters in eq.(1) are given in table I. We study eq.(1) by means of the Poincaré section technique i.e we sample the solution at $t = i \times T$, $i = 1, \infty$, where $T = 1/f$ is the driving period.

Recently, using techniques of the center manifold theory a two dimensional flow was derived from eq.(1) [7]. This method is a more refined version of the adiabatic elimination of fast variables [7,15]. Firstly, in this method a suitable change of coordinates in the system is made. Then, an approximation is introduced based on an appropriate smallness parameter,

which gives rise to a set of two coupled nonlinear equations [7,15]. The center manifold equations (CME) and the 4-LM give similar solutions for a broad range of parameters [7]. However, one can not know a priori that the CME and the 4-LM have close critical exponents η since, no theory exists that relates the corresponding exponents in both equations.

The CME are the following [7]:

$$\begin{aligned}\frac{dY}{dt} &= -k(t)Y(1+bY)(1+aY) + \Delta Y(1+bY) \\ \frac{d\Delta}{dt} &= -\gamma_+ \Delta - 2\Delta Y + \gamma_p Q + k(t)(\tilde{\gamma} + 2aY^2)\end{aligned}\quad (2)$$

Here, $Y = I/(Z+1)$ and $\Delta = (N_2 + M_2 - N_1 - M_1)/(Z+1)$. $\gamma_+ = (\gamma_1 + \gamma_2)/2$, $\gamma_p = 2\gamma_1\gamma_2/(\gamma_1 + \gamma_2)$, $\tilde{\gamma} = (\gamma_1 - \gamma_2)^2/2(\gamma_1 + \gamma_2)$, $a = 2Z/\Gamma_R$, $b = a(k(t)/\Gamma_R - 1)$, where $\Gamma_R = \gamma_R + \gamma'_R$. For the CME, η can be calculated in terms of the eigenvalues [1], since these equations can be reduced to a two dimensional invertible map. In eq.(2) the third dimension appears as a result of the modulated damping rate $k(t)$.

Finally, there is a simpler and popular model for the CO_2 laser known as the two-level model (2-LM) [13]. This model is described by the following set of equations :

$$\begin{aligned}\frac{d\tilde{I}}{dt} &= -k(t)\tilde{I} + \tilde{\Delta}\tilde{I} \\ \frac{d\tilde{\Delta}}{dt} &= -\gamma\tilde{\Delta} - 2\tilde{\Delta}\tilde{I} + \gamma Q\end{aligned}\quad (3)$$

where, $\tilde{I}$ and $\tilde{\Delta}$ stand for the laser intensity and population inversion respectively. γ is a small parameter and was changed from $\gamma = 10^3$ to $\gamma = 10^5$ Hz in order to fit the results of early experiments [13]. Recently, however it was shown that the 2-LM predicts incorrect behavior for the CO_2 laser in the transient and chaotic regimes of laser operation [5,6].

III. THE CRITICAL EXPONENTS

Firstly, we describe those crises for which the eigenvalues of the unstable orbits are determined. To this end we have carried out a study of the Poincaré sections in the neighborhood

of the crises reported here. In Fig 1a, we see the bifurcation diagram for the effective population inversion $\Delta = (N_2 + M_2 - N_1 - M_1)/(Z+1)$ versus the modulation amplitude m in the 4-LM. Here, the arrows indicate three different crises in this model. We observe that starting from low values of m a period doubling cascade that ends in chaos takes place. The basic period of this cascade is one. After the onset of chaos, merging of chaotic bands take place and at the end of the merging crises two big attractor pieces in the Poincaré section merge at $m = m_1 \approx 0.117760$ due to the collision of the attractor with the unstable orbit of period one P_1 . The merging of attractors are homoclinic crises [1]. In this figure we also observe that two successive crises occur at $m = m_{31} \approx 0.15162$ and at $m = m_{32} \approx 0.16028$. The crisis at $m = m_{31}$ is of attractor destruction type while that at $m = m_{32}$ is of attractor widening type. Both crises occur due to the collision of the existing attractor with the unstable orbit of period three P_3 . These two successive crises due to the same unstable orbit, in our case P_3 , are known as formation of sequential horseshoes in chaotic attractors [9,16]. Both are homoclinic crises [9,16]. In Fig. 1b we show the corresponding crises in the CME. We see that the bifurcation diagrams in Figs. 1a and 1b are basically the same.

Near the crises for one-dimensional maps, dissipative two-dimensional maps or three dimensional dissipative flows, the characteristic time τ is found theoretically to scale with the control parameter m as follows [1]:

$$\tau = a |m - m_c|^{-\eta} \quad (4)$$

Here $\eta > 0$ is the critical exponent, m_c is the critical control parameter and a is a constant. In a two-dimensional invertible map and for a homoclinic tangency the critical exponent η is given by :

$$\eta = \frac{\ln |\beta_2|}{2 \ln |\beta_1 \beta_2|} \quad (5)$$

Where β_1 is the expanding eigenvalue ($|\beta_1| > 1$) and β_2 is the contracting eigenvalue ($|\beta_2| < 1$) [1].

Figs. 2a and 2b show that the scaling dependence given by eq.(3) holds in the 4-LM for the attractor merging crisis at $m = m_1$ and for the attractor destruction crisis at $m = m_{31}$

respectively. The corresponding slopes η of the straight lines at $m = m_1$ and $m = m_{31}$ are $\eta_1^{(mod)} = 0.502 \pm 1.3 \times 10^{-2}$ and $\eta_{31}^{(mod)} = 0.529 \pm 1.2 \times 10^{-2}$ respectively. If our map was a highly contracting two-dimensional map then by setting $\beta_2 \rightarrow 0$ we obtain $\eta = \frac{1}{2}$ from eq.(4).

Table II shows the eigenvalues for the corresponding unstable orbits at the crises at $m = m_1, m_{31}$ and m_{32} in the 4-LM. Table III contains the same information but for the CME. As regards the 4-LM there are two eigenvalues of order $O(1)$, one expansive and another contractive. Then follows an eigenvalue of order $O(10^{-3} - 10^{-8})$ and finally two highly contractive eigenvalues whose order of magnitude is equal or less than $O(10^{-33} - 10^{-34})$. The roundoff errors in the algorithm limit the estimation of the true value of the last two eigenvalues. We have found that this spectrum of eigenvalues for the unstable orbits is typical for other crises that we have considered in the 4-LM. The CME have instead one expanding eigenvalue of order $O(1)$ and one highly contractive eigenvalue, whose order of magnitude is close to λ_3 in the 4-LM. Similarly, this spectrum of eigenvalues is also typical for other crises in the CME.

As mentioned above crises at m_1, m_{31} and m_{32} are homoclinic. If in the 4-LM only the most contracting and the only expanding eigenvalues played the leading role at crises then it would be enough to replace $\beta_1 \rightarrow \lambda_1$ and $\beta_2 \rightarrow \lambda_{4,5}$ to obtain $\eta \approx \frac{1}{2}$ in eq.(4). Namely, with the ansatz $\beta_1 \rightarrow \lambda_1$, $\beta_2 \rightarrow \lambda_{4,5}$, we obtain $0.5 < \eta_1^{(theor)} < 0.504$ at $m = m_1$, $0.5 < \eta_{31}^{(theor)} < 0.509$ at $m = m_{31}$ and $0.5 < \eta_{32}^{(theor)} < 0.510$ at m_{32} . In this manner a good agreement between $\eta_{1,31}^{(mod)}$ and $\eta_{1,31}^{(theor)}$ is found. On the other hand, if the only expanding eigenvalue and the contracting eigenvalue λ_3 were the only relevant eigenvalues at the crises then, $\beta_1 \rightarrow \lambda_1$ and $\beta_2 \rightarrow \lambda_3$ and $\eta_1 = 0.550$ and $\eta_{31} = 0.538$ and the agreement with $\eta^{(mod)}$ is not as good as in the previous ansatz.

This ansatz suggests that near the crises the most contracting eigenvalues play a relevant role in highly dissipative systems. In the 4-LM, other crises due to the unstable orbits of period four, five and six within the interval $0 < m < 0.25$ have their spectrum of eigenvalues similar to those of table II. Therefore, we believe that the dynamics at other homoclinic

crises must be the same as at $m = m_1$ and at $m = m_{31}$ and here the critical exponents are too $\eta \approx \frac{1}{2}$.

On the other hand, the CME predict critical exponents $\eta \approx \frac{1}{2}$. This can be proved by inserting the eigenvalues given in table III into eq.(4). We obtain $\eta_1 = 0.550$, $\eta_{31} = 0.507$ and $\eta_{32} = 0.512$ for m_1, m_{31} and m_{32} respectively. These values η are close to $\eta^{(mod)}$. The ubiquity of a highly contracting eigenvalue at all considered crises in the CME and the similarity between the CME and the 4-LM as shown in Figs. 1a and 1b suggest that at homoclinic and heteroclinic crises $\eta \approx \frac{1}{2}$ in the multidimensional 4-LM. This similarity between the 4-LM and the CME dynamics and the existence of critical exponents in the CME, may be considered as an indirect proof that the power-law scaling holds in the multidimensional 4-LM.

Let us compare the above mentioned results with those of the 2-LM with modulated losses. This model was extensively studied theoretically [10,11]. It was shown that for $\gamma \approx 10^3$ Hz as claimed in early experiments [13], the behavior of the 2-LM is that of a conservative system with a small dissipative perturbation [10,11]. Therefore, a small external periodic modulation may stabilize the damped oscillations of the system [10,11]. On the other hand, in a conservative two-dimensional map the eigenvalues of saddles are $|\lambda_1| > 1$ and $|\lambda_2| < 1$ with $|\lambda_1 \lambda_2| = 1$. In order to fit the results of the experiments γ was chosen to vary from 10^4 to 10^5 Hz [5,6]. However, recent studies [5,6] proved that even these new values cannot reproduce important features of the experiment.

As stated above the behavior of a dynamical system depends critically on its periodic orbits. Therefore, it is interesting to know the typical eigenvalue spectrum predicted by the 2-LM. This spectrum corresponds to the parameter region of m around which the first attractor destruction crisis occurs in the 2-LM. For $\gamma = 1.95 \times 10^4$ Hz we obtain for the period one and period two unstable orbits the following spectrum. For saddles between $0.016 < m < 0.02$, $|\lambda_1| \sim O(1) > 1$ and $|\lambda_2| \sim O(1) < 1$ e.g. for $m = 0.019$ $|\lambda_1| \approx 1.5$ and $|\lambda_2| \approx 0.4$. On the other hand for $\gamma = 1.95 \times 10^5$ Hz, for the period one orbit the eigenvalues of the saddle at $m = 0.155$ are $|\lambda_1| \approx 1.3$ and $|\lambda_2| \approx 0.13$. While for the period

two orbit the eigenvalues of the saddle at $m = 0.154$ are $|\lambda_1| \approx 1.3$ and $|\lambda_2| \approx 5 \times 10^{-3}$. These eigenvalues are typical for $0.15 < m < 0.16$.

It is known that in the 2-LM, the attractor originated from the period one orbit undergoes a first attractor destruction crisis caused by the unstable orbit of period two [10,6]. Instead in the 4-LM as well as in the CME, the bifurcation diagram is close to that of the logistic map [17]. The latter is certainly at odds with the 2-LM predictions. Moreover, for $\gamma = 1.95 \times 10^4$ Hz, the contracting eigenvalues for the 2-LM, are too big when compared to those of the CME. Therefore, the predicted η is substantially different from $1/2$. On the other hand for $\gamma = 1.95 \times 10^5$ Hz, the contracting eigenvalues of the 2-LM for period one are two orders of magnitude larger those for the CME, for the same range of values of m . On the basis of the above study, we believe that an experiment can be carried out in the following way. Since, sweeping the modulation depth m and measuring the attractor lifetimes is followed by strong noise in the experiment [18], the straightforward method of finding the eigenvalues seems more appropriate to test our results. To this end we proceed to use embedding techniques [8]. As a result we form the delay coordinate vector $\vec{\varphi}(t) = (\varphi(t), \varphi(t - T_d), \dots, \varphi(t - (Q - 1)T_d))$, where T_d is the delay time and Q is the embedding dimension [8]. $\varphi(t)$ is the laser intensity. Next, in order to obtain a discrete time series from $\vec{\varphi}(t)$ we can use a Poincaré surface of section but in φ -space. Let Z_n be the points in the $\vec{\varphi}$ -space surface of section for a given value of the control m . In order to find the eigenvalues of an unstable periodic orbit, say of period one, we must find a matrix A such that

$$Z_{n+1} = AZ_n + C$$

for all possible pairs Z_n and Z_{n+1} lying close to some Z^* , which is the period one orbit. A is the least-squares fit matrix and is an approximation to the Jacobian matrix A of the exact map from where we obtain the eigenvalues of the periodic orbit [19].

The calculation of periodic orbits and eigenvalues was performed using standard methods for the appropriate n -th iterate of the Poincaré map [20,21]. To determine the characteristic times for those crises at $m = m_1$ and at $m = m_{31}$, a binary partition was chosen as suggested

by Fig. 1a. We used 300 initial conditions localized at $t = 0$ in the core ($m = m_1$) or old ($m = m_{31}$) attractors. We choosed as the critical bifurcation parameter m_c , that with the least error for $\eta_1^{(mod)}$ and $\eta_{31}^{(mod)}$ when fitting data to a straight line as shown in Figs. 2.

IV. CONCLUSIONS

Our results can be divided in two parts. First, we have shown that a multidimensional dynamical system known as the four-level model (4-LM) which describes the CO_2 laser with modulated losses predicts a power-law scaling for the characteristic times in three different crises. These are attractor merging, attractor destruction and attractor widening crises. The mean lifetimes of chaotic attractors at crises are related to the critical exponents. Our study suggests that the only expanding and the most contracting eigenvalues play a relevant role in the dynamics at the crises in the 4-LM. Our work was based on a direct determination of the critical exponents and the corresponding eigenvalues at the crises in the 4-LM. Moreover, the center manifold equations (CME) derived from the 4-LM predict that one of its two eigenvalues is highly contracting. As a result the critical exponents of the CME are in agreement with those of the 4-LM. The spectra of the eigenvalues in the 4-LM are qualitatively the same at all crises that we have considered. The same takes place in the CME. This property of the spectra of the eigenvalues and the fact that the dynamics of the 4-LM and the CME is similar suggest that the existence of critical exponents $\eta \approx \frac{1}{2}$ is a generic property for all homoclinic and heteroclinic crises in the 4-LM, at least within the range of control parameters that we have studied ($0 < m < 0.25$ where m is the modulation amplitude).

Second, the simpler and popular model for the CO_2 laser known as the two-level model has eigenvalues for its unstable periodic orbits that differ qualitatively from those predicted by the CME. The comparison is made in the parameter region of m where the first attractor destruction crisis in the two-level model takes place. This region is interesting since, it allows to find easily relevant differences in the above mentioned models in terms of their bifurcation

diagrams and with those of the experiment [6]. In the experiment, embedding techniques are necessary to calculate the eigenvalues of the unstable periodic orbits. We conclude that a practical way to find the attractor lifetimes is to obtain these eigenvalues in the experiment.

We believe that the present work may be useful in the study of crisis in related multidimensional laser systems which show high dissipation. Finally, our numerical calculations indicate that it would be useful to have a theoretical estimate of the critical exponents (attractor lifetimes) of a multidimensional system in terms of those of its center manifold version.

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TABLES

TABLE I. Parameters of the four-level model for the CO_2 laser $\mu = 10^6 s^{-1}$.

Parameter	Numerical value	Parameter	Numerical value
γ_R	7.0μ	γ'_R	0.7μ
γ_1	0.08μ	γ_2	0.01μ
k_0	23.15μ	ω	100 kHz
Q	1.4256	φ	$\frac{3\pi}{2}$
Z	10		

TABLE II. Eigenvalues λ_i at the crises $m = m_1$, $m = m_{31}$ and $m = m_{32}$ in the four-level model (eq.(1) in the text). All the eigenvalues are real numbers.

Control parameter	$m = m_1 = 0.11770$	$m = m_{31} = 0.15162$	$m = m_{32} = 0.161$
$ \lambda_1 $	1.88	3.79	4.41
$ \lambda_2 $	0.65	0.28	0.28
$\log_{10}(\lambda_3)$	-3	-8	-8
$\log_{10}(\lambda_{4,5})$	≤ -33	≤ -34	≤ -34

TABLE III. Eigenvalues λ_i at the crises $m = m_1$, $m = m_{31}$ and $m = m_{32}$ for the center manifold equations (eq.(2) in the text). All the eigenvalues are real numbers.

Control parameter	$m = m_1 = 0.115$	$m = m_{31} = 0.151$	$m = m_{32} = 0.161$
$ \lambda_1 $	1.88	1.41	1.71
$\log_{10}(\lambda_2)$	-3	-10	-10

FIG. 1. (a) Bifurcation diagram for the inversion $\Delta = (N_2 - N_1 + M_2 - M_1)/(Z + 1)$ versus the modulation amplitude m in the four-level model (4-LM). (b) The same as (a) but for the center manifold equations (CME).

FIG. 2. Power-law scaling for the characteristic times τ near crises in the four-level model (4-LM). Plots (squares) of $\ln \tau$ versus $\ln \delta m$ for (a) $\delta m = m - m_1$ and (b) $\delta m = m - m_{31}$. The straight lines are the best fit to the data. The measured critical exponents of the 4-LM are $\eta^{(mod)} \approx \frac{1}{2}$ in both cases.

Fig.1a

Fig.1b

Fig.2