Bosonic Realization of Boundary Operators in
$SU(2)$-invariant Thirring Model

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Abstract
Boundary operators and boundary states in $SU(2)$-invariant Thirring model are considered from
the point of view of bosonization and oscillator realizations of bulk and boundary Zamolodchikov-Faddeev
algebras.

1 Introduction
The integrability of quantum theory implies that one can obtain the exact correlation function or form
factors for arbitrary many local operators in the theory. The recent investigations on such theories
have lead to important progresses. In exact solvable lattice models [1]-[6], the free boson realization of
$\eta$-deformed affine algebras [7] has provided a bosonization of type I and type II vertex operators [8]-
[11] which satisfy Zamolodchikov-Faddeev (ZF) algebra, and hence the correlation functions of six-vertex
model or $XXZ$-chain are successfully calculated. In quantum field theories, e.g. Sine-Gordon and $SU(2)$-
invariant Thirring models, Lukyanov [12] constructed two kinds of generators of ZF algebra—using free
bosons and $\eta$-deformed oscillators [13, 14]—which behave quite similar to the type I and type II operators
mentioned above, and also calculated the form factors [15, 16] in these theories. Another important aspect
of recent progresses in integrable models is the work of Ghoshal and Zamolodchikov [17], which proposed
the theories of integrable boundaries and boundary operators. Jimbo et al [18, 19] obtained the $\eta$-boson
realization for the boundary operators of $XXZ$-model and the corresponding multipoint correlation
functions, and it is the present work which is intended to the study of boundary operators and boundary
states in $SU(2)$-invariant Thirring model and their boson realization.

2 $SU(2)$-invariant Thirring model in the bulk: an overview
Let us first review Lukuyanov’s boson realization of ZF algebra and the corresponding $S$-matrix in the
case of $SU(2)$-invariant Thirring model [12].

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As mentioned in the introduction, in integrable quantum field theories, there are two kinds of operators, i.e. local and asymptotic operators, among each there is a subset satisfying ZF algebra.

The local generators $Z_{ab}^{\dagger}(\beta)$ (here and after we shall adopt the convention that operators with a prime denote local operators and those without a prime denote asymptotic operators) satisfy the ZF algebra

$$Z_{ab}^{\dagger}(\beta_1)Z_{cd}^{\dagger}(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2)Z_{ab}^{\dagger}(\beta_2)Z_{cd}^{\dagger}(\beta_1),$$

where the $S$-matrix $S_{ab}^{cd}(\beta)$ is determined by the well-known Yang-Baxter equation, unitarity and crossing symmetry, and summation over repeated indices is understood. Explicitly, the $S$-matrix reads

$$S_{ab}^{cd}(\beta) = S'(\beta) \left( \begin{array}{cccc} 1 & \frac{-\beta}{i\pi + \beta} & \frac{i\pi}{i\pi + \beta} & \frac{-\beta}{i\pi + \beta} \\ \frac{-\beta}{i\pi + \beta} & 1 & \frac{i\pi}{i\pi + \beta} & \frac{-\beta}{i\pi + \beta} \\ \frac{i\pi}{i\pi + \beta} & \frac{-\beta}{i\pi + \beta} & 1 & \frac{-\beta}{i\pi + \beta} \\ \frac{-\beta}{i\pi + \beta} & \frac{i\pi}{i\pi + \beta} & \frac{-\beta}{i\pi + \beta} & 1 \end{array} \right),$$

$$S'(\beta) = \frac{\Gamma \left( \frac{i\pi - \beta}{2i\pi} \right) \Gamma \left( \frac{2i\pi + \beta}{2i\pi} \right)}{\Gamma \left( \frac{i\pi + \beta}{2i\pi} \right) \Gamma \left( \frac{2i\pi - \beta}{2i\pi} \right)}.$$

The ZF algebra (1) can be realized through the bosonic field $\phi'(\beta)$ with the commutation relation

$$[\phi'(\beta_1), \phi'(\beta_2)] = \ln S'(\beta_2 - \beta_1),$$

as follows,

$$Z^+_{ab}(\beta) = V'(\beta) \equiv e^{i \phi'(\beta)} \cdot,$$

$$Z^-_{ab}(\beta) = i (\chi' V'(\beta) + V'(\beta) \chi'),$$

where $\beta$ is the rapidity of the boson $\phi'$, i.e. the momentum $p'$ of $\phi'$ is related to $\beta$ via $p' = (p_0, p_1) = m'(ch_0, sh_0)$, $\chi'$ is defined through

$$\langle u'|v \rangle \equiv \eta' \langle u| \int_{C'} \frac{d\gamma}{2\pi} \tilde{V}'(\gamma) |v \rangle,$$

$$\tilde{V}'(\gamma) \equiv e^{-i \phi'(\gamma)} \cdot,$$

$$\tilde{\phi}'(\gamma) = \phi'(\gamma + i \frac{\pi}{2}) + \phi'(\gamma - i \frac{\pi}{2}),$$

in which $|u\rangle$ and $|v\rangle$ are some states in the (dual) Fock space of the boson $\phi'$, $\eta'$ is an irrelevant constant, and the integration contour $C'$ is taken such that it encloses only the poles originated from the action of $\tilde{V}'$ on the right-handed state $|v\rangle$ clockwise.

Notice that, instead of considering the actions of local operators onto the Hilbert space $\mathcal{H}$, we now consider their actions on the Fock space $\mathcal{F}$ which is a proper subspace of $\mathcal{H}$. To specify the Fock space $\mathcal{F}$ more clearly, one needs the unique $SU(2)$ invariant vacuum state $|0\rangle$ under which the two-point function of the boson $\phi'(\beta)$ reads

$$\langle 0| \phi'(\beta_1) \phi'(\beta_2)|0 \rangle = -\ln S'(\beta_2 - \beta_1).$$ (3)
The consistence of (2) with (3) implies that $g'(\beta)$ is the Riemann-Hilbert factor of $S'(\beta)$ which is analytic in the lower half plane ($\Im \beta < 0$),

$$S'(\beta) = \frac{g'(-\beta)}{g'(\beta)}, \quad g'(\beta) = k^{1/2} \frac{\Gamma \left( \frac{2i\pi - \beta}{2i\pi} \right)}{\Gamma \left( \frac{i\pi - \beta}{2i\pi} \right)}.$$ 

For later use we also present the following two-point functions,

$$\langle 0| \bar{g}'(\beta_1) g'(\beta_2) |0\rangle = \ln w'(\beta_2 - \beta_1),$$
$$\langle 0| \bar{g}'(\beta_1) g'(\beta_2) |0\rangle = - \ln \bar{g}'(\beta_2 - \beta_1),$$

where

$$w'(\beta) = \left[ g(\beta + \frac{i\pi}{2}) g(\beta - \frac{i\pi}{2}) \right]^{-1} = k^{-1} \frac{2\pi}{i(\beta - \frac{i\pi}{2})},$$
$$\bar{g}'(\beta) = \left[ \bar{w}'(\beta + \frac{i\pi}{2}) \bar{w}'(\beta - \frac{i\pi}{2}) \right]^{-1} = k^{3} \frac{\beta(\beta - i\pi)}{2\pi}.$$ 

The locality of the operators $Z'_a(\beta)$ is best manifested by the following specific operator products,

$$C_{ab} Z'_a(\beta + i\pi) Z'_b(\beta) = i,$$
$$Z'_a(\beta - i\pi) Z'_b(\beta) = iC_{ab},$$

where $C_{ab} = \sigma_1$ (Pauli matrix) is the charge conjugation matrix, and we shall also use its inverse $C^{ac}$: $C^{ac} C_{ab} = \delta^c_a$.

To end this review section, let us mention that the asymptotic operators $Z_a(\beta)$ also satisfy an FZ algebra similar to (1), but now with the $S$-matrix replaced by

$$S^{'ed}_{ab}(\beta) = - S^{'ed}_{ab}(-\beta),$$

and their asymptotic behavior is manifested by the following operator product,

$$iZ_a(\beta_2) Z_b(\beta_1) = \frac{C_{ab}}{\beta_2 - \beta_1 - \frac{i\pi}{2}} + ...$$

Reasonably, these asymptotic operators should be regarded as appropriate (weak) limits of the local operators $Z'_a(\beta)$ as $|t| \to \infty$. However such limit must be taken in a nontrivial way sense the transformation from the set of local operators to the set of asymptotic operators is an improper unitary transformation.

### 3 Boundary operators in $SU(2)$-invariant Thirring model

In this and the forthcoming sections we shall concentrate on the $SU(2)$-invariant Thirring model in the presence of reflecting boundaries.

In the presence of boundaries, one has the following boundary reflection equation
\[ Z'_a(\beta)|B\rangle = R'_a(\beta)Z'_a(-\beta)|B\rangle \]

with the “boundary state” \(|B\rangle\). This state should be considered to be included in the Fock space \(\mathcal{F}\), which is generated by the “boundary operator” \(\exp(\Psi_-)\) from the vacuum state \(|0\rangle\), i.e.

\[ |B\rangle = \exp(\Psi_-)|0\rangle, \]

and the relation (5) is an extension of the ZF algebra (1).

The consistence of (5) and (1) leads to the boundary Yang-Baxter equation [17, 20, 21]

\[
R''_{a_2}(\beta_2)S_{c_1 d_2}^{c_1 d_2}(\beta_1 + \beta_2)R''_{c_1}(\beta_1)S_{d_1 d_2}^{d_1 d_2}(\beta_1 - \beta_2) \\
= S_{a_1 c_2}^{a_1 c_2}(\beta_1 - \beta_2)R''_{c_1}(\beta_1)S_{d_1 d_2}^{d_1 d_2}(\beta_1 + \beta_2)R''_{d_2}(\beta_2)
\]

for the boundary reflection matrix \(R''_a(\beta)\). This equation, while supplied with the the boundary unitarity and boundary crossing symmetry conditions [17]

\[
R''_a(\beta)R''_b(-\beta) = \delta^b_a, \\
C^{ac}R''_c(-\frac{i\pi}{2} - \beta) = S^{ab}_{c d}(2\beta)C^{cd}R''_c(-\frac{i\pi}{2} + \beta)
\]

give rise to the following solution,

\[
R''_a(\beta) = \frac{R'(\beta)}{-\sqrt{AD_1 D_2 + A^2\beta} - B \begin{pmatrix} -A\beta - B & 2D_1 \beta \\ 2D_2 \beta & A\beta - B \end{pmatrix}},
\]

\[
R'(\beta) = \frac{\Gamma \left( \frac{2\mu - \beta}{2i\pi} \right) \Gamma \left( \frac{2\mu + \beta}{2i\pi} \right) \Gamma \left( \frac{\mu + \beta + i\pi}{2i\pi} \right) \Gamma \left( \frac{\mu - \beta + i\pi}{2i\pi} \right)}{\Gamma \left( \frac{2\mu + \beta}{2i\pi} \right) \Gamma \left( \frac{2\mu - \beta}{2i\pi} \right) \Gamma \left( \frac{\mu + \beta + i\pi}{2i\pi} \right) \Gamma \left( \frac{\mu - \beta + i\pi}{2i\pi} \right)},
\]

\[
\mu = \frac{B}{\sqrt{AD_1 D_2 + A^2}},
\]

where the constants \(A, B, D_1, D_2\) are parameters related to the boundary state \(|B\rangle\).

In the rest of this article, we shall restrict ourselves to the relatively simple case of \(D_1 = D_2 = 0\), which corresponds to the diagonal \(R''\)-matrix

\[
R''_a(\beta) = R'(\beta) \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}
\]

with \(\mu = B/A\).

Similar to \(S'(\beta)\), \(R'(\beta)\) can also be Riemann-Hilbert factorized,

\[
R'(\beta) = \frac{f'(\beta, \mu)}{f'(\beta', \mu')}
\]

\[
f'(\beta, \mu) = \frac{\Gamma \left( \frac{2\mu - \beta}{2i\pi} \right) \Gamma \left( \frac{\mu - \beta + i\pi}{2i\pi} \right)}{\Gamma \left( \frac{2\mu + \beta}{2i\pi} \right) \Gamma \left( \frac{\mu + \beta + i\pi}{2i\pi} \right)},
\]

\[
f'(\beta', \mu') = \frac{\Gamma \left( \frac{2\mu' - \beta}{2i\pi} \right) \Gamma \left( \frac{\mu' + \beta + i\pi}{2i\pi} \right)}{\Gamma \left( \frac{2\mu' + \beta}{2i\pi} \right) \Gamma \left( \frac{\mu' - \beta + i\pi}{2i\pi} \right)}.
\]
where \( f' (\beta, \mu) \) is analytic in the lower half \( \beta \)-plane, \( i.e. \Im \beta < 0 \).

For diagonal boundary reflection matrix, the boundary reflection equation (5) becomes (here no summation is taken over \( a \))

\[
Z'_a (\beta) [B] = R^a(\beta) Z'_a (-\beta) [B], \quad a = (+, -).
\]

If \( a = + \), we have from (7-8) that

\[
f' (\beta, \mu) Z'_+ (\beta) [B] = (\beta \rightarrow -\beta).
\]

In order to study the property of the boundary operator \( \Psi_- \), we first introduce the following decomposition of the bosonic field \( \phi'(\beta) \),

\[
\phi'(\beta) = \phi'_+ (\beta) + \phi'_- (\beta),
\]

where

\[
\begin{align*}
\phi'_+ (\beta) |0\rangle &= 0 = \langle 0 | \phi'_- (\beta), \\
[\phi'_+ (\beta_1), \phi'_- (\beta_2)] &= -\ln g'(\beta_2 - \beta_1), \\
[\phi'_0, \phi'_\pm (\beta)] &= 0, \\
[\phi'_0, \phi'_0] &= 0.
\end{align*}
\]

These commutation relations imply the following two-point functions,

\[
\begin{align*}
\langle 0 | \phi'_+ (\beta_1) \phi'_- (\beta_2) |0\rangle &= -\ln g'(\beta_2 - \beta_1), \\
\langle 0 | \phi'_0 \phi'_\pm (\beta_2) |0\rangle &= 0, \\
\langle 0 | \phi'_0 \phi'_0 (\beta_2) |0\rangle &= 0.
\end{align*}
\]

We shall show in the next section that, while appropriately regularized, the only nontrivial effect of \( \phi'_0 \) is to decompose the Fock space into different sectors. The fact that \( \phi'_0 \) is independent of \( \beta \) will also be clear in the next section.

Now let us make the ansatz

\[
\begin{align*}
\langle 0 | \Psi_- &= 0, \\
[\Psi_-, \phi'_- (\beta)] &= 0, \quad [\Psi_-, \phi'_0] = 0, \\
[\Psi_-, \phi'_+ (\beta)] &= \kappa_- \phi'_- (-\beta) + \frac{1}{2} \gamma_- (-\beta), \\
[\gamma_- (\beta), \text{ everything}] &= 0.
\end{align*}
\]

for the operator \( \Psi_- \). Substituting the above ansatz into equation (9), we get

\[
\kappa_- = -1, \quad \exp \left( -\frac{i}{2} \gamma_- (\beta) \right) = g'(2\beta)^{1/2} f' (\beta, \mu).
\]

Let us check the consistency of the above ansatz and result with the boundary reflection equation (8) for \( a = - \). This means that we should check the equality

\[
\begin{align*}
\langle 0 | \Psi_- &= 0, \\
[\Psi_-, \phi'_- (\beta)] &= 0, \quad [\Psi_-, \phi'_0] = 0, \\
[\Psi_-, \phi'_+ (\beta)] &= \kappa_- \phi'_- (-\beta) + \frac{1}{2} \gamma_- (-\beta), \\
[\gamma_- (\beta), \text{ everything}] &= 0.
\end{align*}
\]
\[
(\mu - \beta)f'(\beta, \mu)Z'_{\gamma}(\beta)|B\rangle = (\beta \rightarrow -\beta).
\] (11)

From equation (10), we have
\[
e^{i\phi' + \beta}|B\rangle = f'(-\beta, \mu)e^{i\phi' - \beta}|B\rangle,
\]
e\[e^{-i\phi' - \beta}|B\rangle = -\frac{(2\beta)^{3/2}}{4\pi^2} e^{-i\phi' - \beta}|B\rangle.\] (12)

Using the explicit form of $Z_{\gamma}(\beta)$ and equation (12), we find that equation (11) is equivalent to
\[
\int_{C} \frac{d\gamma}{2\pi i} \frac{e^{i(\mu - \beta)(\mu + \gamma + \frac{i\pi}{2})\gamma}}{(\gamma - \beta - \frac{i\pi}{2})(\gamma - \beta + \frac{i\pi}{2})(\gamma + \beta + \frac{i\pi}{2})} e^{-i\phi' - (\gamma - i\phi' - (\gamma - \beta - \frac{i\pi}{2})|B\rangle = (\beta \rightarrow -\beta),}
\] (13)

where the integration contour $C'$ encloses the poles $\gamma_1 = \beta - \frac{i\pi}{2}$ and $\gamma_2 = -\beta - \frac{i\pi}{2}$ clockwise but does not enclose the pole $\gamma_3 = \beta + \frac{i\pi}{2}$. The last equation can be directly verified by calculating the residues of the left hand side at the poles $\gamma_1, \gamma_2$ and $\gamma_3$ respectively, and then subtracting from the sum of the first two residues the last one, which results in an expression which is symmetric under $\beta \rightarrow -\beta$.

Now let us mention that the same procedure as above can be applied to obtained the left-handed boundary operator $e^{\Psi^*}$, which is used to construct the left-handed boundary state $\langle B\rangle = \langle 0|e^{\Psi^*}$, which satisfies the boundary reflection equation
\[
\langle B\rangle Z'^*_{a}(\beta) = \langle B\rangle Z'^*_{a}(-\beta) R'^*_{a}(-\beta),
\] (14)

where
\[
Z'^*_{a}(\beta) = C_{ba}Z'_{a}(\beta + i\pi),
\]
\[
R'^{b}_{a}(\beta) = R^{b}_{a}(\beta) = R^{*}(\beta) \begin{pmatrix} e^{-\beta} & 0 \\ \frac{\beta}{i\pi} & 1 \end{pmatrix},
\]
\[
R^{*}(\beta) = \frac{\Gamma \left( \frac{2\pi - \beta}{2i\pi} \right) \Gamma \left( \frac{2\pi}{2i\pi} \right)}{\Gamma \left( \frac{2\pi + \beta}{2i\pi} \right) \Gamma \left( \frac{2\pi}{2i\pi} \right)} \Gamma \left( \frac{\beta - 3\pi}{2i\pi} \right) \Gamma \left( \frac{\beta - 3\pi}{2i\pi} \right)
\]

As before, $R^{*}(\beta)$ can be factorized according to the Riemann-Hilbert problem
\[
R^{*}(\beta) = \frac{f^{*'}(-\beta, \mu)}{f^{*'}(\beta, \mu)},
\]
\[
f^{*'}(\beta, \mu) = \frac{\Gamma \left( \frac{3\pi - \beta}{2i\pi} \right)}{\Gamma \left( \frac{3\pi + \beta}{2i\pi} \right)} \Gamma \left( \frac{\beta + i\pi}{2i\pi} \right) \Gamma \left( \frac{\beta - 3\pi}{2i\pi} \right) \Gamma \left( \frac{\beta - 3\pi}{2i\pi} \right)
\]

As in the case of right-handed operators, we make the ansatz
\[
\Psi_{\pm}|0\rangle = 0,
\]
\[
[\Psi_{+}, \phi_{+}'(\beta)] = 0, \quad [\Psi_{+}, \phi_{-}'] = 0,
\]
\[
[\Psi_{+}, \phi_{-}'(\beta)] = \kappa_{+} \phi_{+}'(-\beta + 2i\pi) + \frac{1}{2} \gamma_{+}(\beta - i\pi),
\]
\[
[\gamma_{+}(\beta), \text{everything}] = 0.
\] (15) (16) (17) (18)
From the boundary reflection equation (14) with $a = -$, we get

$$f^*(-\beta, \mu)\langle 0|e^{\Phi}Z'_+(\beta + i\pi) = (\beta \leftarrow -\beta)$$

hence

$$\kappa_+ = 1, \quad \exp\left(\frac{i}{2}\gamma_+(\beta)\right) = f^*(\beta, \mu)g^*(2\beta)^{-1/2}. \quad (20)$$

Furthermore, in analogy to equation (12), we have

$$\langle B|^{i\phi'-(\beta+i\pi)} = f^*(-\beta, \mu)\langle B|^{i\phi'+(-\beta+i\pi)},$$

$$\langle B|^{\beta}(\beta) \sim \frac{2i\pi - 2\beta}{\mu - \beta + \frac{ix}{2}}(B|^{i\phi'+(-\beta+i\pi)}.$$  

where $\sim$ means “equals up to normalization constant”. The reflection equation in the case of $a = +$

$$\langle B|Z''_+(\beta + i\pi) = \langle B|Z''_+(-\beta + i\pi)\frac{\mu + \beta}{\mu - \beta}R''(-\beta)$$

then is equivalent to

$$\int_{C^*} \frac{d\gamma}{2\pi} \frac{i\pi(\mu - \beta)(2\gamma - 2i\pi)}{(\gamma + \beta - \frac{3i\pi}{4})(\gamma - \mu - \frac{i\pi}{4})(\gamma - \beta - \frac{3i\pi}{4})(\gamma - \beta - \frac{i\pi}{4})}$$

where the contour $C^*$ encloses only the pole $\gamma_1 = \beta + \frac{3i\pi}{4}$ clockwise but does not enclose the poles $\gamma_2 = -\beta + \frac{3i\pi}{4}$, $\gamma_3 = \mu + \frac{i\pi}{4}$ and $\gamma_4 = \beta + \frac{3i\pi}{4}$. This equality can be verified using exactly the same technique as used to verify the equality (13).

Before ending this section we would like to interpretate the functions $\gamma_\pm(\beta)$ as appropriate vacuum expectation values under the vacuum state $|0\rangle$. From equations (10) and (18) we can easily see that

$$\langle 0|\phi'_+(\beta)P|\rangle = \frac{1}{2}\gamma_+(-\beta),$$

$$\langle 0|\phi'_-(\beta)|\rangle = \frac{1}{2}\gamma_+(\beta - i\pi).$$

On the other hand, defining $F = [\Psi_+, \Psi_-]$, we can easily calculate the following relations using the ansatzs (10) and (18),

$$[F, \phi'_+(\beta)] = -\phi'_+(\beta + 2i\pi) - \frac{1}{2}\gamma_+(\beta - i\pi),$$

$$[F, \phi'_-(\beta)] = \phi'_-(\beta - 2i\pi) - \frac{1}{2}\gamma_-(\beta - 2i\pi). \quad (21)$$

Thus the functions $\gamma_\pm(\beta)$ are also equivalent to the following vacuum expectation values

$$\langle 0|\phi'_+(\beta)F|0\rangle = \frac{1}{2}\gamma_+(-\beta - i\pi),$$

$$\langle 0|F\phi'_-(\beta)|0\rangle = -\frac{1}{2}\gamma_-(\beta - 2i\pi).$$
Notice the peculiar effect of $F$ on $\phi'_\pm$ (see equation (21)). Besides changing the signs of $\phi'_\pm(\beta)$ and shifting by scalar functions, $F$ also shift the rapidities of $\phi'_\pm(\beta)$ by $\pm 2i\pi$. It is possible that behind this phenomenon there lies some deep reasoning. However at this moment we cannot say anything about that.

4 Boundary operators versus $q$-oscillators

The bosonic field $\phi'(\beta)$ essentially carries singularities. Therefore, it cannot be expanded into an infinite sum of oscillator modes as the usual free field does. However, we can perform an oscillator realization by the ultraviolet regularization introduced by Lukuyanov [12]. To this end the first thing to do is to introduce the regularization parameter $\epsilon$ such that

$$-\frac{\pi}{\epsilon} \leq |\beta| \leq \frac{\pi}{\epsilon}$$

and consider the “regularized field” $\phi'_\epsilon$. Then the boson $\phi'$ can be viewed as proper limit of $\phi'_\epsilon$ as $\epsilon \to 0$.

In analogy to (2), we have the following commutation relation,

$$[\phi'_1(\beta_1), \phi'_2(\beta_2)] = \ln S'_\epsilon(\beta_2 - \beta_1),$$

where

$$S'_\epsilon(\beta) = \exp \left(\frac{-i\epsilon\beta}{2}\right) \frac{g'_\epsilon(-\beta)}{g'_\epsilon(\beta)},$$

$$g'_\epsilon(\beta) = (1 - q)^{1/2} \frac{\Gamma_q \left(\frac{2i\pi - \beta}{2i\pi}\right)}{\Gamma_q \left(\frac{i\pi - \beta}{2i\pi}\right)}, \quad (q = \exp(-2i\pi))$$

$$\Gamma_q(x) = (1 - q)^{1-x} \prod_{k=1}^{\infty} \frac{1 - q^k}{1 - q^{k+x-1}}.$$

The oscillator expansion of the regularized boson $\phi'_\epsilon(\beta)$ reads

$$\phi'_\epsilon(\beta) = \phi'_{\epsilon\phi}(\beta) + \phi'_{\epsilon+}(\beta) + \phi'_{\epsilon-}(\beta),$$

where

$$\phi'_{\epsilon\phi}(\beta) = -\frac{1}{\sqrt{2}}(Q - \epsilon\beta P),$$

$$i\phi'_{\epsilon+}(\beta) = \sum_{m=1}^{\infty} \frac{a'_m}{\text{sh}(\pi m)} \exp(i\pi\epsilon\beta),$$

$$i\phi'_{\epsilon-}(\beta) = \sum_{m=1}^{\infty} \frac{a'_{-m}}{\text{sh}(\pi m)} \exp(-i\pi\epsilon\beta),$$

$$[P, Q] = -i,$$

$$[a'_m, a'_n] = \text{sh} \frac{\pi m}{2} \text{sh} \pi m e^{-\frac{\pi |m| \epsilon}{2}} \delta_{m+n, 0}. \quad (22)$$
Here the Fock space $\mathcal{F}_\epsilon$ is defined via the “vacuum state” $|0\rangle_p$ on which the oscillators act as follows [8],

$$a'_m|0\rangle_p = 0, \quad (p > 0)$$
$$P|0\rangle_p = p|0\rangle_p.$$  

Notice that the action of $e^Q$ on the state $|0\rangle_p$ shifts the eigenvalue of $P$ by 1 and does not affect the actions of $a'_m$. So the space $\mathcal{F}_\epsilon$ has the structure

$$\mathcal{F}_\epsilon = \oplus_p \mathcal{F}_p,$$

where the “vacuum state” of $\mathcal{F}_p$ is $|0\rangle_p$. In the limit of $\epsilon \to 0$, the space $\mathcal{F}_\epsilon$ tends to $\mathcal{F}$, with the vacuum $|0\rangle_0 \to |0\rangle$.

The generators of ZF algebra in the presence of ultraviolet regularization read [3, 12]

$$Z'_{\epsilon+}(\beta) = e^{\frac{i\epsilon}{2}} \chi'_\epsilon(\beta) = e^{\frac{i\epsilon}{2}} \left( e^{-\frac{\epsilon}{2}} \chi'_{\epsilon}(\beta) + e^{\frac{\epsilon}{2}} V'_{\epsilon}(\beta) \right)$$
$$Z'_{\epsilon-}(\beta) = i e^{-\frac{\epsilon}{2}} \left( e^{-\frac{i\epsilon}{2}} \chi'_{\epsilon}(\beta) + e^{\frac{i\epsilon}{2}} V'_{\epsilon}(\beta) \chi'_{\epsilon} \right),$$

which satisfy ZF algebra with the “regularized” $S$-matrix

$$S'_{\epsilon+\epsilon}(\beta) = S'_{\epsilon-\epsilon}(\beta) = S'_\epsilon(\beta),$$
$$S'_{\epsilon+\epsilon}(\beta) = S'_{\epsilon-\epsilon}(\beta) = -S'_\epsilon(\beta) \frac{\text{sh} \frac{\epsilon}{2}}{\text{sh} \frac{\epsilon(1+\beta)}{2}},$$
$$S'_{\epsilon+\epsilon}(\beta) = S'_{\epsilon-\epsilon}(\beta) = -S'_\epsilon(\beta) \frac{\text{sh} \frac{\epsilon}{2}}{\text{sh} \frac{\epsilon(1+\beta)}{2}}.$$  

Correspondingly, the diagonal “ultraviolet regularized” $R'$-matrix is

$$R'_{\epsilon,\epsilon}(\beta) = R'_\epsilon(\beta) \begin{pmatrix} 1 & 0 \\ \frac{\text{sh} \frac{\epsilon(1+\beta)}{2}}{\text{sh} \frac{\epsilon(1+\beta)}{2}} & 0 \end{pmatrix},$$

where

$$R'_\epsilon(\beta) = \frac{\Gamma_{q^2} \left( \frac{2\pi \beta}{2\pi} \right) \Gamma_{q^2} \left( \frac{2\pi \beta + 2\pi}{2\pi} \right) \Gamma_q \left( \frac{\mu + \beta + i\pi}{2\pi} \right) \Gamma_q \left( \frac{\mu - \beta + 2i\pi}{2\pi} \right)}{\Gamma_{q^2} \left( \frac{2\pi \beta + 2\pi}{2\pi} \right) \Gamma_{q^2} \left( \frac{2\pi \beta}{2\pi} \right) \Gamma_q \left( \frac{\mu - \beta + i\pi}{2\pi} \right) \Gamma_q \left( \frac{\mu + \beta + 2i\pi}{2\pi} \right)}.$$

In analogy to equation (7), we have the following Riemann-Hilbert problem,

$$R'_\epsilon(\beta) = \frac{f'_\epsilon(-\beta, \mu)}{f'_\epsilon(\beta, \mu)}$$
$$f'_\epsilon(\beta, \mu) = \frac{\Gamma_{q^2} \left( \frac{2\pi \beta}{2\pi} \right) \Gamma_q \left( \frac{\mu - \beta + i\pi}{2\pi} \right)}{\Gamma_{q^2} \left( \frac{2\pi \beta + 2\pi}{2\pi} \right) \Gamma_q \left( \frac{\mu + \beta + 2i\pi}{2\pi} \right)},$$
where $f'_b(\beta, \mu)$ is analytic in the lower half $\beta$-plane.

The ansatz for $\Psi_-$ now reads

$$\Psi_- = \sum_{n=1}^{\infty} \frac{n \exp \left( \frac{\pi i n}{2} \right)}{2 \sh \frac{\pi i n}{2} \sh \pi \epsilon} \alpha_n a_n^2 + \sum_{n=1}^{\infty} \frac{n \exp \left( \frac{\pi i n}{2} \right)}{\sh \frac{\pi i n}{2}} \lambda_n a_n^2,$$

(24)

where $\alpha_n$, $\lambda_n$ are parameters to be determined. This form of ansatz for $\Psi_-$ was first introduced by the paper [18] in which similar problem for XXZ-spin chain was considered. In the same paper the rational limit of XXX-spin chain was also taken. Here we adopt this ansatz to achieve an oscillator realization of boundary operators in a field theoretic model.

Following the fundamental commutation relation (22) of the oscillators we have

$$e^{-\Psi_-} a'_n e^{\Psi_-} = a'_n + \alpha_n a'_n + \lambda_n \sh \pi \epsilon,$$

$$e^{-\Psi_-} a'_n e^{\Psi_-} = a'_n, \quad n > 0.$$

Substituting equations (23), (24-25) into (9), we get

$$\alpha_n = -1,$$

$$\lambda_n = -\frac{q^{\frac{\pi i n}{2}}}{n \left( 1 + q^{\frac{\pi}{2}} \right)} + \theta_n \frac{q^{\frac{\pi}{2}} - q^{\frac{-\pi}{2}}}{n(1 + q^{\frac{\pi}{2}})},$$

(26)

where

$$\theta_n = \begin{cases} 0, & m \text{ odd} \\ 1, & m \text{ even} \end{cases}$$

The regularized form of equation (11) reads

$$\sh \frac{i \epsilon (\mu - \beta)}{2} f'_b(\beta, \mu) Z'_\beta(\beta) |B \rangle = (\beta \rightarrow -\beta).$$

(27)

Using exactly the same procedures as in the non-regularized case we can show that the solution (24,26) for the boundary operator $\Psi_-$ is consistent with equation (27). Similarly, making the ansatz

$$\Psi_+ = \sum_{n=1}^{\infty} \frac{n \exp \left( \frac{\pi i n}{2} \right)}{2 \sh \frac{\pi i n}{2} \sh \pi \epsilon} \sigma_n d_n^2 + \sum_{n=1}^{\infty} \frac{n \exp \left( \frac{\pi i n}{2} \right)}{\sh \frac{\pi i n}{2}} \rho_n d_n^2$$

for $\Psi_+$, we find the following consistent coefficients,

$$\sigma_n = -q^n,$$

$$\rho_n = -\frac{q^{\frac{\pi i n}{2}}}{n \left( 1 + q^{\frac{\pi}{2}} \right)} + \theta_n \frac{q^{\frac{\pi}{2}} - q^{-\pi}}{n(1 + q^{\frac{\pi}{2}})}.$$
5 Reflection equations for asymptotic operators

Having finished the construction of boundary operators using the reflection equations of local ZF generators, let us now turn to the other kind of ZF generators, i.e., the asymptotic generators. As mentioned in Section 2, these operators also satisfy ZF algebra (1), with a different $S$-matrix given by equation (4). Moreover, they can also be bosonized using a boson $\phi$ which is closely related to the free boson $\phi'$ used in the last section. Therefore the operators $Z_a(\beta)$ can also effectively act on the Fock space of the free boson $\phi'$, and hence there arise the necessity of checking the consistency between the boundary operators obtained above and the boundary reflection equations for the operators $Z_a(\beta)$.

Let us proceed in some more detail. First, using the boundary Yang-Baxter equation, boundary unitarity and the boundary crossing symmetry

$$C^{ae} R^b_e(-\beta + \frac{i\pi}{2}) = S^{ab}_{cd}(2\beta) C^{de} R^c_d(\beta + \frac{i\pi}{2})$$

corresponding to the asymptotic operators, we obtain the (diagonal) boundary reflection matrix

$$R^b_a(\beta) = R(\beta) \left( \begin{array}{cc} 1 & 0 \\ \frac{\nu + \beta}{\nu - \beta} & 0 \end{array} \right),$$

$$R(\beta) = \frac{\Gamma \left( \frac{\nu + \beta + i\pi}{2\pi} \right) \Gamma \left( \frac{\nu + \beta + 2i\pi}{2\pi} \right) \Gamma \left( \frac{\nu + \beta + 3i\pi}{2\pi} \right)}{\Gamma \left( \frac{\nu - \beta + i\pi}{2\pi} \right) \Gamma \left( \frac{\nu - \beta + 2i\pi}{2\pi} \right) \Gamma \left( \frac{\nu - \beta + 3i\pi}{2\pi} \right)} \left( \frac{\nu - \beta + 2i\pi}{2\pi} \right)^{-1},$$

$$f(\beta, \nu) = \frac{\Gamma \left( \frac{\nu - \beta + 3i\pi}{2\pi} \right) \Gamma \left( \frac{\nu - \beta + 2i\pi}{2\pi} \right)}{\Gamma \left( \frac{\nu - \beta + i\pi}{2\pi} \right) \Gamma \left( \frac{\nu - \beta + 2i\pi}{2\pi} \right)}.$$

In order that the operators $Z_a(\beta)$ and $Z'_a(\beta)$ have the same boundary states we require that the boundary parameters $\mu$ and $\nu$ be related as

$$\nu = \mu - \frac{i\pi}{2}.$$

Lukuyanov also gave the bosonization formula for the operators $Z_a(\beta)$, which read

$$Z_+(\beta) = V(\beta) \equiv e^{i\beta} \chi V(\beta) \chi,$$

$$Z_-(\beta) = i \chi V(\beta) + V(\beta) \chi,$$

where $\chi$ is defined through the matrix elements

$$\langle u | \chi | v \rangle = \xi <u | \int_C \frac{d\gamma}{2\pi} \bar{V}(\gamma) | v \rangle,$$

$$\bar{V}(\gamma) = :e^{i\beta(\gamma)} : = \frac{1}{2} e^{i\beta(\gamma + i\pi) + i\beta(\gamma - i\pi)} :,$$

and the principles for choosing the integration contour $C$ are the same for choosing the contour $C'$, and the boson $\phi$ is connected to $\phi'$ via
\[ \phi_+ (\beta) = -\phi_+ (\beta - \frac{i\pi}{2}), \]
\[ \phi_- (\beta) = -\phi_- (\beta + \frac{i\pi}{2}). \] (28)

Using equations (12) and (28) we can get
\[ e^{i\phi_+ (\beta)} |B\rangle = \frac{g'(-2\beta + i\pi)}{f'(-\beta + \frac{i\pi}{2})} e^{i\phi_- (-\beta)} |B\rangle, \]
\[ e^{-i\phi_+ (\beta)} |B\rangle \sim \frac{2\beta}{\mu + \beta} e^{-i\phi_- (-\beta)} |B\rangle. \]

Using the method of Sections 3 and 4 we can verify that the boundary state \(|B\rangle\) again solve the boundary reflection equation
\[ Z_\alpha (\beta) |B\rangle = R^\phi_\alpha (\beta) Z_\alpha (-\beta) |B\rangle \]
for asymptotic operators with diagonal reflection matrix \(R\). Similarly we can check that the boundary state obtained from the operator \(e^{\Phi_+}\) satisfy the left-handed reflection equation
\[ \langle B|Z_\alpha (\beta + i\pi) = \langle B|Z_\alpha (-\beta + i\pi) R^\phi_{\alpha} (-\beta). \]

This finishes the full proof for the consistency between the boundary states and the boundary reflection equations for the asymptotic operators.

6 Conclusion

In this article we derived the explicit expression for the boundary operators (states) using the technique of bosonization via deformed bosonic fields and oscillators. This is just a partial result for a more involved but self-contained project of calculating the multi-point correlation functions and form factors in the boundary scattering theory. We hope to present the full result elsewhere. We also hope to consider some different models such as Sine-Gordon model, etc. The more general problems such as the connections between deformed bosonic fields and deformed Virasoro algebra [14], the boundary Knizhnik-Zamolodchikov equation and form factor axioms [19] are all worth of further study.

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References


