QUARK CONFINEMENT IN 2+1 DIMENSIONAL
PURE YANG-MILLS THEORY

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ABSTRACT

We study the quark confinement problem in 2+1 dimensional pure Yang-Mills theory using Euclidean instanton methods. The instantons are regularized and dressed Wu-Yang monopoles. The dressing of a monopole is due to the mean field of the rest of the monopoles. We argue that such configurations are stable to small perturbations unlike the case of singular, undressed monopoles. Using exact non-perturbative results for the 3-dim. Coulomb gas, where Debye screening holds only for arbitrarily low temperatures, we show in a self-consistent way that a mass gap is dynamically generated in the gauge theory. In a sense the pure Yang-Mills theory generates a dynamical Higgs effect.
1 Introduction

The problem of quark confinement is one of the “old” unsolved problems in theoretical physics. Despite intense activity over the past two decades, and several approaches to the problem, it is surprising how little we know about this phenomenon. Lattice gauge theories, together with the theory of renormalization group (which provides the basic conceptual framework for all other approaches as well) is the only known quantitative and reliable method of attacking this problem and Monte Carlo simulations have indeed almost “demonstrated” confinement in pure gauge theories in four dimensions. However there is the nagging feeling that we can “demonstrate” confinement, even calculate relevant quantities to some degree of accuracy - but we still don’t “understand” confinement. Despite impressive progress lattice methods remain a black box. More specifically we do not understand in full generality whether (if any) some specific type of gauge field configurations are responsible for confinement and whether we can arrive at a consistent picture of the vacuum of strongly coupled gauge theories.

Physical mechanisms for confinement are, of course, as old as the idea itself. One of the most significant ideas, proposed by Nambu [1] Mandelstam [2] and ’t Hooft [3] is that of dual superconductivity - that the gauge theory vacuum is a condensate of magnetic monopoles. By the dual version of Meissner effect, quarks would then be naturally confined. The formulation of this idea gave rise to deep results concerning duality between electric and magnetic fluxes and its implications to the phase diagram of general gauge theories. The idea is very convincing, and lattice results indeed seem to support it: but is rather difficult to establish in usual gauge theories. Recently, however, there has been progress in supersymmetric gauge theories where topological properties of the moduli space of vacua have been used to argue for the existence of such a monopole condensate [4]. This approach is essentially Hamiltonian: one tries to obtain a picture of the vacuum wave functional.

A complementary viewpoint stems from the Euclidean approach to the problem. This is the idea that Euclidean instantons essentially disorder the vacuum and lead to color confinement. The only known successful implementation of this idea is the classic work of Polyakov [5] who showed that confinement indeed occurs by this mechanism in three dimensional $SU(2)$ Yang-Mills theory coupled to adjoint representation Higgs field. The Higgs breaks the gauge group to $U(1)$. The instantons of this model are the ’t Hooft-Polyakov monopoles. (In the three dimensional Euclidean theory these are of course not solitons, but tunnelling configurations). When the mass scale of this symmetry breaking - the mass of the $W$ boson, $m_W$ is large compared to the mass scale set by the dimensional gauge coupling, a dilute gas of monopoles provides a self-consistent picture. The resulting monopole plasma leads to Debye screening. Wilson loops in the fundamental representation obey an area law and a careful treatment shows that the adjoint
representation Wilson loops obey a perimeter law [6] \(^1\). This is exactly what one expects. The argument can be extended to \(SU(N)\) theory as well.

In this paper we extend this approach to pure Yang-Mills theory. This is a much more nontrivial system for several reasons. The most important reason is that this theory has only one length scale set by the gauge coupling \(g\) and one does not have the luxury of having another length scale \(m_W\) to enable us to control a semiclassical approximation. The classical monopole configurations are singular configurations in the continuum limit and hence the renormalization of the monopole gas is very nontrivial.

It is possible to regulate the singularity by modifying the fields inside a “core” of some size \(\lambda\). The classical action now depends on \(\lambda\). However, if the fluctuations around a regularised monopole solution are decomposed in terms of representations of the direct product of spin and isospin groups, the even parity S-wave fluctuations are unstable [7], because the background magnetic field has a long range Coulomb tail.

However the fluctuation problem in the Yang-Mills theory should be performed not around a single monopole solution, but around the neutral plasma of monopoles which populate the vacuum. These monopoles have long range Coulomb interactions: other monopoles affect the field around a typical monopole in a nontrivial way. More significantly the monopole positions and charges are fluctuating variables, which make the charge density field of the monopoles a dynamical variable. Such a fluctuation problem is difficult to solve exactly.

In this work we address this question in an approximation guided by the physics of the problem. We incorporate the fluctuations of the charge density \(\rho(x)\) by invoking known rigorous results of the three dimensional Coulomb gas due to Brydges [8]. In [8] it has been shown that for a given arbitrarily low temperature there is a chemical potential (fugacity) such that the correlation of the density operator cluster. In other words there is Debye screening at all temperatures. The field around a given charge thus decays exponentially over a debye length as opposed to a power law decay.

Our strategy is as follows. The aim is to show that Debye screening is self consistently realized in the plasma of magnetic monopoles. The main complication in a sense is that the fugacity of the Coulomb gas is itself a functional of the density of monopoles \(\rho(x)\). However using the results of [8] we can argue that for a given value \(g^2\) of the gauge coupling there exists a fugacity and in fact a mean density \(\bar{\rho}(x)\) of monopoles for which the plasma has a finite Debye length. For such a \(\bar{\rho}(x)\), the mean ‘magnetic field’ is also screened with a fall off given by the Debye length. In this sense a monopole configuration in a plasma does not have a Coulomb tail and such a configuration which incorporates this collective property we call a ‘dressed’ monopole. Small fluctuations around a dressed monopole are expected to be stable for reasons similar to the Yang-Mills-Higgs theory. There

\(^1\)The details of the original treatment of Polyakov is erroneous in this regard.
the presence of the Higgs fields in the stability operator cancels the long range tail of the magnetic field of a single monopole. Hence in the pure Yang-Mills theory one seems to have a dynamical Higgs effect that is produced by the monopole plasma.

Our discussion of the confinement problem gives a picture of the dominant configurations in the Euclidean framework. Feynman [9] has given qualitative arguments for the ground state wave function of this gauge theory in analogy with his work on the roton spectrum in liquid helium. It would be interesting to relate these two approaches.

The plan of this paper is as follows. In Section II we discuss the topology of the vacuum in both the Georgi-Glashow model and the pure gauge theory. In Section III we review the generation of mass gap and string tension in the Yang-Mills Higgs system, reiterating our earlier treatment of the model which corrects a crucial error in Polyakov’s work and correctly shows that only “N-ality” is confined. In Section IV we discuss self-consistent Debye screening of the monopole plasma in the pure Yang-Mills theory. Section V is devoted to conclusions. In Appendix I we state the main result of [8].

2 Topology of the vacuum

In this section we shall review the topological properties of the vacuum of the Georgi-Glashow models and the pure Yang-Mills theory in 2+1 dimensions.

The pure Yang-Mills theory is described by the Lagrangian density

\[ \mathcal{L} = -\frac{1}{2g^2} \, \text{tr} \, F_{\mu\nu} F^{\mu\nu} \]  

(2.1)

where \( g^2 \) is the dimensional coupling constant. The gauge group is SU(2). \( F_{\mu\nu} \) is the field strength related to potentials \( A_\mu \) by

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \]  

(2.2)

The Georgi-Glashow model has an additional scalar field \( \Phi(x) \) in the adjoint representation of the gauge group SU(2), the Lagrangian being

\[ \mathcal{L} = -\frac{1}{2g^2} \, \text{tr} \, F_{\mu\nu} F^{\mu\nu} - \text{tr}(\nabla_\mu \Phi)^2 - \frac{\lambda}{4} \left( 2\text{tr} \, \Phi^2 - \eta^2 \right)^2 \]  

(2.3)

where \( \nabla_\mu \) denotes the covariant derivative

\[ \nabla_\mu \equiv \partial_\mu + i[A_\mu, \ ] \]  

(2.4)

We shall often write the matrices \( \phi, A_\mu \) in terms of their components:

\[ A_\mu = \sum_{a=1}^{3} A^a_\mu \frac{\tau^a}{2}, \quad \Phi = \sum_{a=1}^{3} \Phi^a \frac{\tau^a}{2} \]  

(2.5)
(In the following, whenever a field appears without a Roman index, it denotes the corresponding matrix.)

In this paper we shall always discuss the theory in Euclidean three-dimensional space. We will sometime use the word “vacuum” to denote the dominant field configuration in the path integral.

2.1 Topological excitation in the Georgi-Glashow model

To discuss the vacuum configuration it is essential to fix a gauge. We will often work in the unitary gauge defined by

$$\left[\Phi, \tau_3\right] = 0$$

(2.6)

The generic form of the scalar field is:

$$\Phi = \frac{1}{2} \Phi^x x^x = \frac{1}{2} |\Phi| \begin{pmatrix} \cos \Theta & \sin \Theta e^{-i \Psi} \\ \sin \Theta e^{i \Psi} & -\cos \Theta \end{pmatrix}$$

(2.7)

where $|\Phi| = \sqrt{\bar{\Phi} \Phi}$ and $\Theta, \Psi$ are polar angles in isospace:

$$\begin{align*}
\Phi^1 &= |\Phi| \sin \Theta \cos \Psi \\
\Phi^2 &= |\Phi| \sin \Theta \sin \Psi \\
\Phi^3 &= |\Phi| \cos \Theta
\end{align*}$$

(2.8)

The gauge condition (2.6) breaks the symmetry to $U(1)$ since if $\Omega(x)$ is a gauge function implementing (2.6) so does

$$\begin{pmatrix} e^{i \gamma(x)} & 0 \\
0 & e^{-i \gamma(x)} \end{pmatrix} \Omega(x).$$

(2.9)

Furthermore there is a remaining discrete symmetry in this gauge. This is the permutation group $S_2$, called the Weyl group in this context. The elements of the Weyl group may be written as $(1, i \tau_1)$. Thus if $\Omega(x)$ implements the gauge condition (2.6) so does $i \tau_1 \Omega(x)$.

A choice of $\Omega(x)$ is given by

$$\Omega(x) = \begin{pmatrix} e^{i \Psi} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} e^{-i \Psi} \\
-\sin \frac{\theta}{2} e^{i \Psi} & \cos \frac{\theta}{2} \end{pmatrix}$$

(2.10)

In this gauge $\Phi$ is diagonal:

$$\Phi' = \Omega \Phi^x \Omega^+ = \text{diag} \left( \frac{1}{2} |\Phi|, -\frac{1}{2} |\Phi| \right)$$

(2.11)

The weyl group permutes the diagonal elements.

It is clear that the unitary gauge cannot be implemented at all points. This is because $\Omega(x)$ is not defined at the points where $\Phi^x(x) = 0$. The condition
\( \Phi^a(x) = 0 \) denotes three equations in three space-time variables. Hence this is satisfied at isolated points. At these points the gauge transformation leading to the unitary gauge becomes singular.

It is well known that zeroes of the Higgs field are nothing but magnetic monopoles [10]. Let us first discuss the situation in a gauge in which \( A_\mu(x), \Phi(x) \) are regular functions of space. Let \( \vec{x}_0 \) denote a zero of \( \Phi(x) \), i.e.

\[
\Phi(\vec{x}_0) = 0
\]  
(2.12)

Consider a large sphere \( S_R^2 \) centered at \( \vec{x}_0 \). Then, with \( \tilde{\Phi}^a \equiv \Phi^a/|\Phi| \),

\[
\tilde{\Phi}^a : S_R^2 \rightarrow S_\Phi^2
\]  
(2.13)

defines a map from \( S_R^2 \) on to the unit sphere in isospin space, denote by \( S_\Phi^2 \), formed by the tips of unit isovectors \( \tilde{\Phi}^a \). The Kronecker index of this map, specifying the homotopy class is given by [10]

\[
n = \frac{2i}{8\pi} \, Tr \int_{S_R^2} \epsilon_{ijk} \left\{ \tilde{\Phi} \left[ \partial_j \tilde{\Phi}, \partial_k \tilde{\Phi} \right] \right\} \, d^2 \sigma_i
\]  
(2.14)

\( n \) is necessarily an integer and the magnetic change of the monopole at \( \vec{x}_0 \) is given by \( q \)

\[
q = n/g
\]  
(2.15)

In this gauge the field configuration around a monopole reflects the topology of the homotopy group \( \Pi_2(S_2) \).

Let us illustrate the above discussion by the case of a 't Hooft-Polyakov monopole [11]. This is a classical solution of the model (2.3). In the regular gauge, a monopole at the origin is described by

\[
\Phi(x) = \frac{1}{2r} x^a r^a H(r) \\
A_\mu(x) = \epsilon_{\mu a b} x^a \tau_b \frac{1}{2r^2} (1 - K(m \omega r))
\]  
(2.16)

where \( H(r) \) and \( K(m \omega r) \) are structure functions. As \( r \rightarrow \infty \), \( H(r) \rightarrow 1, K(r) \rightarrow 0 \), while at \( r = 0 \), \( H(r) = 0, K(r) = 1 \), thus making \( \Phi(r = 0) = 0 \). From (2.14) it follows that \( n = 1 \), i.e. \( q = \frac{1}{g} \).

To see why (2.16) denotes a magnetic monopole at \( r = 0 \) it is convenient to go the unitary gauge (2.6). In terms of spherical polar coordinates in 3-dim. space, (2.16) becomes, for large \( r \):

\[
\Phi = \frac{1}{2} \left( \begin{array}{cc}
\cos \theta & e^{-i\phi} \sin \theta \\
e^{i\phi} \sin \theta & -\cos \theta
\end{array} \right)
\]  
(2.17)

Hence, from (2.10) the gauge transformation implementing the unitary gauge is

\[
\Omega(x) = \left( \begin{array}{cc}
e^{i\phi} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-\sin \frac{\theta}{2} & e^{-i\phi} \cos \frac{\theta}{2}
\end{array} \right)
\]  
(2.18)
which is clearly singular at $r = 0$, and indeed $r = 0$ is the position of the monopole. In this gauge the vector potential becomes:

\[
A_\mu(x) = \frac{1}{2} \begin{pmatrix} D_\mu(x) & 0 \\ 0 & -D_\mu(x) \end{pmatrix}
\]

\[
D_\mu(x) = -\left(1 - \frac{z}{r}\right) \partial_\mu \tan^{-1}\left(\frac{y}{x}\right)
\]

(2.19)

Here $x, y$ and $z$ denote the cartesian coordinates and then $D_\mu(x)$ is the usual Dirac potential for a monopole at $r = 0$ having a string singularity along the positive $z$-axis. The configuration is thus a monopole with respect to the unbroken invariance group $U(1)$ in the unitary gauge.

What is the topology of the monopole in the unitary gauge? To see this we go back to (2.14). Under the singular gauge transformation (2.18) the right hand side of (2.14) is not invariant. In fact, for large $r$

\[
tr \left\{ \Phi \left[ \partial_\mu \Phi, \partial_\nu \Phi \right] \right\} \sim \Omega \rightarrow \frac{1}{2} \left( \left[ \partial_\mu, \partial_\nu \right] \Omega \right) \Omega^{-1}
\]

(2.20)

where

\[
\Phi'(x) = \Omega(x)\Phi(x)\Omega^{-1}(x) \sim \frac{1}{2} \tau^3 \text{ as } r \rightarrow \infty.
\]

(2.21)

The monopole charge $q$ and hence $n$ is of course invariant. This is explicitly seen by evaluating the surface integral of (2.21) with $\Omega(x)$ given by (2.18). Only the second term contributes since at large $r$ $\Phi'$ is constant; and one has

\[
n = \frac{1}{4\pi} \int \epsilon_{ijk} \left( [\partial_j, \partial_k] \Phi \right) \cos^2 \frac{\theta}{2} d^2\sigma_i
\]

(2.22)

$[\partial_j, \partial_k] \Phi$ is non-vanishing only on the string singularity along $Z$-axis. Hence, with the understanding that the surface integral above is over $S^2$ with a small patch near $\theta = 0$ removed, one can convert it into a line integral along a small loop $C$ enclosing the string:

\[
n = \frac{1}{2\pi} \oint_C \partial_\mu \Phi \, dx^\mu = 1 \text{ identically.}
\]

(2.23)

Thus in the unitary gauge the field reflects the topology of $\pi_1(U(1))$ corresponding to a map from points on a loop enclosing the string, to points on the unbroken $U(1) \subset SU(2)$. The topology is transferred from the Higgs field to the gauge field.

### 2.2 Topology of the vacuum in pure Yang-Mills theory

In the above discussion the fact that the Higgs field $\Phi(x)$ is a dynamical field is irrelevant in determining the topology of magnetic monopoles. In the pure
Yang-Mills theory one can carry out an entirely analogous analysis by replacing
the field $\Phi(x)$ by a local adjoint operator $X$ constructed from the gauge fields
alone. A detailed discussion along these lines has been made by ’t Hooft [12].
The unitary gauge is now defined by

$$[X, \tau_3] = 0 \quad (2.24)$$

Writing $X = \lambda_1 + \epsilon$ where $\epsilon = \sum_{a=1}^{3} \epsilon_a \frac{\tau_a}{2}$ and 1 is the unit $2 \times 2$ matrix, one sees
that in this gauge $X$ is diagonal. Points where $\epsilon = 0$ (i.e., where eigenvalues of $X$
coincide) are monopoles of the $U(1)$ left invariant by the gauge choice (2.24).

Thus in three space dimensions pointlike excitations having the topology of
magnetic monopoles are present in the pure Yang-Mills theory in the unitary
gauge. Treating these as 2+1 dimensional instantons we shall develop a variational
picture of the vacuum state. We shall use the terms monopoles, instanton
or pseudo-particles interchangeably.

3 Confinement in the 2+1 dimensional Georgi-Glashow model

We now briefly discuss the salient features of the mechanism of quark confinement
in the Georgi-Glashow model described by (2.3) (for details see Ref. [5] and [6]).
This is necessary since certain aspects of Polyakov treatment relating to screening
theory are in error.

3.1 Dilute gas approximation

From Section II we know that the vacuum contains monopoles. These monopoles
are best described in the unitary gauge. In this gauge, the gauge field $A_\mu$ due to
a monopole at $r = 0$ is given for large $r$ by

$$A_\mu = \frac{1}{2g} \left( \begin{array}{cc} qD_\mu & 0 \\ 0 & -qD_\mu \end{array} \right) \quad (3.1)$$

where $D_\mu(x) = -\left(1 - \frac{x}{r}\right) \partial_\mu \tan^{-1} \left(\frac{x}{2}\right)$; the same as in (2.19). $q$ is the monopole
charge. Requiring the absence of a string singularity leads to the quantization
condition

$$q = 0, \pm, \pm 2 \ldots \quad (3.2)$$

Half integer charges can be present if the gauge group in (2.3) is $SO(3)$ rather
than $SU(2)$.

For reasons to be discussed later we are interested in monopoles with $q = \pm 1$
($q = 0$ is the trivial case). These are the ’t Hooft-Polyakov monopoles. The
configuration in the regular gauge is given in (2.16). In the unitary gauge the fields are given by:

\begin{align*}
A^1_{\mu} &= \frac{1}{2} \left[ q A^3_{\mu} A^1_{\mu} + i q A^2_{\mu} - q A^3_{\mu} \right] \\
\Phi &= q \frac{H(vr)}{r} \frac{\tau^3}{2}
\end{align*} 

(3.3a, 3.3b)

where \( q = \pm 1 \) and

\begin{align*}
A^1_{\mu} &= -\frac{K(rm_W)}{r} \left[ \dot{\phi} \cos \phi + \hat{\theta} \sin \phi \right] \\
A^2_{\mu} &= \frac{K(rm_W)}{r} \left[ -\dot{\phi} \sin \phi + \hat{\theta} \cos \phi \right] \\
A^3_{\mu} &= -\frac{1}{r} \tan \frac{\theta}{2} \left[ \dot{\phi} \right]_{\mu} = D_{\mu}
\end{align*} 

(3.4, 3.5)

where \((r, \theta, \phi)\) denote polar coordinates in space-times. The functions \( K(rm_W) \), \( H(vr) \) obey well known differential equations for (38a,b) to be a classical Euclidean solution. For \( r \gg m_W^{-1} \) one has \( K(rm_W) \simeq 0 \) and \( H(vr) \simeq 1 \). \( m_W^{-1} \) denotes the “size” of the monopole.

Note that given a configuration of monopoles and anti-monopoles, the Weyl group changes each monopole into an anti-monopole and vice-versa. In principle one may fix the gauge further to remove this discrete degeneracy. We, however, prefer not to do so and average over the Weyl group. Then one may freely perform a sum over all the \( q = \pm 1 \).

The long distance properties of the Yang-Mills-Higgs system are now described in terms of a “classical” field configuration due to a neutral dilute gas of these monopoles, and small fluctuations around this “plasma”. For our description to be consistent we must have

(a) a dilute gas; i.e. if \( R_{ab} \) denotes the typical distance between monopoles\(^2\)

\[ R_{ab} \gg 1/m_W \] 

(3.6)

(b) fluctuations around the plasma configurations must be small, so that a mean field description in terms of the mean field of the monopole plasma makes sense. This condition essentially means that there must be a large number of monopoles within a volume whose size is of the order of \( \xi \), \( \xi \) being the correlation length in the plasma.

\(^2\)This shows why monopoles with \( q = \pm 1 \) alone are important. A \( q = 2 \) monopole may be considered as two \( q = 1 \) monopoles on top of each other. However, in a dilute gas the monopoles are always far apart, and this never happens.
We shall see that these conditions are met in a self-consistent manner if the mass $m_W$ is very large compared to $g^2$.

Apart from these consistency requirements the monopole plasma must of course be stable against fluctuations. This poses no problem, since it is known that due to the presence of Higgs fields, the otherwise unstable $S$-wave fluctuations are stable for a range of parameters [7].

We now proceed to evaluate the partition function of the system.

Outside the monopole cores, the classical fields are simply given by

$$\tilde{A}_\mu = \sum_a q_a D_\mu (x - x_a) \frac{r^3}{2}$$

$$\tilde{\Phi} = \sum_a q_a r^3 \frac{1}{2}$$

where $q_a$ is charge of the monopole at position $x_a$. In evaluating the partition function we expand the fields around the classical configuration (3.7):

$$A_\mu = \tilde{A}_\mu + a_\mu$$

$$\Phi = \tilde{\Phi} + \Phi$$

We fix the background gauge on the fluctuations

$$\partial_\mu a_\mu + i \left[ \tilde{A}_\mu, a_\mu \right] + \left[ \tilde{\Phi}, \Phi \right] = 0$$

The path integral may be now written as a grand canonical partition function of a gas of monopoles:

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} J^N Q_N$$

where

$$Q_N = \sum_{\{q_a\}} \int \prod d x_a \exp \left( - \frac{2\pi}{g^2} \sum \frac{q_a q_b}{|x_a - x_b|} \right)$$

and the fugacity $J$ is:

$$J = \frac{16}{\sqrt{\pi}} g^6 s^{3/2} e^{-s} \left( \frac{\det D^2}{\det -\partial^2} \right)^{-1/2} \left( \frac{\det \Delta_{FP}}{\det -\partial^2} \right)$$

$D^2$ denotes the small fluctuation operator around single monopole filed, $\Delta_{FP}$ is the Fadeev operator, and $s$ is the action of a single monopole.

In (3.10) - (3.12) we have treated the various zero modes of $D^2$ by the standard procedure of collective coordinates [13]. The Weyl degeneracy of the unitary gauge condition has been accounted for by summing over $q_a = 0, \pm 1$ for each space time point, as noted earlier.

The Coulomb gas of eqns. (3.10)-(3.12) may be expressed as a massive scalar field theory using the sine-Gordon transform:
\[ Z = \int \mathcal{D}\chi(x) \exp \left[ -\frac{g^2}{32\pi^2} \int d^6x \left\{ (\nabla \chi)^2 - 2M^2(1 - \cos \chi) \right\} \right] \] (3.13)

where

\[ M^2 = \frac{16\pi^2 J}{g^2}. \] (3.14)

Let us now examine self-consistency of the above scheme. When \( J \) is small, so that the scalar field theory is weakly coupled, \( M \) is precisely the mass gap. Equation (3.13) expresses the long distance behavior of the theory in a dual representation. The classical action of a single monopole has the form

\[ s \sim \frac{m_W}{g^2} \] (3.15)

When \( m_W/g^2 \gg 1 \), it follows from (3.12) that \( J/g^6 \ll 1 \). Further the “classical” piece in \( J \) dominates over the contribution from one-loop fluctuations. Thus the scalar field theory (3.13) is indeed weakly coupled; \( M \) is the mass gap and from (3.14) \( m/g^2 \) is small.

Since \( M \) is the mass gap, the correlation length \( \xi = 1/M \). Also, \( J \) denotes the probability of occurrence of a single monopole and hence the number of monopoles in a Debye volume of size \( \xi \), \( N_\xi \) is given by

\[ N_\xi = \frac{J}{M^3} \sim \frac{g^2}{M} \gg 1 \] (3.16)

where we have used (3.14).

We immediately see that there are large number of monopoles in the Debye volume and our assumption of a mean field description is self-consistently verified [5]. The situation here is identical to the Debye-Huckel theory of electrolytes in the limit of high temperatures where the Debye length is large and the smoothly varying potential field \( \chi \) satisfies the classical sine-Gordon equation.

### 3.2 Wilson loops and confinement of N-ality

Confinement of external quarks in the model may be discussed in terms of screening theory of the monopole plasma. This has been discussed in detail in Ref. [6]. Here we repeat the basic arguments for completeness.

The Wilson loop average is given by

\[ \langle W(c) \rangle = \frac{1}{2} \langle Tr \ P \exp \left( i \oint A_\mu dx_\mu \right) \rangle \] (3.17)

where \( A_\mu = A^a_\mu \frac{T^a}{2} \) when the external quarks are in the fundamental representation of \( SU(2) \) and \( A_\mu = A^a_\mu T^a \) when the quarks are in the adjoint representation. Here \( T^a \) are the generators of the adjoint representation.
The “classical” contribution from the Wilson loop factors out:

\[ \langle W(c) \rangle = (W(c))_{c\ell} [\omega(c)]_{g} \]  

where \((W(c))_{c\ell} = \frac{1}{2} \langle Tr P \exp \left( i \oint A_{\mu} T_{\mu} dx_{\mu} \right) \rangle\), the average being evaluated in the ensemble given by (3.13). \([\omega(c)]_{g}\) obeys a Perimeter law:

\[ [\omega(c)]_{g} \sim \exp(-\alpha P) \]  

where \(\alpha\) is a constant and \(P = \text{perimeter of loop}\).

To evaluate \((W(c))_{c\ell}\) we introduce the density variable

\[ \rho(x) \equiv \sum q_{a} \delta(x - x_{a}) \]  

so that (3.7) becomes

\[ \tilde{A}_{\mu} = \frac{1}{2} r^{3} \int \rho(y) D(x - y) d^{3}y \equiv \frac{1}{2} r^{3} A_{\mu} \]  

Then, for Wilson loop in fundamental representation

\[ \langle W(c) \rangle_{c\ell}^{\text{fund}} = \langle \exp \left( \frac{i}{2} \oint A_{\mu} dx_{\mu} \right) \rangle \]  

while in the adjoint representation

\[ \langle W(c) \rangle_{c\ell}^{\text{adj}} = \frac{1}{2} + \langle \exp \left( i \oint A_{\mu} dx_{\mu} \right) \rangle \]  

Using stokes' theorem one may rewrite \(\oint A_{\mu} dx_{\mu} = \int_{S} H_{\lambda} d\sigma^{\lambda}\) where \(S\) is the minimal surface spanning the loop and

\[ H_{\lambda} \equiv \frac{1}{2} \epsilon_{\mu\nu\lambda} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) = \int \rho(y) \frac{(x - y)_{\lambda}}{|x - y|^{3}} d^{3}y \]  

One thus has

\[ \oint A_{\mu} dx_{\mu} = \int \rho(y) \eta(y) d^{3}y \]  

where

\[ \eta(y) = \int d\sigma x^{\lambda} \frac{(x - y)_{\lambda}}{|x - y|^{3}} \]  

is the solid angle subtended by the loop at the point \(y\). Physically \(\eta(y)\) is the magnetic scalar potential due to a dipole layer of unit strength on sheet \(S\).
\( \langle W(e) \rangle_{\ell} \), evaluated in the ensemble given by (3.13) becomes; for quarks in the fundamental representation:

\[
\langle W(e) \rangle_{\ell}^{\text{fund}} = \int \mathcal{D} \chi(y) \exp \left[ -\frac{g^2}{32\pi^2M} \int dy \left( \left( \nabla \left( \chi - \frac{\eta}{2} \right) \right)^2 - 2(1 - \cos \chi) \right) \right]
\]

(3.27)

where we have scaled the distances: \( y \equiv Mx \).

Since \( M \) is small, (3.27) may be evaluated by stationary phase approximation. The saddle point equation is:

\[
\nabla^2 \chi = \frac{1}{2} \nabla^2 \eta + \sin \chi
\]

(3.28)

We consider a large Wilson loop in the \( y_1 - y_2 \) phase. Then by symmetry \( \chi, \eta \) are functions of \( y_3 \) alone.

The function \( \eta(y) \) has a discontinuity of 4\( \pi \) when the point \( y \) crosses the surface \( S \). In a real physical situation the dipole layer is not infinitely thin. As the thickness approaches zero \( \eta'(y_3) \) becomes singular at \( y_3 = 0 \) and zero elsewhere. For \( \langle W(e) \rangle_{\ell} \) in (3.27) to remain finite one must then demand the boundary conditions:

\[
\chi(y_3 = 0) = \frac{1}{2} \eta(y_3 = 0) \; , \; \chi'(y_3 = 0) = \frac{1}{2} \eta'(y_3 = 0)
\]

(3.29)

so that \( \chi(y_3 = +\epsilon) - \chi(y_3 = -\epsilon) = 2\pi \) as \( \epsilon \to 0 \).

With this boundary condition eqn. (3.29) has the solution

\[
\tilde{\chi}(y_3) = \begin{cases} 
4 \tan^{-1}(e^{-y_3}) & y_3 > 0 \\
-4 \tan^{-1}(e^{y_3}) & y_3 < 0
\end{cases}
\]

(3.30)

This means that in the presence of a fundamental representation Wilson loop a dipole sheet is formed on the minimal surface spanning the loop. There is a concentration of action on the surface, and one is naturally led to an area law for \( \langle W(e) \rangle_{\ell}^{\text{fund}} \). This is indeed found by using (3.30) to evaluate \( \langle W(e) \rangle_{\ell}^{\text{fund}} \) from equations (3.27) in stationary phase approximation one easily obtains

\[
\langle W(e) \rangle_{\ell}^{\text{fund}} = \exp(-\sigma A)
\]

(3.31)

where

\[
\sigma = \frac{g^2 M}{32\pi^2} \int_{-\infty}^{+\infty} dy_3 \left\{ \left( \frac{d\tilde{\chi}}{dy_3} \right)^2 + 2(1 - \cos \tilde{\chi}) \right\} = \frac{3g^2 M}{8\pi^2} \ldots
\]

(3.32)

and \( A = \text{Area of the loop in physical units} \).

Combining (3.31), (3.18) and (3.19) one has

\[
\langle W(e) \rangle_{\ell}^{\text{fund}} = \exp(-\sigma A - \alpha P)
\]

(3.33)
when the loop is large $P \ll A$ and one has an area low:
\begin{equation}
\langle W(c) \rangle_{\text{fund}} = \exp(-\sigma A)
\end{equation}
indicating that quarks in the fundamental representation are confined.

When the quarks are in the adjoint representation one has (3.23), (3.24) and (3.25):
\begin{equation}
\langle W(c) \rangle_{\text{ad}} = \int \mathcal{D}\chi(y) \exp \left[ -\frac{g^2}{32\pi^2 M} \int d^3y \left\{ (\nabla(\chi - \eta))^2 - 2(1 - \cos \chi) \right\} \right]
\end{equation}
The saddle point equation for large loops becomes
\begin{equation}
\frac{d^2\chi}{dy_3^2} = \nabla^2\eta + \sin \chi(y_3)
\end{equation}
and the boundary condition is now
\begin{equation}
\chi(\pm) - \chi(\mp) = 4\pi.
\end{equation}
The only solution satisfying this boundary condition is
\begin{equation}
\chi(y_3) = \begin{cases} 
2\pi & y_3 > 0 \\
-2\pi & y_3 < 0.
\end{cases}
\end{equation}
However, for large but finite loops (3.38) is valid only near the center of the loop and near $y_3 = 0$. To satisfy the continuity requirement for $\chi(y_3)$ for points on $y_3 = 0$ plane, but outside the loop, $\chi$ necessarily has finite gradients over unit (dimensionless) distances all along the area of the loop. This costs an action proportional to the area, and hence one has
\begin{equation}
\langle W(c) \rangle_{\text{ad}} \sim \exp(-\sigma' A)
\end{equation}
$\sigma'$ being some constant. Using (3.23) and (3.19) we have:
\begin{equation}
\langle W(c) \rangle_{\text{ad}} = \left( \frac{1}{2} + e^{-\sigma' A} \right) e^{-\alpha P}
\end{equation}
for large loops $e^{-\sigma' A} \ll \frac{1}{2}$ and one has a perimeter law:
\begin{equation}
\langle W(c) \rangle_{\text{ad}} \sim e^{-\alpha P}
\end{equation}
This shows that quarks in the adjoint representation are not confined.

The picture of confinement discussed in this section is valid for any SU(N) gauge group. The mechanism is essentially Debye screening in a gas of “non-Abelian” monopoles belonging to the adjoint representation of a dual group *SU(N) [6].
4 Confinement in pure Yang-Mills theory

From our discussion in Section 2, monopole configurations are present in pure Yang-Mills theory as well. This fact depends only on the fact that the gauge group is $SU(2)$ and space is 3-dimensional. However unlike the Georgi-Glashow model, in the absence of the Higgs field there is no second scale (i.e. $m_W$) which gives a definite size of these monopoles. There are classical solutions which correspond to monopoles of zero size. These are Wu-Yang monopoles with a singularity at the location of the monopole. We will regulate these monopoles by assigning a size $\lambda$, which is explained in the next subsection. We will then construct an expansion around a plasma of such monopoles and as explained in the introduction argue that these monopoles are stable against fluctuations. For our discussions we shall assume that our theory has a ultraviolet cutoff, e.g. a space lattice with lattice spacing ‘a’, so that $g_0^2 = ag^2 \to 0$ and $\lambda \gg a$.

4.1 The monopole solution

We will consider monopoles which have some size $\lambda$. The field due to single monopole at $\vec{x} = 0$ is given in the unitary gauge by:

$$A_\mu^a(x) = \frac{1}{2} \left[ q \hat{A}_\mu^3 - iq \hat{A}_\mu^2, \hat{A}_\mu^1 - i q \hat{A}_\mu^3 - \hat{A}_\mu^1 - i q \hat{A}_\mu^2 \right]$$

(4.1a)

with

$$A_\mu^1(x) = - \frac{K(r/\lambda)}{r} \left[ \hat{\phi} \cos \phi + \hat{\theta} \sin \phi \right]$$

(4.1b)

$$A_\mu^2(x) = - \frac{K(r/\lambda)}{r} \left[ \hat{\phi} \sin \phi + \hat{\theta} \cos \phi \right]$$

(4.1c)

$$A_\mu^3(x) = \frac{1}{r} \tan \frac{\theta}{2} \left[ \hat{\phi}_\mu = D_\mu \right]$$

in spherical coordinates, here $K(r/\lambda)$ is a structure function regulating the fields at $r = 0$. The function $K(r/\lambda)$ goes to 1 as $r \to 0$ as follows

$$K(r/\lambda) \sim 1 - \frac{r^2}{\lambda^2} \quad \text{for } r \to 0$$

(4.2)

while for $r > \lambda$, $K(r/\lambda)$ vanishes rapidly. For all practical purposes $K(r/\lambda) = 0$ for $r \geq \lambda$. As shown by Banks, Myerson and Kogut [15], one can choose such a $K(r/\lambda)$ so that the configuration (4.1) is a classical solution. The action of single monopole is

$$s = \frac{4\pi}{g^2} \int_0^\infty dr \left[ \left( \frac{dK}{dr} \right)^2 + \left( \frac{K^2 - 1}{2r^2} \right) \right]$$

(4.3)

Note that the monopole field is abelian far outside the monopole core. Consider a dilute gas of such monopoles. In this case only need to consider $q = \pm 1$. 

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(Diluteness means that if \( x_a, x_b \) denote monopole positions \( (x_a - x_b) \gg \lambda \).) The field configuration in the regions much beyond the core of the monopoles is:

\[
A^\text{el}_\mu(x) = \sum_{a=1}^{N} \tilde{A}_\mu(x - x_a)
\]  

(4.4)

where \( x_a \) denotes a monopole position and \( N \) is the total number of monopoles. The total action of this gas is

\[
S_{cl} = N s + \frac{2\pi}{g^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|}
\]  

(4.5)

Finally we record the single monopole field configuration in the radial gauge

\[
A^a_\mu = -\frac{\epsilon_{\mu aj}x^j}{r^2}(1 - K(r/\lambda))
\]  

(4.6)

### 4.2 The path integral

The monopole configuration is used to evaluate the path integral by saddle point method. One expands the field around \( A^\text{el}_\mu \):

\[
A_\mu = A^\text{el}_\mu + g_0 a_\mu
\]  

(4.7)

The WKB parameter is the dimensionless coupling, \( g_0^2 = g^2 a \) where \( a \) is the lattice spacing. Then for small \( g_0 \) the fluctuations are small and may be treated in the gaussian approximation. The path integral may be formally written as:

\[
Z = \int \prod_x dA_\mu(x) \exp(-S_{cl}) \exp \left(-\int a D^2 a d^3 x \right)
\]  

(4.8)

where \( S_{cl} \) is given by (4.5). Here \( D^2 \) is the stability operator for small fluctuations:

\[
\int a D^2 a d^3 x \equiv \int \mathcal{L}''(A_{el}) a^2 d^3 x
\]  

(4.9)

where \( \mathcal{L}''(A_{cl}) \) is the second functional derivative of the Lagrangian density evaluated at \( A_{el} \).

Equation (4.8) as it stands is meaningless since \( D^2 \) has a zero eigenvalue for each symmetry of the original Lagrangian. The local gauge symmetry is fixed by requiring the fluctuations to satisfy the background gauge condition

\[
\nabla_\mu(A^\text{el}) a_\mu(x) = 0
\]  

(4.10)

where \( \nabla_\mu(A^\text{el}) \) is the covariant derivative evaluated at the configuration \( A^\text{el} \):

\[
\nabla_\mu(A_{el}) \equiv \partial_\mu + i \left[ A^\text{el}_\mu, \right]
\]  

(4.11)
This gives rise to the usual Fadeev-Popov determinant \((\det \Delta_{FP})\). The zero modes arising from the breaking of the global translation invariance and the global \(U(1)\) are replaced by integration over corresponding collective coordinates. Finally we have to sum over all \(N\) with the standard division by \(N!\). One finally has the formal expressions

\[
Z = \sum_{N=0}^{\infty} \frac{1}{N!} Q_N
\]

\[
Q_N = \sum_{\{\varphi\}} \left(\frac{8}{\sqrt{\pi}} s^{3/2}\right)^N \int \prod_{s=1}^{N} dx_se^{-S_{\varphi}} \left(\frac{\det D^2}{\det -\partial^2}\right)^{-1/2} \left(\frac{\det \Delta_{FP}}{\det -\partial^2}\right) 
\]

### 4.3 Instability of the Single Undressed Monopole

In the usual semiclassical method the dilute instanton gas is non-interacting and one writes

\[
\det D^2 = (\det d^2)^N
\]

where \(d^2\) is the stability operator for the single monopole configuration.

We will see, however, that the \(d^2\) has negative eigenvalues signifying an instability of a single monopole. In the background gauge (4.10) the operator \(d^2\) has the form:

\[
d^2 = \delta_{\mu\nu} \nabla_\alpha \left(\hat{A}_{\alpha\ell}\right) \nabla^\alpha \left(\hat{A}_{\alpha\ell}\right) + i \left[\hat{F}_{\mu\nu}^{\alpha\ell}, \right]
\]

where \(\hat{A}_{\alpha\ell}\) is defined in (4.1) and \(\hat{F}_{\mu\nu}^{\alpha\ell}\) is the corresponding field. The Wu-Yang case corresponds to \(K = 0\) in (4.1). Following Yoneya [7] we shall use a spherical basis in the product space (space-time) \(\otimes\) (isospin space). In this product space \(\hat{A}_{\alpha\ell}\) and the fluctuations \(a_\mu\) become tensors. Then the unstable modes of \(d^2\) are given by

(a) Odd parity \(S\) waves:

\[
\phi_1 = a_{rr}, \quad \phi_4 = \frac{1}{\sqrt{2}} (a_{\phi\phi} + a_{\theta\theta})
\]

(b) Even parity \(S\) wave:

\[
\bar{\phi}_4 = \frac{1}{\sqrt{2}} (\phi_{\phi\phi} - \phi_{\theta\theta})
\]

The tensor indices on \(\phi\) refer to the abovementioned spherical basis. The corresponding eigenvalue equations are: (See Yoneya, Ref. [7])

\[
\left( -\frac{d^2}{dr^2} + \frac{3K^2 - 1}{r^2} \right) \left( r \bar{\phi}_4 \right) = \alpha_c^2 (r \bar{\phi}_4)
\]
\[
\frac{2K^2}{r^2} \phi_1 - \sqrt{2} \left( \frac{K}{r} \frac{d}{dr} + \frac{K - rK'}{r^2} \right) \phi_4 = \alpha_0^2 \phi_1 \tag{4.16}
\]
\[
\sqrt{2} \left( \frac{1}{r^2} \frac{d}{dr} (rK) - \frac{K - rK'}{r} \right) \phi_1 + \left( \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{K^2 - 1}{r^2} \right) \phi_4 = \alpha_0^2 \phi_4 \tag{4.17}
\]
\[
\left( \frac{d}{dr} + \frac{2}{r} \right) \phi_1 = \sqrt{2} \frac{K}{r} \phi_4 \tag{4.18}
\]

For the Wu-Yang monopole \( K = 0 \) and the instability is obvious from (4.15) and (4.17). In this case both \( r \phi_4 \) and \( r \phi_4 \) obey the equation
\[
\left( - \frac{d^2}{dr^2} - \frac{1}{r^2} \right) \psi = \alpha^2 \psi, \quad \psi = r \phi_4, r \phi_4. \tag{4.19}
\]

The above equation is the Schrödinger equation for a particle in a spherical potential \( U_0(r) = -\frac{\gamma}{r^\gamma} \) with \( \gamma = 1 \). For such a potential it is known that when \( \gamma > 1/4 \) there are an infinite number of negative eigenvalues \( \alpha^2 < 0 \) with \( \alpha^2 = 0 \) as a limit point [16]. (This corresponds to the fact that the fact that the particle falls to the center).

When the structure function \( K(r/\lambda) \neq 0 \) the odd parity \( S \)-waves become stable. If we take \( K(r/\lambda) \) falling rapidly to zero beyond \( r = \lambda \), but never becoming exactly zero for any finite \( r \) one can invert (4.18) to express \( \phi_4 \) in tensors of \( \phi_1 \) and insert this into (4.16) to obtain:
\[
\left[ - \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + \frac{K^2 + 1}{r^2} \right] \left( \frac{r \phi_1}{K} \right) = \alpha^2 \left( \frac{r \phi_1}{K} \right) \tag{4.20}
\]

In the corresponding Schrödinger problem the “potential” is now positive everywhere and hence \( \alpha^2 > 0 \). Since one can now rewrite (4.18) as
\[
\phi_1 = \frac{\sqrt{2}}{r^2} \int_0^r y K(y) \phi_4(y) dy \tag{4.21}
\]

stability of \( \phi_4 \) also follows.

However, regularising the field near \( r = 0 \) does not remove the even parity \( S \)-wave instability. In fact (4.15) has a potential that becomes asymptotically \(-\frac{1}{r^2}\) as \( r \rightarrow \infty \). For such a potential there are still an infinite number of negative eigenvalues. This result is independent of the detailed form of the potential near \( r = 0 \). Hence the single regularised monopole is unstable and cannot be treated as a dominant configuration in the path integral.

It should be noted in the presence of Higgs fields the potential is replaced by
\[
\frac{3K^2 + H^2 - 1}{r^2} \tag{4.22}
\]
where \( H(\nu r) \) is the function in (3.3b). As \( r \rightarrow \infty \), \( H(\nu r) = 1 + e^{-\nu r} \), and hence the \( \frac{1}{r^2} \) Coulomb tail is cancelled and we have a screened potential \( e^{-\nu r}/r^2 \), which removes the potential instability.
4.4 Debye Screening in the Monopole Gas

Our main point is that when the instantons are interacting, the fluctuation problem cannot in general be factored into \( N \) copies of the fluctuation problem for an isolated monopole. Rather one should consider the stability of the neutral plasma of monopoles as a whole. The main reason behind this is that we have an integration over the positions of the monopole and the charges, or equivalently a functional integration over the charge density field. The effect of this averaging over the charge density is very nontrivial. The results of [8] show that the charge density field clusters so that the theory of the density field generates a mass gap dynamically. We summarize these results in Appendix I.

In fact the results of [8] specialized to the coincident connected two point function also shows that the fluctuations of the density are finite and bounded, even as the two points approach within a distance \( \lambda \) (monopole core size) of each other. This implies that there is a mean configuration \( \tilde{\rho}(x) \) of the monopoles around which the fluctuations in density are bounded. This mean density in turn determines a mean value of the fugacity. Note that \( \tilde{\rho}(x) \) in general does not satisfy a classical equation. Also since the monopole charge density is the source of the magnetic field, this result implies that the fluctuations of the magnetic field are also bounded and the \( \frac{1}{r^2} \) magnetic field of a single monopole is Debye screened to \( \frac{1}{r^2 \ln r} \), where \( M(g^2, \lambda, z) \) is the non-perturbative mass gap, which depends on the coupling \( g^2 \), the monopole size \( \lambda \) which is effectively the cut-off of the Coulomb gas and the fugacity \( z \). Recall that the source of instability for a single isolated monopole is the long range tail of the Coulomb potential. One might, therefore expect that in the screened neutral plasma a "dressed" monopole whose Coulomb tail has been screened can in fact be stable.

In the following we shall assume that the monopole gas is dilute in the limit \( g_0^2 = a g^2 \to 0 \). Using translation invariance we focus on one monopole at \( x = x_\alpha \) and its neighbourhood. Recall that the fields outside the monopole cores of size \( \lambda \) are abelian. Thus, in the unitary gauge it follows from (4.4) and (4.1) that outside the core of this monopole the field is

\[
\bar{A}_\mu^\text{out} = \sum_{a=1}^{N} \frac{1}{2} q_a \begin{pmatrix}
D_\mu(x - x_a) & 0 \\
0 & -D_\mu(x - x_a)
\end{pmatrix}
\]

where we have set \( x_N = x_\alpha \). Inside the core the effect of the core field of the other monopoles can be ignored and we have

\[
\bar{A}_\mu^\text{in} = \sum_{a=1}^{N-1} \frac{1}{2} q_a \begin{pmatrix}
D_\mu(x - x_a) & 0 \\
0 & -D_\mu(x - x_a)
\end{pmatrix}
+ \frac{1}{2} \begin{pmatrix}
q D_\mu(x - x_\alpha) & W^- (x - x_\alpha) \\
W^+ (x - x_\alpha) & -q D_\mu(x - x_\alpha)
\end{pmatrix}
\]

where \( \vec{W}_\mu = \vec{A}_\mu^1 \pm i q_\alpha \vec{A}_\mu^2 \) and \( \vec{A}_\mu^1, \vec{A}_\mu^2 \) are as in (4.1). Introducing the charge
density
\[ \rho(x) \equiv \sum_{\alpha=1}^{N} q_{\alpha} \delta(x - x_{\alpha}) \]  
and assuming that in the mean \( \rho_{\text{N}} \simeq \rho_{\text{N}-1} \) we can rewrite (4.23a) and (4.24) as
\[ A_{\mu}(x; x_{\alpha}, [\bar{\rho}]) = \int d^{3} y \frac{\pi^{3}}{2} \rho(y) D_{\mu}(x - y) \]
\[ + \theta(c - |x - x_{\alpha}|) \left( D_{\mu}(x - x_{\alpha}) W_{\mu}^{-}(x - x_{\alpha}) \right) (4.26) \]
In the above expression the sharp \( \theta \) function may be replaced by a smoother version. The equation (4.26) represents a “dressed” monopole configuration, if we replace as per our previous discussion the density \( \rho(y) \) in (4.26) by its mean value \( \bar{\rho}(y) \).

Since \( \lambda \ll |x_{a} - x_{b}| \) and anticipating the fact that the dilute gas of “dressed” monopoles is only weakly interacting we can use the standard approximation to write so that
\[ \det D^{2} \simeq \prod_{\alpha} \left( \det D^{2}[\bar{\rho}, x_{\alpha}] \right) \]  
where \( D^{2}[\bar{\rho}, x_{\alpha}] \) now denotes the stability operator for a single dressed monopole at \( x = x_{\alpha} \). In fact we have, by translation invariance
\[ \det D^{2} \simeq (\det D^{2}[\bar{\rho}])^{N} \]  
(4.28)

Also the classical action of a single regularised dressed monopole is no longer given by (4.3), but should be
\[ \tilde{s} \simeq \frac{4\pi}{g^{2}} \left( \int_{0}^{\lambda} dr \left( \left( \frac{dK}{dr} \right)^{2} + \left( K^{2} - 1 \right) \frac{3r^{2}}{2} \right) \right) + \int_{\lambda}^{\infty} dr e^{-2Mr/r^{2}} \]  
(4.29)
Note that \( \tilde{s} = \tilde{s}(g^{2}, \lambda, z) \).

### 4.5 The Sine-Gordon Transform and Dynamical generation of Mass Gap

We can now rewrite the theory in terms of a sine-Gordon model. The path integral is written as
\[ Z = \sum_{N=0}^{\infty} \frac{1}{N!} \left( \frac{8g^{6}}{\sqrt{\pi}} \right)^{\frac{3}{2}} \left( e^{-\frac{2}{g^{2}}} \right)^{N} \sum_{\{a\} = i=1}^{N} \prod d\bar{x}_{i} \exp \left( -\frac{2\pi}{g^{2}} \sum_{\{a\} = i=1}^{N} q_{a} q_{b} \frac{1}{|x_{a} - x_{b}|} \right) \left( \Theta[\bar{\rho}] \right)^{N} \]  
(4.30)
where we have defined \( \Theta[\bar{\rho}] \) (functional of the charge densities) as
\[ \Theta[\bar{\rho}] = \left( \frac{\det D^{2}[\bar{\rho}]}{\det -\partial^{2}} \right)^{-1/2} \left( \frac{\det \Delta_{FP}(\bar{\rho})}{\det -\partial^{2}} \right) \]  
(4.31)
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We then have
\[
Z = \sum_{N=0}^{\infty} \frac{1}{N!} \bar{J}^N \prod_{i=1}^{N} \int d\vec{r}_i \exp \left( -\frac{2\pi}{g^2} \sum \frac{q_a q_b}{|x_a - x_b|} \right)
\]  
(4.32)
where the mean fugacity \( \bar{J} \) is given by the formula
\[
\bar{J} = \frac{8}{\sqrt{\pi}} \frac{g^6}{32\pi^2} e^{-\frac{4}{\bar{J}}} \Theta[\bar{p}]
\]  
(4.33)

We can now perform the sine-Gordon transform as in section 3, and we write (4.31) as
\[
Z = \int \mathcal{D}_\lambda(x) \exp \left[ -\frac{g^2}{32\pi^2} \int d^3 x \left\{ (\nabla \chi)^2 - 2\bar{M}^2 (1 - \cos \chi) \right\} \right]
\]  
(4.34)
where
\[
\bar{M}^2 = \frac{16\pi^2 \bar{J}}{g^2}.
\]  
(4.35)

It is important to emphasize in accordance with the discussion of [8], that the quadratic term \( \int d^3 x (\nabla \chi)^2 \) in (4.33) must be understood in a regularised sense, so that the Coulomb potential between the monopoles is valid only up to distances greater than the core size \( \lambda \). In this sense \( \lambda \) is the cut off (lattice spacing) of the sine-Gordon theory.

### 4.6 Stability of the dressed monopole

We now discuss, in some more detail than previously, whether the function \( \Theta[\bar{p}] \) which is used in the definition of the fugacity \( \bar{J} \) in (4.33) is well defined. This issue is important because we have already indicated in Section 4.3 that as we average over \( \rho(x) \) the operator \( D^2[\bar{p}] \) has negative eigenvalues when \( \rho \) corresponds to a single isolated monopole.

We will now argue that \( D^2[\bar{p}] \) is a positive operator. Recall the form of \( D^2[\bar{p}] \) in the unitary gauge, outside the core of the dressed monopole which we can choose to be at \( x = 0 \)
\[
D^2[\bar{p}] = -\delta_{\mu\nu} \nabla_\alpha (A(x, [\bar{p}])) \nabla^\alpha (A(x, [\bar{p}])) + i \left[ F^{(3)}_{\mu\nu}(x, [\bar{p}]), \right]
\]  
(4.36)
where \( F^{(3)}_{\mu\nu}(x, [\bar{p}]) \) is related to the magnetic field by \( B^{(3)}_{\lambda} = \epsilon_{\mu\nu\lambda} F^{(3)}_{\mu\nu} \) and \( B^{(3)}_\lambda \) is given by
\[
B^{(3)}_\lambda = \frac{\partial}{\partial x_\lambda} \int d^3 y \frac{\bar{p}(y)}{|x - y|}
\]  
(4.37)
Debye screening means that
\[
\bar{B}^{(3)}_\lambda = \frac{x^\lambda}{r^3} e^{-Mr}
\]  
(4.38)
and $M$ is the mass gap related to the Debye length $l_D$ by $M = \frac{1}{l_D}$.

For a nonzero mass gap, the field outside the monopole core cannot be transformed to the radial form of \eqref{4.6}. However close to the core and distances much smaller than the Debye length $l_D$ the field is close to the single monopole field outside the core and may be cast in the radial gauge. Furthermore at distances much larger than the Debye length, the field is close to zero and once again one may cast the gauge potential in the radial gauge trivially (i.e. with $K(r) = 1$). As mentioned in Section (4.3) the source of the even parity S-wave instability is the long range Coulomb field of the monopole. Since screening cuts off this Coulomb field and replaces it approximately by an exponential, one expects stability.

The situation is in fact similar to that of the Yang-Mills-Higgs system in some respects. Recall (4.22) that the S-wave stability of the 't Hooft-Polyakov monopole is ensured by the fact the Higgs field rises exponentially to 1 beyond the core and cancels the negative $\frac{1}{r}$ tail of the gauge field, preventing the potential in the Schrödinger problem from being negative at large distances. In our problem the gauge field itself falls of exponentially to zero and thus the potential in \eqref{4.15} is positive at large distances. In this sense we have a dynamical Higgs effect. A rough estimate based on the considerations of the previous paragraph indicate that this indeed happens for monopole sizes less than the Debye length, thus ensuring self consistency of our procedure.

5 Conclusions

We have argued that in 2+1 dimensional pure Yang-Mills theory Debye screening in a gas of regularized and dressed Wu-Yang monopoles provides a consistent picture of quark confinement. We have used the results of \cite{8} that in a three dimensional Coulomb gas the charge density field always clusters, leading to Debye screening even for arbitrarily low temperatures, i.e. \( g_0^2 \to 0 \). Our line of argument has been self-consistent in nature, because Debye screening in turn implies a screened magnetic field and hence the stability operator around a single dressed monopole is expected to have no negative eigenvalues. The mass gap thus obtained is non-perturbative and unlike the Yang-Mills-Higgs system is not necessarily small. A related issue is that the mean configuration $\tilde{r}(x)$ is in general non-classical and hence the associated scale of potential $\tilde{\chi}(x)$ does not satisfy a classical equation like \eqref{3.28}. Hence the explicit evaluation of the Wilson loops is not as easily done. However on general grounds the existence of a mass gap leads to qualitative conclusions that are similar to the case of the Yang-Mills-Higgs system.
6 Appendix I

In this appendix we state the main results of [8] on the 3-dim. Coulomb gas. Theorem 2.1 in Brydges [8], adapted to our notation states that:

Given any \( c_1 > 0 \), there exists \( c_2 > 0 \) such that for \( \frac{1}{c_1} \leq g^2 \lambda \) and \( z \leq \frac{1}{2} c_2 g^6 \) (\( z \) is the fugacity) the correlation functions of the density operator exits and clusters exponentially, i.e. there exist strictly positive constants \( M(z, g^2, \lambda), \ c' = c'(n') \) such that for \( n_1 < n' \)

\[
\left| \langle \Pi_{i=1}^{n_1} \rho(x_i) \Pi'_{j=n_1+1}^{n_1'} \rho(x_j + a) \rangle - \langle \Pi_{i=1}^{n_1} \rho(x_i) \rangle \langle \Pi'_{j=n_1+1}^{n_1'} \rho(x_j) \rangle \right| \\
\leq c' \exp \left( -\frac{2}{z n_1 n_1'} \sum_{i=1}^{n_1} \sum_{j=1}^{n_1'} |x_i - x_j + a| \cdot M \right) \quad (6.1)
\]

\( M(z, g^2, \lambda) \) is the mass gap whose inverse is the Debye screening length. In the limit \( g^2 \lambda \to \infty \) one has the classical Debye-Huckel limit \( \frac{M}{g^2} \to 0 \). If we apply (6.1) to the 2-point function of the density, for separation of the order \( \lambda \), the monopole core size, which is the lattice spacing for the Coulomb gas we get

\[
|\langle \Delta \rho(0) \Delta \rho(\lambda) \rangle| \leq c' e^{-M \lambda} \quad (6.2)
\]

where \( \Delta \rho(x) = \rho(x) - \langle \rho(x) \rangle \). (6.2) says that the fluctuations of \( \rho(x) \) are bounded and finite. This means that there is a mean density \( \bar{\rho}(x) \) that dominates the partition function.

References


