Extended Dualization: a method for the Bosonization of Anomalous Fermion Systems in Arbitrary Dimension

José Luis Cortés†, Elena Rivas† and Luis Velázquez*

†Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain.
*Departamento de Matemática Aplicada, Universidad de Zaragoza, 50015 Zaragoza, Spain.

24 March 1995

Abstract

The technique of extended dualization developed in this paper is a path-integral method for the bosonization of quantized fermion systems in arbitrary dimension $D$. In its original (minimal) form, dualization is restricted to models wherein it is possible to define a dynamical quantized conserved charge. We generalize the usual dualization prescription to include systems with dynamical non-conserved quantum currents.

Bosonization based on this extended dualization requires the introduction of an additional rank 0 (scalar) field together with the usual antisymmetric tensor field of rank $(D-2)$. Our generalized dualization prescription permits one to clearly distinguish the arbitrariness in the bosonization from the arbitrariness in the quantization of the system.

We study the bosonization of the most general quantization of the massive Thirring model in arbitrary dimension. Dualization permits one to bosonize this model trivially by invoking the bosonization of the free massive Dirac fermion.

We also apply our extended dualization to the bosonization of the Chiral Schwinger model. For this model, minimal dualization is inadequate. We show that two independent scalar fields are required to describe the chiral current in the most general quantization of the Chiral Schwinger model.
1 Introduction

Bosonization is a procedure for converting a given fermion field theory into its bosonic equivalent. This equivalence is to be understood at the quantum level, as an equivalence between the Green functions of the two quantum theories. The importance of bosonization is clear, since it permits one to investigate quantum fermion systems by using bosonic techniques, which are always more powerful and better developed.

The bosonization program started a considerable time ago, but for a long while the known procedures were very restrictive, only applying to theories in two dimensional ($D = 2$) spacetime [1, 2, 3, 4].

A new direction of inquiry has been recently opened by applying dualization techniques to the purpose of bosonization [5, 6, 7, 8]. Dualization can be applied to arbitrary spacetime dimension $D$, and also to arbitrary fermion field theories, provided that a conserved quantum charge can be defined. We will call this original prescription “minimal dualization”.

Given a $D$-dimensional fermion system with a dynamical conserved quantum current, minimal dualization guarantees the existence of a bosonized version. The explicit bosonic action is obtained by integrating out an auxiliary gauge field. The bosonic variable is an antisymmetric tensor field of rank $D - 2$, a $(D - 2)$-form, denoted $\Lambda^{(D-2)}$.

Minimal dualization rewrites the conserved quantized current density $J^{(1)}$, in terms of the real field $\Lambda^{(D-2)}$ as

$$J^{(1)} = * d \Lambda^{(D-2)}.$$  \hspace{1cm} (1)

Here $d$ is the exterior differential and $*$ the Hodge star operation.

Alternatively, in explicit components,

$$J_\mu (x) = \epsilon_{\mu_1 \mu_2 \cdots \mu_{D-1}} \partial^{\mu_1} \Lambda^{\mu_2 \cdots \mu_{D-1}} (x).$$  \hspace{1cm} (2)

For $D = 2$ minimal dualization [5, 7] gives the same results as conventional bosonization [2].

In this paper we will extend the minimal dualization procedure to include anomalous quantum fermion systems. We define an anomalous system as a
system with a non-conserved quantum dynamical current. In that case, the current density $\mathcal{J}^{(1)}$ is not necessarily conserved.

In extended dualization, the bosonic equivalent action depends on the previously introduced antisymmetric rank $(D-2)$ field, plus a rank 0 real scalar field $\lambda^{(0)}$. The relation between the current density and the bosonic fields is modified to

$$i\mathcal{J}^{(1)} = d\lambda^{(0)} + i*d\Lambda^{(D-2)}.$$  \hspace{1cm} (3)

If the quantization of the fermion fields is compatible with the conservation of the current $\mathcal{J}^{(1)}$, we will see that the scalar field $\lambda$ is equal to zero and we recover the usual minimal dualization.

Both (1) and (3) imply that for $D > 2$, the tensor field $\Lambda^{(D-2)}$ is not uniquely defined. If two $(D-2)$-forms $\Lambda$ and $\Lambda'$ are related by

$$\Lambda' - \Lambda = d\chi^{(D-3)},$$  \hspace{1cm} (4)

for some antisymmetric $(D-3)$-form, $\chi^{(D-3)}$, then these two $(D-2)$-forms yield the same current. The bosonic action has, by construction, a gauge symmetry. The $(D-2)$-form $\Lambda$ is a so-called “gauge form” (generalization of a gauge field) [7].

In Section 2, we will give a full description of the extended dualization method, and show that it permits the bosonization of anomalous fermion theories in arbitrary dimension. We will restrict the discussion in this paper to abelian dualization. Non-abelian dualization will be the subject of a subsequent paper.

Extended dualization is a generalization of minimal dualization to include anomalous quantum systems. It also exhibits other virtues that deserve mention here. Within our extended approach, we can reinterpret minimal dualization in a very simple way. This reinterpretation clearly shows that there is a considerable amount of freedom involved in the dualization procedure. This arbitrariness in the bosonization is clearly differentiated from the arbitrariness in the quantization of the fermion system.

Although minimal dualization allows one to (at least in principle) bosonize any arbitrary fermion theory with a dynamical conserved current, it does not guarantee the corresponding bosonic action to be easily tractable or even local. The use of the above mentioned arbitrariness in dualization, that we
identify in this paper, will be very convenient in order to find the most tractable form of bosonization. For instance, we know that systems of non-relativistic fermions at positive density yield a well behaved bosonic action using minimal dualization [7]. However, even for the relatively simple case of a free massive Dirac fermion, the mass makes dualization nontrivial.

The situation we will encounter in this paper is the following:

- For $D = 2$, one can study the high energy (small fermion mass) or low energy (large mass) regime independently. In the high energy regime dualization reproduces all the well known results of conventional $D = 2$ bosonization [2, 9, 10]. We will see in this paper that the low energy regime can also be studied using an equivalent bosonic formulation obtained by extended dualization.

- For $D \geq 3$, we explore the low energy regime. The resulting bosonic action turns out to be local. In order to go beyond the low energy limit, one needs to use the freedom in dualization. This situation will be treated in a subsequent paper.

Dualization has the nice property that it permits one to obtain relationships among a wide class of models. We will apply this property to obtain the bosonic equivalent of the $D$-dimensional massive Thirring model (and other subsidiary models) very simply, as a consequence of the bosonization of the massive Dirac fermion.

We remark that, in contradistinction to minimal dualization, extended dualization allows one to bosonize the most general quantization of these models (which is, in general, anomalous).

The high energy regime in $D = 2$ is a special case that we consider separately in Section 2.

Section 3 will deal with the low energy regime, in arbitrary dimension $D$. For large fermion mass, we give all the results of the extended dualization of a free massive Dirac fermion, and related offspring such as the massive Thirring model, for their most general quantization.

One should mention here that some other work has been done in $D > 2$ bosonization using frameworks different from dualization. In particular, for the abelian massive Thirring model in dimension $D = 3$, the results in [11]
are a particular case of the bosonic action obtained in this paper using extended dualization. On the other hand, the methods described in [11] are quite convoluted and, more importantly, are very much tied to the peculiarities of the Thirring model. The dualization prescription, in contrast, is a nice and simple technique that allows one to study a wide range of quantum fermion systems, among which the Thirring model is just a particularly simple example.

Section 4 is devoted to the dualization of the simplest model without non-anomalous quantization, the Chiral Schwinger model. Minimal dualization is inapplicable to that model, and we will see the necessity of introducing the extra scalar field $\lambda$ to get the most general bosonization of the model.

We will conclude, in Section 5, with a brief summary of the main results on extended abelian dualization.

2 Extended dualization: Bosonization

2.1 Abelian extended dualization

Consider a quantum system of fermions in a flat euclidean spacetime of arbitrary dimension $D$. The partition function is given by

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_f(\psi, \bar{\psi}, \phi)},$$

where $S_f$ includes any fermion self-interaction, and interactions with other external fields or source terms (generically denoted by $\phi$).

We will start by describing a trick to modify the partition function (5). It consists of the introduction of a path-integral representation of the identity into the partition function. We call this trick “extended dualization”. Later we will establish the relationship between such manipulations and the bosonization of the fermion system.

Consider the following integral representation of the identity:

$$1 = \int \mathcal{D}A \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{F(A, \psi, \bar{\psi}, \phi)} e^{(A, d\lambda) + i\bar{\phi} d\Lambda}.$$

In a general curved spacetime with non-trivial topology, the extended dualization procedure has to be modified, as it is indicated in appendix A.
Here $A^{(1)}$ is an auxiliary 1-form and the inner scalar product of forms\(^2\) has been used. \(F(A, \phi, \bar{\psi}, \phi)\) is an arbitrary scalar functional depending on the original fields and the non-physical vector field $A^{(1)}$, subject to the condition

\[ F(A = 0, \psi, \bar{\psi}, \phi) = 0. \] (7)

Furthermore, $A^{(D-2)}$ is an antisymmetric rank $(D-2)$ tensor field and $\lambda^{(0)}$ is a rank 0 scalar field. $\mathcal{D}[\Lambda]$ denotes the measure on the space of gauge orbits

\[ [\Lambda] = \{ \Lambda': \Lambda' - \Lambda = d\chi^{(D-3)} \}. \] (8)

We relegate the proof of (6) to Appendix A.

Now introduce the identity, in its path-integral form (6), into the fermion partition function (5). So far we have nothing but an equivalence among path-integral expressions,

\[ \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \ e^{-S_f} = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \mathcal{D}[\Lambda] \ e^{-S_f + F(A, \psi, \bar{\psi}, \phi) + [A, d\Lambda + i* d\Lambda]}. \] (9)

Now we want to extract from this identity a bosonic system in terms of which we could calculate Green functions of the fermionic theory. For the purpose of bosonization we will first suppose that the change in the order of integration leaves results unaltered. In the right hand side of (9) we will try to perform the integration over the fermions and the auxiliary field $A$, and leave an expression in terms of $\Lambda$ and $\lambda$ (which will be the fields of the bosonized action).

Define the bosonic action $S_b$,

\[ e^{-S_b(\Lambda, \lambda, \phi)} = \int \mathcal{D}[A] \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \ e^{-S_f + F(A, \psi, \bar{\psi}, \phi) + [A, d\Lambda + i* d\Lambda]}. \] (10)

---

\(^2\)Given two rank $k$-forms $a^{[k]} = \frac{1}{k!} a_{\mu_1 \ldots \mu_k} (x) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$, and $b^{[k]}$, we define the inner scalar product

\[ (a, b) \equiv \int d^D x \ a_{\mu_1 \ldots \mu_k} (x) b^{\mu_1 \ldots \mu_k} (x). \]

Then

\[ (A, d\lambda) = \int d^D x A^\mu (x) \partial_\mu \lambda(x), \quad (A, i* d\Lambda) = i \epsilon_{\mu_1 \ldots \mu_{D-1}} \int d^D x A^\mu (x) \partial_\mu \Lambda^{\mu_1 \ldots \mu_{D-1}}(x). \]
By construction, we have

\[ Z = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \quad e^{-S_f(\psi, \overline{\psi}, \phi)} = \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \quad e^{-S_b(\Lambda, \lambda, \phi)}. \quad (11) \]

Therefore, for any fermionic action \( S_f(\psi, \overline{\psi}, \phi) \), we have introduced an equivalent bosonic action \( S_b(\Lambda, \lambda, \phi) \). The partition function of the quantum system, originally given in terms of the fermionic field \( \psi \), admits a bosonic representation in terms of the fields \( \Lambda^{(D-2)} \) and \( \lambda^0 \).

It immediately follows from its definition that the bosonic action is invariant under an abelian gauge transformation of the form \( \Lambda \rightarrow \Lambda + d\chi^{(D-3)} \). One says that the form \( \Lambda^{(D-2)} \) is a gauge form, which is to be integrated over the space of gauge orbits.

So far, the bosonic action \( S_b(\Lambda, \lambda, \phi) \) defined in (10) includes an arbitrary functional \( F(A, \psi, \overline{\psi}, \phi) \). It is a matter of delicacy and judgement to choose \( F \) in such a way that the bosonic action, and the identification between Green functions of the fermion and boson models, become tractable.

In the case of a fermion system whose interaction involves a single current density \( \mathcal{J}^{(1)}(\psi, \overline{\psi}) \), there is a particularly natural and economic choice of the functional \( F(A, \psi, \overline{\psi}, \phi) \):

\[ F(A, \psi, \overline{\psi}, \phi) = -i(\Lambda, \mathcal{J}). \]

If the dynamics of the system involves more than one current density, a simple generalization is to introduce as many auxiliary vector fields and bosonic fields as there are different dynamical currents. For simplicity, we will restrict attention to a single dynamical current density.

The convenience of the choice (12) is clear if, in (10), one first integrates out the auxiliary field \( A \). The current then has a nice and simple bosonic equivalent

\[ \int \mathcal{D}A \quad e^{-i(A, \mathcal{J})} e^{\{A, d\lambda + i* d\Lambda\}} = \delta(-i\mathcal{J} + d\lambda + i* d\Lambda). \quad (13) \]

Therefore, the bosonic action defined by

\[ e^{-S_b(\Lambda, \lambda, \phi)} = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\overline{\psi} \quad e^{-S_f(\Lambda, \lambda, \phi) + i(\Lambda, \mathcal{J}) + \{A, d\lambda + i* d\Lambda\}}, \quad (14) \]
makes the identification of the bosonic equivalent of the fermion current density very simple
\[ i \mathcal{J}(x) = d\lambda(x) + i * d\Lambda(x). \] (15)

**Remark:** From the point of view of finding equivalent representations of the given partition function (11), other choices for the functional \( F(A, \psi, \bar{\psi}, \phi) \) [different from (12)] are equally valid. This arbitrariness in dualization, which differs from the quantization arbitrariness [inherent in the fermionic integral in (10)], will be exploited in certain cases to simplify the bosonic action. For example, the functional given by (12) will be seen to be very efficient when the fermionic effective action of the system under consideration is quadratic, otherwise more complicated choices than (12) have to be taken in order to perform the integration over the auxiliary vector field \( A^{(1)} \).

The price to pay for such extra freedom to choose \( F \) is that the identification of the bosonic equivalent of the current density will be more complicated than (15).

In general, for \( D > 2 \) abelian systems beyond the low energy regime, and for \( D \)-dimensional non-abelian systems in any energy regime, one will require some additional contributions to (12) in order to find a tractable bosonic action. This situation will be discussed in a subsequent paper.

## 2.2 Properties of bosonization by extended dualization

Let us concentrate on the properties of abelian bosonization defined by relation (14).

**Property 2.2.1** \( S_f \) and \( S_b \) related by (14), describe the same partition function,
\[ Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_f(\psi, \bar{\psi}, \phi)} = \int \mathcal{D}[A] \mathcal{D}\lambda \ e^{-S_b(A, \lambda, \phi)}. \] (16)

This property follows by construction (assuming the order of integration can be exchanged).

The expression for the fermi fields in the bosonic theory will, in general, be very complicated [3, 7]. However, fermion bilinears like \( \mathcal{J}^{(1)} \) have simple expressions.
Property 2.2.2 Any correlation function of the current $i \mathcal{J}(x)$ in the fermionic theory is related to a correlation function of $[d\lambda + i* d\Delta(x)]$ in the bosonic model by

$$\langle i \mathcal{J}^\mu(x_1) ... i \mathcal{J}^\mu(x_n) \rangle_f = \langle [d\lambda + i* d\Delta]^\mu(x_1) ... [d\lambda + i* d\Delta]^\mu(x_n) \rangle_b. \quad (17)$$

The proof is simple, doing some path-integral manipulations and changing the order of integrals, one finds

$$\langle i \mathcal{J}^\mu(x_1) ... i \mathcal{J}^\mu(x_n) \rangle_f \equiv Z^{-1} \int \mathcal{D} \psi \overline{\mathcal{D}} \overline{\psi} \mathcal{D} \mathcal{A} \mathcal{D} \mathcal{D} \mathcal{A} [\Lambda] \mathcal{D} \lambda \quad e^{-S_f(\psi, \overline{\psi}, \phi)} \prod^n_{i=1} i \mathcal{J}^\mu(x_i) \quad (18)$$

$$= Z^{-1} \int \mathcal{D} \psi \overline{\mathcal{D}} \overline{\psi} \mathcal{D} \mathcal{A} \mathcal{D} \mathcal{D} \mathcal{A} [\Lambda] \mathcal{D} \lambda \quad e^{-S_f - i (A, \mathcal{J}) + (A, d\lambda + i* d\Delta)} \prod^n_{i=1} \mathcal{J}^\mu(x_i) \quad (18)$$

$$= Z^{-1} \int \mathcal{D} \psi \overline{\mathcal{D}} \overline{\psi} \mathcal{D} \mathcal{A} \mathcal{D} \mathcal{D} \mathcal{A} [\Lambda] \mathcal{D} \lambda \quad e^{-S_f - i (A, \mathcal{J}) + (A, d\lambda + i* d\Delta)} \prod^n_{i=1} \left( \frac{\delta}{\delta A_{\mu_i}(x_i)} \right) e^{+ (A, d\lambda + i* d\Delta)} \quad (18)$$

$$= Z^{-1} \int \mathcal{D} [\Lambda] \mathcal{D} \lambda \int \mathcal{D} \mathcal{A} \mathcal{D} \psi \overline{\mathcal{D}} \overline{\psi} \quad e^{-S_f - i (A, \mathcal{J}) + (A, d\lambda + i* d\Delta)} \prod^n_{i=1} [d\lambda + i* d\Delta]^\mu(x_i) \quad (18)$$

$$= Z^{-1} \int \mathcal{D} [\Lambda] \mathcal{D} \lambda \quad e^{-S_b} \prod^n_{i=1} [d\lambda + i* d\Delta]^\mu(x_i) \quad (18)$$

$$\equiv \langle \prod^n_{i=1} [d\lambda + i* d\Delta]^\mu(x_i) \rangle_b. \quad (18)$$

Property 2.2.3 Consider a fermion–boson dual description of a given quantum system, where the actions $S_f(\psi, \overline{\psi}, \phi)$ and $S_b(\Lambda, \lambda, \phi)$ are related by extended dualization. Any modified system obtained from the original one by adding current interaction terms in the fermionic description $S'_f = S_f + S_{int}(i \mathcal{J})$ has a bosonic dual given by $S'_b = S_b + S_{int}(d\lambda + i* d\Delta)$

Again some path integral manipulations yield

$$e^{-S_b[L, \lambda, \phi]} \equiv \int \mathcal{D} \mathcal{A} \mathcal{D} \psi \overline{\mathcal{D}} \overline{\psi} \quad e^{-S'_f(\psi, \overline{\psi}, \phi) - i (A, \mathcal{J}) + (A, d\lambda + i* d\Delta)} \quad (19)$$

$$= \int \mathcal{D} \mathcal{A} \mathcal{D} \psi \overline{\mathcal{D}} \overline{\psi} \quad e^{-S_f(\psi, \overline{\psi}, \phi) - S_{int}(i \mathcal{J}) - i (A, \mathcal{J}) + (A, d\lambda + i* d\Delta)} \quad (19)$$
As a consequence of this property we can trivially dualize a large class of systems obtained from a given one, by adding suitable current density interaction terms. Examples are,

- Current–current interaction terms, such as

  \[ S_{\text{int}} = \frac{1}{2} \int d^D x d^D y J_\mu(x) V_{\mu\nu}(x, y) J_\nu(y) = \frac{1}{2} (J, V J), \]  

  modify the original bosonized action \( S_b \) by the term

  \[ -\frac{1}{2} (d\lambda + i * d\Lambda, V[d\lambda + i * d\Lambda]). \]  

Therefore, quartic interactions of fermions become quadratic in terms of the bosonic fields.

- Source terms for the currents, \( j^{(1)} \), and interactions with abelian gauge fields \( a^{(1)} \), of the form

  \[ S_{\text{int}} = (i J, j + a), \]  

  lead in the bosonic action to

  \[ (d\lambda + i * d\Lambda, j + a). \]  

**Remark:** Note that properties 2.2.2 and 2.2.3 could also be obtained by first integrating over the auxiliary vector field \( A^{(1)} \), and then using the resulting delta function (13).

**Property 2.2.4** If the quantum field theory admits a quantized dynamical conserved fermion current density, then the scalar field \( \lambda^{(0)} \) is equal to zero, and one recovers minimal dualization.
Proof: Let \( \mathcal{J}^{(1)}(\psi, \overline{\psi}) \) be a quantized dynamical conserved current density of the system, and let us use this current density to define the bosonized version of the theory:

\[
e^{-S_b(\Lambda, \lambda)} = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{-S_f-i(\mathcal{A}, \mathcal{J})+(\mathcal{A}, \mathcal{d}\lambda+i\lambda d\Lambda)}.
\]

(24)

Now define the “effective action”

\[
e^{-\Gamma(A)} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{-S_f-i(\mathcal{A}, \mathcal{J})}.
\]

(25)

First note that, with this dualization, the quantum current density is conserved iff the effective action is gauge invariant.

\[
d*\mathcal{J}^{(1)} = 0 \iff \Gamma(A) = \Gamma(A + d\alpha).
\]

(26)

Let us study the consequences of this gauge symmetry on the bosonic action,

\[
e^{-S_b(\Lambda, \lambda)} = \int \mathcal{D}A \ e^{-\Gamma(A)+(\mathcal{A}, \mathcal{d}\lambda+i\lambda d\Lambda)}.
\]

(27)

Any non–harmonic one-form \( A^{(1)} \) can be cast into the form

\[
A^{(1)} = d\zeta^{(0)} + i*d\varphi^{(D-2)}.
\]

(28)

In terms of the fields \( \zeta^{(0)} \) and \( \varphi^{(D-2)} \) we have,

\[
e^{-S_b(\Lambda, \lambda)} \propto \int \mathcal{D}\zeta \mathcal{D}\varphi \ e^{-\Gamma(\varphi+(d\zeta+i*d\varphi, \lambda \lambda+i\lambda d\Lambda)}
\]

(29)

\[
= \int \mathcal{D}\zeta \ e^{(d\zeta, \lambda)} \int \mathcal{D}\varphi \ e^{-\Gamma(\varphi-(d\varphi,d\Lambda)}
\]

\[
\propto \delta(\lambda) \int \mathcal{D}\varphi \ e^{-\Gamma(\varphi-(d\varphi,d\Lambda)}
\]

\[
\propto \delta(\lambda) \int \mathcal{D}[A] \ e^{-\Gamma(A)+(A, i*d\Lambda)}.
\]

Here \( \mathcal{D}[A] \) is the integral over the space of gauge orbits.

The scalar field \( \lambda^{(0)} \) trivially disappears because it is set to zero by the delta function. We can use

\[
e^{-S_b(\Lambda)} = \int \mathcal{D}[A] \ e^{-\Gamma(A)+(A, i*d\Lambda)},
\]

(30)
to describe the partition function, originally introduced in terms of fermions. This is the result of minimal dualization [5, 7, 8].

Remark: Note that the extended dualization presented in this paper, first enlarges minimal dualization because it permits the discussion of anomalous theories. Second, simplifies minimal dualization because it is introduced by an integral representation of the identity. And third, it allows to identify a big freedom in the dualization procedure (associated to the choice of the functional $F$). This arbitrariness is clearly differentiated from the possible arbitrariness in the path-integral quantization.

2.3 $D = 2$ free massless Dirac fermion and related models

The conventional bosonization of the $D = 2$ free Dirac fermion is well known. It is obtained by using a quantization where the vector-like fermion current density $\bar{\psi} \gamma_\mu \psi$ is conserved. The result is that the free Dirac fermion in $D = 2$ is equivalent to a free scalar boson $\Lambda$. The bosonized version of the vector-like fermion current is $\epsilon_{\mu \nu} \partial^\nu \Lambda$. Minimal dualization agrees with conventional bosonization [5, 7]. However, using all the arbitrariness in the quantization, one gets a gauge non-invariant result for the fermionic path-integral [12, 13]. In order to extract the bosonization for the most general quantization of the system, we need to use an extended dualization.

The partition function is defined by

$$Z_{\text{free}} = \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \ e^{-S_f^{\text{free}}}.$$  \hspace{1cm} (31)

Here $S_f^{\text{free}}(\psi, \bar{\psi})$ stands for the free Dirac action in flat euclidean spacetime:

$$S_f^{\text{free}}(\psi, \bar{\psi}) = -\bar{\psi} \gamma^\mu \partial_\mu \psi.$$  \hspace{1cm} (32)

In $D = 2$ the antisymmetric form $\Lambda^{(D-2)}$ reduces to a scalar field (a zero-form). The bosonic action obtained by dualization is given in terms of the

\footnote{Our conventions are:

$$\gamma_5 = \gamma_\mu, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu \nu}, \quad \gamma_5^1 = \gamma_5.$$

In $D = 2$ we use $\gamma_\mu \gamma_5 = i\epsilon_{\mu \nu} \gamma^\nu$.}
two scalar fields $\Lambda$ and $\lambda$ by

$$
e^{-S^{free}_b(\Lambda, \lambda)} = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{-S^{free}_f - i(\mathcal{A}, J) + (A, d\lambda + i \sigma dA)}$$

(33)

Here we have dualized using the vector-like current $J_\mu(x) = \overline{\psi}(x) \gamma_\mu \psi(x)$.

We introduce the effective action,

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{i \overline{\psi} \gamma^\mu (\partial_\mu - i A_\mu) \psi},$$

(34)

in terms of which the bosonic action reads

$$S^{free}_b(\Lambda, \lambda) = - \log \int \mathcal{D}A \ e^{-\Gamma(A) + (A, d\lambda + i \sigma dA)}.$$  

(35)

The effective action $\Gamma(A)$ can be calculated exactly. However, we will use a symmetry argument to obtain its value. Consider a local and arbitrary transformation over the fermion fields

$$\psi \rightarrow e^{i(\alpha + \beta \gamma_5)} \psi,$$

$$\overline{\psi} \rightarrow \overline{\psi} e^{-i(\alpha - \beta \gamma_5)}.$$  

(36)

Under such transformation the effective action has the following property

$$\Gamma(A) = \Gamma(A - d\alpha + i * d\beta) - \log J(\alpha, \beta, A),$$

(37)

where $J(\alpha, \beta, A)$ is the finite Jacobian [12] corresponding to the fermion transformation (36).

The most general expression for this Jacobian can be read, for example, from [13]. It has the form

$$\log J(\alpha, \beta, A) = \frac{i}{\pi} [\xi(\beta, \epsilon_{\mu \nu} \partial_\mu A_\nu) + i \eta(\alpha, \partial_\mu A_\mu)] - \frac{1}{2\pi} [\xi(d\beta, d\beta) + \eta(d\alpha, d\alpha)].$$

(38)

Here $\xi$ and $\eta$ are parameters introduced to describe the particular quantization employed. They satisfy the relation $\xi + \eta = 1$.

The particular case ($\eta = 0, \xi = 1$) corresponds to a gauge invariant quantization. This special case is equivalent to minimal dualization [5, 7].
Property (37) implies that the bosonic action satisfies

\begin{equation}
S_b^{\text{free}}(\Lambda, \lambda) = S_b^{\text{free}} \left( \Lambda + \frac{\xi}{\pi} \beta, \lambda + \frac{\eta}{\pi} \alpha \right) - \frac{1}{2\pi} \left[ \xi (d\beta, d\beta) + \eta (d\alpha, d\alpha) \right] - (d\beta, d\Lambda) - (d\alpha, d\lambda) .
\end{equation}

As a consequence:

\begin{align}
S_b^{\text{free}}(\Lambda, \lambda) - \frac{\pi}{2\xi} (d\Lambda, d\Lambda) & \quad \text{is invariant under} \quad \Lambda \rightarrow \Lambda + \frac{\xi}{\pi} \beta, \quad \lambda \rightarrow \lambda, \\
S_b^{\text{free}}(\Lambda, \lambda) - \frac{\pi}{2\eta} (d\lambda, d\lambda) & \quad \text{is invariant under} \quad \lambda \rightarrow \lambda + \frac{\eta}{\pi} \alpha, \quad \Lambda \rightarrow \Lambda .
\end{align}

Therefore we get the final result

\begin{equation}
S_b^{\text{free}}(\Lambda, \lambda) = \frac{\pi}{2\xi} (d\Lambda, d\Lambda) + \frac{\pi}{2\eta} (d\lambda, d\lambda)
\end{equation}

Scaling to canonical variables, \( \Lambda \rightarrow \sqrt{\frac{\xi}{\pi}} \Lambda \) and \( \lambda \rightarrow \sqrt{\frac{2}{\pi}} \lambda \), we find that the free Dirac fermion in \( D = 2 \) bosonizes into two free scalars. By construction we have the following identification for the fermion currents

\begin{align}
\bar{\psi} \gamma_\mu \psi & \quad \longleftrightarrow \quad \sqrt{\frac{\xi}{\pi}} \epsilon_{\mu\nu} \partial^\nu \Lambda - \sqrt{\frac{\eta}{\pi}} i \partial_\mu \lambda , \\
\bar{\psi} \gamma_\mu \gamma_5 \psi & \quad \longleftrightarrow \quad \sqrt{\frac{\eta}{\pi}} \epsilon_{\mu\nu} \partial^\nu \lambda - \sqrt{\frac{\xi}{\pi}} i \partial_\mu \Lambda .
\end{align}

The gauge invariant result corresponds to \( \lambda = 0 \) and \( \xi = 1 \).

Other related models can be trivially bosonized once we have expression (41) in hand. In particular, consider the fermionic action given by

\begin{equation}
S_f = S_f^{\text{free}} + (i J, j) + \frac{g}{2} (J, J),
\end{equation}

with a source term \( j^{(1)} \) for the current density, and a Thirring-like interaction of strength \( g \).
Using the general properties of extended dualization we have

\[ S_h(\Lambda, \lambda) = S_{h}^{\text{free}}(\Lambda, \lambda) + (j, d\lambda + i*d\Lambda - \frac{g}{2}(d\lambda + i*d\Lambda, d\lambda + i*d\Lambda) \]

\[ = \frac{1}{2} \left( \frac{\pi}{\xi} + g \right) (d\Lambda, d\Lambda) + \frac{1}{2} \left( \frac{\pi}{\eta} - g \right) (d\lambda, d\lambda) + (j, d\lambda + i*d\Lambda). \]

The net effect of the fermion self-interaction is a change of the kinetic coefficients for the scalar fields.

The stability of the Thirring model requires

\[-\frac{\pi}{\xi} < g < \frac{\pi}{\eta}. \tag{45}\]

This is a generalization of the usual condition \(-\pi < g\). See [2].

In the range of stability, we rescale \(\Lambda\) and \(\lambda\) to canonical variables

\[ \Lambda \rightarrow \frac{\beta}{2\pi} \Lambda, \quad \lambda \rightarrow \frac{\gamma}{2\pi} \lambda, \tag{46}\]

where \(\beta\) and \(\gamma\) are constants defined by

\[ \frac{4\pi}{\beta^2} = \frac{1}{\xi} + \frac{g}{\pi}, \quad \frac{4\pi}{\gamma^2} = \frac{1}{\eta} - \frac{g}{\pi}. \tag{47}\]

In terms of the canonical variables we have

\[ S_h(\Lambda, \lambda) = \frac{1}{2} (d\Lambda, d\Lambda) + \frac{1}{2} (d\lambda, d\lambda) + (j, \frac{\gamma}{2\pi} d\lambda + \frac{\beta}{2\pi} i*d\Lambda). \tag{48}\]

The identification of currents is

\[ \overline{\psi}\gamma_\mu \psi \leftrightarrow \frac{\beta}{2\pi} \epsilon_{\mu\nu} \partial^\nu \Lambda - \frac{\gamma}{2\pi} i\partial_\mu \lambda, \tag{49}\]

\[ \overline{\psi}\gamma_\mu \gamma_5 \psi \leftrightarrow \frac{\gamma}{2\pi} \epsilon_{\mu\nu} \partial^\nu \lambda - \frac{\beta}{2\pi} i\partial_\mu \Lambda. \]

The particular case \((\beta^2 = 4\pi \xi, \gamma^2 = 4\pi \eta)\) corresponds to the free Dirac fermion.
2.4 $D = 2$ free massive Dirac fermion and related models

The partition function is defined by

$$Z_{\text{free}}(m) = \int D\psi D\overline{\psi} \ e^{-S_{\text{free}}^f(m; \psi, \overline{\psi}),}$$

while the free massive Dirac action has the form

$$S_{\text{free}}^f(m; \psi, \overline{\psi}) = -[\overline{\psi} \gamma^\mu \partial_\mu \psi + m \overline{\psi} \psi].$$

Consequently, the bosonic action for a free massive Dirac fermion has the following path integral expression

$$S_{\text{free}}^b(m; \Lambda, \lambda) = \int DAD\psi D\overline{\psi} \ e^{-S_{\text{free}}^f-m(A, A) + (A, d\lambda + i\delta d\lambda)}$$

$$= \int DAD\psi D\overline{\psi} \ e^{\overline{\psi} \gamma^\mu (\partial_\mu - iA_\mu) \psi + m \overline{\psi} \psi + (A, d\lambda + i\delta d\lambda)}.$$

Now, introduce the effective action,

$$e^{-\Gamma_m(A)} = \int D\psi D\overline{\psi} \ e^{\overline{\psi} \gamma^\mu (\partial_\mu - iA_\mu) \psi + m \overline{\psi} \psi},$$

in terms of which

$$S_{\text{free}}^b(m; \Lambda, \lambda) = \int DA \ e^{-\Gamma_m(A) + (A, d\lambda + i\delta d\lambda)}.$$

For small mass $m$, the bosonic action $S_{\text{free}}^b(m; \Lambda, \lambda)$ can be calculated by a perturbative expansion in the mass term. This procedure is well known in the $D = 2$ bosonization folklore [9, 10]. The main steps of the calculation are exhibited in Appendix B, and the final result is

$$S_{\text{free}}^b(m; \Lambda, \lambda) = S_{\text{free}}^b(0; \Lambda, \lambda) - 2m\Lambda_{UV} \int d^2x \cos \left[ \frac{2\pi}{\xi} \Lambda(x) \right]$$

$$= \frac{\pi}{2\xi}(d\Lambda, d\Lambda) + \frac{\pi}{2\eta}(d\lambda, d\lambda) - 2m\Lambda_{UV} \int d^2x \cos \left[ \frac{2\pi}{\xi} \Lambda(x) \right].$$

Here $\Lambda_{UV}$ is an ultraviolet cutoff required to regularize the integral over the auxiliary field $A$. 

15
Adding other interactions and sources to the bosonized free massive Dirac fermion is, as before, trivial. Take the fermionic action to be

$$S_f = S^\text{free}_f(m; \overline{\psi}, \psi) + (i \mathcal{J}, j) + \frac{g}{2}(\mathcal{J}, \mathcal{J}).$$ \quad (56)

The corresponding bosonic action is given by

$$S_b(m; \Lambda, \lambda, j) = \frac{1}{2} \left( \frac{\pi}{\xi} + g \right) (d\Lambda, d\Lambda) + \frac{1}{2} \left( \frac{\pi}{\eta} - g \right) (d\lambda, d\lambda)$$

$$- 2m \Lambda_{UV} \int d^2x \cos \left[ \frac{2\pi}{\xi}\Lambda(x) \right] + (j, d\lambda + i*d\Lambda).$$ \quad (57)

Rescaling [to the canonical variables introduced in (46), (47)] we have

$$S_b(m; \Lambda, \lambda, j) = \frac{1}{2} (d\Lambda, d\Lambda) + \frac{1}{2} (d\lambda, d\lambda)$$

$$- 2m \Lambda_{UV} \int d^2x \cos \left[ \frac{\beta}{\xi}\Lambda(x) \right] + (j, \frac{\gamma}{2\pi}d\lambda + \frac{\beta}{2\pi}i*d\Lambda).$$ \quad (58)

The identification of the fermion current densities is not modified from the massless case (49), neither is the stability condition (45).

This is a generalization of the well known result [2, 9], that in the small mass limit, the massive Thirring model is equivalent to the sine-Gordon model. In our analysis there is also an additional scalar field $\lambda$, that plays a role in the identification of the fermionic current densities

$$\overline{\psi}\gamma_\mu\psi \leftrightarrow \frac{\beta}{2\pi}\epsilon_\mu^\nu \partial^\nu\Lambda - \frac{\gamma}{2\pi}i\partial_\mu\lambda,$$ \quad (59)

$$\overline{\psi}\gamma_\mu\gamma_5\psi \leftrightarrow \frac{\gamma}{2\pi}\epsilon_\mu^\nu \partial^\nu\lambda - \frac{\beta}{2\pi}i\partial_\mu\Lambda.$$

The parameter values $\beta^2 = 4\pi\xi$, and $\gamma^2 = 4\pi\eta$, correspond to the massive free Dirac fermion system. In particular, setting $\beta^2 = 4\pi$, and $\gamma^2 = 0$, corresponds to minimal dualization.

### 3 The $D$-dimensional massive Thirring model: Large mass

The massive Thirring model is one of the simplest nontrivial fermion systems suitable for dualization. Other approaches have been used to bosonize the
massive Thirring model [11]. However, dualization makes the bosonization a lot simpler. Additionally, while other approaches rely on rather specific properties of the Thirring model, dualization applies to any fermionic system. Even more, with extended dualization we can derive the bosonic dual of the most general quantization of the Thirring model.

The case of $D = 2$ in the high energy limit (small fermion mass) is a rather special situation. It has been already discussed (together with the free massive Dirac fermion) in the previous section. In this section we will concentrate in the low energy (large fermion mass) limit for arbitrary dimension $D$.

For definiteness

$$S_{T}^{Th}(m; \psi, \bar{\psi}) = S_{f}^{free}(m; \psi, \bar{\psi}) + \frac{g}{2}(J, J).$$  \hspace{1cm} (60)

Here $S_{f}^{free}(m; \psi, \bar{\psi})$ is the free massive Dirac action defined in (51), while $J$ is the vector-like current density $J_{\mu}(x) = \bar{\psi}(x)\gamma_{\mu}\psi(x)$.

The quantum system is defined by the partition function which, including a source $j^{(1)}$ for the current, reads

$$Z_{T}(j) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \quad e^{-S_{T}^{Th}(iJ, j)}.$$  \hspace{1cm} (61)

As we have pointed out previously, the extended dualization of the Thirring model proceeds as follows

$$Z_{T}(j) = \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \quad e^{-S_{b}^{Th}(m; \lambda, j)}.$$  \hspace{1cm} (62)

The bosonic action $S_{b}^{Th}$ is given by

$$S_{b}^{Th}(m; \lambda, \Lambda, j) = S_{b}^{free}(m; \Lambda, \lambda) + \frac{g}{2} [(d\Lambda, d\Lambda) - (d\lambda, d\lambda)] + (j, d\lambda + i*d\Lambda).$$  \hspace{1cm} (63)

Therefore, to obtain the bosonic representation of the massive Thirring model, the only serious calculation to do is the bosonic action for the free massive Dirac fermion $S_{b}^{free}(m; \Lambda, \lambda)$

$$S_{b}^{free}(m; \Lambda, \lambda) = \int \mathcal{D}A \quad e^{-\Gamma_{m}(A) + \Lambda d\lambda + i* d\Lambda}.$$  \hspace{1cm} (64)

Next we will evaluate the effective action $\Gamma_{m}(A)$ given by (53). In general, for arbitrary mass and dimension, $\Gamma_{m}(A)$ is a very complicated functional.
However, in the limit of large fermionic mass, $m \to \infty$, the effective action becomes quadratic in the field $A$ and we can give an explicit expression for the bosonic action $S_b^{\text{free}}(m \not\to \infty; \Lambda, \lambda)$. We will consider the two cases for the quantization of the fermionic system where the vector-like current is, or is not, conserved.

### 3.1 Minimal dualization: Low energy limit

In the large mass limit, the gauge invariant effective action $\Gamma_m(A)$ is quadratic in the field $A$, and can be cast into the form

$$\Gamma_m(A) = \frac{1}{2} \int d^D x \ A^\mu(x) C^{\mu\nu}(\partial, m) A^\nu(x)$$

The tensorial differential operator $C^{\mu\nu}(\partial, m)$ depends on the spacetime dimension as follows

$$C^{\mu\nu}_2(\partial, m) = \frac{1}{6\pi^2 m^2} \left[ 1 + \frac{\partial^2}{5m^2} + O \left( \frac{1}{m^4} \right) \right] (\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2)$$
$$C^{\mu\nu}_3(\partial, m) = \frac{\text{sign}(m)}{8\pi^2} \left[ 1 + O \left( \frac{1}{m} \right) \right] \epsilon_{\mu\nu\rho} \partial^\rho$$
$$C^{\mu\nu}_4(\partial, m) = K_D \left[ 1 + O \left( \frac{1}{m^2} \right) \right] (\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2)$$

Here $K_D$ is a coefficient depending on the regularization method required to have a well defined fermionic integral.

To obtain the bosonic action we integrate over the gauge orbits of the auxiliary gauge field $A$. Since the effective action is gauge invariant, the scalar field $\lambda$ is set to zero, and the dualized action depends only on the tensor field $A^{(D-2)}$

$$S_b^{(T,h)}(\Lambda, j) = \frac{g}{2} (d\Lambda, d\Lambda) + i(j, *d\Lambda) - \log \int \mathcal{D}[A] \ e^{-\frac{1}{2}(A,CA)+(A,i*\Lambda)}.$$  

We pick a representative field from every gauge orbit by going to the Landau gauge. ($\partial_\mu A_\mu = 0$, that is to say, $A = i* d\varphi^{(D-2)}$.) Subject to this condition, the quadratic form $C^{(D)}(\partial, m)$ is invertible and we get

$$\int \mathcal{D}[A] \ e^{-\frac{1}{2}(A,CA)+(A,i*\Lambda)} = e^{-\frac{1}{2}(s d\Lambda, C^{-1}s d\Lambda)}.$$
The bosonic action is now
\begin{equation}
S_b^{(Th)}(m \not\to \infty; \Lambda, j) = \frac{g}{2}(d\Lambda, d\Lambda) + i(j, *d\Lambda) + \frac{1}{2}(*_d\Lambda, C^{-1} * d\Lambda). \tag{69}
\end{equation}

In the Landau gauge, the inverse of the quadratic operators are
\begin{align*}
[C_{\mu\nu}(\partial, m)]^{-1} &= -6\pi m^2 \left[1 - \frac{\partial^2}{5m^2} + O\left(\frac{\partial^4}{m^4}\right)\right] \delta_{\mu\nu},
[C_{\mu\nu}(\partial, m)]^{-1} &= -\frac{8\pi^2}{\text{sign}(m)} \left[1 + O\left(\frac{\partial}{m}\right)\right] \frac{1}{\partial^2} \epsilon_{\mu\nu\rho} \partial^\rho, \tag{70}
[C_{\mu\nu}(\partial, m)]^{-1} &= -\frac{1}{K_D} \left[1 + O\left(\frac{\partial^2}{m^2}\right)\right] \delta_{\mu\nu}.
\end{align*}

The bosonic action has, by construction, an abelian gauge symmetry for the tensor field $\Lambda$. This implies that we can impose the Landau gauge condition on the tensor field $\Lambda$ itself. In components this reads $\partial^\mu \Lambda_{\mu_1, \mu_2, \ldots, \mu_D = 3} = 0$, so that we have
\begin{align*}
(*d\Lambda, \frac{1}{\partial^2} * d\Lambda) &= -(\Lambda, \Lambda), \tag{71} \\
(*d\Lambda, \frac{1}{\partial^2} * d\ast d\Lambda) &= -(\Lambda, \ast d\Lambda).
\end{align*}

The final results for the Thirring model in the infinite mass limit are:

**D=2** The bosonic action is a free boson $\Lambda$ with a mass that is proportional to the fermion mass and to the inverse of the Thirring coupling $g$. For $g \sim O(m^0)$ one has
\begin{equation}
S_b^{(Th)}(m \not\to \infty; \Lambda, j) = \frac{1}{2} \left[g + \frac{5\pi}{6}\right] (d\Lambda, d\Lambda) + \frac{1}{2} 6\pi m^2(\Lambda, \Lambda) + i(j, *d\Lambda). \tag{72}
\end{equation}

The mass for the scalar $\Lambda$ is $M^2 = 6\pi m^2 / [g + \frac{5\pi}{6}]$.

**D=3** The bosonic action includes a gauge field $\Lambda_{\mu}$, with an abelian Chern–Simons term
\begin{equation}
S_b^{(Th)}(m \not\to \infty; \Lambda, j) = \frac{1}{2} g(d\Lambda, d\Lambda) + \frac{8\pi^2}{\text{sign}(m)} (\Lambda, *d\Lambda) + i(j, *d\Lambda). \tag{73}
\end{equation}

The gauge field $\Lambda_{\mu}$ has a topological mass $[14]$ given by $M = 8\pi^2 / g$.  

19
\[ D \geq 4 \] The bosonic field is a rank \((D-2)\) antisymmetric form \(A_{\mu_1 \ldots \mu_{D-2}}\), and the action becomes
\[
S_b^{(Th)}(m \nearrow \infty; \Lambda, j) = \frac{1}{2} g (d\Lambda, d\Lambda) + \frac{1}{K_D} (\Lambda, \Lambda) + i(j, *d\Lambda). \tag{74}
\]

Therefore, in terms of the gauge form \(\Lambda^{(D-2)}\), the bosonic action is local in the infinite mass limit.

Now suppose that the Thirring coupling \(g\) is chosen such that \(gK_D = 1/tm^2\) for \(t \sim O(m^0)\). Scaling \(\Lambda\) to canonical variables we have
\[
S_b^{(Th)}(m \nearrow \infty; \Lambda, j) = \frac{1}{2} (d\Lambda, d\Lambda) + tm^2 (\Lambda, \Lambda) + \frac{i}{\sqrt{g}} (j, *d\Lambda). \tag{75}
\]

The mass for the form \(\Lambda\) is, in this case, proportional to the fermion mass (as in the \(D = 2\) case).

### 3.2 Extended dualization: Low energy limit

If one uses a regularization where the vector-like current density is not conserved, the most general result for the effective action in the infinite mass limit is
\[
\Gamma_m(A) = \frac{1}{2} \kappa_D(A, A), \tag{76}
\]
with a regularization dependent coefficient \(\kappa_D\).

The bosonic action is given by
\[
S_b^{(Th)}(m \nearrow \infty; \Lambda, \lambda, j) = \frac{g}{2} \left[(d\Lambda, d\Lambda) - (d\lambda, d\lambda)\right] + i(j, d\lambda + i*d\Lambda)
- \log \int \mathcal{D}A \ e^{-\frac{1}{2} \kappa_D(A, A) + (A, d\lambda + i*d\Lambda)} \tag{77}
\]
The bosonic action contains two fields \(\Lambda^{(D-2)}\) and \(\lambda^{(0)}\) with opposite kinetic coefficients
\[
S_b^{(Th)}(m \nearrow \infty; \Lambda, \lambda, j) = \frac{1}{2} \left(g + \frac{1}{\kappa_D}\right) \left[(d\Lambda, d\Lambda) - (d\lambda, d\lambda)\right] + (j, d\lambda + i*d\Lambda). \tag{78}
\]
4 Chiral Schwinger model: bosonization

The Chiral Schwinger model is the simplest model wherein the dynamical current is always non-conserved. The bosonization of the Chiral Schwinger model requires the use of extended dualization. We will see that the bosonic equivalent of the Chiral Schwinger model includes two independent scalar fields, coupled to the abelian gauge field. The bosonic equivalent of the chiral density current involves the two scalar fields.

The partition function for the Chiral Schwinger model is defined by

$$Z_{CSM}(j) = \int \mathcal{D}a \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{-S_{CSM}(\psi, \overline{\psi}, a) - i \langle j, \overline{\mathcal{J}} \rangle}.$$  \hspace{1cm} (79)

The action has the form

$$S_{CSM}(\psi, \overline{\psi}, a) = \frac{1}{4} (da, da) - \overline{\psi} \gamma^{\mu} \left[ \partial_{\mu} - i e \frac{1-\gamma_{5}}{2} a_{\mu} \right] \psi.$$  \hspace{1cm} (80)

Here $a^{(1)}$ stands for an abelian gauge field and the chiral current is

$$\mathcal{J}^{ch}_{\mu} = \overline{\psi} \gamma_{\mu} \frac{1-\gamma_{5}}{2} \psi.$$  \hspace{1cm} (81)

As a result of integrating out the fermion fields we get an effective action

$$\Gamma_{CSM}(a) = -\log \int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{\overline{\psi} \gamma^{\mu} \left[ \partial_{\mu} - i e \frac{1-\gamma_{5}}{2} a_{\mu} \right] \psi}.$$  \hspace{1cm} (82)

Evaluating this effective action, in its most general form, one finds [15, 16]

$$\Gamma_{CSM}(a) = \frac{e^{2}}{2\pi} \left[ \xi(a, a) - \left( a^{+}_{\mu}, \frac{\partial_{\nu}}{a_{\nu}} a^{+}_{\mu} \right) \right].$$  \hspace{1cm} (83)

Here $a^{+}_{\mu} = \frac{1}{2} (\delta_{\mu} + i e_{\mu}) a_{\nu}$. The arbitrariness in the quantization procedure is reflected in the presence of the local term $(a, a)$, and this arbitrariness is parameterized by the coefficient $\xi$.

This is the result of conventional bosonization [15]. As a consequence of integrating out the fermion fields, the gauge field acquires a mass term, depending on the arbitrary parameter $\xi$. This effective action can be expressed in a local way by introducing an additional scalar field.
Bosonization via extended dualization goes beyond this conventional result since it permits one to exhibit the bosonic equivalent of the chiral current.

We see in (83) that the effective action is never gauge invariant for any value of $\xi$. Therefore the chiral current, $J_{ch}$, is never conserved for any possible quantization of the Chiral Schwinger model. Extended dualization is the only option.

The partition function (79) can be expressed in terms of a bosonic action

$$Z_{CSM}(j) = \int \mathcal{D}a \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{-S_{b}^{CSM}(\Lambda, \lambda, a, j)}.$$  \hspace{1cm} (84)

The bosonic action, using the extended dualization prescription, is given by

$$e^{-S_{b}^{CSM}(\Lambda, \lambda, a, j)} = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_{CSM}(\psi, \bar{\psi}, a) - i(j, J_{ch}) - i(A, J_{ch}) + (A, d\lambda + i*d\Lambda)}.$$ \hspace{1cm} (85)

After integrating over the fermions one gets

$$S_{b}^{CSM}(\Lambda, \lambda, a, j) = \frac{1}{4}(da, da) + (ea + j, d\lambda + i*d\Lambda) - \log \int \mathcal{D}A \ e^{-\Gamma_{CSM}(A) + (A, d\lambda + i*d\Lambda)}.$$ \hspace{1cm} (86)

The gaussian integral over the auxiliary vector field $A$ is straightforward and the result for the bosonic action is

$$S_{b}^{CSM}(\Lambda, \lambda, a, j) = \frac{1}{4}(da, da) + (ea + j, d\lambda + i*d\Lambda)$$

$$+ \frac{\pi(4\xi - 1)}{8e^{2}\xi^{2}}(d\Lambda, d\Lambda) - \frac{\pi(4\xi + 1)}{8e^{2}\xi^{2}}(d\lambda, d\lambda) - \frac{\pi}{4e^{2}\xi^{2}}(d\Lambda, d\lambda).$$ \hspace{1cm} (87)

Introducing the canonical variables

$$\theta = \frac{\sqrt{\pi}}{2e\xi} \{(1 - 2\xi)\Lambda + (1 + 2\xi)\lambda\},$$ \hspace{1cm} (88)

$$\phi = \frac{\sqrt{\pi}}{e}(\Lambda - \lambda),$$
the bosonic action becomes

\[ S_b^{CSM}(\theta, \phi, a, j) = \frac{1}{4}(da, da) + \frac{1}{2}(d\phi, d\phi) + \frac{1}{2}(d\phi, d\phi) \]

\[ + \frac{e}{2\sqrt{\pi}} (\epsilon a + j, d [2\xi(\theta + \phi) - \phi]) \]

\[ + \frac{e}{2\sqrt{\pi}} (\epsilon a + j, i \ast d [2\xi(\theta + \phi) + \phi]). \] (89)

The bosonic action includes the gauge field \( a^{(1)} \), plus two scalar fields \( \theta \) and \( \phi \), coupled to this gauge field.

The bosonic equivalent for the chiral current is given by

\[ i \bar{\psi} \gamma_\mu \frac{1}{2} - \gamma_\nu \psi \quad \longrightarrow \quad \frac{e}{2\sqrt{\pi}} \partial_\mu [2\xi(\theta + \phi) - \phi] \]

\[ + i \frac{e}{2\sqrt{\pi}} \xi^{\mu\nu} \partial_\nu [2\xi(\theta + \phi) + \phi]. \] (90)

The importance of this result is that bosonization based on extended dualization allows one to calculate correlation functions of the chiral current in the Chiral Schwinger model. We see in (90) that for the most general quantization of the chiral Schwinger model, the chiral current depends on two independent scalar fields.

5 Conclusions

Our purpose in this paper has been to show how \( D \)-dimensional anomalous quantum fermionic systems can be put in correspondence with bosonic models. We have introduced a constructive determination of the bosonic equivalent of a given anomalous fermion system, called extended dualization. The bosonic action includes both a scalar field and a rank \((D-2)\) antisymmetric form as fundamental fields. Our extended duality transformation is a generalization of minimal dualization. The last one applies only for systems in the presence of a dynamical conserved quantum charge and the only relevant bosonic field is the \((D-2)\)-form.

We have seen that for a given fermionic system the bosonic counterpart is not unique. A large freedom is involved in the dualization transformation. One should choose the most convenient dualization in order to have
the most tractable bosonic action and correlation functions for the relevant operators. In this paper we have explored one of the most simple options that is quite efficient for abelian bosonization when the fermionic effective action is quadratic.

A very nice property of dualization is that it permits one to study a wide class of systems as trivial modifications of a much simpler one. We have examined the bosonization of the $D$-dimensional massive Thirring model (in the limit of low energies) by interpreting it as a trivial modification of the bosonization of the $D$-dimensional free massive Dirac fermion.

We have applied the extended dualization procedure to determine the bosonized version of the Chiral Schwinger model and have shown that the bosonic equivalent of the chiral current is, for the most general quantization of the model, calculable in terms of two independent scalar fields.

The considerable freedom in the extended dualization prescription introduced in this paper opens up the possibility of applying this freedom in order to dualize more complicated fermi systems (with or without anomalies), such as abelian systems in more than two dimensions beyond the low energy regime, and non-abelian systems.

Acknowledgments

We wish to thank F. Falceto for discussions. This work was partially supported by the CICYT (proyecto AEN 94-0218). The work of E.R. has been supported by a contract with the spanish government under the program “contratos para la incorporación de doctores y tecnólogos a grupos de investigación en España”.

24
Appendix A

In this appendix we will prove the identity

\[ 1 = \int \mathcal{D}A \mathcal{D} \lambda \ e^{F(A, \psi, \bar{\psi}, \phi)} e^{(A, d \lambda + i \delta d \lambda)}. \]  

(91)

We assume that the fields are sufficiently well-behaved at the boundary that integration by parts is valid. Then:

- Integrating out the field \( \lambda \), we have
  \[ dA = 0. \]  
  (92)

- To integrate out the field \( \lambda \) we need to perform an analytic continuation \( \lambda \rightarrow i \lambda \), and (at the end of the calculation) we need to return to the physical region of interest. This implies
  \[ d \ast A = 0. \]  
  (93)

Therefore, the result of integrating out the bosonic fields \( \Lambda \) and \( \lambda \) is

\[ \int \mathcal{D}[\Lambda] \mathcal{D} \lambda \ e^{(A, d \lambda + i \delta d \lambda)} = \delta(dA) \delta(d \ast A), \]  

(94)

so that the auxiliary vector field must be a harmonic one-form \( A^{(1)}_h \). That is, \( \Delta A_h = 0 \), where \( \Delta \) is the Laplacian that acts on one-forms in the \( D \)-dimensional space under consideration.

Therefore

\[ \int \mathcal{D}A \mathcal{D}[\Lambda] \ e^{F(A, \psi, \bar{\psi}, \phi)} e^{(A, d \lambda + i \delta d \lambda)} = \int \mathcal{D}A_h e^{F(A_h, \psi, \bar{\psi}, \phi)}. \]  

(95)

In simple cases where spacetime has trivial topology, such as \( R^n \) or \( S^n \), there are no harmonic one-forms, and therefore the identity (91) follows, provided only that

\[ F(A = 0, \psi, \bar{\psi}, \phi) = 0. \]  

(96)

In more complicated spacetimes with nontrivial topology, a modification of (91) is necessary, taking into account the space of harmonic one-forms.
We generalize the path-integral representation of the identity in the following way: we replace the integration over the space of 1-forms $A^{(1)}$ by an integration over the space of orbits

$$[A] = \{ A' : A' - A = A_h, \; \Delta A_h = 0 \}. \quad (97)$$

The generalization of (91) is obtained by taking a quotient over the space of harmonic one-forms, and then averaging the functional $F$ over all harmonic one-form transformations,

$$1 = \int \mathcal{D}[A] \mathcal{D}[\Lambda] \mathcal{D}\lambda \quad e^{\int \mathcal{D}A_h F(A + A_h, \psi, \overline{\psi}, \phi) e^{(A, d\lambda + i\omega d\Lambda)}}. \quad (98)$$

The proof is as follows: Integrating over the bosonic fields, we have

$$\int \mathcal{D}[A] \mathcal{D}[\Lambda] \mathcal{D}\lambda \quad e^{\int \mathcal{D}A_h F(A + A_h, \psi, \overline{\psi}, \phi) e^{(A, d\lambda + i\omega d\Lambda)}} = e^{\int \mathcal{D}A_h F(A_h, \psi, \overline{\psi}, \phi)}. \quad (99)$$

where translation invariance for the integration over harmonic 1-forms has been used.

The identity (98) follows, provided that

$$\int \mathcal{D}A_h F(A_h, \psi, \overline{\psi}, \phi) = 0. \quad (100)$$

This is the generalization, in the case of nontrivial harmonic one-forms, of the previous condition $F(A = 0, \psi, \overline{\psi}, \phi) = 0$.

If the functional $F(A, \psi, \overline{\psi}, \phi)$ is linear in the auxiliary vector field $A$, as it is the case for the simplest choice $F(A, \psi, \overline{\psi}, \phi) = -i(A, J)$ [in (12)], the condition (100) is trivially satisfied and we have

$$\int \mathcal{D}A_h F(A + A_h, \psi, \overline{\psi}, \phi) = F(A, \psi, \overline{\psi}, \phi) \int \mathcal{D}A_h + \int \mathcal{D}A_h F(A_h, \psi, \overline{\psi}, \phi) = F(A, \psi, \overline{\psi}, \phi) \quad (101)$$

Therefore, in this case, the only new component to the dualization for a nontrivial topology comes from the explicit representation of the integration over harmonic orbits for the auxiliary vector field $A^{(1)}$ in (98). The simplest example, a two dimensional fermion in a cylinder, has been done in detail in [5].

26
Appendix B

This appendix is devoted to a sketch of the technical details of the $D = 2$ path integral calculation of

$$e^{-S^f_r(m; \lambda)} = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{\overline{\psi} \gamma^\mu (\partial_\mu - iA_\mu) \psi + m\overline{\psi} \psi + (A, d\lambda + i\pi \Lambda)},$$

for small fermion mass, $m$.

We will proceed by performing a perturbative expansion in the mass, but first we rearrange this expression in a more convenient form.

Consider the following local transformation for the fermion fields:

$$\psi \rightarrow e^{i(\zeta + \varphi \gamma_5)} \psi, \quad \overline{\psi} \rightarrow \overline{\psi} e^{-i(\zeta - \varphi \gamma_5)}.$$  \hspace{1cm} (103)

The parameters of this transformation are the two scalar fields, $\zeta$ and $\varphi$, used to write the auxiliary vector field, $A = d\zeta + i*d\varphi$.

Under such a transformation we have

$$e^{-S^f_r(m; \lambda)} = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\overline{\psi} \ J(A) \ e^{\overline{\psi} \gamma^\mu \partial_\mu \psi + m\overline{\psi} \psi - 2i\varphi \gamma_5 \psi + (A, d\lambda + i\pi \Lambda)}.$$  \hspace{1cm} (104)

The Jacobian $J(A)$ associated with the finite transformation (103) can be read off from the general result (38) in the particular case $\alpha = \zeta, \beta = -\varphi$. One finds

$$J(A) = \frac{1}{2\pi} [\xi(d\varphi, d\varphi) + \eta(d\zeta, d\zeta)].$$  \hspace{1cm} (105)

Because the mass term depends only on the field $\varphi$, the integration over $\zeta$ can be easily done. Apart from trivial constants we get

$$S^f_r(m; \lambda, \lambda) = \frac{\pi}{2\eta} (d\lambda, d\lambda)$$  \hspace{1cm} (106)

$$-\log \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{\overline{\psi} \gamma^\mu \partial_\mu \psi + m\overline{\psi} \psi - 2i\varphi \gamma_5 \psi + \frac{\xi}{2\pi} (d\varphi, d\varphi) - (d\varphi, d\varphi)}.$$  

Rescaling, to the canonical variable $\varphi \rightarrow \sqrt{\xi} \varphi + \frac{1}{\xi} \Lambda$, we have

$$S^f_r(m; \lambda, \lambda) = \frac{\pi}{2\eta} (d\lambda, d\lambda) + \frac{\pi}{2\xi} (d\lambda, d\lambda)$$  \hspace{1cm} (107)

$$-\log \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \ e^{\overline{\psi} \gamma^\mu \partial_\mu \psi + m\overline{\psi} \psi - 2i\left(\sqrt{\xi} \varphi + \frac{1}{\xi} \Lambda\right) \gamma_5 \psi + \frac{1}{2\pi} (d\varphi, d\varphi)}.$$
Therefore
\[ S_{\phi}^{\text{free}}(m; \Lambda, \lambda) = S_{\phi}^{\text{free}}(m = 0; \Lambda, \lambda) - \log \int \mathcal{D}\varphi \ e^{+\frac{1}{2}(d\varphi, d\varphi) - \Gamma_m[\sqrt{-\xi} + \xi \Lambda]}, \]

where the effective action $\Gamma_m$ is the result of the fermionic integral
\[ e^{-\Gamma_m[\varphi]} = \int \mathcal{D}\varphi\mathcal{D}\overline{\varphi} \ e^{\overline{\varphi} \gamma^a \partial_a \varphi + m \overline{\varphi} e^{-2i \theta / 2} \psi \varphi}. \]

So far we have done nothing more than rearrange the integral in a convenient way. In order to perform (109) we will do a small mass perturbation. Introducing $\sigma_\pm \equiv \overline{\varphi} / 2 (1 \pm \gamma_5) \psi$, one gets
\[ e^{-\Gamma_m[\chi]} = \sum_{i=0}^{\infty} \frac{m^{2i}}{i!} \int \left[ \prod_{k=1}^{\infty} d^2 x_k d^2 y_k \right] e^{2i \sum_k [\theta(y_k) - \theta(x_k)]} \int \mathcal{D}\varphi\mathcal{D}\overline{\varphi} e^{\overline{\varphi} \gamma^a \partial_a \varphi \sum_k \sigma_-(x_k) \sigma_+(y_k)}. \]

Using Zinn-Justin [9, p. 680, eq. (A28.14)], one sees
\[ \int \mathcal{D}\varphi\mathcal{D}\overline{\varphi} e^{\overline{\varphi} \gamma^a \partial_a \varphi \sum_k \sigma_-(x_k) \sigma_+(y_k)} = \left( \frac{1}{2\pi} \right)^{2i} \prod_{k<l} \frac{|z_k - z_l|^2 |z_k' - z_l'|^2}{\prod_{k,l} |z_k - z_l'|^2} \equiv K_i(x, y), \]

where $z_k = x_k^0 + i x_k^1, z_k' = y_k^0 + i y_k^1$.

This implies
\[ S_{\phi}^{\text{free}} = \frac{\pi}{2\eta} (d\lambda, d\lambda) + \frac{\pi}{2\xi} (d\Lambda, d\Lambda) \]
\[ -\log \sum_{i=0}^{\infty} \frac{m^{2i}}{i!} \int \left[ \prod_{k=1}^{\infty} d^2 x_k d^2 y_k \right] K_i(x, y) e^{2i \sum_k [\Lambda(y_k) - \Lambda(x_k)]} \int \mathcal{D}\varphi \ e^{\frac{1}{2}(d\varphi, d\varphi)} \prod_{k=1}^i e^{2i \sqrt{-\xi} \varphi(y_k)} \prod_{k=1}^i e^{-2i \sqrt{-\xi} \varphi(x_k)}. \]

Now using Zinn-Justin [9, p. 664, eq. (28.13)],
\[ \int \mathcal{D}\theta \ e^{-\frac{1}{2}(d\theta, d\theta) + i \sum_i \epsilon_i \theta(x_i)} \propto \begin{cases} 0 & \text{for } \sum_i \epsilon_i \neq 0, \\ \prod_i \epsilon_i \Lambda_{UV} |x_i - x_j|^{-\epsilon_{ij} / 2\alpha} & \text{for } \sum_i \epsilon_i = 0 \end{cases} \]

(113)
for an ultraviolet cutoff $\Lambda_{UV}$ that appears when one regularizes the free boson propagator. The fact that this correlation function is zero unless the coefficients satisfy the condition $\sum_i c_i = 0$ is a result of invariance under constant translations of the field $\theta$.

The integral over the scalar fields $\varphi$ results in

$$\int \mathcal{D}\varphi \ e^{\frac{i}{2}(\varphi, \varphi)} \prod_{k=1}^i e^{2i\sqrt{T}y_k} \prod_{k=1}^i e^{-2i\sqrt{T}y_k} \propto (\Lambda_{UV})^{2i} K_i^{-1}(x, y).$$

(114)

The final result is

$$S_{f}^{\text{free}} (m; \Lambda, \lambda) = \frac{\pi}{2\eta} (d\lambda, d\lambda) + \frac{\pi}{2\xi} (d\Lambda, d\Lambda)$$

$$- \log \sum_{i=0}^\infty \frac{(m\Lambda_{UV})^{2i}}{2i!} \left( \frac{2i}{i} \right) \left( \int d^2 x e^{i2\pi\Lambda} \right)^i \left( \int d^2 x e^{-i2\pi\Lambda} \right)^i.$$ 

(115)

Consider now the partition function in the presence of a source $j$:

$$Z = \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{-S_{f}^{\text{free}} (m; \Lambda, \lambda) - (j, d\lambda + i* d\Lambda)}.$$ 

(116)

Invoking invariance under constant translations of the field $\Lambda$, we can show that the result of (113) for $\sum_i c_i \neq 0$ generalizes, in the presence of the source $j$, to

$$Z = \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{-S_{f}^{\text{free}} (m; \Lambda, \lambda) - (j, d\lambda + i* d\Lambda)}$$

$$\sum_{i=0}^\infty \frac{(m\Lambda_{UV})^{2i}}{2i!} \sum_{k=0}^i \frac{2i}{k} \left( \int d^2 x e^{i2\pi\Lambda} \right)^k \left( \int d^2 x e^{-i2\pi\Lambda} \right)^{2i-k}$$

$$= \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{-S_{f}^{\text{free}} (m; \Lambda, \lambda) - (j, d\lambda + i* d\Lambda)}$$

$$\sum_{i=0}^\infty \frac{(m\Lambda_{UV})^{2i}}{2i!} \sum_{k=0}^i \frac{2i}{k} \left( \int d^2 x e^{i2\pi\Lambda} \right)^k \left( \int d^2 x e^{-i2\pi\Lambda} \right)^{2i-k}$$

$$= \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{-S_{f}^{\text{free}} (m; \Lambda, \lambda) - (j, d\lambda + i* d\Lambda)}$$

$$\sum_{i=0}^\infty \frac{(m\Lambda_{UV})^{2i}}{2i!} \left( \int d^2 x e^{i2\pi\Lambda} + \int d^2 x e^{-i2\pi\Lambda} \right)^i$$

$$= \int \mathcal{D}[\Lambda] \mathcal{D}\lambda \ e^{-S_{f}^{\text{free}} (m; \Lambda, \lambda) - (j, d\lambda + i* d\Lambda)}$$

$$\sum_{i=0}^\infty \frac{(m\Lambda_{UV})^{2i}}{i!} \left( \int d^2 x e^{i2\pi\Lambda} + \int d^2 x e^{-i2\pi\Lambda} \right)^i.$$
= \int \mathcal{D}[\Lambda] \mathcal{D} \lambda \quad e^{-\frac{2\pi}{\xi} \int \left( d\lambda, d\lambda \right) - \frac{2\pi}{\xi} \left( d\Lambda, d\Lambda \right) - \left( j, d\lambda + i d\Lambda \right)}
\exp \left( 2m \Lambda_{UV} \int d^2 x \cos \left( \frac{2\pi}{\xi} \Lambda \right) \right)

This result is a generalization of the well known [2, 3] equivalence of a massive Dirac fermion in $D = 2$ to the sine–Gordon theory, for the particular value $\xi = 1$. 
References


