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The BRST Quantization of Second-Class Constrained Systems

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Abstract The problem of quantization of second-class constrained systems where the constraint functions, denoted by $\chi_\alpha = (G_a, C_a)$, are such that the G_a 's are first class among themselves, is considered in the context of the BRST formalism based on path integrals. In this letter it is shown that (i) there exists a first-class constrained system classically equivalent to the original second-class one; (ii) there is a gauge fermion implementing the canonical gauge conditions, $C_a = 0$, such that the BRST quantization of the first-class constrained system is equivalent with the quantization of the original second-class constrained one; (iii) the local measure in the path integral contains k -th order derivatives of the canonical Hamiltonian associated to the original system with respect to the functions C_a ($k \geq 2$). It is also derived the Lagrangian form of the path integral corresponding to the second-class constrained system under discussion in the special case of the original Hamiltonian being at most quadratic in the functions C_a . These results are exemplified on three representative models.

1. Introduction

The BRST method represents the most powerful quantization method for first-class constrained Hamiltonian systems. It was also extended to systems possessing besides first-class constraints second class ones, as shown in [1]. There, the quantization is accomplished without a preliminary elimination of the second-class constraints (by passing to the Dirac bracket), but mainly relies on the presence in the theory of the first-class constraints. In other words, the formalism described in [1] is no longer applicable to systems having only second-class constraints. The same result as in [1] (the same path integral) is derived in [2] and [3] by making use of canonical methods. This work intends to clarify how the BRST method can be also applied to systems possessing only second-class constraints in a way to be explained below. In this paper we adopt for simplicity the notation of finite-dimensional analytical mechanics, but the analysis can be extended straightforwardly to field theory.

Let H be the canonical Hamiltonian of a system subject to second-class constraints, $\chi_\alpha = 0$. We assume the functions $\chi_\alpha = (G_a, C_a)$ such that the Poisson brackets of G_a 's and G_b 's satisfy the relations

$$[G_a, G_b] = C_{ab}^c G_c. \quad (1)$$

The corresponding equations of motion read

$$\dot{F} \approx [F, H]^*, \quad (2)$$

where

$$[F_1, F_2]^* = [F_1, F_2] - [F_1, \chi_\alpha] C^{\alpha\rho} [\chi_\rho, F_2]. \quad (3)$$

is the Dirac bracket among the functions F_1 and F_2 . The symbol $\approx$ denotes weak equality (an equality holding when $\chi_\alpha = 0$). In (1) $C^{\alpha\rho}$ represents the elements of the inverse of the matrix

$$C_{\alpha\rho} = [\chi_\alpha, \chi_\rho]. \quad (4)$$

Equation (2) can be derived from a variational principle based on the action

$$\tilde{S}_0[q, p, \mu] = \int dt \left(\dot{q}^i p_i - H - \mu^\alpha \chi_\alpha \right), \quad (5)$$

the variables μ^α being auxiliary [4]. The path integral correspondent to action (5) is given by [5]

$$Z = \int Dq Dp D\mu \left(\det C_{\alpha\beta} \right)^{1/2} \exp(i\tilde{S}_0). \quad (6)$$

Under these specifications, we are able to formulate the problems to be investigated in the sequel:

a) Does exist a Hamiltonian

$$\bar{H} = H + \text{"extra terms in } q\text{'s and } p\text{'s"} \quad (7)$$

such that $\bar{H}$ is first-class with respect to the constraints $G_a = 0$?

b) Furthermore, if the answer to the previous question is positive, is the first-class constrained system with the Hamiltonian $\bar{H}$ and the constraints $G_a = 0$ equivalent at the classical level to the original second-class one?

c) Does exist a gauge-fixing fermion Ψ implementing the canonical gauge conditions $C_a = 0$ such that the corresponding path integrals to be equal, $Z\Psi = Z$?

Here, $Z\Psi$ denotes the path integral obtained from the gauge-fixed action correspondent to the solution of the master equation [5]

$$(S, S) = 0 \quad (8)$$

together with the boundary condition

$$S^{(0)}[q, p, v] = \int dt (\dot{q}^i p_i - \bar{H} - v^a G_a) \quad (9)$$

where

$$\text{antigh}(S^{(0)}) = 0 \quad (10)$$

We shall explicitly show that the answers to the above questions are affirmative. Thus, we can indeed find a Hamiltonian $\bar{H}$ satisfying

$$[\bar{H}, G_a] = f^b_a G_b \quad (11)$$

such that the first-class constrained system described by it together with the constraints $G_a = 0$ to be classically equivalent (equivalent at the level of equations of motion) with the original one by making use of a specific canonical gauge. This particular canonical gauge-fixing turns to be represented exactly by the conditions $C_a = 0$. It will be also indicated that the gauge fermion Ψ implementing the gauge conditions $C_a = 0$ with

$$(12) \quad gh(\Psi) = -1 \quad ; \quad \varepsilon(\Psi) = 1 \quad (12)$$

is such that the path integral associated to the original system coincides with the one associated to the first-class system (to be treated accordingly the BRST technique). The conclusion inferred from the prior affirmations is that we can join to the original second-class constrained system a first-class one for which the BRST quantization using the gauge fermion Ψ leads to the same path integral as the one corresponding to the original system. This is the meaning of applying the BRST method to the second-class constrained systems, $\chi_\alpha = (G_a, C_a)$. In (12) the notations gh and ε signify the ghost number, respectively the Grassmann parity. Concerning the BRST quantization in the path integral formalism we shall follow the same lines as in [5].

The work is organised in five sections. Section 2 is devoted to proving the existence of $\bar{H}$ satisfying (1) and to evidencing the equivalence at the classical level between the first-class constrained system and the original one. In Section 3 it will be applied the BRST technique to action (9) (describing the first-class constrained system) and proved the existence of Ψ implementing the gauge condition $C_a = 0$ such that $Z = Z\Psi$. Section 4 will keep us busy with writing down the Lagrangian form of $Z\Psi$ in two cases of practical interest. In Section 5 are exposed three models as an illustration of the results proved in the theoretical part of the paper.

2. Substituting second-class constrained systems with equivalent first-class ones

Let's consider a system having the canonical Hamiltonian H , described by N canonical pairs (q^i, p_i) , and subject to second-class constraints $\chi_\alpha = 0$, where $\chi_\alpha = (G_a, C_a)$ such that the constraint functions G_a satisfy (1). The Grassmann parities of the canonical variables and constraint functions are given by

$$\varepsilon(q^i) = \varepsilon(p_i) = \varepsilon_i \quad (13)$$

$$\varepsilon(\chi_\alpha) = \varepsilon_\alpha \quad (14)$$

The Hamiltonian H is a bosonic function. Because the constraints $\chi_\alpha = 0$ are second-class it follows that

$$\det(C_{\alpha\beta}) \neq 0 \quad (15)$$

Using the notation

$$\Delta_{ab} = [C_a, G_b] \quad (16)$$

we obtain in a straightforward manner

$$\det(C_{\alpha\beta}) = (\det(\Delta_{ab}))^2 \quad (17)$$

Since the matrix Δ_{ab} is invertible (relation (15) and (17)), it possesses an inverse Δ^{ab} such that

$$\Delta_{ab} \Delta^{bc} = \delta_a^c \quad (18)$$

We will take the case where Δ_{ab} doesn't depend on the canonical variables. Under these assumptions, the next theorem is the central result of this section.

Theorem 1. Let H be the canonical Hamiltonian of the system subject to the second-class constraints $\chi_\alpha = 0$. Then, there exists a function $\bar{H} = H + \text{"extra terms in } q\text{'s and } p\text{'s"}$ such that $\bar{H}$ is first-class with respect to the constraints $G_a = 0$

$$[\bar{H}, G_a] = f^b_a G_b \quad (19)$$

with f^b_a some functions of q 's and p 's.

Proof. The proof is intended to be constructive, finally obtaining the concrete form of "extra terms in q 's and p 's". Let's take $\bar{H}$ of the form

$$\bar{H} = H + \lambda^\alpha \chi_\alpha = H + \bar{\lambda}^a C_a + \lambda^a G_a \quad (20)$$

with $\lambda^\alpha = (\lambda^a, \bar{\lambda}^a)$ functions on phase-space. Thus, proving the theorem reduces to proving the existence of some functions λ^α such that $\bar{H}$ given by (20) satisfies equations (19). Introducing (20) in (19) one gets

$$[H, G_a] + \bar{\lambda}^b [C_b, G_a] + (-)^{\varepsilon_a \bar{\varepsilon}_b} [\bar{\lambda}^b, G_a] C_b + \lambda^b [G_b, G_a] + (-)^{\varepsilon_a \varepsilon_b} [\lambda^b, G_a] G_b = f^b_a G_b \quad (21)$$

$$\text{Here} \quad \varepsilon_a = \varepsilon(G_a) \quad (22 a)$$

$$\bar{\varepsilon}_a = \varepsilon(C_a) \quad (22 b)$$

Choosing λ^a such that

$$f^b_a = \lambda^c C_{ca}^b + (-)^{\varepsilon_a \varepsilon_b} [\lambda^b, G_a] \quad (23)$$

we obtain from (21) and due to relations (1)

$$[H, G_a] + \bar{\lambda}^b [C_b, G_a] + (-)^{\varepsilon_a \bar{\varepsilon}_b} [\bar{\lambda}^b, G_a] C_b = 0 \quad (24)$$

At this stage it is enough to prove the existence of some solutions $\bar{\lambda}^b$ for equations (23). We search a solution of (23) of the form

$$\bar{\lambda}^b = \sum_{k=0}^{\infty} u^{ba_1 \dots a_k} C_{a_1} C_{a_2} \dots C_{a_k} = u^b + u^{ba_1} C_{a_1} + u^{ba_1 a_2} C_{a_1} C_{a_2} + \dots \quad (25)$$

where $u^{ba_1 \dots a_k}$ are functions on phase-space. If one introduces (24) in (23) and identifies the coefficients of the same powers in C^a 's one gets the following tower of equations (corresponding respectively to powers $0, 1, \dots, k, \dots$)

$$[H, G_a] + u^b [C_b, G_a] = 0 \quad (26 a)$$

$$[u^b, G_a] + 2u^{ba_1} [C_{a_1}, G_a] = 0 \quad (26 b)$$

$$[u^{ba_1 \dots a_{k-1}}, G_a] + (-)^{\rho(a_1, \dots, a_{k-1})} (k+1) u^{ba_1 \dots a_k} [C_{a_k}, G_a] = 0 \quad (26 c)$$

where

$$\rho(a_1, \dots, a_k) = \bar{\varepsilon}_{a_k} (\bar{\varepsilon}_{a_1} + \dots + \bar{\varepsilon}_{a_{k-1}}) \quad , \quad \text{for } k \geq 2 \quad (26 d)$$

Multiplying equation (26 c) by Δ^{am} and summing over a it follows, also taking into account (18)

$$u^{ba_1 \dots a_{k-1} m} = -\frac{(-)^{\rho(a_1, \dots, a_{k-1})}}{k+1} [u^{ba_1 \dots a_{k-1}}, G_a] \Delta^{am} \quad , \quad \text{for } k \geq 2 \quad (27)$$

Reprising the same operations on equation (26 a) and (26 b) it results

$$u^m = -[H, G_a] \Delta^{am} \quad (28 a)$$

$$u^{ba_1} = \frac{1}{2} [[H, G_a] \Delta^{ab}, G_c] \Delta^{ca_1} \quad (28 b)$$

Relations (27) and (28) output recursively the coefficients from expansion (25)

$$u^{ba_1 \dots a_k} = \frac{(-)^{k+1+\theta(a_1, \dots, a_k)}}{(k+1)!} \left[\dots \left[[H, G_{m_{k+1}}] \Delta^{m_{k+1} b}, G_{m_k} \right] \Delta^{m_k a_1}, \dots, G_{m_1} \right] \Delta^{m_1 a_k} \quad (29 a)$$

with

$$\theta(a_1, \dots, a_k) = \sum_{1 \leq i \leq j \leq k} (\bar{\varepsilon}_{a_i} \bar{\varepsilon}_{a_j}) \quad (29 b)$$

For $k=0$ and $k=1$ the definitions of the correspondent θ 's are extended as

$$\theta(a_0) = 0 \quad (29 c)$$

$$\theta(a_1) = 0 \quad (29 d)$$

Finally, combining (20), (25), (28) we end the proof of the theorem by obtaining $\bar{H}$ as

$$\bar{H} = \sum_{k=0}^{\infty} \frac{(-)^{k+1+\theta(a_1, \dots, a_k)}}{(k+1)!} \left[\dots \left[[H, G_{m_{k+1}}] \Delta^{m_{k+1} a_{k+1}}, G_{m_k} \right] \Delta^{m_k a_1}, \dots, G_{m_1} \right] \Delta^{m_1 a_k} \times \\ \times C_{a_1} C_{a_2} \dots C_{a_{k+1}} + H + \lambda^a G_a \quad (30)$$

q.e.d.

A natural consequence of elements Δ_{ab} not depending on the canonical variables is that the ones with mixed indices (i.e., $\bar{\varepsilon}_a \neq \varepsilon_b$) are all equal to zero. Then, the way one should read formulas (26)-(30) is that the summation is performed there only over the indices corresponding to constraints of the same Grassmann parity, so that $\theta(a_1, \dots, a_k)$ takes the simple form

$$\theta(a_1, \dots, a_k) = 0, \quad \text{for } \{\bar{\varepsilon}_{a_i}\}_{i=1,2,\dots,k} = 0 \quad (31 a)$$

and

$$\theta(a_1, \dots, a_k) = \frac{k(k-1)}{2}, \quad \text{for } \{\bar{\varepsilon}_{a_i}\}_{i=1,2,\dots,k} = 1 \quad (31 b)$$

The equations of motion of the first-class constrained system (with the Hamiltonian $\bar{H}$ and constraints $G_a = 0$) can be derived from a variational principle based on action (9), and are given by

$$\dot{F} = [F, \bar{H}] + v^a [F, G_a] (-)^{\varepsilon_a \varepsilon_F} \quad (32 a)$$

$$G_a = 0 \quad (32 b)$$

Imposing on the first-class constrained system the canonical gauge conditions

$$C_a = 0 \quad (33)$$

we get, demanding their consistency ($\dot{C}_a = 0$), and also using (32 a), the following equations

$$[C_a, \bar{H}] + v^b \Delta_{ab} (-)^{\varepsilon_a \bar{\varepsilon}_b} = 0 \quad (34)$$

From them we derive the multiplier functions v^a as Δ_{ab} has an inverse. If one introduces the solution of (34) in (9) one has

$$\stackrel{(0)}{S}[q, p, v(q, p)] = \bar{S}_0[q, p] = \int dt (\dot{q}^i p_i - \bar{H} + (-)^{\varepsilon_a \bar{\varepsilon}_b} \Delta^{ab} [C_b, \bar{H}] G_a) \quad (35)$$

The variational principle based on action (35) leads us to the equations of motion

$$\dot{F} \approx [F, \bar{H}]^* \quad (36)$$

The constraint functions (G_a, C_a) being second-class, from (30) and the properties of Dirac bracket, one obtains

$$[F, \bar{H}]^* \approx [F, H]^* \quad (37)$$

Combining equations (36) and (37) one recovers equation (2). We conclude thus that at the classical level the first-class constrained system having the Hamiltonian $\tilde{H}$ and the constraints $G_a = 0$ is equivalent to the original second-class constrained system in the gauge $C_a = 0$.

In the end of this section let's focus on action (9). It is invariant under the following gauge transformations

$$\delta_\epsilon q^i = (-)^{\epsilon_i} \epsilon^a \epsilon_a [q^i, G_a] \quad (38 \text{ a})$$

$$\delta_\epsilon p_i = (-)^{\epsilon_i} \epsilon^a \epsilon_a [p_i, G_a] \quad (38 \text{ b})$$

$$\delta_\epsilon v^a = \dot{\epsilon}^a - \epsilon^b f^a_b + v^c \epsilon^b C_{bc}^a \quad (38 \text{ c})$$

With the help of relations (21)-(24), (30), and (38) it is easy to see that

$$\delta_\epsilon \tilde{H} = 0 \quad (39)$$

$$\delta_\epsilon [(\lambda^a + v^a) G_a] = \dot{\epsilon}^a G_a \quad (40)$$

where $\tilde{H}$ is defined by

$$\tilde{H} = \bar{H} - \lambda^a G_a \quad (41)$$

and $\bar{H}$ is given by (30).

Relation (40) shows that is permissible to perform locally a coordinate transformation of the type

$$(q^i, p_i) \rightarrow (Q^a, P_a, z^\Delta, \bar{p}_\Delta) \quad (42)$$

such that

$$P_a = G_a \quad (43 \text{ a})$$

$$[\bar{p}_\Delta, z^{\Delta'}] = -\delta_\Delta^{\Delta'} \quad (43 \text{ b})$$

$$[z^\Delta, z^{\Delta'}] = 0 \quad (43 \text{ c})$$

$$[\bar{p}_\Delta, \bar{p}_{\Delta'}] = 0 \quad (43 \text{ d})$$

In (43 b-d) the variables z^Δ and $\bar{p}_\Delta$ are the rest of coordinates on phase-space. We take z^Δ and $\bar{p}_\Delta$ to satisfy

$$[z^\Delta, P_a] = 0 \quad (44)$$

$$[\bar{p}_\Delta, P_a] = 0 \quad (45)$$

As a result of transformations (42) and (43), action (9) becomes

$$S_0[Q, P, z, \bar{p}, \bar{v}] = \int dt (\dot{Q}^a P_a + \dot{z}^\Delta \bar{p}_\Delta - \tilde{H}(Q, P, z, \bar{p}) - \bar{v}^a P_a) \quad (46)$$

with

$$\bar{v}^a = v^a + \lambda^a(Q, P, z, \bar{p}) \quad (47)$$

The gauge invariances of action (46) are given by

$$\delta_\epsilon Q^a = \epsilon^a \quad (48 \text{ a})$$

$$\delta_\epsilon P_a = 0 \quad (48 \text{ b})$$

$$\delta_\epsilon z^\Delta = 0 \quad (48 \text{ c})$$

$$\delta_\epsilon \bar{p}_\Delta = 0 \quad (48 \text{ d})$$

$$\delta_\epsilon \bar{v}^a = \dot{\epsilon}^a \quad (48 \text{ e})$$

Relying on relations (47) and (48) one has

$$C_{bc}^a(Q, P, z, \bar{p}) \equiv 0 \quad (49)$$

As Δ_{ab} 's were supposed not to depend on the canonical variables it is simply to check that one can make the choice

$$Q^a = \Delta^{ab} C_b \quad (50)$$

Without loss of generality we can consider $G_a = 0$ primary constraints and $C_a = 0$ secondary ones for the second-class constrained system. It is then clear that the canonical Hamiltonian $\bar{H}$ does not depend on P_a 's and so $\tilde{H}$ also doesn't. The dependence of $\tilde{H}$ on P_a 's in (46) is then trivial.

We emphasise that the form of action (46) is implied by relations (39) and (40), and not by the fact that the transformation (42) is canonical. Although it is not canonical in general, its Jacobian is equal to ± 1 . Indeed, we have

$$\frac{D(Q, P, z, \bar{p})}{D(q, p)} = \pm 1 \quad (51)$$

and also

$$\frac{D(\bar{v})}{D(v)} = 1 \quad (52)$$

Relations (50) and (30) imply after performing usual computation

$$\tilde{H}(Q, z, \bar{p}) = H(Q, z, \bar{p}) + \sum_{k=0}^{\infty} \frac{\overrightarrow{\partial}^{k+1} H(Q, z, \bar{p})}{\partial Q^{a_1} \dots \partial Q^{a_{k+1}}} \frac{(-)^{k+1}}{(k+1)!} Q^{a_1} \dots Q^{a_{k+1}} \quad (53)$$

We shall represent $H(Q, z, \bar{p})$ as a series of powers in Q^a 's with coefficients functions of z^Δ 's and $\bar{p}_\Delta$'s

$$H(Q, z, \bar{p}) = H(0, z, \bar{p}) + \sum_{j=1}^{\infty} \frac{1}{j!} \left. \frac{\overrightarrow{\partial}^j H(Q, z, \bar{p})}{\partial Q^{a_1} \dots \partial Q^{a_j}} \right|_{Q=0} Q^{a_1} \dots Q^{a_j} \quad (54)$$

In (53) and (54) the symbol $\overrightarrow{\partial}$ denotes, as usually, the right derivative. Because of the previous assumption on G_a 's being primary and C_a 's secondary it follows that the term proportional to Q^a in (54) is absent. One must also demand

$$\left. \frac{\vec{\partial}^2}{\partial Q^a \partial Q^b} H(Q, z, \bar{p}) \right|_{Q=0} = -\Delta_{ab} \quad (55)$$

due to relations (50). The sign minus in (55) is taken for convenience. Replacing now (54) in (53), the latter becomes

$$\tilde{H}(Q, z, \bar{p}) = H(0, z, \bar{p}) \equiv h(z, \bar{p}) \quad (56)$$

3. The BRST Quantization of the first-class constrained system

In this section we shall apply the BRST technique to the first-class constrained system. We shall take for the sake of simplicity all the constraints $P_a = 0$ to be primary also for the first-class system. Let's apply now the BRST antibracket-antifield formalism to action (46). In agreement with the general formalism [5], one associates to all the variables in the theory antifields, respectively denoted by Q_a^* , P^{*a} , z_{Δ}^* , $\bar{p}^{*\Delta}$ and $\bar{v}_a^*$. The Grassmann parities and the ghost numbers of the antifields are given by

$$\varepsilon(\Phi_A^*) = \varepsilon(\Phi^A) + 1; \quad gh(\Phi_A^*) = -gh(\Phi^A) - 1, \quad (57a)$$

where Φ^A and Φ_A^* are the original variables, respectively their antifields.

As the gauge transformations (48) are irreducible and the constraint functions P_a are abelian, the minimal ghost spectrum will contain only the ghosts η^a . The number of ghosts is equal with the number of the first-class constraints. Thus, we have

$$\varepsilon(\eta^a) = \varepsilon_a + 1; \quad gh(\eta^a) = 1. \quad (57b)$$

Under these conditions, the minimal solution of the master equation reads

$$S' = \bar{S}_0 + \int dt (Q_a^* \dot{\eta}^a + \bar{v}_a^* \dot{\eta}^a) \quad (58)$$

where $\bar{S}_0$ is given by (46). Adding the non-minimal sector $(B^a, B_a^*, \bar{\eta}_a^*, \bar{\eta}^a)$ as in [5], the non-minimal solution of the master equation takes the simple form

$$S = S' + \int dt (\bar{\eta}_a^* B^a) \quad (59)$$

The solution (59) is invariant at the following BRST transformations

$$sQ^a = \eta^a, \quad (60a) \quad sz_{\Delta}^* = 0, \quad (60e) \quad s\bar{v}^a = \eta^a, \quad (60i)$$

$$sP_a = 0, \quad (60b) \quad s\bar{p}_{\Delta}^* = 0, \quad (60f) \quad s\bar{v}_a^* = P_a, \quad (60j)$$

$$sQ_a^* = \dot{P}_a, \quad (60c) \quad sB^a = 0, \quad (60g) \quad s\bar{\eta}^a = B^a, \quad (60k)$$

$$sP_a^* = -\dot{Q}^a + \bar{v}^a, \quad (60d) \quad sB_a^* = -\bar{\eta}_a^*, \quad (60h) \quad s\bar{\eta}_a^* = 0. \quad (60l)$$

Further, we need to implement the canonical gauge conditions (33). They are equivalent to the conditions

$$Q^a = 0, \quad (61)$$

due to relations (50), as $\det(\Delta_{ab}) \neq 0$. The next theorem is the main result of this section.

Theorem 2. There exists a set of functions $f^a(Q)$ such that

$$\frac{1}{2} \Delta_{ab} f^a(Q) f^b(Q) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \frac{\vec{\partial}^{(k+1)} H(Q, z, \bar{p})}{\partial Q^a \partial Q^b \partial Q^{a_1} \dots \partial Q^{a_{k+1}}} Q^a \partial Q^{a_1} \dots \partial Q^{a_{k+1}} \quad (62)$$

Proof. We shall expand $f^a(Q)$ in a series of powers in Q^a 's

$$f^a(Q) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\vec{\partial}^j f^a(Q)}{\partial Q^{c_1} \dots \partial Q^{c_j}} \bigg|_{Q=0} Q^{c_1} \dots Q^{c_j} \quad (63)$$

From (54) one gets

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \frac{\vec{\partial}^{(k+1)} H(Q, z, \bar{p})}{\partial Q^a \partial Q^b \partial Q^{a_1} \dots \partial Q^{a_{k+1}}} Q^a \partial Q^{a_1} \dots \partial Q^{a_{k+1}} = -\sum_{k=2}^{\infty} \frac{1}{k!} \frac{\vec{\partial}^k H(Q, z, \bar{p})}{\partial Q^{b_1} \dots \partial Q^{b_k}} \bigg|_{Q=0} Q^{b_1} \dots Q^{b_k} \quad (64)$$

Substituting (63) in (62) and due to (64) one obtains after identifying the coefficients of powers 0, 1, and 2 in Q^a 's

$$\frac{1}{2} \Delta_{ab} f^a(0) f^b(0) = 0, \quad (65)$$

$$\frac{1}{2} \Delta_{ab} \left(f^a(0) \frac{\vec{\partial} f^b(Q)}{\partial Q^{b_1}} \bigg|_{Q=0} + \frac{\vec{\partial} f^a(Q)}{\partial Q^{b_1}} \bigg|_{Q=0} f^b(0) \right) = 0, \quad (66)$$

$$\begin{aligned} \frac{1}{2} \Delta_{ab} \left(\frac{1}{2!} f^a(0) \frac{\vec{\partial}^2 f^b(Q)}{\partial Q^{b_1} \partial Q^{b_2}} \bigg|_{Q=0} + \frac{\vec{\partial} f^a(Q)}{\partial Q^{b_1}} \bigg|_{Q=0} \frac{\vec{\partial} f^b(Q)}{\partial Q^{b_2}} \bigg|_{Q=0} + \right. \\ \left. + \frac{1}{2!} \frac{\vec{\partial}^2 f^a(Q)}{\partial Q^{b_1} \partial Q^{b_2}} \bigg|_{Q=0} f^b(0) \right) = -\frac{1}{2!} \frac{\vec{\partial}^2 H(Q, z, \bar{p})}{\partial Q^{b_1} \partial Q^{b_2}} \bigg|_{Q=0}, \quad (67) \end{aligned}$$

$$\frac{1}{2} \Delta_{ab} \left(\frac{1}{3!} f^a(0) \frac{\vec{\partial}^3 f^b(Q)}{\partial Q^{b_1} \partial Q^{b_2} \partial Q^{b_3}} \bigg|_{Q=0} + \frac{1}{2!} \frac{\vec{\partial} f^a(Q)}{\partial Q^{b_1}} \bigg|_{Q=0} \frac{\vec{\partial}^2 f^b(Q)}{\partial Q^{b_2} \partial Q^{b_3}} \bigg|_{Q=0} + \right.$$

$$\left. + \frac{1}{2!} \frac{\vec{\partial}^2 f^a(Q)}{\partial Q^{b_1} \partial Q^{b_2}} \bigg|_{Q=0} \frac{\vec{\partial} f^b(Q)}{\partial Q^{b_3}} \bigg|_{Q=0} + \frac{1}{3!} \frac{\vec{\partial}^3 f^a(Q)}{\partial Q^{b_1} \partial Q^{b_2} \partial Q^{b_3}} \bigg|_{Q=0} f^b(0) \right) =$$

$$= \frac{1}{3!} \frac{\vec{\partial}^3 H(Q, z, \bar{p})}{\partial Q^b \partial Q^b \partial Q^b} \Big|_{Q=0} \quad (68)$$

Equations (65)-(68) allow one to find $f^a(0)$, $\frac{\vec{\partial} f^a(Q)}{\partial Q^b} \Big|_{Q=0}$, $\frac{\vec{\partial}^2 f^a(Q)}{\partial Q^b \partial Q^c} \Big|_{Q=0}$ of the form

$$f^a(0) = 0, \quad (69)$$

$$\frac{\vec{\partial} f^a(Q)}{\partial Q^b} \Big|_{Q=0} = \delta^a_b, \quad (70)$$

$$\frac{1}{2!} \frac{\vec{\partial}^2 f^a(Q)}{\partial Q^b \partial Q^b} \Big|_{Q=0} = -\frac{1}{3!} \Delta^{(ab)} \frac{\vec{\partial}^3 H(Q, z, \bar{p})}{\partial Q^b \partial Q^b \partial Q^b} \Big|_{Q=0}, \quad (71)$$

$\Delta^{(ab)}$ being the inverse of the matrix

$$\Delta_{(ab)} = \frac{1}{2} (\Delta_{ab} + \Delta_{ba}). \quad (72)$$

Due to solutions (69)-(71), one obtains for $k \geq 3$

$$\begin{aligned} \frac{1}{k!} \frac{\vec{\partial}^k f^c(Q)}{\partial Q^c \partial Q^c \dots \partial Q^c} \Big|_{Q=0} &= -\Delta^{(cm)} \left(\frac{1}{(k+1)!} \frac{\vec{\partial}^{k+1} H(Q, z, \bar{p})}{\partial Q^m \partial Q^c \partial Q^c \dots \partial Q^c} \Big|_{Q=0} + \right. \\ &\left. + \frac{1}{2!} \Delta_{ab} \sum_{j=2}^{k-1} \frac{1}{j!(k-j+1)!} \frac{\vec{\partial}^j f^a(Q)}{\partial Q^m \partial Q^c \partial Q^c \dots \partial Q^c} \Big|_{Q=0} \frac{\vec{\partial}^{k-j+1} f^b(Q)}{\partial Q^c \partial Q^c \dots \partial Q^c} \Big|_{Q=0} \right) \end{aligned} \quad (73)$$

The last formula gives the coefficients of power order k in Q^a 's in terms of the coefficients of power orders less than k . Making use of (71) and (73) one can output recursively the solution of equation (62).

q.e.d.

Thus, the first terms in expansion (63) are as follows

$$f^a(Q) = Q^a - \frac{1}{3!} \Delta^{(ab)} \frac{\vec{\partial}^3 H(Q, z, \bar{p})}{\partial Q^b \partial Q^b \partial Q^b} \Big|_{Q=0} Q^b \partial Q^b + \dots \quad (74)$$

The above solution, as well as the last theorem should be understood in the same manner as Theorem 1, i.e. taking into account the fact that we assumed Δ_{gh} independent of the variables on the phase-space

From (63) and (69) it follows that $f^a(Q) = 0$ imply conditions (61), $Q^a = 0$. We choose the gauge-fixing fermion as

$$\Psi = -\int dt \left(\bar{\eta}^a \Delta_{ab} f^b(Q) \right) \quad (75)$$

Eliminating the antifields in the theory with the aid of relations

$$\Phi_A^* = \frac{\overleftarrow{\delta} \Psi}{\delta \Phi^A}, \quad (76)$$

we get due to (59) the gauge-fixed action

$$S\Psi = \int dt \left(\dot{Q}^a P_a + \dot{z} \Delta \bar{p}_\Delta - h(z, \bar{p}) - \bar{v}^a P_a - \bar{\eta}^a \left(\Delta_{ab} \frac{\overleftarrow{\partial} f^b(Q)}{\partial Q^c} \right) \eta^c - B^a \Delta_{ab} f^b(Q) \right) \quad (77)$$

Action (77) is invariant under the following BRST transformations

$$\begin{aligned} sQ^a &= \eta^a, & (78a) \quad s\bar{z} &= 0, & (78c) \quad s\bar{v}^a &= \bar{\eta}^a, & (78e) \quad s\bar{\eta}^a &= B^a, & (78g) \\ sP_a &= 0, & (78b) \quad s\bar{p}_\Delta &= 0, & (78d) \quad sB^a &= 0, & (78f) \quad s\eta^a &= 0, & (78h) \end{aligned}$$

The path integral associated to action (77) is given by

$$Z\Psi = \int DQDPdzD\bar{p}D\bar{\eta}D\eta DBD\bar{v} \exp(iS\Psi) \quad (79)$$

If one integrates in (79) over the variables Q^a , P_a , $\bar{\eta}^a$, η^a , B^a and $\bar{v}^a$, one obtains

$$Z\Psi = \int DzD\bar{p} \exp \left(i \int dt \left(\dot{z} \Delta \bar{p}_\Delta - h(z, \bar{p}) \right) \right), \quad (80)$$

which expresses the path integral correspondent to the first-class constrained system in terms of independent variables. If in expression (6) one performs transformation (42) and integrates over the multipliers μ^α , the Q^a 's, and P_a 's taking into account relations (43) and (50) one derives the following form of the path integral

$$Z = \int DzD\bar{p} \exp \left(i \int dt \left(\dot{z} \Delta \bar{p}_\Delta - H(0, z, \bar{p}) \right) \right) \quad (81)$$

If one compares (80) and (81) relying on (56), one gets

$$Z = Z\Psi \quad (82)$$

In order to derive the gauge-fixed action (77) we applied the BRST antibracket-antifield formalism. We shall explicitly prove in the end of this section that one can reobtain this result by applying the standard Hamiltonian BRST formalism [5]. Accordingly the general formalism (as the constraint functions are irreducible) we associate to every constraint function P_a a ghost γ^a with $\varepsilon(\gamma^a) = \varepsilon_a + 1$, $gh(\gamma^a) = 1$. We also introduce the ghost momenta π_a , with $\varepsilon(\pi_a) = \varepsilon(\gamma^a)$ and $gh(\pi_a) = -gh(\gamma^a)$ such that

$$[\pi_a, \gamma^b] = (-)\varepsilon_a [\gamma^b, \pi_a] = -\delta_a^b \quad (83)$$

The non-minimal sector (consisting of cohomologically trivial variables) is taken to contain the variables B^a , P_a^B , B^a , P_a^B , $\bar{\gamma}_a$, $P(\bar{\gamma})^a$, $\bar{\gamma}_a^1$, $P(\bar{\gamma}^1)^a$. These variables are canonically conjugated in pairs, namely B^a , P_a^B , $\bar{\gamma}_a$ and $\bar{\gamma}_a^1$ are momenta while the remaining ones are respectively their conjugated variables. Their ghost numbers and Grassmann parities are as follows

$$gh(B^a) = gh(B^I a) = gh(P_a^B) = gh(P_a^{B^I}) = 0 \quad , \quad (84)$$

$$\varepsilon(B^a) = \varepsilon(B^I a) = \varepsilon(P_a^B) = \varepsilon(P_a^{B^I}) = \varepsilon_a \quad , \quad (85)$$

$$gh(\bar{\gamma}_a) = gh(\bar{\gamma}_a^I) = -1 \quad , \quad (86)$$

$$gh(P(\bar{\gamma})^a) = gh\left(P\left(\bar{\gamma}^I\right)^a\right) = 1 \quad , \quad (87)$$

$$\varepsilon(\bar{\gamma}_a) = \varepsilon(\bar{\gamma}_a^I) = \varepsilon(P(\bar{\gamma})^a) = \varepsilon\left(P\left(\bar{\gamma}^I\right)^a\right) = \varepsilon_a + 1 \quad . \quad (88)$$

The non-minimal BRST charge reads

$$\Omega = \int dt \left(P_a \dot{\gamma}^a + P(\bar{\gamma})^a B^a + P\left(\bar{\gamma}^I\right)^a B^I a \right) \quad . \quad (89)$$

We take the Hamiltonian gauge-fixing fermion of the form

$$\Psi_H = \int dt \left(\pi_a \left(-\dot{Q}^a + \Delta_{ab} f^b(Q) \right) + P_a^{B^I} (\pi_a - \bar{\gamma}_a) + P_a^B \left(\bar{\gamma}_a^I + \dot{\bar{\gamma}}_a \right) + P_a^{B^I} \dot{\bar{\gamma}}_a^I \right) \quad . \quad (90)$$

The BRST-invariant extension of $h(z, \bar{p})$ is exactly itself because

$$[\Omega, h(z, \bar{p})] = 0 \quad . \quad (91)$$

Then, the gauge-fixed action reads

$$S_{\Psi}^H = \int dt \left(\dot{Q}^a P_a + \dot{z} \Delta_{\bar{p}} - h(z, \bar{p}) + P_a \Delta_{ab} f^b(Q) + P_a^{B^I} (P_a - B^a) - \right. \\ \left. - \pi_a \Delta_{ab} \frac{\overrightarrow{\partial} f^b(Q)}{\partial Q^c} \gamma^c + P_a^B B^I a - \bar{\gamma}_a^I P(\bar{\gamma})^a + (\bar{\gamma}_a - \pi_a) P\left(\bar{\gamma}^I\right)^a \right) \quad (92)$$

The correspondent path integral will be given by

$$Z_{\Psi}^H = \int DQ DP Dz D\bar{p} D\pi D\gamma DB DP^B DB^I DP^B D\bar{\gamma} DP(\bar{\gamma}) D\bar{\gamma}^I DP\left(\bar{\gamma}^I\right) \exp\left(i S_{\Psi}^H\right) \quad . \quad (93)$$

Integrating now in (93) over the Q 's, P 's, π 's, γ 's, B 's, P^B 's, B^I 's, P^{B^I} 's, $\bar{\gamma}$'s, $P(\bar{\gamma})$'s, $\bar{\gamma}^I$'s, and

$P\left(\bar{\gamma}^I\right)$'s, one recovers (80).

4. The Lagrangian form of the path integral under simplifying assumptions

In this section we shall derive the Lagrangian form of the path integral (79) under some simplifying assumptions. Performing in (79) the inverse transformation

$$(Q, P, z, \bar{p}) \rightarrow (q, p) \quad , \quad (94)$$

it is easy to observe from (40) that

$$Z\Psi = \int Dq Dp D\bar{\eta} D\eta DB Dv \exp \left(i \int dt \left(\dot{q}^i p_i - \bar{H} - v^a G_a - B^a \Delta_{ab} f^b(C) - \right. \right. \\ \left. \left. - \bar{\eta}^a \left(\Delta_{ab} \frac{\overleftarrow{\partial} f^b(C)}{\partial C_m} \Delta_{mc} \right) \eta^c \right) \right) \quad , \quad (95)$$

$f^a(C)$ being obtained from $f^a(Q)$ with the help of relations (50). Integration over $\bar{\eta}$'s and η 's in (95) leads to

$$Z\Psi = \int Dq Dp DB Dv \det \left(\Delta_{ab} \frac{\overleftarrow{\partial} f^b(C)}{\partial C_m} \Delta_{mc} \right) \exp \left(i \int dt \left(\dot{q}^i p_i - \bar{H} - v^a G_a - B^a \Delta_{ab} f^b(C) \right) \right) \quad . \quad (96)$$

Using (71) and (73), it follows from (96) that if the canonical Hamiltonian $\bar{H}$ of the original system is more than quadratic in the functions C_a , the local measure

$$\mu = \det \left(\Delta_{ab} \frac{\overleftarrow{\partial} f^b(C)}{\partial C_m} \Delta_{mc} \right) \quad , \quad (97)$$

depends nontrivially on q 's and p 's even in the case of Δ_{ab} not depending on them. If the functions C_a depend only on q 's we can however integrate over p 's, B 's, and v 's in (96) in order to output the Lagrangian form of the path integral. We shall accomplish this task in two important cases.

(i) Let's consider the case of the second-class constrained system possessing the primary constraints $G_a = 0$ and also secondary ones $C_a = 0$. We realize then in (96) the replacement

$$f^a \rightarrow f^a + \frac{1}{2} B^a \quad . \quad (98)$$

This change doesn't affect the BRST invariance of the path integral as $S B^a = 0$. Integrating now over B 's and v 's in (96), it results with the aid of (62)

$$Z\Psi = \int Dq Dp \prod_n \delta(G_n) \det \left(\Delta_{ab} \frac{\overleftarrow{\partial} f^b(C)}{\partial C_m} \Delta_{mc} \right) (\det \Delta)^{-\frac{1}{2}} \exp \left(i \int dt \left(\dot{q}^i p_i - H - \lambda^a G_a \right) \right) \quad . \quad (99)$$

Because the canonical Hamiltonian is built only in the presence of the primary constraints, the integration over p 's in (99) gives

$$Z\Psi = \int Dq \det \left(\Delta_{ab} \frac{\overleftarrow{\partial} f^b(C)}{\partial C_m} \Delta_{mc} \right) (\det \Delta)^{-\frac{1}{2}} \exp \left(i \int dt L(q, \dot{q}) \right) \quad , \quad (100)$$

where $L(q, \dot{q})$ is the Lagrangian of the original second-class system. If the canonical Hamiltonian H is at most quadratic in terms of the secondary constraint functions C_a (e.g. the massive Yang-Mills theory) one has

$$\frac{\overleftarrow{\partial} f^b(C)}{\partial C_m} = \Delta^{bm} \quad , \quad (101)$$

and thus the Lagrangian path integral becomes

$$Z\Psi = \int Dq \left(\det C_{\alpha\beta} \right)^{\frac{1}{4}} \exp(i \int dt L(q, \dot{q})) \quad (102)$$

It is necessary to mention that relation (102) takes place even when the constraint functions C_a also depend on the p 's

because in this case $\left(\frac{\overrightarrow{\partial}^k H}{\partial C_{m_1} \dots \partial C_{m_k}} \right) \Big|_{C=0} = 0$, for any $k > 2$) relation (101) is also valid.

(ii) Let's pass now to the case of all second-class constraints being primary. Then, we no longer need to eliminate the factor $\prod_a \delta(G_a)$ from the local measure with the help of transformation (98) as the canonical Hamiltonian H will be constructed now in the presence of the constraints $G_a = 0$ as well as of $C_a = 0$. Integrating in (96) over B 's, v 's and subsequently over p 's one obtains

$$Z\Psi = \int Dq \left(\det C_{\alpha\beta} \right)^{\frac{1}{2}} \exp(i \int dt L(q, \dot{q})) \quad (103)$$

The last relation also holds when the functions C_a depend on the p 's because for systems having only primary

second-class constraints relation (101) takes place (for such systems, one always has $\frac{\overrightarrow{\partial}^k H}{\partial C_{m_1} \dots \partial C_{m_k}} \Big|_{C=0} = 0$, for any $k > 2$).

5. Examples

In this section we shall exemplify the results derived previously exposing some models characteristic to the above investigated situations.

5.1. The massive Yang-Mills theory

The Lagrangian action describing the massive Yang-Mills fields is

$$S_0^L[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{2} M^2 A_\mu^a A_a^\mu \right) \quad (104)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{bc}^a A_\mu^b A_\nu^c \quad (105)$$

Computing the canonical momenta, π_a^μ , one obtains the primary constraints

$$\pi_a^0 = 0 \quad (106)$$

The canonical Hamiltonian has the form

$$H = \int d^3x \left(\frac{1}{2} \pi_i^a \pi_{ia} + \frac{1}{4} F_{ij}^a F_a^{ij} - A_0^a D^i \pi_{ia} + \frac{1}{2} M^2 A_\mu^a A_a^\mu \right) \quad (107)$$

where

$$D^i \pi_{ia} = \partial^i \pi_{ia} - f_{ab}^c A_i^b \pi_c^i \quad (108)$$

The consistency conditions on the primary constraints imply the secondary ones, namely

$$-D^i \pi_{ia} + M^2 A_a^0 = 0 \quad (109)$$

There are no further constraints.

The constraint functions π_a^0 satisfy the relations

$$\left[\pi_a^0, \pi_b^0 \right] = 0 \quad (110)$$

It is easy to check that the constraint functions π_a^0 and $-D^i \pi_{ia} + M^2 A_a^0$ are second-class. Accordingly the previous sections, one makes the following identifications

$$G_a = \pi_a^0 \quad (111)$$

$$C_a = -D^i \pi_{ia} + M^2 A_a^0 \quad (112)$$

$$\Delta_{ab} = [C_a, G_b] = M^2 \delta_{ab} \quad (113)$$

From (53) one deduces

$$\tilde{H} = \int d^3x \left(\frac{1}{2} \pi_i^a \pi_{ia} + \frac{1}{4} F_{ij}^a F_a^{ij} + \frac{1}{2} M^2 A_i^a A_a^i - \frac{1}{2M^2} (D^i \pi_{ia}) (D^j \pi_j^a) \right) \quad (114)$$

We observe that the role of independent variables z^Δ and $\bar{p}_\Delta$ in the general theory is played here by A_1^a respectively π_a^1 . The canonical Hamiltonian can be rewritten as

$$H = \int d^3x \left(\frac{1}{2} \pi_i^a \pi_{ia} + \frac{1}{4} F_{ij}^a F_a^{ij} - \frac{1}{2M^2} (D^i \pi_{ia}) (D^j \pi_j^a) + \frac{1}{2} M^2 A_i^a A_a^i + \frac{1}{2M^2} C_a C^a \right) \quad (115)$$

We conclude that

$$\frac{\partial^2 H}{\partial C_a \partial C_b} \Big|_{C=0} = \frac{1}{M^2} \delta^{ab} = \Delta^{ab} \quad (116)$$

$$\frac{\partial^k H}{\partial C_{m_1} \dots \partial C_{m_k}} \Big|_{C=0} = 0 \quad , \quad \text{for any } k > 2 \quad (117)$$

The variables Q^a have the form

$$Q^a = \frac{1}{M^2} \delta^{ab} \left(-D^i \pi_{i b} + M^2 A_b^{(0)} \right) \quad (118)$$

Using (117) and (118) one obtains

$$f^a(C) = \frac{1}{M^2} \delta^{ab} C_b \quad (119)$$

Thus, we are under the conditions of Section 4, case (i). It follows that

$$Z\Psi = \int DA_\mu^a \det(M \delta_{ab}) \exp \left(i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{M^2}{2} A_\mu^a A_a^\mu \right) \right) \quad (120)$$

The Hamiltonian form of the path integral in terms of the independent variables will be, accordingly to (80)

$$Z\Psi = \int DA_i^a D\pi_a^i \exp \left(i \int d^4x \left(\dot{A}_i^a \pi_a^i - \frac{1}{2} \pi_a^i \pi_{i a} - \frac{1}{4} F_{ij}^a F_a^{ij} - \frac{1}{2} M^2 A_i^a A_a^i + \frac{1}{2M^2} (D^i \pi_{i a}) (D^j \pi_j^a) \right) \right) \quad (121)$$

Expressions (120) and (121) coincide with the ones given in [3]. We note that in this subsection, as well as in the next one, we haven't taken explicitly into account the distributional character of the constraint functions for the sake of notation simplicity.

5.2. The Dirac field

The starting point is the action

$$S_0[\Psi, \bar{\Psi}] = \int d^4x \left(\frac{i}{2} \bar{\Psi}_\alpha (\gamma^\mu)_\beta^\alpha \partial_\mu \Psi^\beta - \frac{i}{2} (\partial_\mu \bar{\Psi}_\alpha) (\gamma^\mu)_\beta^\alpha \Psi^\beta - m \bar{\Psi}_\alpha \Psi^\alpha \right) \quad (122)$$

The canonical momenta will be given by

$$\pi_\alpha = \frac{i}{2} \bar{\Psi}_\beta (\gamma^0)_\alpha^\beta \quad (123)$$

$$\bar{\pi}^\alpha = \frac{i}{2} (\gamma^0)_\beta^\alpha \Psi^\beta \quad (124)$$

so, we obtain the primary constraints

$$\pi_\alpha - \frac{i}{2} \bar{\Psi}_\beta (\gamma^0)_\alpha^\beta = 0 \quad (125)$$

$$\bar{\pi}^\alpha - \frac{i}{2} (\gamma^0)_\beta^\alpha \Psi^\beta = 0 \quad (126)$$

In deriving (123) and (124) we have used the right derivative. The canonical Hamiltonian reads

$$H = \int d^3x \left(-\frac{i}{2} \bar{\Psi}_\alpha (\gamma^j)_\beta^\alpha \partial_j \Psi^\beta + \frac{i}{2} (\partial_j \bar{\Psi}_\alpha) (\gamma^j)_\beta^\alpha \Psi^\beta + m \bar{\Psi}_\alpha \Psi^\alpha \right) \quad (127)$$

There are no further constraints; the constraint functions are second-class. Taking

$$G_\alpha = \pi_\alpha - \frac{i}{2} \bar{\Psi}_\beta (\gamma^0)_\alpha^\beta \quad (128)$$

$$C^\alpha = \bar{\pi}^\alpha - \frac{i}{2} (\gamma^0)_\beta^\alpha \Psi^\beta \quad (129)$$

we observe that both (128) and (129) are first-class among themselves

$$[G_\alpha, G_\beta] = 0 \quad (130)$$

$$[C^\alpha, C^\beta] = 0 \quad (131)$$

It follows that

$$[C^\alpha, G_\beta] = i (\gamma^0)_\beta^\alpha \quad (132)$$

We are exactly under the conditions of Section 4, case (ii). Then, the Lagrangian form of the path integral will be given by

$$Z_K = \int D\bar{\Psi}_\alpha D\Psi^\alpha \det(\gamma^0) \exp \left(i \int d^4x \left(\frac{i}{2} \bar{\Psi}_\alpha (\gamma^\mu)_\beta^\alpha \partial_\mu \Psi^\beta - \frac{i}{2} (\partial_\mu \bar{\Psi}_\alpha) (\gamma^\mu)_\beta^\alpha \Psi^\beta - m \bar{\Psi}_\alpha \Psi^\alpha \right) \right) \quad (133)$$

Usually, the factor $\det(\gamma^0)$ doesn't appear in textbooks of Quantum Field Theory because

$$\det(\gamma^0) = 1 \quad (134)$$

in any representation of the Dirac matrices. Previously, we denoted the gauge-fixing fermion by K . In order to write down the path integral in terms of the independent variables, we perform the following transformations

$$Q^\alpha = i (\gamma^0)_\beta^\alpha \bar{\pi}^\beta + \frac{1}{2} \Psi^\alpha \quad (135)$$

$$P_\alpha = \pi_\alpha - \frac{i}{2} \bar{\Psi}_\beta (\gamma^0)_\alpha^\beta \quad (136)$$

$$z^\alpha = -i (\gamma^0)_\beta^\alpha \bar{\pi}^\beta + \frac{1}{2} \Psi^\alpha \quad (137)$$

$$\bar{p}_\alpha = \pi_\alpha + \frac{i}{2} \bar{\Psi}_\beta (\gamma^0)_\alpha^\beta \quad (138)$$

In terms of the new variables, (127) takes the form

$$H = \int d^3x \left(-\frac{1}{2} \bar{p}_\alpha (\gamma^0)_\beta^\alpha (\gamma^j)_\lambda^\beta \partial_j z^\lambda + \frac{1}{2} (\partial_j \bar{p}_\alpha) (\gamma^0)_\beta^\alpha (\gamma^j)_\lambda^\beta z^\lambda - i m \bar{p}_\alpha (\gamma^0)_\beta^\alpha z^\beta \right) \quad (139)$$

Taking into account (53), one has

$$\tilde{H} = H \quad (140)$$

In the end we get

$$Z_K = \int D\bar{p}_\alpha Dz^\alpha \exp \left(i \int d^4x \left(\bar{p}_\alpha z^\alpha + \frac{1}{2} \bar{p}_\alpha (\gamma^0)_\beta^\alpha (\gamma^j)_\lambda^\beta \partial_j z^\lambda - \frac{1}{2} (\partial_j \bar{p}_\alpha) (\gamma^0)_\beta^\alpha (\gamma^j)_\lambda^\beta z^\lambda + \right. \right. \\ \left. \left. + i m \bar{p}_\alpha (\gamma^0)_\beta^\alpha z^\beta \right) \right) . \quad (141)$$

One reobtains easily (133) from (141) if one computes z^α and $\bar{p}_\alpha$ with the aid of (137) and (138) when the relations $Q^\alpha = 0$ and $P_\alpha = 0$ hold. One then has

$$z^\alpha = \Psi^\alpha , \quad (142)$$

$$\bar{p}_\alpha = i \bar{\Psi}_\beta (\gamma^0)_\alpha^\beta . \quad (143)$$

Due to the Grassmannian character of the variables, it results that

$$D\bar{p}_\alpha Dz^\alpha = \det(\gamma^0) D\bar{\Psi}_\alpha D\Psi^\alpha . \quad (144)$$

Substituting (142)-(144) in (141) one recovers (133). This agrees with the fact that Ψ^α and $\bar{\Psi}_\alpha$ are canonically conjugated pairs on the reduced phase-space, as deduced in the case of canonical quantization methods.

5.3. A simple model

The example to be considered below exhibits a simple model having the canonical Hamiltonian more than quadratic in the functions C_a . We begin with the Lagrangian action

$$S_0[q] = \int dt \left(-\frac{1}{2!} (q^1)^2 - \frac{1}{3!} (q^1)^3 - \frac{1}{4!} (q^1)^4 + L_0(q^i, \dot{q}^i) \right) , \quad (145)$$

where the index i is valued from 2 to n and $L_0(q^i, \dot{q}^i)$ is regular. The variables $q^1, q^2, \dots, q^n$ are bosonic. The canonical momenta have the form

$$p_1 = 0 , \quad (146)$$

$$p_i = \frac{\partial L_0(q, \dot{q})}{\partial \dot{q}^i} . \quad (147)$$

The canonical Hamiltonian reads

$$H = \frac{1}{2!} (q^1)^2 + \frac{1}{3!} (q^1)^3 + \frac{1}{4!} (q^1)^4 + g(q^i, p_i) , \quad (148)$$

with $g(q^i, p_i)$ the Hamiltonian corresponding to $L_0(q^i, \dot{q}^i)$. It is simply to see that (146) is the only primary constraint. The consistency condition on this constraint implies the secondary constraint

$$-q^1 = 0 . \quad (149)$$

There are no further constraints. The above constraint functions are second-class. We take

$$G = p_1 , \quad (150)$$

$$C = -q^1 . \quad (151)$$

With the aid of (53) we get

$$\tilde{H} = g(q^i, p_i) . \quad (152)$$

The variables Q^a and P_a from the general theory are

$$Q = q^1 , \quad (153)$$

$$P = p_1 . \quad (154)$$

It is simply to see from (62) that

$$f(q^1) = q^1 \left(1 + \frac{1}{3} q^1 + \frac{1}{12} (q^1)^2 \right)^{\frac{1}{2}} . \quad (155)$$

We are again under the conditions of Section 4, case (i), but $f(q^1)$ does not satisfy (101). The Lagrangian path integral reads

$$Z_\Psi = \int Dq \mu(q^1) \exp \left(i \int dt \left(-\frac{1}{2!} (q^1)^2 - \frac{1}{3!} (q^1)^3 - \frac{1}{4!} (q^1)^4 + L_0(q^i, \dot{q}^i) \right) \right) , \quad (156)$$

where

$$\mu(q^1) = i \frac{\partial}{\partial q^1} \left((q^1)^2 + \frac{1}{3} (q^1)^3 + \frac{1}{12} (q^1)^4 \right)^{\frac{1}{2}} . \quad (157)$$

In terms of the independent variables the path integral takes the simple form

$$Z_\Psi = \int Dp_i Dq^i \exp \left(i \int dt (\dot{q}^i p_i - g(q^i, p_i)) \right) . \quad (158)$$

The local nontrivial measure (157) is due to the fact that

$$\left. \frac{\partial^3 H}{\partial (q^1)^3} \right|_{q^1=0} \neq 0 , \quad \left. \frac{\partial^4 H}{\partial (q^1)^4} \right|_{q^1=0} \neq 0 . \quad (159)$$

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