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ON THE QUANTUM RADIATION FROM MIRRORS MOVING SIDEWAYS

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ABSTRACT

Radiation from moving mirrors, a pure quantum effect, is well understood for mirrors that advance or recede. Here the simplest complementary scenarios are explored, where the mirror is not displaced: half-spaces shifting parallel to their surface, and spheres rotating around a diameter. Perfect mirrors do not then radiate at all; hence this preliminary study (for a scalar field) is confined to order $(1 - 1/n^2)^2$ for media with a constant (nondispersive) refractive index $n \approx 1$; and also to leading order in (mirror velocity $\equiv \mathbf{v}$)/ c . The (photon-pair) emission probabilities, the radiative reaction force, and the radiated power are determined for a half-space with arbitrary $\mathbf{v}$. For simple-harmonic $\mathbf{v}$ the single-photon spectrum and angular distribution are given explicitly. The drag exerted on a fixed half-space by a moving one vanishes for constant $\mathbf{v}$, though no reason from any symmetry is apparent. If $a\Omega \gg 1$, then a sphere of radius a oscillating harmonically at frequency Ω radiates power at the same rate per unit area of its surface as would a half-space with the same velocity; but if $a\Omega \ll 1$, the power is less by a factor $(a\Omega)^6$, a measure of the difference between effectively infinite and finite-size mirrors. An appendix evaluates the finite-temperature reaction and drag forces at constant $\mathbf{v}$. Remarkably, this drag on a fixed half-space is independent of its distance from the moving one.

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1. INTRODUCTION

1.1. Background and motivation

The radiation emitted by moving mirrors is a pure quantum phenomenon. For a plane mirror moving perpendicularly to its surface the theory is by now fairly well understood; the literature can be traced, for instance, from references [1], [2]. Perfectly reflecting materials present a serious challenge to the theorist, because inside them the canonical fields are not defined, whence perpendicular displacements of such surfaces cannot be dealt with by standard canonical quantum field theory; the attendant difficulties were clearly explained by Moore in the very first paper on the subject [3]. Imperfect reflectors however are quite amenable to relatively standard and elementary methods, allowing perfect reflection to be dealt with as a limiting case, where the reflectivity tends to unity at the end of the calculation [1], [2]. To discuss such perpendicular motion it has become traditional to adopt minimalist models featuring a scalar field in just one dimension (1D), which do indeed suffice to exhibit the most crucial points. (Notable exceptions are the 3D approaches of Ford and Vilenkin [4] and of Maia Neto [5].)

Here we aim to explore the complementary processes of radiation by mirrors moving sideways, so that the region of space they occupy never changes. We call such motions “lateral”; the mirror-induced radiation they entail we call “lateral

MIR” (by a slight abuse of language), and shall consider two examples: a half-space moving parallel to its bounding surface, and a sphere rotating around a diameter. It must be stated at the outset that such MIR is too weak, by many orders of magnitude, to be measured with presently available techniques. We study it because we believe that it is interesting in principle, and that some of the requisite ideas may turn out to have applications elsewhere (eg to the drag exerted by a moving on a nearby stationary half-space, as discussed in section 4 below).

The main difference between lateral and normal motions is that *perfect* mirrors moving laterally do not radiate into the adjacent vacuum at all, because the matching conditions on the fields depend only on the geometry of the surface, while perfect reflection excludes the fields from the interior, ie from the only region where the motion might affect them. Hence lateral displacements are free of the difficulties uncovered by Moore [3]; moreover, the essentials of such radiation can be studied, at least qualitatively, even for very weakly reflecting media, in what we call the Born approximation. That is what we shall do in this paper, leaving open the question how lateral MIR varies with increasing reflectivity, and how emission into the vacuum¹ peaks and eventually vanishes as the refractive index n rises.

For this initial orientation we adopt the simplest possible model. There can be no lateral motion in 1D; hence we work in 3D, but with a scalar field, and with essentially the same nondispersive medium as in an earlier paper [1] which

we shall cite as BE. We refer to BE for a full discussion of the physics and of the basic equations of the model. The “dielectric” response is described by a constant $\varepsilon = n^2$. The price for the simplicity afforded by constant n includes the need for a frequency cutoff $1/\alpha$; it turns out that renormalization terms do but the irreversibly radiated power does not depend on α . We maintain also some other simplifications in BE, by treating the velocity $\mathbf{v}(t)$ of the mirror as a prescribed classical function, and by disregarding the zero-point energy of the quantized field. One consequence is that we shall see only part of the mirror’s self-mass. How to lift these restrictions is explored in [2], while ways to enforce a prescribed velocity are discussed in [6]. Lastly, we continue to work only to leading order in $\mathbf{v}/c$, where $\mathbf{v}$ is the the velocity of the mirror²; and the system is supposed to be at zero temperature, except for an aside in Appendix B.

However, we simplify well beyond BE by adopting, throughout, the Born approximation already motivated above: we work only to leading order in the combination $(1 - 1/n^2)$, which enters the Hamiltonian naturally. The expectation values of forces, energies, and radiated power are all quadratic in this combination³.

1.2. Preview

The rest of this paper is laid out as follows. Section 2 recalls or adapts, from BE, the Hamiltonian and some other basics of the model, and spells out the relative simplicities of the Born approximation. Section 3, our weightiest, deals

with a single half-space. Section 3.1 allows for arbitrary velocity $\mathbf{v}(t)$, deriving the emission probabilities, the total emitted energy, the radiated power, and the radiative reaction force. For the special case of simple-harmonic $\mathbf{v}(t)$, section 3.2 then uses the Golden Rule to rederive, more directly and much more simply, the total power, as well as its distribution in frequency and direction. Section 3.3 discusses the distribution of the reaction force over the material, mainly with a view to gathering clues (which turn out to be inconclusive) as to just where the MIR actually comes from. Section 4 determines the drag experienced by a fixed half-space due to a similar half-space separated from it by a distance a , and moving as envisaged in section 3.1. (In the Born approximation, the emission from and the reaction force experienced by the moving mirror are the same as if it were alone.) To evaluate the Green’s functions that give the forces as functionals of $\mathbf{v}(t)$ is not altogether trivial, though the general result for two half-spaces (section 4) turns out to deliver also the result for just one (section 3); the technicalities of these calculations are relegated to Appendix A.

In order to discover how MIR is affected by changing from an infinitely extended to a finite mirror, section 5 determines the radiation from a sphere oscillating harmonically about a fixed diameter, again using the Golden Rule. The total power is found, and also its distribution in frequency and direction. For large spheres or high frequencies (wavelengths much smaller than the radius), the sphere behaves essentially like a half-space, in the sense that, on either, surface elements having the same velocity radiate the same power per unit area. By

contrast, for small spheres or low frequencies the expression for the total power carries an extra factor (radius/wavelength)⁶ : it appears that reduction from a large to a small mirror (or equivalently from high to low frequency for a mirror of fixed finite size) reduces the power by six orders of magnitude.

Section 6.1 reviews our conclusions, and complements the present preview by doubling as a retrospective directory to the main technical results. Section 6.2 considers, briefly and tentatively, to what extent the results from our model afford plausible guidance about MIR from reflectors with more realistic optical properties. Finally, Appendix B sketches the general theory at finite temperature, ie for systems immersed in black-body radiation. This excursion is motivated mainly by observing that both the reaction force in section 3.1 and the drag force in section 4 lack components proportional to $\mathbf{v}$, obligatorily by Lorentz invariance in the first case, but for no apparent reason in the second. At finite temperature there are such contributions, though the underlying Green's functions are so complicated that we do not press them for any further information.

2. THE MODEL

2.1. The equations of the model

For the reflecting material we adopt the straightforward 3D generalization of the 1D scalar model discussed rather fully in BE, and simply quote the results we shall need⁴. Operators in Hilbert space wear hats.

With media at rest, the field equation reads

$$\left\{ \varepsilon(\mathbf{r}) \frac{\partial^2}{\partial t^2} - \nabla^2 \right\} \hat{\phi} = 0, \quad (2.1)$$

where the response function ε is equal to the constant n^2 inside and to 1 outside the material. (We use ε when wishing to emphasize the position- and later the time-dependence of the response, and n^2 otherwise.) Across surfaces, $\hat{\phi}$ and its normal derivative are continuous. One can think of $\partial \hat{\phi} / \partial t$ and of $\nabla \hat{\phi}$ as loosely analogous to the electromagnetic $\mathcal{E}$ and $\mathcal{B}$ fields. The force per unit volume exerted by the field on the material is given by

$$\hat{\mathbf{f}} = -(1 - 1/n^2) \nabla^2 \hat{\phi} \nabla \hat{\phi}. \quad (2.2)$$

Finally, the field canonically conjugate to $\hat{\phi}$ is

$$\hat{\Pi} \equiv \varepsilon \partial \hat{\phi} / \partial t, \quad (2.3)$$

and the Hamiltonian density reads

$$\hat{\mathcal{H}}_{rad} = \frac{1}{2} \left\{ \hat{\Pi}^2 / \varepsilon(\mathbf{r}) + (\nabla \hat{\phi})^2 \right\}. \quad (2.4)$$

*With moving media*⁵, the dielectric function generally varies with time, because at different times the medium generally occupies different regions of space.

Then one has

$$\hat{H}(t) = \hat{H}_{rad}(t) + \hat{H}_{int}(t) = \int d^3\mathbf{r} \hat{\mathcal{H}} \equiv \int d^3\mathbf{r} \{ \hat{\mathcal{H}}_{rad}(t) + \hat{\mathcal{H}}_{int}(t) \}, \quad (2.5)$$

$$\hat{\mathcal{H}}_{rad}(t, \mathbf{r}) \equiv \frac{1}{2} \left\{ \hat{\Pi}^2 / \varepsilon(\mathbf{r}, t) + (\nabla \hat{\phi})^2 \right\}, \quad \hat{\mathcal{H}}_{int}(t, \mathbf{r}) \equiv -[1 - 1/\varepsilon(\mathbf{r}, t)] \mathbf{v}(t) \cdot \hat{\Pi} \nabla \hat{\phi}. \quad (2.6)$$

up to corrections, which we neglect, of higher order in v/c . The technical problems characteristic of normal motions [1] arise from the crucial interplay between amplitudes stemming from the time-variations of $\hat{H}_{rad}(t)$ and of $\hat{H}_{int}(t)$. For the lateral motions considered in the present paper, ε is constant in time⁶, whence $\hat{H}_{rad}$ too becomes a constant, and transitions are induced exclusively by $\hat{H}_{int}(t)$, remaining time-dependent solely through its factor $\mathbf{v}(t)$.

The field is quantized through the standard equal-time commutation rule

$$[\hat{\phi}(\mathbf{r}, t), \hat{\Pi}(\mathbf{r}', t)] = i\hbar \delta(\mathbf{r} - \mathbf{r}'), \quad (2.7)$$

irrespective of any movement of the medium. For mirrors at rest we expand $\hat{\phi}$ and $\hat{\Pi}$ in terms of the appropriate normal-mode solutions of (2.1), labelled symbolically by a single index λ , and normed so that the expansion coefficients become the usual annihilation and creator operators $\hat{a}_\lambda^+$, $\hat{a}_\lambda$ (with $[\hat{a}_\lambda^+, \hat{a}_{\lambda'}] = \delta_{\lambda\lambda'}$

and $[\hat{a}_\lambda, \hat{a}_{\lambda'}] = 0$), diagonalizing the Hamiltonian

$$\hat{H}_{rad} = \sum_\lambda \omega_\lambda (\hat{a}_\lambda^+ \hat{a}_\lambda + 1/2). \quad (2.8)$$

In fact we shall work in the Schrödinger picture, whence the arguments t disappear from (2.7).

Here we should explicate, following BE, that by “photons” we always mean excitations of the Hamiltonian $\hat{H}_{rad}$ appropriate to the system with the mirrors in their present positions and at rest. (Although important in principle, for purely lateral motions both these provisos of course lapse in practice, as is evident from the comments on eq (2.6) above.) Correspondingly, by “the radiated energy $E(t)$ ” we shall mean the expectation value of $\hat{H}_{rad}$ in the state vector at time t (less the zero-point energy).

Since $\hat{H}_{int}$ is quadratic in the fields, mirror-induced photons are evidently emitted in pairs; loosely speaking therefore MIR is (“multimode”) squeezed, though here we shall not study the correlations that this implies. In our shorthand notation, the two-photon states are normed according to

$$\begin{aligned} |\lambda\lambda'\rangle &= |\lambda'\lambda\rangle \equiv \hat{a}_\lambda^+ \hat{a}_{\lambda'}^+ |0\rangle, \\ \langle\mu\mu' | \lambda\lambda'\rangle &= \delta_{\mu\lambda} \delta_{\mu'\lambda'} + \delta_{\mu\lambda'} \delta_{\mu'\lambda}, \end{aligned} \quad (2.9)$$

where $|0\rangle$ is the no-photon (“vacuum”) state. Sums over complete sets of two-photon states then read $\frac{1}{2} \sum_{\lambda\lambda'} \dots$, with the factor 1/2 to avoid double-counting.

Our first task is to solve the Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \{\hat{H}_{rad} + \hat{H}_{int}(t)\} |\psi(t)\rangle, \quad (2.10)$$

$$|\psi(-\infty)\rangle = |0\rangle. \quad (2.11)$$

To leading order in v , we look for the solution in the form

$$|\psi(t)\rangle \simeq |0\rangle + \frac{1}{2} \sum_{\lambda\lambda'} c_{\lambda\lambda'}(t) \exp[-i(\omega_\lambda + \omega_{\lambda'})t] |\lambda\lambda'\rangle, \quad (2.12)$$

with the requisite physical information carried by the coefficients $c_{\lambda\lambda'}$. For brevity we shall generally write ω_λ as ω , and $\omega_{\lambda'}$ as ω' .

2.2. The Born approximation

As explained in section 1, we aim for the present to study lateral MIR only to leading order in the parameter

$$(1 - 1/n^2) \simeq (n^2 - 1) \simeq 2(n - 1), \quad (2.13)$$

calling this the Born approximation. It simplifies the calculations enormously, because $\hat{H}_{int}$ itself is already of this order, so that everywhere else one must set $n = 1$, reducing all the normal modes to plane waves. Suppose for instance that

the material occupies the half-space $z \geq 0$, with

$$\varepsilon(\mathbf{r}) = \theta(-z) + \theta(z)n^2 \quad (\text{half-space}), \quad (2.14)$$

where θ is the Heaviside step-function. Then the Born approximation relieves one of the need to write different expressions for modes incident from the left and from the right (whether below or above the critical angle for total internal reflection), apart from choosing the appropriate sign for K_z in the exponents. In other words, all we need is elementary perturbation theory to leading order in $\hat{H}_{int}$, based on the standard Schrödinger-picture plane-wave expansions for unbounded space:

$$\hat{\phi}(\mathbf{r}) = \int d^3K \frac{1}{4\pi^{3/2}} \frac{1}{\omega^{1/2}} \hat{a}_{\mathbf{K}} \exp(i\mathbf{K} \cdot \mathbf{r}) + Hc, \quad (2.15)$$

$$\hat{\Pi}(\mathbf{r}) = -i \int d^3K \frac{1}{4\pi^{3/2}} \omega^{1/2} \hat{a}_{\mathbf{K}} \exp(i\mathbf{K} \cdot \mathbf{r}) + Hc, \quad (2.16)$$

where $\omega \equiv |\mathbf{K}|$, and Hc stands for Hermitean conjugate. Accordingly, we have

$$\lambda \rightarrow \mathbf{K}, \quad \delta_{\lambda\lambda'} \rightarrow \delta(\mathbf{K} - \mathbf{K}'), \quad \sum_{\lambda} \dots \rightarrow \int d^3K \dots, \quad (2.17)$$

and in (2.10) we replace

$$\hat{H}_{rad} \rightarrow \hat{H}_o \equiv \frac{1}{2} \int d^3r \{\hat{\Pi}^2 + (\nabla \hat{\phi})^2\} = \int d^3K \omega \{\hat{a}_{\mathbf{K}}^\dagger \hat{a}_{\mathbf{K}} + 1/2\} \quad (\text{Born approximation}). \quad (2.18)$$

3. HALF-SPACE

3.1. General theory

Here one must solve the Schrödinger equation (2.10, 2.11) with $\hat{H}_{int}$ given by (2.5) and (2.14). Elementary time-dependent perturbation theory yields the solution in the form (2.12) with the emission amplitudes

$$c_{\lambda\lambda'}(t) = -i \int_{-\infty}^t dt' \langle \lambda\lambda' | \hat{H}_{int}(t') | 0 \rangle \exp[i(\omega + \omega')t']. \quad (3.1)$$

We define

$$\begin{aligned} \mathbf{K} &\equiv (\mathbf{k}, q), & K_z &\equiv q, & (K_x, K_y) &\equiv \mathbf{k}, & |K| &\equiv \omega, \\ q &= \omega \cos \vartheta, & (k_x, k_y) &= \omega \sin \vartheta (\cos \varphi, \sin \varphi), \end{aligned} \quad (3.2)$$

and recall that the mirror velocity $\mathbf{v}$ is parallel to the surface, ie to the xy -plane.

Then one finds straightforwardly that

$$\langle \lambda\lambda' | \hat{H}_{int}(t) | 0 \rangle = -i \frac{(1 - 1/n^2)}{4\pi} \frac{\delta(\mathbf{k} + \mathbf{k}')}{\sqrt{\omega\omega'}} \frac{(\omega - \omega')}{(q + q')} \mathbf{v}(t) \cdot \mathbf{k}, \quad (3.3)$$

where, as throughout the present paper, we set

$$\omega \equiv \sqrt{k^2 + q^2}, \quad \omega' \equiv \sqrt{k'^2 + q'^2}. \quad (3.4)$$

In (3.3) we have taken advantage of the delta function to replace a factor $(\omega \mathbf{k}' + \omega' \mathbf{k})$ by $-(\omega - \omega') \mathbf{k}$; the factor $-i/(q + q')$ stems from the singular integral (generalized function) $\int_0^\infty dz \exp[i(q + q')z]$. (Any component of this integral proportional to $\delta(q \pm q')$ can be dropped here, because $|q| = |q'|$ jointly with $k = k'$ entails $\omega = \omega'$, whence the factor $(\omega - \omega')$ eliminates such contributions.) Thus we obtain

$$c_{\lambda\lambda'}(t) = -\frac{(1 - 1/n^2)}{4\pi} \frac{\delta(\mathbf{k} + \mathbf{k}')}{\sqrt{\omega\omega'}} \frac{(\omega - \omega')}{(q + q')} \mathbf{k} \cdot \mathbf{J}_{\omega\omega'}(t), \quad (3.5)$$

where

$$\begin{aligned} \mathbf{J}_{\omega\omega'}(t) &\equiv \int_{-\infty}^t dt' \mathbf{v}(t') \exp[i(\omega + \omega')t'] \\ &= \exp[i(\omega + \omega')t] \int_0^\infty d\tau \mathbf{v}(t - \tau) \exp[-i(\omega + \omega')\tau]. \end{aligned} \quad (3.6)$$

The spectrum and intensity of lateral MIRE are governed by the transition probabilities $|\langle \lambda\lambda' | \psi(t) \rangle|^2 = |c_{\lambda\lambda'}(t)|^2$, so that (3.5) is our central result. That these probabilities are unchanged by $(q, q') \rightarrow (-q, -q')$ is of course peculiar to the Born approximation: to higher orders in $(1 - 1/n^2)$ emission out of and into the material half-space would cease to be identical.

The delta function in $c_{\lambda\lambda'}$ stems directly from translational invariance parallel to the surface, and in expressions involving the intensities one appeals to the standard equivalence

$$\delta(\mathbf{k} + \mathbf{k}') \delta(\mathbf{k} + \mathbf{k}') = \delta(\mathbf{k} + \mathbf{k}') \int \frac{d^2 r}{(2\pi)^2} \exp[i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}] = \delta(\mathbf{k} + \mathbf{k}') \int \frac{d^2 r}{(2\pi)^2}$$

in order to replace

$$\delta^2(\mathbf{k} + \mathbf{k}') \rightarrow \delta(\mathbf{k} + \mathbf{k}') \times A/(2\pi)^2, \quad (3.7)$$

where A is the (effectively infinite) area of the surface. Physical significance attaches only to energy, power, or force *per unit area*, and we shall write such quantities as, for instance, E/A .

The mean radiated energy is given by

$$E(t) \equiv \langle \psi(t) | \hat{H}_0 | \psi(t) \rangle - \langle 0 | \hat{H}_0 | 0 \rangle = \frac{1}{2} \sum_{\lambda\lambda'} |c_{\lambda\lambda'}(t)|^2 (\omega + \omega'). \quad (3.8)$$

With

$$Z \equiv (1 - 1/n^2)^2 / 64\pi^3, \quad (3.9)$$

substitution from (3.5) yields

$$E(t)/A = \frac{Z}{2\pi} \int d^2k \int \int dq dq' \frac{(\omega - \omega')^2 (\omega + \omega')}{\omega \omega' (q + q')^2} | \mathbf{k} \cdot \mathbf{J}_{\omega\omega'}(t) |^2 \exp[-\alpha(\omega + \omega')]. \quad (3.10)$$

The exponential is a cutoff factor. On occasion we may omit such factors for brevity, but shall display them if it is appropriate to stress that they are necessary for convergence. Wherever possible we take the limit $\alpha \rightarrow 0$, and in any case consider only the regime where α is much smaller than any time scale characterizing $\mathbf{v}(t)$.

Similarly we determine the mean total force experienced by the material,

$$\mathbf{F}(t) \equiv \langle \psi(t) | \hat{\mathbf{F}} | \psi(t) \rangle, \quad \hat{\mathbf{F}} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \int_{mat} dz \hat{\mathbf{f}}(\mathbf{r}), \quad (3.11)$$

$$\mathbf{F}(t) = 2 Re \frac{1}{2} \sum_{\lambda\lambda'} \langle 0 | \hat{\mathbf{F}} | \lambda\lambda' \rangle c_{\lambda\lambda'}(t) \exp[-i(\omega + \omega')t], \quad (3.12)$$

where one needs

$$\langle 0 | \hat{\mathbf{F}} | \lambda\lambda' \rangle = -i \frac{(1 - 1/n^2)}{4\pi} \delta(\mathbf{k} + \mathbf{k}') \frac{(\omega^2 - \omega'^2)}{\sqrt{\omega\omega'}} \mathbf{k} \int_{mat} dz \exp[i(q + q')z]. \quad (3.13)$$

Here we have taken advantage of the delta function to replace a factor $(\omega^2 \mathbf{k}' + \omega'^2 \mathbf{k})$ by $-(\omega^2 - \omega'^2) \mathbf{k}$, similarly to the way we reached (3.3) above. The integral $\int_{mat}$ runs over the interior of the material, in this case from 0 to ∞ . Recalling the other comments à propos of (3.3) we get

$$\langle 0 | \hat{\mathbf{F}} | \lambda\lambda' \rangle = \frac{(1 - 1/n^2)}{4\pi} \delta(\mathbf{k} + \mathbf{k}') \frac{(\omega^2 - \omega'^2)}{\sqrt{\omega\omega'}} \frac{\mathbf{k}}{(q + q')} . \quad (3.14)$$

Comparison with (the complex conjugate of) (3.3) shows that [1]

$$-i(\omega + \omega') \langle 0 | \hat{H}_{int} | \lambda\lambda' \rangle = \mathbf{v} \cdot \langle 0 | \hat{\mathbf{F}} | \lambda\lambda' \rangle. \quad (3.15)$$

Substitution into (3.12) from (3.5) and (3.14) followed by some rearrangement

leads to

$$\mathbf{F}(t)/A = \int_0^\infty d\tau G(\tau) \mathbf{v}(t - \tau), \quad (3.16)$$

with the Green's function defined by

$$G(\tau) \equiv -Z \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \cos[(\omega + \omega')\tau] \\ \times \frac{(\omega - \omega')^2 (\omega + \omega')}{\omega \omega' (q + q')^2} \exp[-\alpha(\omega + \omega')]. \quad (3.17)$$

The evaluation of G is dealt with in Appendix A; eventually one obtains

$$G(\tau) = -Z \int_0^\infty d\Omega \cos(\Omega\tau) \exp(-\alpha\Omega) \frac{2}{45} \Omega^4 = -\frac{2Z}{45} \left(\frac{\partial}{\partial \tau} \right)^4 \frac{\alpha}{\tau^2 + \alpha^2}. \quad (3.18)$$

We substitute into (3.16), define $\mathbf{v}^{(s)} \equiv d^s \mathbf{v} / dt^s$, and integrate by parts to find

$$\mathbf{F}(t)/A = -\frac{2Z}{45} \left\{ \frac{2}{\alpha^3} \mathbf{v}^{(1)}(t) - \frac{1}{\alpha} \mathbf{v}^{(3)}(t) + \int_0^\infty d\tau \mathbf{v}^{(4)}(t - \tau) \frac{\alpha}{\tau^2 + \alpha^2} \right\}. \quad (3.19)$$

If (as we assume) the velocity varies only on time scales much longer than α , then in the last term one can implement

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty d\tau \mathbf{v}^{(4)}(t - \tau) \frac{\alpha}{\tau^2 + \alpha^2} = \mathbf{v}^{(4)}(t) \int_0^\infty d\tau \frac{\alpha}{\tau^2 + \alpha^2} = \frac{\pi}{2} \mathbf{v}^{(4)}(t). \quad (3.20)$$

Thus we obtain the end result

$$\mathbf{F}(t)/A \simeq -\frac{(1 - 1/n^2)^2}{2^5 \cdot 3^2 \cdot 5 \cdot \pi^3} \frac{\hbar}{c^4} \left\{ \frac{2}{\alpha^3} \mathbf{v}^{(1)}(t) - \frac{1}{\alpha} \mathbf{v}^{(3)}(t) + \frac{\pi}{2} \mathbf{v}^{(4)}(t) \right\}, \quad (3.21)$$

where $\hbar$ and c have been re-instated. Since only the last term on the right changes sign under time-reversal, it alone must prove responsible for true radiation, ie for the irreversible transfer of energy from mirror to field. The reaction on a single isolated mirror can have no components proportional to $\mathbf{v}(t)$, because they would persist even when $\mathbf{v}$ is constant, contrary to Lorentz (as to Galilean) invariance.

It is worth deriving (3.21) in a different way, with the slightly different status of an incipient asymptotic series in inverse powers of the time scales characteristic of $\mathbf{v}(t)$. To this end one reverts to (3.16) and writes $\mathbf{v}(t - \tau)$ as a Taylor series around t , obtaining the moment expansion⁷

$$\mathbf{F}(t)/A = \sum \frac{(-1)^n}{n!} \mathbf{v}^{(n)}(t) \mathcal{M}_n, \quad \mathcal{M}_n \equiv \int_0^\infty d\tau \tau^n G(\tau). \quad (3.22)$$

Here the sum is understood to run over $n = 0, 1, 2, \dots$, up to some maximum value where the integrals $\mathcal{M}_n$ start to diverge.

Related to E and $\mathbf{F}$ but more interesting probably than either is the mean radiated power W , defined and given by

$$W(t) \equiv dE/dt = -\mathbf{v} \cdot \mathbf{F}(t). \quad (3.23)$$

The best way to check the second of these relations is to obtain dE/dt directly from (3.10), noting that

$$\frac{d}{dt} |\mathbf{k} \cdot \mathbf{J}_{\omega\omega'}(t)|^2 = 2 [\mathbf{k} \cdot \mathbf{v}(t)] \mathbf{k} \cdot \int_0^\infty d\tau \mathbf{v}(t-\tau) \cos[(\omega + \omega')\tau]. \quad (3.24)$$

Indeed the most informative view of the radiated energy is as the integral of (3.23) rearranged as

$$\begin{aligned} \frac{dE/dt}{A} = \frac{(1 - 1/n^2)^2}{2^5 \cdot 3^2 \cdot 5 \cdot \pi^3} \left\{ \frac{d}{dt} \left[\frac{1}{\alpha^3} \mathbf{v}^2 - \frac{1}{\alpha} \left(\mathbf{v} \cdot \mathbf{v}^{(2)} - \frac{1}{2} \mathbf{v}^{(1)2} \right) + \frac{\pi}{2} \left(\mathbf{v} \cdot \mathbf{v}^{(3)} - \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \right) \right] \right. \\ \left. + \frac{\pi}{2} \mathbf{v}^{(2)2} \right\}. \end{aligned} \quad (3.25)$$

The total energy radiated in motion from rest to rest is unaffected by the time-derivative, being given by

$$E(\infty)/A = \frac{(1 - 1/n^2)^2}{2880\pi^2} \frac{h}{c^4} \int_{-\infty}^{\infty} dt \left(\frac{d^2 \mathbf{v}(t)}{dt^2} \right)^2, \quad (\mathbf{v}(\pm\infty) = 0). \quad (3.26)$$

As anticipated, it stems wholly from the last term in (3.21); and it remains finite in the no-cutoff limit. The other terms in (3.21) are renormalization effects; the first is evidently the contribution of radiative reaction to the inertial self-mass of the mirror⁸, while the second is analogous to the Schott term in the radiation from a classical charged particle (see eg [9]).

3.2. The Golden Rule for a harmonically-oscillating mirror

For simple-harmonic motion, say in the x-direction, with

$$v_x = v_0 \cos(\Omega t), \quad v_y = 0, \quad (3.27)$$

the mean power follows from (3.26), with $\cos^2(\Omega t) \rightarrow 1/2$, and with $\int dt \rightarrow T$ interpreted as the duration of the process, so that $E/T \rightarrow \bar{W}$ (overbars indicate a time average). Alternatively it can be found from first principles through the Golden Rule of time-dependent perturbation theory; this gives the mean (and time-independent) transition probability from $|0\rangle$ to $|\lambda\lambda'\rangle$ as

$$\frac{d}{dt} |c_{\lambda\lambda'}|^2 \equiv \gamma_{\lambda\lambda'} = 2\pi |\langle \lambda\lambda' | \hat{H}_{eff} | 0 \rangle|^2 \delta(\Omega - \omega - \omega'). \quad (3.28)$$

where the matrix element is got from (3.3) by replacing $\mathbf{v}(t) \rightarrow \mathbf{v}_0/2$. Thus, after setting $(\omega^2 - \omega'^2) = (q^2 - q'^2)$ as here one may, we obtain

$$\gamma_{\lambda\lambda'}/A = \frac{Z}{2} v_0^2 \delta(\mathbf{k} + \mathbf{k}') \delta(\Omega - \omega - \omega') \frac{k_x^2 (q - q')^2}{\Omega^2 \omega \omega'}, \quad (3.29)$$

whence

$$\frac{\bar{W}}{A} = \frac{\Omega}{2A} \sum_{\lambda\lambda'} \gamma_{\lambda\lambda'} = \frac{\pi Z}{90} v_0^2 \Omega^4 = \frac{(1 - 1/n^2)^2 h v_0^2 \Omega^4}{5760 \pi^2 c^4}, \quad (3.30)$$

which tallies as it should with the result from (3.26). The sum $\sum_{\lambda\lambda'}$ is evaluated with the methods of Appendix A. (The writer is grateful to Clive North for an

independent check.)

To assess such radiation rates, we reason that typical photons will have energies comparable to $\hbar\Omega$, and estimate the number emitted from an area $(2\pi c/\Omega)^2 \sim (\text{wavelength})^2$, in a time interval $(2\pi/\Omega) \sim (\text{one period})$. One sees that, apart from a rather small numerical coefficient, this number is of order $(v_0/c)^2$, ie small only because the mirror moves (by assumption) far below the speed of light.

Finally, one can determine the single-photon number and power spectra $\Gamma_\lambda \equiv \sum_{\lambda'} \gamma_{\lambda\lambda'}$ and $\bar{W}_\lambda \equiv \omega_\lambda \Gamma_\lambda$ (so that $\bar{W} = (1/2) \sum_\lambda \bar{W}_\lambda$). In terms of the variables defined in (3.2) the accessible phase-space region turns out to be given by

$$\omega/\Omega \equiv \rho \leq 1/(1 + \sin \vartheta), \quad (3.31)$$

and for the fully differential power spectrum one eventually obtains

$$\frac{d\bar{W}/A}{d\omega d\varphi d\cos\vartheta} = Z v_0^2 \frac{\hbar\Omega^3}{c^4} \frac{[(1-\rho)^2 + \rho^2(\cos^2\vartheta - \sin^2\vartheta)]}{\sqrt{(1-\rho)^2 - \rho^2\sin^2\vartheta}} \sin^2\vartheta \cos^2\varphi. \quad (3.32)$$

Note the (of course integrable) singularity on the boundary.

3.3. The force distribution and the sources of MIR

The distribution of the total mean force over the halfspace, ie the expectation value

$$\mathbf{f}(t, z) \equiv \langle \psi(t) | \hat{\mathbf{f}}(\mathbf{r}) | \psi(t) \rangle, \quad (3.33)$$

can be approached from $\mathbf{F} = \int_0^\infty dz \mathbf{f}(z, t)$: one starts by observing that the matrix elements (3.14) of $\hat{\mathbf{F}}$ yield those of $\hat{\mathbf{f}}$ on reversing the integration in (3.13), ie on multiplication by $(q + q')/i$. The calculation is very similar to that of the drag force between two half-spaces⁹, whose results are given in the next section and justified in Appendix A: we skip the details and simply quote the conclusion.

$$\mathbf{f}(t, z)/A = \int_0^\infty d\tau h(\tau, z) \mathbf{v}(t - \tau). \quad (3.34)$$

$$h(\tau, z) = \frac{Z}{15} \frac{\partial}{\partial z} \left\{ \frac{1}{z^3} \frac{\partial}{\partial \tau} \left[\frac{\alpha}{(\tau - z)^2 + \alpha^2} - \frac{\alpha}{(\tau + z)^2 + \alpha^2} \right] + \frac{1}{z^2} \frac{\partial^2}{\partial \tau^2} \left[\frac{\alpha}{(\tau - z)^2 + \alpha^2} + \frac{\alpha}{(\tau + z)^2 + \alpha^2} \right] \right\}. \quad (3.35)$$

Here as in BE, the main reason for studying $\mathbf{f}$ as a function of z is that it seems to be our only source of clues as to where the mirror-induced photons actually come from. In both cases¹⁰, as $\alpha \rightarrow 0$ the singularity structure of $\mathbf{f}(z, t)$ at $z = 0$ becomes quite exceptionally intricate: volume-distributed contributions to $\mathbf{F}$ cannot be separated in any unique way from those truly supported just by the surface, and the operations $\lim_{\alpha \rightarrow 0}$ and $\int_0 dz$ fail to commute. While with fixed finite α it is easy to verify that by integrating (3.35) over z one recovers our result for G and thence for $\mathbf{F}$, this option is closed if the limit is taken first.

Here we do confine ourselves to the no-cutoff limit, which suffices to establish the force density at depths well below α , ie well away from the singular surface region. That in turn suffices to demonstrate explicitly that the total force is not

a pure surface phenomenon. Accordingly we discard from (3.35) terms leading to $\lim_{\alpha \rightarrow \infty} \alpha / [\alpha^2 + (\tau + z)^2] = \pi \delta(\tau + z)$, since both τ and z are nonnegative; and eventually find

$$\mathbf{f}(t, z \gg \alpha)/A = -\frac{\pi Z}{15} \left\{ \frac{3}{z^4} \mathbf{v}^{(1)}(t - z) + \frac{3}{z^3} \mathbf{v}^{(2)}(t - z) + \frac{1}{z^2} \mathbf{v}^{(3)}(t - z) \right\}. \quad (3.36)$$

(As anticipated above, one cannot now recover the total force by integrating this density over z .)

Unfortunately there are two opposite plausibility arguments from (3.36) to the spatial provenance of MIR. On the one hand, the local power density is $-\mathbf{v} \cdot \mathbf{f}$, and it seems reasonable to infer that MIR is emitted where the energy flows from the material into the field; in that case some of the photons must originate from the bulk. This was the view taken in BE. On the other hand, the fact that $\mathbf{f}(t, z)$ features the $\mathbf{v}^{(s)}$ precisely at the retarded time $(t - z)$ might be taken to suggest that at time t we are simply looking at forces exerted on the material by radiation all of which has come precisely from the surface. Pending some other criteria, there appears to be no compelling reason to choose either of these interpretations over the other.

4. THE DRAG EXERTED BY A MOVING ON A STATIONARY HALF-SPACE

Consider two half-spaces, both with the same refractive index, their surfaces parallel and separated by a distance a . When both are at rest, they experience an attractive Casimir force (normal to their surfaces)

$$F_C/A = C(n)\hbar c/a^4, \quad (4.1)$$

where $C(n)$ is a numerical coefficient¹¹.

We take the right-hand half-space to be just the one dealt with in section 3: it extends over $z \geq 0$, and moves rigidly with a prescribed velocity $\mathbf{v}(t)$ parallel to the surface. The left-hand half-space is fixed, and occupies the region $z \leq -a$. The mean reaction force on the right-hand half-space will be written $\mathbf{F}_R$, and its Green's function as G_R . In general $\mathbf{F}_R$ depends on a , and differs from the reaction called $\mathbf{F}$ in section 3, but in the Born approximation they are the same: $\mathbf{F}_R(t, a) \simeq \mathbf{F}_R(t) \simeq \mathbf{F}(t)$. What we wish to study is the mean force $\mathbf{F}_L(t, a)$ that the fields due to the moving half-space exert on the other. Evidently $\mathbf{F}_L$ is given by eqs (3.11-3.13), but with $\int_{mat} dz$ now running from $-\infty$ to $-a$.

It might help to spell out some fairly obvious preliminary observations. First, Lorentz invariance does not now forbid $\mathbf{v}$ -proportional contributions, neither to $\mathbf{F}_L$ nor to $\mathbf{F}_R$; it insists only that they should be equal and opposite. Second,

dimensional considerations suggest that the order of magnitude of dynamical corrections to the Casimir force (whether for perpendicular or for lateral motions) might be obtainable from (4.1) by replacing c with v . Thus one might plausibly expect to find $|\mathbf{F}_L|/A \sim \hbar v/a^4$. If so, then one would need to go on to consider whether the absence of c is fortuitous, or an indication that such effects are truly non-relativistic (even though they might not seem so when reached by our particular route).

Once embarked on the calculation, one arrives straightforwardly at expressions analogous to (3.16, 3.17):

$$\mathbf{F}_L(t, a)/A = \int_0^\infty d\tau G_L(\tau, a) \mathbf{v}(t - \tau), \quad (4.2)$$

$$G_L(\tau, a) = Z \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \cos[(\omega + \omega')\tau + (q + q')a] \\ \times \frac{(\omega - \omega')^2(\omega + \omega')}{\omega\omega'(q + q')^2} \exp[-\alpha(\omega + \omega')]. \quad (4.3)$$

As for a single half-space, lateral $\mathbf{v}$ elicits purely lateral forces: it does not compete with the zero-point effects responsible for the classic Casimir attraction F_C . We note the change of sign relative to G_R , and the identity $G_L(\tau, 0) = -G_R(\tau)$, whence the total force $(\mathbf{F}_L + \mathbf{F}_R)$ on the two half-space vanishes with their separation. Thus, in the Born approximation, $\mathbf{F}_L(t, a)$ yields $\mathbf{F}_R$. But we note also that in general, ie for $a > 0$, $\mathbf{F}_L$ and $\mathbf{F}_R$ are by no means equal and opposite: they are not directly related by Newton's third law, and it would be seriously

misleading to talk of “the drag *between* two half-spaces”. Hence the cumbersome title of this section.

The evaluation of the integral (4.3) is sketched in Appendix A. It proceeds through a double Fourier representation, and eventually one finds

$$G_L(\tau, a) = Z \int_0^\infty d\Omega \exp(-\alpha\Omega) \cos(\Omega\tau) \\ \times \int_{-\infty}^\infty dp \cos(pa) \theta(\Omega^2 - p^2) \frac{1}{30} \Omega(\Omega^2 - p^2), \quad (4.4)$$

whence

$$G_L(\tau, a) = \frac{Z}{15} \left\{ \frac{1}{a^3} \frac{\partial}{\partial \tau} \left[\frac{\alpha}{(\tau - a)^2 + \alpha^2} - \frac{\alpha}{(\tau + a)^2 + \alpha^2} \right] \right. \\ \left. + \frac{1}{a^2} \frac{\partial^2}{\partial \tau^2} \left[\frac{\alpha}{(\tau - a)^2 + \alpha^2} + \frac{\alpha}{(\tau + a)^2 + \alpha^2} \right] \right\}. \quad (4.5)$$

While it is easy to see that as $a \rightarrow 0$ eq(4.4) reduces to (the negative of) (3.18) as it should, some care is needed with the limit before (4.5) will reproduce (3.21).

In fact we are not much interested in distances comparable to α ; this is fortunate because the limits $a \rightarrow 0$ and $\alpha \rightarrow 0$ do not commute, making $\mathbf{F}_L$ at such small separations rather awkward to study. (The difficulty is similar to what section 3.3 found regarding the force density, where it is the operation $\int_0 dz$ that fails to commute with $\alpha \rightarrow 0$.) Hence we concentrate firmly on the regime $a \gg \alpha$, by taking the limit $\alpha \rightarrow 0$ at fixed nonzero a (after which it becomes impossible to recover the identity $G_L(\tau, 0) = -G_R(\tau)$). In this limit $\alpha/[(\tau + a)^2 + \alpha^2] \rightarrow \pi\delta(\tau + a)$

vanishes because both τ and α are nonnegative, and one finds

$$G_L(\tau, a \gg \alpha) = \frac{\pi Z}{15} \left\{ \frac{1}{a^3} \delta'(\tau - a) + \frac{1}{a^2} \delta''(\tau - a) \right\}, \quad (4.6)$$

where primes denote differentiation with respect to the argument.

Finally, substitution into (4.2) yields, in ordinary units,

$$\mathbf{F}_L(\tau, a \gg c\alpha)/A = \frac{\pi \hbar Z}{15c} \left\{ \frac{1}{a^3} \mathbf{v}^{(1)}(t - a/c) + \frac{1}{a^2 c} \mathbf{v}^{(2)}(t - a/c) \right\}. \quad (4.7)$$

Accordingly, the plausible expectations sketched at the beginning of this section are not borne out, at least not by our model in the Born approximation and at zero temperature: our $\mathbf{F}_L$ is a pure quantum effect, and has no component proportional to $\mathbf{v}$. (However, Appendix B will find that at finite T such components do occur both in $\mathbf{F}_L$ and in $\mathbf{F}_R$.)

The literature contains three attempts [10], [11], [12] to calculate the drag force, with three results quite different from (4.7) and also from each other. Judged by the comments in section 6.2 below, not all these discrepancies need be unreasonable, since the three approaches model the material in quite different ways.

5. SPHERES

The infinite half-spaces discussed in sections 3 and 4 are of course very artificial, especially in our model with its total disregard of absorption. Hence the present section deals with (the simplest example of) the lateral motion of finite bodies, applying the Golden Rule to find the power radiated by a rigidly rotating sphere, of radius a , with its centre fixed at the origin. MIR is produced only by nonuniform rotation, because otherwise the velocity $\mathbf{v}$ of the material at any point fixed in space is constant, so that

$$\hat{H}_{int}(t) \equiv -(1 - 1/n^2) \int d^3r \theta(a - r) \mathbf{v}(t, \mathbf{r}) \cdot \hat{\Pi}(\mathbf{r}) \nabla \hat{\phi}(\mathbf{r}) \quad (5.1)$$

becomes time-independent and thus incapable of inducing transitions.

We consider only simple-harmonic oscillations around a fixed axis with angular velocity $\mathbf{\Lambda}$, where

$$\mathbf{\Lambda}(t) = \mathbf{\Lambda}_0 \cos(\Omega t), \quad \mathbf{v}(t, \mathbf{r}) = \mathbf{\Lambda}(t) \times \mathbf{r}. \quad (5.2)$$

The polar axis is chosen along $\mathbf{\Lambda}_0$. We shall need the form-factor Q defined by

$$\begin{aligned} (4\pi a^3/3) \cdot Q(\kappa a) &\equiv \int d^3r \theta(a - r) \exp(-i\kappa \cdot \mathbf{r}), \\ Q(\xi) &= (3/\xi) j_1(\xi) = (3/\xi^3)(\sin \xi - \xi \cos \xi), \end{aligned} \quad (5.3)$$

so that

$$Q(\xi \ll 1) \sim \{1 - \xi^2/10 + \dots\}, \quad Q(\xi \gg 1) \sim -(3/\xi^2) \cos \xi. \quad (5.4)$$

Proceeding straightforwardly, one finds

$$\langle \lambda \lambda' | \hat{H}_{int}(t) | 0 \rangle = -i \frac{(1 - 1/n^2)(\omega - \omega')}{12\pi^2} \frac{\Lambda(t) \cdot \mathbf{K} \times \mathbf{K}'}{\sqrt{\omega \omega'}} \frac{a^5 F'(\kappa a)}{\kappa a}, \quad (5.5)$$

$$\kappa \equiv |\mathbf{K} + \mathbf{K}'|. \quad (5.6)$$

In (5.5), the Golden Rule replaces $\hat{H}_{int}(t) \rightarrow \hat{H}_{eff}$ and $\Lambda(t) \rightarrow \Lambda_0/2$. Then the time-averaged total radiated power is given by

$$\begin{aligned} \bar{W} &= \Omega \cdot 2\pi \cdot \frac{1}{2} \sum_{\lambda \lambda'} |\langle \lambda \lambda' | \hat{H}_{eff} | 0 \rangle|^2 \delta(\Omega - \omega - \omega') \\ &= \frac{(1 - 1/n^2)^2 \Omega \Lambda_0^2 a^{10}}{2^6 \cdot 3^2 \cdot \pi^3} \mathcal{L}, \end{aligned} \quad (5.7)$$

$$\mathcal{L} \equiv \int \int d^3 K d^3 K' \frac{(\omega - \omega')^2}{\omega \omega'} (\mathbf{K} \times \mathbf{K}')^2 \delta(\Omega - \omega - \omega') \left\{ \frac{Q'(\kappa a)}{\kappa a} \right\}^2. \quad (5.8)$$

$\mathcal{L}$ can be simplified by changing the integration variables to $a(\mathbf{K} \pm \mathbf{K}')$, and then working away until eventually one obtains

$$\mathcal{L} = \frac{2^2 \cdot \pi^2 \cdot \Omega^9}{3^2 \cdot 5} \int_0^1 d\xi \xi^6 (1 - \xi^2) \left\{ \frac{Q'(\Omega a \xi)}{\Omega a \xi} \right\}^2. \quad (5.9)$$

At high frequencies or equivalently for a large sphere, ie if $\Omega a/c \gg 1$, the integral is dominated by the region where $\Omega a \xi \gg 1$, and we approximate by replacing Q by its high- ξ asymptotic form as given in (5.4). One finds

$$\bar{W} \simeq \frac{(1 - 1/n^2)^2 \Lambda_0^2 (\Omega a)^4}{2^4 \cdot 3^3 \cdot 5 \cdot \pi} \quad (\Omega a) \gg 1. \quad (5.10)$$

As might have been expected, this is just what follows on assuming that every surface element radiates according to (3.30), ie as if it were part of the boundary of a half-space moving with the appropriate local speed governed by (5.2). On that view one would write

$$\bar{W} \simeq \frac{(1 - 1/n^2)^2}{5760\pi^2} \Omega^4 \int_0^\pi 2\pi a^2 d\vartheta \sin(\vartheta) \Lambda_0^2 a^2 \sin^2(\vartheta), \quad (5.11)$$

which does indeed reproduce (5.10).

In the opposite (and the only realistic) low-frequency or equivalently small-sphere regime where $\Omega a/c \ll 1$, $\mathcal{L}$ is dominated by the region where $\Omega a \xi \ll 1$, and we approximate by adopting the low- ξ asymptotic form given in (5.4). Then $\{Q'(\Omega a)/\Omega a\}^2 \simeq 1/25$, and one finds

$$\bar{W} \simeq \frac{(1 - 1/n^2)^2 \Lambda_0^2 (\Omega a)^{10}}{2^3 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot \pi} \quad (\Omega a) \ll 1. \quad (5.12)$$

The denominator equals $5103000\pi \sim 5 \times 10^6 \pi$, producing what must be one of the smallest numerical coefficients on record.

The main purpose in deriving (5.12) was to illustrate how MIR is affected by changing the source from an arbitrarily large to a finite-size body. Since we have just seen that an infinite half-space and a sphere with $(\Omega a) \gg 1$ radiate in essentially the same way, the sensible comparison, as to orders of magnitude, is between the average power from a given sphere with large and small values of Ωa . From (5.10) and (5.12) we see that

$$\frac{\bar{W}_{\Omega a \ll 1}}{\bar{W}_{\Omega a \gg 1}} \simeq \frac{2}{3^{3.5^2.7}} (\Omega a)^6. \quad (5.13)$$

This is a reduction by six orders of magnitude.

As a final illustration we determine the single-photon distribution when $\Omega a \ll 1$. With the doubly-differential transition rate $\gamma_{\lambda\lambda'}$ given by (3.28), the single-photon rate is $dN/d^3K = \Gamma_\lambda = \sum_{\lambda'} \gamma_{\lambda\lambda'}$, and the single-photon power distribution is $\bar{W}_\lambda \equiv \omega \Gamma_\lambda$. Then the low-frequency form of $(Q'/\kappa a)$ eventually yields the distribution

$$\frac{d\bar{W}}{d\omega d\varphi d \cos \vartheta} = \frac{(1 - 1/n^2)^2}{2^3 \cdot 3^3 \cdot 5^2 \cdot \pi^2} \Lambda_6^2 a (\Omega a)^9 f(\rho) \sin^2 \vartheta, \quad (5.14)$$

$$f(\rho) \equiv \rho^4 (1 - \rho)^3 (2\rho - 1)^2, \quad \rho \equiv \omega/\Omega. \quad (5.15)$$

Integrating over frequency (from 0 to Ω) and over solid angle, and dividing by 2, one reproduces the already known result (5.12). Notice the zero in the spectrum at $\omega = \omega' = \Omega/2$.

6. CONCLUSIONS AND COMMENTS

6.1. Summary

The introduction and section 6.2 below take pains to assess the idealizations of our model generally and of the Born approximation within it. Here, by contrast, we aim to highlight the main technical conclusions, stressing that they follow explicitly from explicitly stated premisses, and that they are intended primarily to exhibit some of the basic physics of lateral MIR, rather than to develop formalisms covering a great variety of conceivable scenarios.

For a single half-space moving arbitrarily (subject only to $v/c \ll 1$), the result from which all else follows is the probability amplitude $c_{\lambda\lambda'}(t)$ for the creation of a specified photon pair, given by eq (3.5). It yields the total reaction force $\mathbf{F}(t)$ in (3.21), as a convolution of $\mathbf{v}(t - \tau)$ with a Green's function $G(\tau)$; and thence the total radiated power $W(t) = -\mathbf{v} \cdot \mathbf{F}(t)$ in (3.25). The high-order derivatives of $\mathbf{v}$ featured in $\mathbf{F}$, with their cutoff-dependent coefficients, are common in MIR generally. They entail serious (and unresolved) difficulties if allowed into a classical equation of motion for the mirror; and it is not known whether dispersion can always and fully eliminate them. On the other hand, for a mirror constrained to move harmonically, the time-averaged power (3.30) is problem-free; its fully differential single-photon distribution is given in (3.32).

In the Born approximation the radiation from a moving half-space is unaffected by proximity to another fixed half-space; but the fixed mirror then

experiences a drag force $\mathbf{F}_L$, eq (4.7). Remarkably, $\mathbf{F}_L$ vanishes for constant $\mathbf{v}$, though for $\mathbf{F}_L$, unlike $\mathbf{F}$, this is by no means obligatory under Lorentz invariance.

The phase-space integrations required by the calculations for half-space(s) are nontrivial; they are carried out in Appendix A, exploiting an apparently fortuitous congruence with the relativistic kinematics of two- particle decays.

A half-space is peculiar in that it lacks any parameters characterizing mirror size. Section 5 considered spheres of radius a oscillating harmonically at frequency Ω , the simplest yet hopefully typical examples of MIR from finite-size mirrors. If $a\Omega \gg 1$, a sphere radiates the same power per unit area of its surface as would a half-space with the same tangential velocity; by contrast, if $a\Omega \ll 1$, then the power radiated by the same sphere is less by a factor of order $(\Omega a)^6$. These rates are given by (5.10) and (5.12) respectively. The differential single-photon distribution for $a\Omega \ll 1$ is given in (5.14), (5.15). Characteristically of the Born approximation, it vanishes when the two photons have equal energies, a zero which for a half-space is cancelled only by a coincidence.

These results apply at zero temperature, ie when the initial state is the vacuum. Appendix B derives integral representations for the Green's functions for the radiative-reaction and for the drag forces when the initial state is the mixed state describing black-body radiation at finite temperature. Then the forces persist even when $\mathbf{v}$ is constant, and they are evaluated only for that case. They are equal and opposite, and are given by eqs (B.16)-(B.18). Surprisingly, this part of

the drag force is independent of the separation.

6.2. How representative are such calculations?

We consider briefly how representative our results are likely to prove of lateral MIR from more realistically modelled mirrors. It is obvious that pending actual calculations such comments remain largely speculative.

(i) As explained in the introduction, no lateral MIR can emerge from perfect reflectors; hence in most respects Born approximations (appropriate to weak reflection) should remain plausible qualitative guides to reflectors in general. However, to strictly leading order in $(1 - 1/n^2)$ one obviously cannot describe multiple-reflection and/or resonant effects like those recently studied (for perfect mirrors moving normally) by Law [13].

(ii) Far the most restrictive of our simplifications is the total disregard of dispersion, and of the absorption linked to it, through the dispersion relations, by causality. Then the reflectivity is governed, independently of the frequency, by just the refractive index, ie by a single, constant, and *dimensionless* parameter n . When dimensional parameters become indispensable, we are reduced to the cutoff α , whose physics is distinctly woolly. By contrast, real materials become transparent above some high enough frequency, say Ω_0 , and their response always features dimensional parameter(s) loosely akin to our α . Such parameters can be primarily dispersive, like a characteristic frequency in a dielectric or the plasma frequency in a metal; or primarily absorptive, like a relaxation time gov-

erning some Ohmic conductivity. They stem from dynamical degrees of freedom additional to those of the field, and calculations of MIR are unlikely to prove intelligible unless they take account of these explicitly¹². Meanwhile, about the conjectures that follow the present writer has fewer doubts in the case of materials where Ω_0 is controlled by a dispersive rather than by an absorptive parameter. For normal motion in 1D, a dispersive model has been worked through explicitly elsewhere [2].

(iii) For the mirror's self-mass and for other reversible effects (like the first two terms in the expression (3.21) for the reaction force) the consequences of ignoring dispersion are certainly drastic [2], especially as regards any cutoff dependence. But the situation is less clear with respect to true (irreversible) MIR and the related reactions (eg the third term in (3.21)); or with respect to velocity-proportional components of the finite-temperature drag force between two half-spaces (as discussed in Appendix B).

(iv) When mirror velocities change little over times of order $1/\Omega_0$, the writer suspects (and model calculations suggest [2]) that the irreversible radiation they emit depends mainly on their zero-frequency reflectivity. If that is so, then, merely with some appropriate substitutions for the factors $(1 - 1/n^2)$, our expressions for spectra, angular distributions, and intensities might furnish a rough but reasonable description of MIR emerging from quite general types of materials. Similar optimism might then apply as regards the dissipative part of the radiative reaction experienced by a single mirror.

(v) On the other hand, the drag exerted by a moving on a nearby stationary half-space is likely to be governed, especially at small distances, by physics that for realistic materials is quite different from and much more complicated than any that we have allowed for. Energy and momentum can then be transferred nonradiatively, ie directly between the extra degrees of freedom responsible for the dispersion (eg by nonrelativistic Coulomb forces between the electron gas in one metallic mirror and the electron gas in the other). At close quarters nonradiative transfers will swamp the purely radiative mechanisms that are the only ones incorporated into our nondispersive model¹³. That is why it would be premature to compare in any detail our sections 4 and B.3 with references [10], [11], and [12].

APPENDIX A: THE GREEN'S FUNCTIONS $G(\tau)$ AND $G_L(\tau, a)$

Section 4 explained that

$$G_L(\tau, 0) = -G(\tau), \quad (\text{A.1})$$

so that one need evaluate only $G_L(\tau, a)$. But we start by rewriting both functions as Fourier transforms, in the way that best exhibits the connection between them. The cutoff is irrelevant to these transforms, and will be disregarded.

Thus, setting

$$G(\tau) \equiv -Zg(\tau), \quad G_L(\tau, a) \equiv Zg_L(\tau, a), \quad (\text{A.2})$$

and recalling the definition (3.4) of ω, ω' , we see from (3.17) that

$$g(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dq' \int_0^{\infty} dk k^3 \frac{(\omega - \omega')^2 (\omega + \omega')}{\omega \omega' (q + q')^2} \cos[(\omega + \omega')\tau], \quad (\text{A.3})$$

where the order of integration has been reversed by hindsight. Then, since ω, ω' are nonnegative, we may write

$$g(\tau) = \int_0^{\infty} d\Omega \cos(\Omega\tau) \tilde{g}(\Omega), \quad (\text{A.4})$$

$$\dot{g}(\Omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dq' \int_0^{\infty} dk k^3 \frac{(\omega - \omega')^2 (\omega + \omega')}{\omega \omega' (q + q')^2} \delta(\Omega - \omega - \omega'). \quad (\text{A.5})$$

Similarly, from (4.3),

$$g_L(\tau, a) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dq' \int_0^{\infty} dk k^3 \frac{(\omega - \omega')^2 (\omega + \omega')}{\omega \omega' (q + q')^2} \cos[(\omega + \omega')\tau + (q + q')a]. \quad (\text{A.6})$$

Expressing the cosine as $\cos[(\omega + \omega')\tau] \cos[(q + q')a] - \sin[(\omega + \omega')\tau] \sin[(q + q')a]$, one sees that the product of the sines vanishes under $\int \int dq dq' \dots$, because this

part of the integrand is odd under $(q, q') \rightarrow (-q, -q')$. Accordingly we may write

$$g_L(\tau, a) = \int_0^{\infty} d\Omega \int_{-\infty}^{\infty} dp \cos(\Omega\tau) \cos(pa) \tilde{g}_L(\Omega, p), \quad (\text{A.7})$$

$$\tilde{g}_L(\Omega, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dq' \frac{\delta(p - q - q')}{(q + q')^2} \int_0^{\infty} dk k^3 \frac{(\omega - \omega')^2 (\omega + \omega')}{\omega \omega'} \delta(\Omega - \omega - \omega'). \quad (\text{A.8})$$

Comparison between (A.3) and (A.6) and between (A.5) and (A.8) then manifests both of the equivalent relations

$$g(\tau) = g_L(\tau, 0), \quad \dot{g}(\Omega) = \int_{-\infty}^{\infty} dp \tilde{g}_L(\Omega, p). \quad (\text{A.9})$$

The crucial trick for evaluating (A.8) is to integrate over k first, and to recognize from the delta function that in effect one is faced with the familiar kinematics of the decay from rest of a parent particle of mass Ω into two daughters of (rest) masses $|q|, |q'|$, having equal and opposite momenta each of magnitude k^* , where $k = k^*(\Omega, p)$ is the positive root of $\omega + \omega' = \Omega$. Hence $\omega = (\Omega^2 + q^2 - q'^2)/2\Omega$, $\omega' = (\Omega^2 + q'^2 - q^2)/2\Omega$, and

$$(|q| + |q'|) \leq \Omega, \quad (\text{A.10})$$

$$\omega - \omega' = (q^2 - q'^2)/\Omega,$$

$$k^{*2} = \{\Omega^2 - (|q| - |q'|)^2\} \{\Omega^2 - (|q| + |q'|)^2\} / 4\Omega^2.$$

The formula

$$\delta(\Omega - \omega - \omega') = \frac{\delta(k - k^*)}{|\partial(\omega + \omega')/\partial k|} = \frac{\delta(k - k^*)}{|k/\omega + k/\omega'|} = \delta(k - k^*) \frac{\omega\omega'}{k(\omega + \omega')} \quad (\text{A.11})$$

allows extensive cancellations, leading to

$$\dot{g}_L(\Omega, p) = (1/p^2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dq' \delta(p - q - q') k^{*2} (\omega - \omega')^2. \quad (\text{A.12})$$

Finally, on substituting from (A.10) and changing variables to $R \equiv (q + q')$,

$S \equiv (q - q')$, the integral factors:

$$\begin{aligned} \dot{g}_L(\Omega, p) &= (1/8p^2\Omega^4) \int_{-\Omega}^{\Omega} dR \delta(p - R) R^2 (\Omega^2 - R^2) \int_{-\Omega}^{\Omega} dS S^2 (\Omega^2 - S^2) \\ \dot{g}_L(\Omega, p) &= \theta(\Omega^2 - p^2) \Omega (\Omega^2 - p^2) / 30, \end{aligned} \quad (\text{A.13})$$

which is the result quoted in (4.4). The integration in (A.9) then yields

$$\dot{g}(\Omega) = \int_{-\Omega}^{\Omega} dp \dot{g}_L(\Omega, p) = 2\Omega^4/45, \quad (\text{A.14})$$

as quoted in (3.18). Of course one can also obtain $\dot{g}(\Omega)$ directly by evaluating (A.5), with help from the same relativistic kinematics as above.

APPENDIX B: FINITE TEMPERATURE

B.1. Preliminaries

In our model, to say that the system is at some finite temperature T means simply that it is pervaded by black-body radiation at T ; in other words, initially the field is in the mixed state described by the thermal-equilibrium density operator¹⁴ $\hat{\rho} = (1/\mathcal{Z}) \exp(-\beta \hat{H}_0)$, with the mean photon occupation numbers given by Planck's formula

$$\langle \hat{n}_\lambda \rangle \equiv \nu_\lambda \equiv 1/[\exp(\beta\omega_\lambda) - 1], \quad \beta \equiv 1/k_B T. \quad (\text{B.1})$$

So far we have considered only the limiting case $T = 0$, ie the initial pure state $|0\rangle$, with $\nu_\lambda = 0$. Then, as a consequence of Lorentz invariance, the reaction force on a single half-space has no component proportional to $\mathbf{v}$. In other words, writing such components generically as $-\eta\mathbf{v}$, so that (cf eq (3.21))

$$\mathbf{F}(t)/A = -\eta\mathbf{v}(t) + (\text{terms linear in derivatives of } \mathbf{v}), \quad (\text{B.2})$$

we have found that $\eta(T = 0) = 0$. By contrast, the black-body radiation selects a preferred inertial frame (with respect to which it is isotropic), whence Lorentz invariance is lost, and it will turn out presently that $\eta(T > 0) \neq 0$, (as for normal motion [16]). Unfortunately, to evaluate the full Green's functions at finite T is

very hard even in the Born approximation, so much so that the coefficients like η are all that we shall determine explicitly.

At this point the reader should be warned that with regard to the drag force $-\eta \mathbf{v}$ our nondispersive/nonabsorptive model is at its least representative, lacking as it does any means to equilibrate the radiation with the material. This is best seen by envisaging a half-space with constant velocity. Any real material would eventually entrain the radiation, except perhaps in some boundary layer, and then it would cease to be obvious in what (if any) sense one could assign the same temperature to the moving halfspace as to adjacent vacuum or to another nearby but stationary half-space. It is only in the Born approximation that one can sidestep such questions, since they affect only terms of higher than leading order in the interaction between radiation and matter. In fact we shall find that the Born approximation at finite T does exhibit some interesting and unexpected features, whence we do continue with our study somewhat further (in spite of the cautionary remarks at the end of section 6.2).

B.2. Half-space

For a half-space, dimensional plus certain other rather general arguments afford some advance glimpses of η . With the parameters available, there is no loss of generality in writing

$$\eta = (1/\hbar^3 c^4 \beta^4) P(\xi), \quad \xi \equiv \alpha/\hbar\beta, \quad (\text{B.3})$$

where only the one function P of the dimensionless combination ξ remains to be determined. At very low or very high temperature we can be more specific.

At high temperatures ($\xi \gg 1$), we expect η to be proportional to $1/\beta$, because any realizable history $\mathbf{v}(t)$ will effectively admit normal modes only below some kinematically-determined finite frequency, call it ω_0 ; and at $\beta\omega_0 \ll 1$, the mean energies of these modes will become proportional to $1/\beta$. In other words, at high enough temperatures the relevant modes are governed by the equipartition law, and the Planck distribution is adequately approximated by the Rayleigh-Jeans distribution. Given proportionality to $1/\beta$, correct dimensions then dictate

$$\eta \sim 1/\beta \alpha^3 c^4 \quad (\xi \gg 1). \quad (\text{B.4})$$

Being free of Planck's constant, this is presumably a classical effect that could be calculated with no recourse to quantum mechanics at all, ie without any reference to MIR in the first place.

At low temperatures ($\xi \ll 1$), the significant normal modes should be limited by the Planck distribution (B.1) rather than by the kinematics, whence one expects proportionality to the black-body energy density, ie to $1/\beta^4$. Dimensionality then dictates

$$\eta \sim 1/\hbar^3 c^4 \beta^4 \quad (\xi \ll 1), \quad (\text{B.5})$$

independent of the cutoff.

To assess likely orders of magnitude of ξ , we first express the cutoff in terms of a frequency f_0 as $\alpha \equiv 1/2\pi f_0$, in the expectation that f_0 will be comparable to the atomic unit of frequency $f_a \approx 6.58 \times 10^{15}$ Hz. Then one has

$$\xi = (f_a/f_0)(T/300)0.954 \times 10^{-3},$$

so that only $\xi \leq 1$ seems to be realistic.

When it comes to actual calculation, one requires the thermal generalization of section 3. We start with the transitions from an initial state with given occupancies $\{n_\lambda\}$. Since $\hat{H}_{int}$ is quadratic in the fields, only two occupancies change: write these as $n_\lambda, n_{\lambda'}$, or n, n' for short. The selection rules admit only the following matrix elements:

$$\langle n+1, n'+1 | \hat{H}_{int} | n, n' \rangle = X \sqrt{(n+1)(n'+1)} \frac{\delta(\mathbf{k}+\mathbf{k}')(\omega-\omega')}{q+q'} \quad , \quad (\text{B.6})$$

$$\langle n-1, n'-1 | \hat{H}_{int} | n, n' \rangle = -X \sqrt{nn'} \frac{\delta(\mathbf{k}+\mathbf{k}')(\omega-\omega')}{q+q'} \quad , \quad (\text{B.7})$$

$$\langle n+1, n'-1 | \hat{H}_{int} | n, n' \rangle = X \sqrt{(n+1)n'} \frac{\delta(\mathbf{k}-\mathbf{k}')(\omega+\omega')}{q-q'} \quad , \quad (\text{B.8})$$

where $X \equiv -i(1-1/n^2)\mathbf{v}\cdot\mathbf{k}/4\pi\sqrt{\omega\omega'}$. We call the total energy emitted from this initial state $E(t | \{n\})$, and calculate it along the lines of section 3, taking account of the energies $(\pm(\omega+\omega'))$ and $(\omega-\omega')$ respectively emitted in the different types of transitions listed in (B.6)-(B.8).

In order to describe what happens at finite T understood as in section B.1, one simply replaces the n by their thermal averages ν as in (B.1):

$$E(t | \{n\}) \rightarrow E(t | \{\nu\}) \equiv E(t | T). \quad (\text{B.9})$$

In fact the quantity $E(t | T)$ thus obtained is not the energy but the Helmholtz free energy transferred from mirror to field, which is what we require, since at finite temperature forces are given by derivatives of the free energy¹⁵. Accordingly one proceeds from E to dE/dt , and thence to $\mathbf{F}$. Alternatively one can determine $\mathbf{F}$ as the expectation value of $\hat{\mathbf{F}}$ in the mixed state that evolves from the initial thermal-equilibrium state under the action of $\hat{H}_{int}$. Either method requires the integrals $\mathbf{J}_{\omega\omega'}$ defined in (3.6), together with the analogous $\mathbf{J}_{\omega,-\omega'}$. We skip the details, and quote only the expression one eventually finds for the Green's function, ie for the thermal generalization of (3.17):

$$\begin{aligned} G(\tau | T) = & -Z \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \exp[-\alpha(\omega+\omega')] \frac{1}{\omega\omega'} \\ & \times \left\{ \left(\frac{\omega-\omega'}{q+q'} \right)^2 (\omega+\omega')(\nu+\nu'+1) \cos[(\omega+\omega')\tau] \right. \\ & \left. - \left(\frac{\omega+\omega'}{q-q'} \right)^2 (\omega-\omega')(\nu-\nu') \cos[(\omega-\omega')\tau] \right\}. \quad (\text{B.10}) \end{aligned}$$

with ω, ω' from (3.4) as usual, and $\nu \equiv \nu(\omega)$, $\nu' \equiv \nu(\omega')$. The first term inside the braces stems from two-photon emission and two-photon absorption (see eqs

(B.6), (B.7)), while the second term stems from scattering processes where one photon is emitted and one absorbed¹⁶ (see eq (B.8)). The second term has no analogue at zero temperature.

It would require quite disproportionate effort to approximate G in any generally useful manner, and we settle for its zero-moment (see eq(3.22)):

$$\begin{aligned}\eta &= -\mathcal{M}_0(T) \equiv -\int_0^\infty d\tau G(\tau | T) \\ &= Z\pi \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \exp[-\alpha(\omega + \omega')] \frac{1}{\omega\omega'} \\ &\times \left\{ \left(\frac{\omega - \omega'}{q + q'} \right)^2 (\omega + \omega')(\nu + \nu' + 1) \delta(\omega + \omega') - \left(\frac{\omega + \omega'}{q - q'} \right)^2 (\omega - \omega')(\nu - \nu') \delta(\omega - \omega') \right\}.\end{aligned}\quad (\text{B.11})$$

Since $(\omega - \omega')/(q + q')$ remains finite as $(q + q') \rightarrow 0$, the first term vanishes by virtue of $(\omega + \omega')\delta(\omega + \omega') = 0$; thus only scattering processes contribute to η . In the second term we rearrange

$$\frac{(\omega + \omega')^2(\omega - \omega')}{(q - q')^2} = \left[\frac{(\omega^2 - \omega'^2)^2}{(q^2 - q'^2)^2} \right] \left[\frac{(q + q')^2}{\omega - \omega'} \right] = \frac{(q + q')^2}{\omega - \omega'} \quad , \quad (\text{B.12})$$

whence

$$\eta = -Z\pi \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \exp[-\alpha(\omega + \omega')] \frac{(q + q')^2}{\omega\omega'} \left(\frac{\nu - \nu'}{\omega - \omega'} \right) \delta(\omega - \omega'). \quad (\text{B.13})$$

By l'Hôpital's rule the penultimate factor of the integrand becomes¹⁷

$$\lim_{\omega \rightarrow \omega'} \frac{\nu - \nu'}{\omega - \omega'} = \frac{d\nu}{d\omega} = -\frac{\beta}{[2 \sinh(\beta\omega/2)]^2} \quad , \quad (\text{B.14})$$

and the delta function can be used to perform the integration over q' :

$$\delta(\omega - \omega') = \delta(q - q') / |\partial\omega/\partial q| = (\omega / |q|) \delta(q - q'), \quad (\text{B.15})$$

another component proportional to $\delta(q + q')$ being eliminated by the factor $(q + q')^2$. We substitute into (B.13), change the remaining integration variables from (k, q) to (ω, k) , find that the k -integration becomes trivial, and obtain

$$\eta(T) = \frac{16\pi Z}{\beta^4} \int_0^\infty dx \frac{x^4}{\sinh^2 x} \exp[-(4\alpha/\beta)x]. \quad (\text{B.16})$$

At high T , the integrand is dominated by small x , where $x^4/\sinh^2 x \sim x^2$.

With this approximation one finds

$$\eta(T) \simeq \pi Z/2\alpha^3\beta = (1 - 1/n^2)^2/128\pi^2\alpha^3c^4\beta \quad (\alpha/\hbar\beta \gg 1), \quad (\text{B.17})$$

appropriately to the classical regime envisaged in (B.4).

At low T , the exponential in (B.16) can be approximated by unity, and one

finds

$$\eta(T) \simeq 8\pi^5 Z / 15\beta^4 = (1 - 1/n^2)^2 \pi^2 / 120 \beta^4 \hbar^3 c^4 \quad (\alpha/\hbar\beta \ll 1), \quad (\text{B.18})$$

appropriately to the quantum regime envisaged in (B.5).

B.3. The drag exerted by a moving on a stationary half-space

For two half-spaces, dimensional analysis yields little, because of the presence of the dimensional parameter a in addition to α , $\hbar$, and c . An adaptation of section 4 yields the thermal generalization of the Green's function (4.3):

$$\begin{aligned} G_L(\tau, a | T) = & Z \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \exp[-\alpha(\omega + \omega')] \frac{1}{\omega\omega'} \\ & \times \left\{ \left(\frac{\omega - \omega'}{q + q'} \right)^2 (\omega + \omega')(\nu + \nu' + 1) \cos[(\omega + \omega')\tau + (q + q')a] \right. \\ & \left. - \left(\frac{\omega + \omega'}{q - q'} \right)^2 (\omega - \omega')(\nu - \nu') \cos[(\omega - \omega')\tau + (q - q')a] \right\}. \end{aligned} \quad (\text{B.19})$$

As before we replace the cosines of sums by products of cosines: calculate the zero moment which converts the resultant factors $\cos[(\omega \pm \omega')\tau]$ into $\pi \delta(\omega \pm \omega')$; and find again that only the scattering contribution survives. Thus, for

$$\eta_L(a | T) = - \int_0^\infty d\tau G_L(\tau, a | T) \quad (\text{B.20})$$

we obtain

$$\begin{aligned} \eta_L(a | T) = & Z \pi \int_0^\infty dk k^3 \int_{-\infty}^\infty \int_{-\infty}^\infty dq dq' \exp[-\alpha(\omega + \omega')] \frac{(q + q')^2}{\omega\omega'} \\ & \times \left(\frac{\nu - \nu'}{\omega - \omega'} \right) \delta(\omega - \omega') \cos[(q - q')a]. \end{aligned} \quad (\text{B.21})$$

The delta function enforces $q = q'$ exactly as in section B.2; but here this has the peculiar consequence that the cosine reduces to unity, whence η_L becomes independent of a . Accordingly, comparison with (B.13) yields our end result

$$\eta_L(a | T) = -\eta(T), \quad (\text{B.22})$$

with $\eta(T)$ given by eqs (B.16)-(B.18). That the drag force does not vary with the separation is rather strange, and the writer has found no simple physical argument to make it plausible ahead of the detailed calculation. As explained in section 4, it is certainly not a consequence of Newton's third law.

FOOTNOTES

(1) As regards MIR emitted *into the material*, experience discourages guesses about the perfect-reflection limit ($n \rightarrow \infty$ in this case). The surface tension σ of metals supplies one classic cautionary example. According to the hydrodynamic model, where perfect reflection is realized in the limit of infinite plasma frequency, σ diverges in this limit. By contrast, if a plane boundary is treated as perfectly reflecting from the outset, and the Maxwell field quantized afterwards, then it is well known (eg as a concomitant of theories of the Casimir effect) that σ vanishes.

(2) The material is always taken to move rigidly: every point of a halfspace has the same velocity, and every point of a sphere the same angular velocity. Acceleration stresses and any attendant strains are disregarded, not because they need be unimportant to MIR, but because they depend strongly on the specific mechanism producing the acceleration. By contrast, the effects we do study are universal, in the sense that they depend only on $\mathbf{v}(t)$ per se.

(3) By contrast, the expectation values of the radiated fields are only linear. In the 1D model they have been studied very thoroughly to this order by Salamone [7].

(4) We use natural units, $\hbar = 1 = c$, except occasionally for emphasis.

(5) As already stated in section 1, the material is taken to move rigidly, and with a prescribed velocity $\mathbf{v}$. We shall see that under these conditions the total

radiative-reaction force experienced by a half-space (namely $\mathbf{F}(t) \equiv \int_{mat} d^3r \mathbf{f}(\mathbf{r}, t)$, with $\int_{mat}$ taken over all the material at a given time) proves to be an auxiliary variable useful for structuring the discussion of the radiated power. By contrast, to determine the true elastic *response* of a (necessarily deformable) medium, one would need equations allowing for the finite speed of sound (far below c), and in that context $\mathbf{F}(t)$ would lose much of its significance. For similar reasons we sidestep any discussion of the fluctuations of $\mathbf{F}$. By contrast, with perfect reflectors all of the reaction force is exerted on the surface, and $\mathbf{F}$ plays a far more immediate and more important role (see eg [8]).

(6) This dispenses us, in particular, from recourse to the translation operators $\hat{U}$ so essential to BE.

(7) In Appendix B we shall exploit the zero-moment $\mathcal{M}_0$ in order to calculate the finite-temperature drag force.

(8) There are other contributions to the self-mass, from zero-point energies, as discussed (à propos of a dispersive 1D model) in [2].

(9) In particular, there is a close connection, anything but coincidental, between the Green's functions h here and G_L in eq (4.5) below: $h(\tau, z) = \partial G_L(\tau, z)/\partial z$.

(10) Here in 3D the situation is aggravated by the presence of even higher derivatives.

(11) Our scalar model has $C(n) \simeq (1 - 1/n^2)^2 3/1280\pi^2$ in the Born approximation, and $\lim_{n \rightarrow \infty} C(n) = \pi^2/480$.

(12) For the static 1D Casimir effect with absorptive mirrors, this has been done in a remarkable recent paper by Kupiszewska [14]. In a related context, the Fresnel coefficients for reflection from spatially dispersive materials furnish a now classic cautionary example of how theories go wrong when they try to sweep such extra degrees of freedom under the carpet.

(13) A similar effect is well known from the decay rates $\gamma(a)$ of excited molecules held at fixed distances a from a metallic mirror (see eg [15]). For large a the difference $\gamma(a) - \gamma(\infty)$ is explained by readily calculable radiative effects, but the agreement is spoiled by nonradiative energy transfers when a decreases well below a wavelength.

(14) Without the Born approximation $\hat{H}_0$ would revert to $\hat{H}_{rad}$.

(15) The appropriate interplay between field theory (QED) and thermodynamics has been discussed elsewhere [17], in the related context of radiative level shifts.

(16) As regards the convergence of the normal-mode integrals for these scattering contributions, we note the peculiar loss of the usual effective equivalence between, on the one hand, frequency cutoffs like $\exp[-\alpha(\omega + \omega')]$, and, on the other hand, the so-called point-splitting technique. In our example, the latter would try to control divergences by appropriately assigning a small imaginary part to the time variable τ . But in the second term of (B.10) any such congruence must founder on the simple fact that the cutoff features the sum of the frequencies, while the τ -dependent factor features their difference. Right here the discrepancy

is harmless, but similar scattering contributions in more general problems might need to be handled with caution.

(17) This is the only point where one uses the explicit form of the $\langle \hat{n}_\lambda \rangle$. Their thermal equilibrium values ν_λ could be replaced by different expressions, as long as the initial density matrix is diagonal in the occupation numbers.

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