Weyl Anomalies of Strings in Temporal Gauge *

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Abstract

We consider two-dimensional quantum gravity coupled to matters in the temporal gauge, using the Polyakov path integral. We show that the integration over the metric can be explicitly performed under some plausible assumptions. We also discuss that the critical dimensions in string theory may not be determined in the temporal gauge.

I. INTRODUCTION

For the past ten years, string theory has been intensively studied as a candidate of the unified theory. In the development it has been revealed that string theory has too many classical vacua. Although we expect that only one vacuum is selected quantum-mechanically, we can never find the true vacuum if only perturbative approaches are used. Thus, the

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framework beyond perturbation theory is required, and string field theory should be one of
the strong candidates.

Much effort has been devoted to searching for a satisfactory string field theory [1–6]. However, it is proved to be very difficult to construct it, especially for closed strings. Although the light-cone gauge string field theory [1] is consistently formulated, the lack of manifest Lorentz covariance makes it difficult to get an insight into the underlying structure of string theory. Therefore a theory with the manifest Lorentz covariance is desired. Zwiebach has proposed such a theory [6]. At present, more researches seem to be required to get non-perturbative information from his theory.

Recently, a new formulation has been proposed [7] as a second-quantized string theory with \( 0 \leq c \leq 1 \). Let us briefly explain the main features of the \( c = 0 \) case, for simplicity. The field operators of a string are the creation operator \( \Psi(l)^\dagger \) and the annihilation operator \( \Psi(l) \), which creates and annihilates a loop with length \( l \), respectively. These operators satisfy the commutation relation

\[
[\Psi(l), \Psi(l')^\dagger] = \delta(l - l').
\]

The Hamiltonian of this theory is given by

\[
\mathcal{H} = \int dl dl' (l + l') \Psi(l)\Psi(l')^\dagger \Psi(l + l')
+ g \int dl dl' l' \Psi(l + l')^\dagger \Psi(l) \Psi(l')
+ \int dl \rho(l) \Psi(l),
\]

where \( g \) is the string coupling constant, and \( \rho(l) \) is the amplitude for the process in which a loop with length \( l \) vanishes. The vacuum state \( |0\rangle \) is defined as

\[
\Psi(l)|0\rangle = 0.
\]

Then, the amplitude for \( n \) loops can be expressed as

\[
\lim_{D \to \infty} \langle 0| \exp[-D\mathcal{H}] \Psi(l_1)^\dagger \Psi(l_2)^\dagger \cdots \Psi(l_n)^\dagger |0\rangle,
\]

\( \cdot \)
where $D$ is interpreted as the geodesic distance from the incident $n$ loops, which was first introduced in the transfer-matrix formalism of $2D$ quantum gravity based on the dynamical triangulation [8]. These amplitudes are proved to satisfy the Schwinger-Dyson equations [9] in the matrix model. Therefore, this theory reproduces all the known results in the $c = 0$ matrix model.

Since one of the main difficulties in constructing string field theory is to decompose each of all amplitudes into a set of propagators and elementary interaction vertices, we expect that this formulation gives an alternative direction toward a satisfactory critical string field theory.

It was shown that the Hamiltonian for the $c = 0$ case can be constructed directly from the transfer-matrix formalism in the dynamical triangulation [10]. The alternative derivation [11] was also given by the stochastic quantization of the matrix model. It was discussed there that the geodesic distance $D$ can be interpreted as the fictitious time in the stochastic quantization.

To derive this Hamiltonian from the continuum theory based on the Polyakov path integral, the temporal gauge was proposed [12] as a gauge-fixing condition. These authors have almost reproduced the Hamiltonian for the $c = 0$ case.

When we consider the critical string theory, the continuum approach seems to be more tractable than the others. So, introducing matter fields into the system of pure gravity considered in the temporal gauge, we intend to search for such a string field Hamiltonian. For that purpose, we need to estimate the integration over the degrees of freedom of the gravity sector, especially the shift function.

It is the purpose of the paper to demonstrate that we can explicitly perform the path integration over the metric, under some plausible assumptions, for cylinder amplitudes with and without matters. As the first step toward the new direction, it is interesting to consider how the critical dimensions emerges in the temporal gauge. In the Polyakov path integral, most of progress has been made in the conformal gauge, where the meaning of the critical dimensions is clear; the central charge of the matters for which the Weyl anomalies coming
both from gravity and matter sectors cancel out each other. Thus in this case we can ignore the dynamical degrees of freedom of gravity, the Liouville modes. However, in the temporal gauge, it is not clear what corresponds to the Liouville modes. Therefore, it is interesting to investigate the Weyl anomalies in this gauge, which is one of the main subjects in the present paper.

This paper is organized as follows: After we review the temporal gauge [12] in sect. II, we will first consider pure gravity in sect. III. We refine the calculation in the paper in a more systematic manner; in particular, it will be shown that integration over the shift function $k(t, x)$ can be made, which is needed in the next section. In sect. IV, introducing matters into the system considered in sect. III, we will explicitly compute the cylinder amplitude with matters, a propagator of closed string [13]. In sect. V, we will give the discussion based on the calculations in the preceding sections.

II. TEMPORAL GAUGE

In the ADM decomposition, a metric $g_{mn}$ on a two-dimensional surface with the coordinates $\xi^m = (\xi^0, \xi^1) = (t, x)$ is parametrized as

$$
[g_{mn}(\xi)] = \begin{pmatrix}
N(\xi)^2 + h(\xi) k(\xi)^2 & h(\xi) k(\xi) \\
- h(\xi) k(\xi) & h(\xi)
\end{pmatrix},
$$

where $N(\xi)$ is the lapse function, $k(\xi)$ the shift function, and $h(t, x)$ the metric on the time slice at $t$.

The temporal gauge [12] is defined as

$$
N(\xi) = 1, \quad \partial_t h(\xi) = 0.
$$

This gauge condition is consistent with the transfer-matrix formalism initiated in [8]. In fact, the first condition (8) allows us to regard the geodesic distance from the boundary directly as the time coordinate $t$. Furthermore, since in the dynamical triangulation, all the
links of triangles are assumed to have equal length, the loop boundaries are also meshed with equal length, and this fact justifies the second condition (9).

Integrating Eq. (9) and setting \( h = l(t)^2 \), we thus have the following parametrization of the metric in the temporal gauge:

\[
\tilde{g}_{mn}(\xi) = \begin{bmatrix}
1 + l(t)^2 k(t, x)^2 & l(t)^2 k(t, x) \\
l(t)^2 k(t, x) & l(t)^2 \\
\end{bmatrix},
\]

where \( l(t) \) can be interpreted as the loop length on the time slice at \( t \).

In this gauge-fixing condition, there remains the following residual gauge symmetry at each time \( t \):

\[
t \to t' = t, \quad (11)
\]
\[
x \to x' = x - \alpha(t), \quad (12)
\]

under which the metric \( \tilde{g}_{mn} \) in the temporal gauge transforms as

\[
\delta_{\text{res}, l}(t) = 0, \quad (13)
\]
\[
\delta_{\text{res}, k}(t, x) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} k(t, x) \right) \alpha(t). \quad (14)
\]

The generator of this transformation is given by \( v^m_{\text{res}} \partial_m = \alpha \frac{\partial}{\partial x} \).

### III. PURE GRAVITY

In this section, we consider pure gravity in the temporal gauge. In particular, we make an explicit integration over the shift function \( k(\xi) \). The manipulation we develop here will enable us to examine the propagator of a string in the next section.

Consider a worldsheet with the topology of cylinder. We call its two boundaries \( C \) and \( C' \). On these boundaries, we impose boundary conditions on loop length as

\[
l(t = 0) = l, \quad l(t = D) = l',
\]

where we use \( t \in [0, D] \) with \( D \) the geodesic distance between \( C \) and \( C' \), as we explained in introduction. The Polyakov path integral for this amplitude is given by
\[
Z(l', l; D) = \int \frac{Dg_{mn}}{\text{Vol}(\text{diff.})} \exp \left[ -\mu_0 \int d^2 \xi \sqrt{g} \right] \\
\times \delta \left( \int \sqrt{g_{mn}} d\xi^m d\xi^n - l \right) \delta \left( \int \sqrt{g_{mn}} d\xi^m d\xi^n - l' \right) \\
\times \delta \left( \int N(g_{mn}) dt - D \right). 
\] (16)

Since the integrand has the reparametrization invariance:
\[
\xi^m \rightarrow \xi'^m = \xi^m - v^m(\xi),
\] (17)
\[
\delta g_{mn}(\xi) = \nabla_m v_n + \nabla_n v_m,
\] (18)
we should factor out this gauge degrees of freedom and will impose the temporal gauge. In this gauge, \( \delta \left( \int N(g_{mn}) dt - D \right) \) actually means \( t \in [0, D] \), and
\[
\delta \left( \int \sqrt{g_{mn}} d\xi^m d\xi^n - l \right) \rightarrow \delta (l(t = 0) - l),
\] (19)
\[
\delta \left( \int \sqrt{g_{mn}} d\xi^m d\xi^n - l' \right) \rightarrow \delta (l(t = D) - l'),
\] (20)
\[
\exp \left[ -\mu_0 \int d^2 \xi \sqrt{g} \right] \rightarrow \exp \left[ -\mu_0 \int dt l(t) \right],
\] (21)
with \( x \in [0, 1] \).

It is useful to introduce two orthonormal tangent vectors \( e_\perp \) and \( e_\parallel \):
\[
(e^m_\perp) = (1, -k), \quad (e^m_\parallel) = (0, l^{-1}),
\] (22)
which are, respectively, in the normal and tangential directions to time slices, and satisfy the following relations:
\[
e^m_\perp e^n_\perp g_{mn} = e^m_\parallel e^n_\parallel \tilde{g}_{mn} = 1,
\] (23)
\[
e^m_\perp e^n_\parallel \tilde{g}_{mn} = 0,
\] (24)
\[
e^m_\perp e^n_\perp + e^m_\parallel e^n_\parallel = \tilde{g}^{mn}.
\] (25)

The basis \( \{ E^\perp, E^\parallel \} \) of the dual cotangent vectors to \( \{ e_\perp, e_\parallel \} \) is then given by
\[
(E^m_\perp) = (1, 0), \quad (E^m_\parallel) = (kl, l).
\] (26)

For a vector \( V^m \), we define
\[ V^\perp = E^\perp_m V^m, \quad V^\parallel = E^\parallel_m V^m. \] (27)

For a cotangent vector \( V^m \), we also define
\[ V^\perp = e^m_\perp V^m, \quad V^\parallel = e^m_\parallel V^m. \] (28)

Thus the differential operators \( \partial^\perp \), \( \partial^\parallel \) in the normal and tangential directions, respectively, are given by
\[ \partial^\perp = e^m_\perp \partial_m = \partial_0 - k(t, x) \partial_1, \]
\[ \partial^\parallel = e^m_\parallel \partial_m = \frac{1}{l(t)} \partial_1. \] (29)

Note that their conjugate operators are given by
\[ \partial^\dagger_\perp = - (\partial^\perp - \omega), \quad \partial^\dagger_\parallel = - \partial^\parallel, \] (30)
\[ \omega = \frac{\partial}{\partial x} k(t, x) - \frac{1}{l(t)} \frac{\partial}{\partial t} l(t), \] (31)
since the conjugation \( ^\dagger \) should be taken here under the inner product \( \langle f_1 | f_2 \rangle \) on the space of the functions on the surface:
\[ \langle f_1 | f_2 \rangle = \int d^2 \xi \sqrt{g} f_1(\xi) f_2(\xi) \]
\[ = \int d^2 \xi l(t) f_1(\xi) f_2(\xi). \] (32) (33)

The measure \( D_g g_{mn} \) is defined by the following norm on the space of infinitesimal deformation \( \delta g_{mn} \) of metric:
\[ \| \delta g_{mn} \|_g^2 = \int d^2 \xi \sqrt{g} g^{m^k} g^{n^l} \delta g_{mn} \delta g_{kl} \]
\[ = \int dt \frac{(l(t))^2}{l(t)} + \frac{1}{2} \int d^2 \xi l(t) \left( l(t) \delta k(t, x) \right)^2 \]
\[ + \int d^2 \xi l(t) \left[ \left( \partial^\perp \delta v^\perp \right)^2 + \left( \partial^\parallel \delta v^\parallel \right)^2 \right], \] (34) (35)

where
\[ \delta v^\perp = \delta v_0 - k(t, x) \delta v_1, \quad \delta v^\parallel = \frac{1}{l(t)} \delta v_1. \] (36)
The norm on the space of tangent vectors on the worldsheet is given by

$$\| \delta v^m \|^2_\beta = \int d^2 \xi \sqrt{g} \, g_{mn} \delta v^m \delta v^n$$

$$= \int d^2 \xi l(t) \left[ (\delta v^1)^2 + (\delta v^/)^2 \right].$$

This defines the measure $D_\beta v^m$ for the generators of the reparametrization transformation connected to the identity.

Changing variables $\{\delta g_{mn}\}$ into physical variables and gauge degrees of freedom, we obtain

$$D_\beta g_{mn} = \prod_i \frac{dl(t)}{\sqrt{l(t)}} D_{i} D_{i} D_{/} D_{/} \left[ \partial_1^+ \partial_1 \right] \text{Det} \left[ \partial_1^+ \partial_1 \right],$$

where we denote by $\tilde{v}^/$ the non-zero modes of $v^/$ for $\partial_1$, and the determinant with a prime only includes non-zero modes. Recall that the measure $D_i k$ is defined by

$$\| \delta k \|^2_l = \frac{1}{2} \int d^2 \xi l(l \delta k)^2.\quad (39)$$

Noting that the generator $v^m_{\text{res.}}$ of the residual symmetry

$$v_{\text{res.}}^+ = 0,\quad (40)$$

$$v_{\text{res.}}^/ = l(t) \alpha(t),\quad (41)$$

is the zero mode for the differential operator $\partial_1^+$, we can decompose $\text{Vol}(\text{diff.})$ as follows:

$$\text{Vol}(\text{diff.}) = \int D_\beta v^m.$$

$$= \int D_i v^1 D_i \tilde{v}^/ D_i v^m_{\text{res.}}.\quad (42)$$

Thus, if we divide the measure $D_\beta g_{mn}$ by $\text{Vol}(\text{diff.})$, the gauge degrees of freedom $\int D_i v^1 D_i \tilde{v}^/$ are eliminated and we obtain

$$Z(l', l; D) = \int \prod_i \frac{dl(t)}{\sqrt{l(t)}} \left\{ \frac{1}{\int D_i v^m_{\text{res.}}} \right\} D_{i} D_{i} \text{Det} \left[ \partial_1^+ \partial_1 \right] \text{Det} \left[ \partial_1^+ \partial_1 \right] \exp \left[ -\mu_0 \int dl l(t) \right]$$

$$\cdot \delta(l(t = 0) - l) \delta(l(t = D) - l') .\quad (43)$$
In order to estimate the determinant $\text{Det}^{\frac{1}{2}} \left[ \partial_\perp \partial_\perp \right]$, we begin with computing the determinant $\text{Det} \left[ \triangle g \right]$ of the Laplacian $\triangle g \left[ l;k \right]$ defined as
\[
\triangle g = -\frac{1}{\sqrt{g}} \partial_m \sqrt{g} g^{mn} \partial_n, \\
= \partial_\perp \partial_\perp + \partial_\parallel \partial_\parallel.
\] (44)

For a generic metric $g_{mn}$, we define $\text{Det}[\triangle g]$ by the following heat kernel regularization which respects the reparametrization invariance:
\[
\ln \text{Det}\triangle g = -\int_0^\infty \frac{d\tau}{\tau} \text{Tr} \exp \left[ -\tau \triangle g \right].
\] (46)

If we perform an infinitesimal Weyl rescaling $g_{mn} \rightarrow e^{2\sigma} g_{mn}$, the determinant $\text{Det} [\triangle g]$ changes by
\[
\delta \ln \text{Det}\triangle g = -2 \int d^2\xi \sqrt{g} \delta\sigma \left( \frac{1}{4\pi\epsilon} + \frac{1}{12\pi} R[g] + O(\epsilon) \right),
\] (47)

where $R[g]$ is the scalar curvature defined as
\[
R[g] = -\frac{1}{2} g^{mn} R^l_{\text{mn}}.
\] (48)

\[
R^l_{\text{mn}} = \partial_k \Gamma^l_{\text{mn}} - \partial_m \Gamma^l_{\text{nk}} + \Gamma^p_{\text{mn}} \Gamma^l_{kp} - \Gamma^p_{\text{mk}} \Gamma^l_{np},
\] (49)

\[
\Gamma^p_{\text{mn}} = \frac{1}{2} g^{pq} \left( \partial_m g_{pq} + \partial_p g_{mq} - \partial_q g_{mn} \right).
\] (50)

Furthermore, if one metric $g_{mn}$ is related to another one $\hat{g}_{mn}$ by $g_{mn} = e^{2\sigma} \hat{g}_{mn}$, then the associated scalar curvatures have the following relation:
\[
R[g] = \triangle g \sigma + R[\hat{g}] e^{-2\sigma}
\] (51)

\[
= e^{-2\sigma} \left( \triangle g \sigma + R[\hat{g}] \right).
\] (52)

Using this equation (52) and Eq.(47), we thus obtain
\[
\text{Det}\triangle g = \Gamma[g] \exp \left[ \frac{1}{4\pi\epsilon} \int d^2\xi \sqrt{g} - \frac{1}{12\pi} \int d^2\xi \sqrt{g} R[g] \frac{1}{\triangle g} R[g] \right],
\] (53)

where the quantity $\Gamma[g]$ should be invariant under both the reparametrization transformation and the Weyl rescaling.
Now that we have the expression (53) for the determinant $\text{Det} \Delta_g$ with a generic metric $g_{mn}$, let us return to the temporal gauge. Since the first term in the exponent in Eq.(53) can be eventually absorbed into the cosmological term, what we need to investigate are the second term in the exponent and the factor $\Gamma[\tilde{g}]$.

As for the second term which we will denote by $A[l;k]$, since the scalar curvature $R[g]$ in the temporal gauge is expressed as

$$R[\tilde{g}] = R[l;k]$$
$$= (\partial_\perp - \omega) \omega$$
$$= -\partial^\perp_\perp \omega,$$

the term $A[l;k]$ turns out to be

$$A[l;k] = \frac{1}{12\pi} \int d^2\xi \sqrt{\tilde{g}} R[\tilde{g}] \frac{1}{\Delta_{\tilde{g}}} R[\tilde{g}]$$
$$= \frac{1}{12\pi} \int d^2\xi \frac{1}{\partial_\perp^2 + \partial_\|^2} \partial^\perp_\perp \omega.$$

Let us then consider $\Gamma[\tilde{g}]$. Here, we would like to know the dependence of this quantity on the loop length $l(t)$ and the shift function $k(t,x)$. It is known that metrics $g_{mn}$ have three kinds of deformations; one under the reparametrization, one under the Weyl rescaling and one associated with the change of Teichmüller parameters. As we mentioned above, since $\Gamma[\tilde{g}]$ should be invariant under both the reparametrization and the Weyl rescaling, it can only depend on the Teichmüller parameter, which is one-dimensional in our case. The deformation $\delta g^T_{mn}$ of the metric associated with the Teichmüller parameter should satisfy the following equations:

$$g^{mn} \delta g^T_{mn} = 0,$$
$$\nabla^a \delta g^T_{mn} = 0.$$

In the temporal gauge, the corresponding equations are written in the following form for the deformations of $\delta l(t)$ and $\delta k(t,x)$:
\[
\left( \partial_\perp - 2\omega + \frac{i(t)}{l(t)} \right) \delta k(t, x) = 0, \tag{60}
\]
\[
l(t) \partial_y \delta k(t, x) = \left( \frac{\partial}{\partial t} - 2\omega \right) \frac{\delta l(t)}{l(t)}, \tag{61}
\]
where we denote by a dot \( \cdot \) the differentiation with respect to \( t \).

Now we try to find a solution of these equations (60), (61). Rewriting Eq.(61), we have
\[
\left( \frac{\partial}{\partial t} + 2 \frac{i(t)}{l(t)} \right) \frac{\delta l(t)}{l(t)} = \frac{\partial}{\partial x} \left( \delta k(t, x) + 2 \frac{\delta l(t)}{l(t)} k(t, x) \right). \tag{62}
\]
From the boundary conditions for \( \delta k(t, x) \) and \( k(t, x) \):
\[
\delta k(t, x = 0) = \delta k(t, x = 1), \tag{63}
\]
\[
k(t, x = 0) = k(t, x = 1), \tag{64}
\]
we find that the L.H.S. and the R.H.S. of Eq.(62) should be equal to zero, since the terms of the L.H.S. depend only on \( t \). Solving these equations, we obtain
\[
\frac{\delta l(t)}{l(t)} = \lambda l(t)^{-2}, \tag{65}
\]
\[
\delta k(t, x) = -2\lambda l(t)^{-2} k(t, x) + c(t), \tag{66}
\]
where \( \lambda \) is a constant, and \( c(t) \) is an arbitrary function depending only on \( t \). However, substituting these into Eq.(60), we can verify that there is only trivial solution; \( \lambda = 0, c(t) = 0 \).

Thus, we may think that the deformations \( \delta l(t) \) and \( \delta k(t, x) \) have nothing to do with the Teichmüller parameter; This parameter can only be related to the geodesic distance \( D \) and the loop lengths \( l, l' \) of the initial and final states. It is thus plausible that we assume the independence of \( \Gamma[\tilde{g}] \) on the loop length \( l(t) \) and the shift function \( k(t, x) \), and we will write
\[
\Gamma[\tilde{g}] = \Gamma[l', l; D]. \tag{67}
\]

In summary, the result is
\[
\begin{align*}
D\epsilon l \triangle \tilde{g} &= \Gamma[l', l; D] \exp \left[ -\frac{1}{4\pi\epsilon} \int dt \, l(t) - A[l, k] \right], \tag{68}
\end{align*}
\]
\[
= \Gamma[l', l; D] \exp \left[ -\frac{1}{4\pi\epsilon} \int dt \, l(t) - \frac{1}{12\pi} \int d^2 \xi \, l \partial_\perp^3 \omega \frac{1}{\partial_\perp^3 \partial_\perp + \partial_\parallel^3 \partial_\parallel} \partial_\perp^3 \omega \right]. \tag{69}
\]
Next, we investigate the determinant $\text{Det}^{\frac{1}{2}} \left[ \partial_+^+ \partial_- \right]$. To do so, we use the following relation for the Laplacian $\Delta_g$ in the temporal gauge [12]:

$$\Delta_g \left[ \beta^{-1} l; k \right] = \partial_+^+ \partial_+ + \beta^2 \partial_+^+ \partial_0,$$

and thus define the determinant $\text{Det} \left[ \partial_+^+ \partial_- \right]$ by

$$\ln \text{Det} \left[ \partial_+^+ \partial_- \right] = \lim_{\beta \to 0} \ln \text{Det} \Delta_g \left[ \beta^{-1} l; k \right].$$

Here we will make use of Eq.(69) to estimate the above. Since we can see from the expression (56) that the scalar curvature $R[g]$ is invariant under the constant rescaling of loop length $l(t)$:

$$R[\beta^{-1} l; k] = R[l; k] \quad (\beta \text{ is a constant}),$$

we obtain

$$\lim_{\beta \to 0} A[\beta^{-1} l; k] = \lim_{\beta \to 0} \frac{1}{12\pi \beta} \int d^2 \xi l \partial_+^+ \frac{1}{\partial_+^+ \partial_+ + \beta^2 \partial_+^+ \partial_0} \partial_+ \omega = \lim_{\beta \to 0} \frac{1}{12\pi \beta} \int d^2 \xi l \omega^2 = \lim_{\beta \to 0} \frac{1}{12\pi \beta} \left[ \int dt \frac{\dot{l}(t)}{l(t)} + \int d^2 \xi l (i \partial_0 k)^2 \right].$$

As for the factor $\Gamma[l', l; D]$, we first suppose to investigate the determinant $\Delta_g$ with loop lengths $\beta l$ and $\beta l'$ at initial and final time, respectively. After that, we scale loop length $l(t)$ as $l(t) \to \beta^{-1} l(t)$. To this end, the factor $\Gamma[l', l; D]$ has the loop lengths $l, l'$ at the initial and final time, respectively.

Thus we obtain

$$\text{Det} \left[ \partial_+^+ \partial_- \right] = \lim_{\beta \to 0} \Gamma[l', l; D] \exp \left[ -\frac{1}{4\pi \beta \epsilon} \int dt \dot{l}(t) - A[\beta^{-1} l, k] \right],$$

$$= \lim_{\beta \to 0} \Gamma[l', l; D] \exp \left[ -\frac{1}{4\pi \beta \epsilon} \int dt \dot{l}(t) - \frac{1}{12\pi \beta} \left\{ \int dt \frac{\dot{l}(t)}{l(t)} + \int d^2 \xi l (i \partial_0 k)^2 \right\} \right].$$

We substitute Eq.(74) into Eq.(43). Renormalizing the first term of exponent in Eq.(74) into the bare cosmological constant $\mu_0$, we denote the renormalized cosmological constant by $\mu$. Then, we have
\[ Z(l', l; D) = \Gamma [l', l; D]^\frac{1}{2} \lim_{\beta \to 0} \int \prod_i \frac{dl(t)}{\sqrt{l(t)}} \left\{ \frac{1}{\int \mathcal{D}_l k} \text{Det} \frac{1}{2} \left[ \frac{\partial}{\partial l} \right] \exp \left[ -\mu \int dt l(t) \right] \right\} \cdot \exp \left[ -\frac{1}{24\pi \beta} \left\{ \int dt \frac{\dot{l}(t)^2}{l(t)} + \int d^2 \xi \ l \ (l \partial_\parallel k)^2 \right\} \right] \cdot \delta \left( l(t = 0) - I \right) \delta \left( l(t = D) - l' \right). \quad (75) \]

As we mentioned in the last section, we have to further fix the residual symmetry. We can see this from the fact that the zero mode \( \delta k_0(t) \) satisfying \( \partial_\parallel \delta k(t, x) = 0 \) does not appear in the integrand of the R.H.S. of Eq.(75). From Eq.(14) and Eq.(41), the following relation is obtained:

\[ \delta_{res, k}(t, x) = \left( \frac{\partial}{\partial t} - k'(t, x) \right) \frac{1}{l(t)} \delta_{v/2, res}(t), \quad (76) \]

where we denote by a prime the differentiation with respect to \( x \). Therefore, we apply the Fadeev-Popov prescription to it; namely, we substitute the identity

\[ 1 = \int \mathcal{D}_{v/2, res} \prod_t l(t)^{-\frac{3}{2}} \prod_t \delta \left( k(t, x_0) - \left( \frac{\partial}{\partial t} - k'(t, x_0) \right) \frac{1}{l(t)} \delta_{v/2, res}(t) \right) \cdot \text{Det}^{-1} \left[ \left( \frac{\partial}{\partial t} - k'(t, x_0) \right) \frac{1}{l(t)} \right] \quad (77) \]

into Eq.(75), where \( x_0 \) is an arbitrary value of \( x \). Furthermore we decompose the measure \( \mathcal{D}_{t} k \) into the part of the zero mode \( \mathcal{D}_{t} k_0 \) and the part of the non-zero mode \( \mathcal{D}_{t} \tilde{k} \). From the definition (39) of the measure \( \mathcal{D}_{t} k \), we can verify that

\[ \int \mathcal{D}_{t} k_0 \cdot 1 = \int \prod_t dk_0(t) \prod_t l(t)^{\frac{3}{2}} \cdot 1 \quad (78) \]

\[ = \int \mathcal{D}_{t} v/2, res \prod_t l(t)^2 \text{Det}^{-1} \left[ \left( \frac{\partial}{\partial t} - k'(t, x_0) \right) \frac{1}{l(t)} \right] \quad (79) \]

Accordingly, the volume factor \( \int \mathcal{D}_{t} v/2, res \) of the residual symmetry in Eq.(75) and the volume factor \( \int \mathcal{D}_{t} v/2, res \) emerging from Eq.(79) cancel out. Thus,

\[ Z(l', l; D) = \lim_{\beta \to 0} \Gamma [l', l; D]^\frac{1}{2} \int \prod_t dl(t) l(t)^{\frac{3}{2}} \mathcal{D}_{t} \tilde{k} \text{Det}^{-1} \left[ \left( \frac{\partial}{\partial t} - k'(t, x_0) \right) \frac{1}{l(t)} \right] \cdot \exp \left[ -\frac{1}{24\pi \beta} \left\{ \int dt \frac{\dot{l}(t)^2}{l(t)} + \int d^2 \xi \ l \ (l \partial_\parallel k)^2 \right\} \right] \cdot \text{Det}^{\frac{1}{2}} \left[ \frac{\partial}{\partial l} \right] \exp \left[ -\mu \int dt l(t) \right] \cdot \delta \left( l(t = 0) - I \right) \delta \left( l(t = D) - l' \right). \quad (80) \]
Since the exponent of the R.H.S. in Eq.(80) means that $\partial_{\parallel} k = 0$ in the limit $\beta \to 0$, we can ignore $k'$ in the determinant $Det^{-1}[\left( \frac{2}{\ell^2} - k'(l, \tau_0) \right)]$ in the same equation. It is thus easy to perform the integration over the non-zero mode $\hat{k}$. This yields $Det^{-\frac{1}{2}}[\frac{1}{12\pi\beta} \partial_{\parallel}^3 \partial_{\parallel}]$, which cancels $Det^{\frac{1}{2}}[\partial_{\parallel}^3 \partial_{\parallel}]$ in the R.H.S. of (80).

After all, we obtain

$$Z(l', l; D) = \lim_{\beta \to 0} \hat{\Gamma}[l', l; D; \beta]^{\frac{1}{2}} \int l(t) \hat{l}(t) dt \exp \left[ -\mu \int dt l(t) - \frac{1}{24\pi^2} \int dt \frac{\dot{l}(t)^2}{l(t)} \right] \cdot \delta(l(t = 0) - l) \delta(l(t = D) - l'),$$

where $\hat{\Gamma}[l', l; D; \beta]^{\frac{1}{2}} = \Gamma[l', l; D]^{\frac{1}{2}} Det^{-1}[\frac{2}{\ell^2}] Det^{\frac{1}{2}}[12\pi\beta]$.

Note that the power of the loop $l(t)$, apart from those exponentiated, in Eq.(81) is different from that in [12]. Ours is three half, while theirs is minus one. This discrepancy will be discussed in sect.V.

**IV. PROPAGATOR**

We now introduce scalar fields into the system considered in the last section. In particular, we pay attention to what corresponds to the Weyl anomalies in this case, which appear in the conformal gauge.

We substitute the path integral for $N$ scalar fields (string coordinates) $X^\mu(\xi)$ ($\mu = 1, \cdots, N$)

$$W[g] = \int Dg X e^{-S[X,g]}$$

into the amplitude (16) in the last section. Here the action $S[X, g]$ is given by

$$S[X, g] = \frac{1}{8\pi} \int d^2\xi \sqrt{g} g^{mn} \partial_m X^\mu \partial_n X^\mu.$$  

This action describes a string propagating in the $N$ dimensional flat Euclidean space-time. So the amplitude under consideration can be regarded as a propagator of such a string [13].

We have to impose boundary conditions on the scalar fields $X^\mu(\xi)$ at the boundaries $C$ and $C'$. Since the string coordinates $X^\mu(\xi)$ map the worldsheet into the space-time,
if two string coordinates can be connected under the reparametrization transformation on the worldsheet, we should regard these as the same string configuration. Thus, up to the reparametrizations, we specify the boundary conditions as follows:

\[ X^\mu(t = 0, x) = X^\mu_0(x) \quad (\text{on } C), \]
\[ X^\mu(t = D, x) = X^\mu_D(x) \quad (\text{on } C'). \]  

Then, the string propagator \( G(l', X_f ; l, X_i ; D) \) is given by

\[
G(l', X_f ; l, X_i ; D) = \int d\Sigma_{i,j}^{\text{diff.}} \int \frac{Dg_{mn}}{\text{Vol}(\text{diff.})} \exp \left[ -\mu_0 \int d^2 \xi \sqrt{g} \right] W[g; X_f, X_i] \\
\times \delta \left( \int_C \sqrt{g_{mn} d\xi^m d\xi^n - l} \right) \delta \left( \int_{C'} \sqrt{g_{mn} d\xi^m d\xi^n - l'} \right) \\
\times \delta \left( \int N(g_{mn}) d\xi - D \right),
\]

where \( d\Sigma_{i,j}^{\text{diff.}} \) denotes integration over the reparametrizations on the boundaries \( C, C' \). Furthermore, for the path integral \( W[g] \) over the string coordinates, we explicitly represent its dependence on the boundary conditions of \( X^\mu(\xi) \) as \( W[g; X_f, X_i] \).

Let us first compute \( W[g; X_f, X_i] \). Let \( \bar{X}^\mu_g \) be the solution of the equation of motion

\[
\Delta_g \bar{X}^\mu_g = 0
\]

satisfying the above boundary conditions \((84,85)\). Then we expand the string coordinates \( X^\mu(\xi) \) around the solution \( \bar{X}^\mu_g(\xi) \) as

\[
X^\mu(\xi) = \bar{X}^\mu_g(\xi) + y^\mu(\xi),
\]

and substitute these into the path integral \( W[g; X_f, X_i] \). Integration now are made over the variables \( y^\mu(\xi) \) satisfying the boundary conditions:

\[
y^\mu(t = 0, x) = y^\mu(t = D, x) = 0,
\]

and the measure is defined as

\[
||\delta y^\mu||^2_g = \frac{1}{8\pi} \int d^2 \xi \sqrt{-g} \delta y^\mu(\xi) \delta y^\mu(\xi).
\]
Furthermore, the action $S[X, g]$ turns out to be

$$S[X, g] = S_{cl} + S[y, g], \quad (91)$$

where the classical action $S_{cl}$ is

$$S_{cl} = S[X^\mu, g]$$

$$= \frac{1}{8\pi} \int dx \left[ \tilde{X}^\mu(\xi) \sqrt{g} \partial_\alpha \tilde{X}^\mu(\xi) \right]_{t=0}^{t=D}. \quad (92)$$

Thus, the path integral $W[g; X_f, X_i]$ is easily performed, and we obtain

$$W[g; X_f, X_i] = e^{-S_{cl}} \int \mathcal{D}_g \exp \left[ -\frac{1}{8\pi} \int d^2\xi \sqrt{g} \mu(\xi) \Delta_\mu \right]$$

$$= e^{-S_{cl}} (\text{Det} \Delta_\mu)^{-\frac{\nu}{2}}. \quad (93)$$

Imposing the temporal gauge on the path integral $W[g; X_f, X_i]$ and using Eq.(99) in the last section, we find

$$W[\tilde{g}; X_f, X_i]$$

$$= \Gamma[l'; l; D]^{-\frac{\nu}{2}} e^{-S_{cl}} \exp \left[ -\frac{N}{8\pi \epsilon} \int dt l(t) - \frac{N}{24\pi} \int d^2\xi l \partial_1^\parallel \omega \left( \partial_1^\parallel + \partial_1^\perp \partial_1^\perp \right) \right]. \quad (95)$$

The second term in the exponent corresponds to the Weyl anomaly from the matters in the conformal gauge.

Substituting this equation (95) into the string propagator (86) and rewriting the remaining part in a similar way as we did in the last section, we can verify that

$$G(l', X_f ; l, X_i ; D) = \lim_{\beta \to 0} \int d^2 \Sigma_{\text{diff}} \prod_i d l(t) l(t) \sqrt{\text{det} \kappa} \Gamma[l'; l; D]^{\frac{2N}{\nu}} e^{-S_{cl}} \cdot \exp \left[ -\frac{1}{24\pi \beta} \left\{ \int dt \frac{\dot{l}^2(t)}{l(t)} + \int d^2\xi l \left( \partial_1^\parallel + \partial_1^\perp \right)^2 \right\} \right]$$

$$\cdot \exp \left[ -\mu_N \int dt l(t) - \frac{N}{24\pi} \int d^2\xi l \partial_1^\parallel \omega \left( \partial_1^\parallel + \partial_1^\perp \partial_1^\perp \right) \right]$$

$$\cdot \text{Det}^{-1} \left[ \left( \frac{\partial}{\partial t} - k'(t, x_0) \right) \right] \cdot \text{Det}^{\frac{\nu}{2}} \left[ \partial_1^\parallel \partial_1^\perp \right]$$

$$\cdot \delta \left( l(t = 0) - l \right) \delta \left( l(t = D) - l' \right), \quad (96)$$
where we denote by $\mu_N$ the renormalized cosmological constant. In this expression, the classical action $S_{cl}$ is

$$ S_{cl} = \frac{1}{16\pi} \int dx \left[ \frac{\partial}{\partial t} \left( l(t) \left( \overline{X}_g^\mu \right)^2 \right) + \omega l(t) \left( \overline{X}_g^\mu \right)^2 \right]_{t=D}^{t=0} \ .$$

(97)

The terms multiplied by $\beta^{-1}$ in the exponent in the R.H.S. of Eq.(96) mean that the following configuration dominates in the limit $\beta \to 0$:

$$ \hat{l}(t) = 0, $$

(98)

$$ k'(t, x) = 0. $$

(99)

Therefore,

$$ \omega \xrightarrow{\beta \to 0} 0, $$

(100)

$$ \Delta_g \xrightarrow{\beta \to 0} - (\partial_0 - k_0(t) \partial_1) (\partial_0 - k_0(t) \partial_1) - l(t)^{-2} \partial_1 \partial_1, $$

(101)

where $k_0(t)$ is the zero mode of $k(t, x)$ for the differential operator $\partial_\mu$. As we can see from these equations, the above-mentioned Weyl anomalies from the scalar fields vanish in this limit; namely

$$ - \frac{N}{12\pi} \int d^2 \xi \int l \frac{1}{\partial_1 \partial_1} \frac{\partial_1 \omega}{\partial_1 \partial_1} \frac{\partial_1 \omega}{\partial_1 \partial_1} \xrightarrow{\beta \to 0} 0. $$

(102)

By similarly calculating the remaining part in the string propagator (96) and integrating out the non-zero mode $\hat{k}(t, x)$, the following result is obtained:

$$ G(l', X_f ; l, X_i ; D) = \lim_{\beta \to 0} \int d\Sigma_{i,j}^{\text{diff.}} \int \prod_t dl(t) l(t)^{\frac{3}{2}} \tilde{\Gamma}[l', l; D]^{\frac{1-M}{2}} e^{-S_{cl}} \cdot \exp \left[ -\mu_N \int dt l(t) - \frac{1}{24 \pi \beta} \int dt \left( \frac{l(t)}{l(t)} \right)^2 \right] \cdot \delta \left( l(t = 0) - l \right) \delta \left( l(t = D) - l' \right). $$

(103)

where $\tilde{\Gamma}[l', l; D]^{\frac{1-M}{2}} = \Gamma[l', l; D]^{\frac{1-M}{2}} Det\left[12 \pi \beta \right] Det^{-1}[\frac{\beta}{24}]$, and the classical action $S_{cl}$ is

$$ S_{cl} = \frac{1}{16\pi} \int dx \left[ l(t) \frac{\partial}{\partial t} \left( \overline{X}_g^\mu \right)^2 \right]_{t=0}^{t=D}. $$

(104)
V. DISCUSSION

In this paper we have considered two-dimensional quantum gravity in the temporal gauge and have demonstrated that we can explicitly perform the path integration over the metric under some plausible assumptions.

In sect.III, we investigated the cylinder amplitude for pure gravity in a different way from that in [12]. As we mentioned at the end of that section, the discrepancy between their result and ours (81) was found in the power of the loop length $l(t)$, apart from those exponentiated. This discrepancy may be explained as a difference in the way to fix the residual symmetry.

Eq.(81) implies that the cylinder amplitude is essentially proportional to the delta function $\delta(l - l')$ in the limit $\beta \to 0$. Then the loop length $l(t)$ should be replaced by the one $l$ at the initial state. So it is not clear how relevant this discrepancy is, until we can compute the function $\Gamma[l', l; D]$.

In sect.IV, we considered a propagator of a string propagating on the $N$ dimensional flat Euclidean space-time. There we have been able to derive what should correspond to the Weyl anomalies from the matters, which finally vanishes in the limit $\beta \to 0$. There are two subtleties in this calculation: first, it is not clear how to integrate over the reparametrizations on the boundaries $C$ and $C'$. Secondly, the validity of our assumption made on $\Gamma[l, l'; D]$ should be examined. As for the latter, it is necessary to establish how the Teichmüller parameter depends on the geodesic distance $D$ and the loop length $l, l'$ of the initial and final states.

Despite these subtleties in this approach, there seems no critical dimensions in the temporal gauge, as we can see from Eq.(103). However, to reach a decisive conclusion as to whether there really exists no critical dimension in the temporal gauge approach, further investigation is needed on the above-mentioned problems. Furthermore, if it turns out to be the case, it is very interesting to examine the mass spectrum of the physical states, especially the graviton ones.
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