One-loop Renormalization of Black Hole Entropy
Due to Non-minimally Coupled Matter

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Abstract

The quantum entanglement entropy of an eternal black hole is studied. We argue that the relevant Euclidean path integral is taken over fields defined on α-fold covering of the black hole instanton. The statement that divergences of the entropy are renormalized by renormalization of gravitational couplings in the effective action is proved for non-minimally coupled scalar matter. The relationship of entanglement and thermodynamical entropies is discussed.

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1 Introduction

According to the thermodynamical analogy one can apply the laws of thermodynamics that are valid for large statistical systems to the description of a single black hole [1]. The key idea of this approach is that a black hole has an intrinsic entropy proportional to the surface area of the event horizon $\Sigma$. This idea has found strong support in Hawking’s discovery [2] of thermal radiation of a black hole that allowed to determine the entropy precisely as $S = A_\Sigma / 4G$. However, the microphysical explanation of the black hole entropy as counting of states is still absent though many attempts have been made (see recent review [3]). One of possible ways is to associate the entropy with a thermal bath of fields propagating just outside the horizon [4]. Recently, it also has been proposed to treat black hole entropy as an entanglement entropy [5], [6]. Starting with the pure vacuum state one traces over modes of quantum field propagating inside the horizon and obtains the density matrix $\rho$. The entropy then is defined by the standard formula $S = -Tr \rho \ln \rho$. It measures the number of inside modes which are considered as internal degrees of freedom of the hole. In a similar manner, Frolov and Novikov [7] suggested to trace over modes outside the horizon.

The calculations for the Rindler space and black holes [8]-[16] have shown that entropy is divergent. This is essentially due to the short-distance correlations between the inside and outside modes.

The purpose of this paper is to demonstrate, following the previous investigations [10], [11], [12], that this divergence is really the ultraviolet one typically appearing in quantum field theory and it can be removed by standard renormalization of the gravitational couplings in the effective action. In the earlier paper [17] we have given a proof of this for a minimally coupled scalar field and noted that the non-minimal coupling needs a special consideration. The reason of this is that there exists a $\delta$-like potential in the field operator due to the scalar curvature $R$ behaving as a distribution on manifold with conical singularity. Below we consider this in more details. We start in Section 2. with formulating the Euclidean path integral which is relevant to calculation of the entanglement entropy of a black hole. In Section 3. we formulate the statement about renormalization
of black hole entropy. The proof of it for non-minimally coupled scalar field is given in Section 4. We conclude in Section 5, with some remarks concerning the relationship of the entanglement (statistical) and thermodynamical entropies. The Appendixes A and B contain basic formulas for curvature tensors and heat kernel expansion on manifolds with conical singularities obtained in previous study.

2 Euclidean path integral for entanglement entropy.

The horizon surface $\Sigma$ naturally separates the whole space-time of a black hole on the regions $R_+$ and $R_-$: the free information exchange between which is impossible. This is obviously due to the fact that the global Killing vector $\xi_t = \partial_t$, generating translations in time, becomes null, $\xi_t^2 = 0$, on the horizon. Therefore, any light signal emitted from any point on the horizon or behind it, never can reach an outside observer. So events happening in the part of the space-time beyond the horizon are unobservable for him in principle. This concerns excitations of quantum fields as well. They are naturally separated on 'visible' (propagating in the region $R_+$) and 'invisible' (propagating in the region $R_-$) modes. The partial loss of information about the microstates composing the concrete macrostate typically appears in statistical description of systems with large number of degrees of freedom. We can see that a similar phenomenon naturally happens for a black hole. This fact certainly lies in the principles of the thermodynamical analogue allowing to apply laws of thermodynamics to a hole.

The situation, when a part of states of the system is unknown, in quantum mechanics is usually described by density matrix. Assume, that the quantum field $\phi$, being considered on the whole space-time, is in pure ground state

$$\Psi_0 = \Psi_0(\phi_+, \phi_-)$$

which is the function of both visible ($\phi_+$) and invisible ($\phi_-$) modes. For an outside observer it is in the mixed state described by the density matrix

$$\rho(\phi_+^1, \phi_+^2) = \int [D\phi_-] \Psi_0^+ (\phi_+^1, \phi_-) \Psi_0(\phi_+^2, \phi_-)$$

(2.2)
where one traces over all invisible modes $\phi_-$. Then the entropy defined as

$$S_{geom} = -Tr \dot{\rho} \ln \dot{\rho}, \quad \dot{\rho} = \frac{\rho}{Tr \rho}$$

is so-called entanglement (or geometric) entropy [5]-[7].

Applying this construction to a black hole, we identify all the invisible modes with internal degrees of freedom and (2.3) with entropy of the hole. The ground state of the black hole is given by the Euclidean functional integral [18] over fields defined on manifold $E'$ which is half-period part of the black hole instanton with metric

$$ds^2_{E'} = \beta_H^2 g(\rho) d\varphi^2 + d\rho^2 + r^2(\rho)d^2 \Omega$$

where angle variable $\varphi$ lies in the interval $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. The inverse Hawking temperature $\beta_H$ is determined by the derivative of the metric function $g(\rho)$ on the horizon ($g(\rho_h) = 0$), $\beta_H = \frac{2}{g'(\rho_h)}$. The $\phi_+$ and $\phi_-$ which enter as arguments in (2.1) are the fixed values at the boundaries $\phi_+ = \phi(\varphi = \frac{\pi}{2}); \phi_- = \phi(\varphi = -\frac{\pi}{2})$, giving the boundary condition in the path integral. The density matrix $\rho(\phi^1_+, \phi^2_+)$ obtained by tracing $\phi_-$-modes is defined by the path integral over fields on the full black hole instanton $E' (-\frac{3}{2} \pi \leq \varphi \leq \frac{1}{2} \pi)$ with cut along the $\varphi = \frac{\pi}{2}$ axis and taking values $\phi^1, \phi^2$ above and below the cut. The trace $Tr \rho$ is obtained by equating the fields across the cut and doing the unrestricted Euclidean path integral on the complete black hole instanton $E$. Analogously, $Tr \rho^n$ is given by the path integral over fields defined on $E_n$, $n$-fold cover of $E$. Thus, $E_n$ is the manifold with abelian isometry (with respect to angle rotation $\partial_\varphi$) with horizon surface $\Sigma$ as stationary set. Near $\Sigma$ the $E_n$ looks as direct product $E_n = \Sigma \otimes \mathcal{C}_n$, where $\mathcal{C}_n$ is two-dimensional cone with angle deficit $\delta = 2\pi(1 - n)$. This construction can be analytically continued to arbitrary (not integer) $n \rightarrow \alpha = \frac{\beta}{\beta_H}$.

Define now the partition function

$$Z(\beta) = Tr \rho^\alpha$$

which is path integral over fields defined on $E_\alpha$, the $\alpha$-sheeted covering of $E$. Then the geometric entropy (2.3) takes the standard thermodynamical form.
being expressed via partition function \( Z \). We see that \( \beta \) plays the role of the inverse temperature. After all calculations one must put \( \beta = \beta_H \) in (2.6). Assuming that the dynamics of matter fields is determined by a differential operator \( \hat{\Delta} \) we obtain that the relevant partition function (2.5) is given by the determinant

\[
Z(\beta) = \det^{-1/2} \hat{\Delta}
\]

considered on \( E_\alpha \). It is essential that \( E_\alpha \) is manifold with conical singularity since namely the singularity produces in the effective action \( W(\alpha) = -\ln Z(\alpha) \) terms proportional to \( (1 - \alpha) \) that contribute to the entropy (2.6).

One can see that the partition function (2.5) looks as a thermal one

\[
Z(\beta) = \text{Tr} e^{-\beta \hat{H}}
\]

with \( \beta \) playing the role of inverse temperature, \( \hat{H} \) being a relevant Hamiltonian. This fact was previously observed in [9] for the Rindler space and was supposed to be general. The relevant Euclidean path integral for the entanglement entropy of the Rindler space was found in [8]. The exact construction of the wave function of a black hole was proposed in [18]. The thermality of the corresponding density matrix was established in [19].

3 The statement.

As defined in previous section entanglement entropy is not free from the ultraviolet divergences. They result from the corresponding divergences of the effective action \( W(\alpha) \). It was shown for the minimally coupled scalar matter [20] that the divergent part of the effective action on \( E_\alpha \) is really sum of volume and surface terms:

\[
W_{\text{div}}(\alpha) = W_{\text{vol}}^{\text{div}} + W_{\text{surf}}^{\text{div}}
\]

The volume term in (3.1) is standard one. It is proportional to \( \alpha \) not contributing to entropy. The second term is given by integral over the singular surface \( \Sigma \). It is proportional
to \((1 - \alpha)\) and hence contributes to the entropy resulting to its divergence [13]. The origin of these divergences lies obviously in the short-distance correlation between 'visible' and 'invisible' modes which is concentrated at the surface \(\Sigma\) separating regions \(R_+\) and \(R_-\).

However, it was proposed in number of papers that the divergences of entropy can be removed by the standard renormalization of the gravitational couplings. Indeed, the higher curvature terms are necessarily generated by quantum corrections. Therefore, such a \(R^2\) terms must be added from the very beginning with some bare constants \((c_{1,B}, c_{2,B}, c_{3,B})\) (tree-level) to absorb the one-loop infinities. The bare (tree-level) gravitational functional thus takes the form\(^1\)

\[
W_{\text{gr}} = \int \sqrt{g} d^4 x \left( -\frac{1}{16\pi G_B} R + c_{1,B} R^2 + c_{2,B} R_{\mu\nu}^2 + c_{3,B} R_{\mu\nu\alpha\beta}^2 \right)
\]  

(3.2)

The corresponding tree-level entropy can be obtained within the procedure considered in the previous section as replica of the action (3.2) on introducing of the conical singularity. The conical singularity at the horizon \(\Sigma\) manifests itself in that a part of an curvature tensor for such a manifold \(E_a\) behaves as a distribution having support on the surface \(\Sigma\) [21], [22] (see Appendix A). Hence, the action (3.2) being considered on \(E_a\) has volume and surface terms:

\[
W_{\text{gr}}[E_a] = W_{\text{gr}}^{\text{vol}}[E_a/\Sigma] + W_{\text{gr}}^{\text{surf}}[\Sigma]
\]

(3.3)

where the volume part is given by integral (3.2) over regular part of the manifold \(E_a\). This part is obviously proportional to \(W_{\text{gr}}^{\text{vol}} \propto \alpha\). So the whole contribution to the entropy comes from the surface term. Using formulas of Appendix A ((A.2)-(A.5)) we obtain finally for the tree-level entropy [17], [22]:

\[
S(G_B, c_{i,B}) = \frac{1}{4G_B} A_\Sigma - \int_\Sigma (8\pi c_{1,B} R + 4\pi c_{2,B} R_{ij}^2 + 8\pi c_{3,B} R_{ijij})
\]

(3.4)

We see that the classical law \(S = \frac{1}{4G} A_\Sigma\) gets modified due to \(R^2\)-terms in the action (3.2). The additional term now depends both on external and internal geometries of the surface.

\(^1\)Of course, we assume an addition to (3.2) due to a classical matter which can be in principle rather complicated.
\( \Sigma \). It should be noted that (3.4) exactly coincides with entropy computed by the Noether charge method of Wald [23].

The main point now is that the divergent part of the entanglement entropy (2.6) is such that its sum with the tree-level entropy (3.4)

\[
S(G_B, c_{i\text{-}B}) + S_{\text{div}}(\epsilon) = S(G_{\text{ren}}, c_{i\text{-}ren})
\]  

(3.5)

takes again the tree-level form \( S(G_{\text{ren}}, c_{i\text{-}ren}) \) expressed through the renormalized constants \( G_{\text{ren}}, c_{i\text{-}ren} \). They are related with the bare constants by usual equations originated from the one-loop renormalization of the action

\[
W_{\text{gr}}(G_B, c_{i\text{-}B}) + W_{\text{div}}(\epsilon) = W_{\text{gr}}(G_{\text{ren}}, c_{i\text{-}ren})
\]  

(3.6)

being considered on regular space-times without horizons.

Thus, divergences of the entanglement entropy are removed by the standard renormalization of the gravitational couplings. So, no special renormalization procedure for entropy is required.

This statement for the Newton constant \( G \) has been advocated in [10], [11] when considered divergences of the entropy of the Rindler space-time. Necessity to renormalize also the higher curvature couplings was argued in [12] for the entropy of the Schwarzschild black hole. For minimal coupling this statement was proved in [17] for general black hole metric. In recent preprint [24] this procedure was checked for the Reissner-Nordström black hole. Below we demonstrate this statement for the non-minimally coupled scalar matter generalizing the result of [17].

4 The heat kernel expansion.

For non-minimally coupled scalar field the curvature directly enters into the matter action:

\[
W_{\text{mat}} = \frac{1}{2} \int [(\nabla \phi)^2 + \xi R \phi^2]
\]  

(4.1)

Considering (4.1) on manifold \( E_\alpha \) we must take into account the \( \delta \)-like contribution of the curvature coming from the conical singularity (see (A.1)) [21], [22]:
\[ R = \tilde{R} + 4\pi(1 - \alpha)\delta_{\Sigma} \] (4.2)

where \( \tilde{R} \) is the regular part of the scalar curvature. Therefore, the quantization of non-minimal matter on \( E_\alpha \) forces with the problem of treating operators with \( \delta \)-like potential. Applying (4.2) to the action (4.1) we obtain that

\[ W_{\text{mat}} = 2\pi(1 - \alpha)\xi \int_{\Sigma} \phi^2 + \frac{1}{2} \int_{E_\alpha} \phi(-\xi)\phi \] (4.3)

where we denote \( \xi = -\xi \tilde{R} \) and assume regularity of the field \( \phi \) on the singular surface \( \Sigma \).

Then, considering the path integral over the scalar field \( \phi \) we get

\[ Z = \int [D\phi] e^{-2\pi(1 - \alpha) \int_{\Sigma} \phi^2 - \frac{1}{2} \int_{E_\alpha} \phi(-\xi)\phi} \] (4.4)

Expanding\(^2\) the first factor in (4.4) by powers of \( (1 - \alpha) \) and omitting higher terms we have

\[ Z = \tilde{Z} \left( 1 - 2\pi\xi(1 - \alpha) < \int_{\Sigma} \phi^2 > \tilde{Z} \right) \] (4.5)

where the average \( < > \) is taken with respect to measure defined by functional integral

\[ \tilde{Z} = \int [D\phi] e^{-\frac{1}{2} \int_{E_\alpha} \phi(-\xi)\phi} \] (4.6)

Equivalently, this can be written as follows

\[ \ln Z = \ln \tilde{Z} - 2\pi\xi(1 - \alpha) < \int_{\Sigma} \phi^2 > \tilde{Z} \] (4.7)

For \( \ln \tilde{Z} \) the following heat kernel expansion is known [20]:

\[ \ln \tilde{Z} = -\frac{1}{2} \ln \det(-\xi) = \frac{1}{2} \int_{\sqrt{s} \leq \infty} ds \text{Tr} \tilde{K}_{E_\alpha}(s), \]
\[ \tilde{K}_{E_\alpha}(s) = e^{-s\xi} = \frac{1}{(4\pi s)^\frac{d}{2}} \sum_n \tilde{a}_n s^n, \quad s \to 0 \] (4.8)

\(^2\)We proceed the perturbation expansion with respect to \( (1 - \alpha) \). The first term of the expansion is well-defined (see (4.11)). The next terms, however, are expected to be ill-defined due to contributions like \( \delta^2(0) \). The indication of this can be found in [25]. In principle, we could use some type of regularization similar to that of [22] to give a sense to such a terms. It should be noted, however, that these terms are irrelevant for the calculation of entropy. I thank D.Fursaev for this remark.
where the coefficients $\tilde{a}_n(x, x)$ generally take the form

$$
\tilde{a}_n(x, x) = \tilde{a}^{st}_n(x, x) + \tilde{a}_{n,\alpha}(x, x)\delta_{\Sigma}
$$

(4.9)

The $\tilde{a}^{st}_n(x, x)$ are standard [26] heat kernel coefficients given by the local functions of curvature tensors (see Appendix B). The second term in (4.9) has support only on the singular surface $\Sigma$, $\tilde{a}_{n,\alpha}(x, x)$ is a local function of projections of a curvature tensors on the subspace normal to $\Sigma$. The exact form of coefficients $\tilde{a}_{n,\alpha}(x, x)$ is given in Appendix B.

On the other hand, by standard arguments we have

$$
<\phi(x)\phi(x')>_{\xi}^{-1} = \int_{\mathbb{R}}^\infty dse^{-st}
$$

(4.10)

Inserting (4.8)-(4.10) into (4.7) we finally obtain

$$
\ln Z = \frac{1}{2} \int_{\mathbb{R}}^\infty \frac{ds}{s} Tr K_{E_n}(s),
$$

$$
Tr K_{E_n}(s) = Tr \tilde{K}_{E_n}(s) - 4\pi \xi(1 - \alpha) s Tr_{\Sigma} \tilde{K}_{E_n}(s)
$$

(4.11)

where the $x$-integration in $Tr_{\Sigma}$ is taken only over the surface $\Sigma$. Identity (4.11) allows us to write the following expansion for the heat kernel $K_{E_n}(s)$:

$$
Tr K_{E_n}(s) = \frac{1}{(4\pi s)^{d/2}} \sum_n a_ns^n,
$$

$$
a_n = \int_{E_n} \tilde{a}_n(x, s) - 4\pi \xi(1 - \alpha) \int_{\Sigma} \tilde{a}_{n-1}(x, s)
$$

(4.12)

Since we are interested only in the first order of $(1 - \alpha)$ we may take $\tilde{a}_{n-1} = \tilde{a}^{st}_{n-1}$ in the r.h.s. of (4.12) neglecting the corresponding $\tilde{a}_{n-1,\alpha}$ term. One can see that $a_n$ has the same volume part $\tilde{a}^{st}_n$ as (4.9) (see (B.3)):

$$
a^{st}_0(x) = 1, \quad a^{st}_1 = \left(\frac{1}{6} - \xi\right) R
$$

$$
a^{st}_2(x) = \frac{1}{180} R_{\mu\nu\alpha\beta}^2 - \frac{1}{180} R_{\mu\nu}^2 - \frac{1}{6} (\frac{1}{5} - \xi) R + \frac{1}{2} \left(\frac{1}{6} - \xi\right)^2 R^2
$$

(4.13)
The difference appears in the surface term. For the few first coefficients we obtain (cf. (B.4)): 

\[ a_{0,0} = 0, \quad a_{1,0} = 4\pi (1 - \alpha) \left( \frac{1}{6} \left( \frac{1}{2\alpha} \right) - \xi \right), \]
\[ a_{2,0} = 4\pi (1 - \alpha) \left( \frac{1}{6} - \xi \right) \frac{1}{(2\alpha)} \frac{1}{\xi} \bar{R} - \frac{\pi}{180} \left( \frac{1}{\alpha^3} \right) (\bar{R}_{ij} - 2\bar{R}_{ijj}) \quad (4.14) \]

where \( \bar{R}_{ii} = \bar{R}_{\mu\nu \rho} n_i^\mu n_i^\nu \) and \( \bar{R}_{ijij} = \bar{R}_{\mu\nu \alpha \rho} n_i^\mu n_i^\nu n_j^\alpha n_j^\rho \).

Now we are ready to calculate the divergences of the effective action \( W_{eff} = -\ln Z \).

In four dimensions the infinite part of the effective action is the following

\[ W_{\text{div}} = -\frac{1}{32\pi^2} \left( \frac{1}{2} \frac{a_0}{\epsilon^2} + \frac{a_1}{\epsilon^2} + 2a_2 \ln \frac{L}{\epsilon} \right) \quad (4.15) \]

where \( L \) is infrared cut-off. Due to the same property (4.9) of the coefficients \( a_n \) (4.12) the \( W_{\text{div}} \) is a sum of volume and surface parts (3.1). Combining volume part of the one-loop action (4.15) with the tree-level one (3.2) we can see that divergences (under \( \epsilon \to 0 \)) are absorbed in the standard renormalization of the coupling constants [26]:

\[ \frac{1}{C_{\text{ren}}} = \frac{1}{G_B} + \frac{1}{2\pi \epsilon^2} \left( \frac{1}{6} - \xi \right), \quad c_{1,\text{ren}} = c_{1,B} - \frac{1}{32\pi^2} \left( \frac{1}{6} - \xi \right)^2 \ln \frac{L}{\epsilon} \]
\[ c_{2,\text{ren}} = c_{2,B} + \frac{1}{32\pi^2} \ln \frac{L}{\epsilon}; \quad c_{3,\text{ren}} = c_{3,B} - \frac{1}{32\pi^2} \frac{1}{90} \ln \frac{L}{\epsilon} \quad (4.16) \]

On the other hand, applying the formula

\[ S_{\text{div}} = (\alpha \partial_\alpha - 1) W_{\text{div}} \big|_{\alpha = 1} \]

we obtain the divergence of the entropy

\[ S_{\text{div}} = \frac{1}{8\pi \epsilon^2} \left( \frac{1}{6} - \xi \right) A_\Sigma + \left( \frac{1}{4\pi} \left( \frac{1}{6} - \xi \right)^2 \int_{\Sigma} \bar{R} - \frac{1}{16\pi} \frac{1}{45} \int_{\Sigma} (\bar{R}_{ii} - 2\bar{R}_{ijij}) \right) \ln \frac{L}{\epsilon} \quad (4.17) \]

We see that the complete entropy which is sum of the tree-level \( S(G_B, c_{i,B}) \) (3.4) and \( S_{\text{div}}(\epsilon) \) (4.17) becomes finite by the same renormalization of the constants (4.16) which renormalizes the effective action. So the identity (3.5) indeed holds.
For the minimal coupling ($\xi = 0$) the expression (4.17) has been obtained in [13]. In the conformal invariant case ($\xi = \frac{1}{6}$) the Newton constant $G$ and the coupling $c_1$ are not renormalized. Correspondingly, there are no area $A_\Sigma$ and $\int_\Sigma R$ contributions to the entropy (4.17) which is remarkably determined by only conformally invariant expression $\int_\Sigma (R_{ii} - 2R_{ij}ij)$.

It should be noted that our proof of the main statement is based on the nice property of the heat kernel coefficients $a_n$. Namely, up to $(1 - \alpha)^2$ terms the exact $a_n$ on manifold $E_\alpha$ occurs to be equal to the standard volume coefficient $\tilde{a}^{st}_n$ considered on manifold $E_\alpha$:

$$a_n(E_\alpha) = \int_{E_\alpha} \tilde{a}^{st}_n(x, x) + O((1 - \alpha)^2)$$

(4.18)

if one applies the formulas of Appendix A for curvatures on $E_\alpha$. Then up to $(1 - \alpha)^2$ the renormalization of entropy (3.5) directly follows from the renormalization of the effective action (3.6).

The curvature terms enter the matter action of the fields of different spins that gives rise to difficulties in operating with entanglement entropy [27]. We believe that our result can be certainly generalized also for these cases.

5 Remarks.

One can look at the entanglement entropy given by the expression (2.6) from quite different point of view. Consider the whole system (gravity plus matter) at arbitrary temperature $T = (2\pi\beta)^{-1}$. Then its partition function is given by the Euclidean functional integral over all fields defined on manifold with abelian isometry along the Killing vector $\partial_\varphi$. They are periodic with period $2\pi\beta$. Assumption that the system includes black hole means that there exists a surface $\Sigma$ (horizon) which is a fixed point of the isometry. Semiclassically, we take a metric satisfying these conditions and evaluate the quantum contribution of matter fields on this background. Then (2.5) and (2.7) are exactly such the partition function with the effective action $W(\beta, g_{\mu\nu}) = -\ln Z$ to be the functional of the temperature $\beta^{-1}$ and the metric $g_{\mu\nu}$. Taking its variation with respect to $\beta$ (when $g_{\mu\nu}$ fixed) gives us the statistical (entanglement) entropy $S_{\text{ent}} = (\beta\partial_\beta - 1)W(\beta, g_{\mu\nu})$ above considered.
On the other hand, taking temperature to be fixed we can find the corresponding equilibrium configuration which is extremum of the effective action $W(\beta, g_{\mu\nu})$. The entanglement entropy then is worth comparing with the thermodynamical entropy\(^3\) of a black hole. The latter is determined by total response of the one-loop free energy $F (\beta F = W)$ of the system being in thermal equilibrium on variation of temperature. So we must compare the free energies of two configurations being in equilibrium at slightly different temperatures. The equilibrium configuration corresponding to the fixed temperature $\beta$ is found from the extreme equation $\frac{\delta W(\beta, g_{\mu\nu})}{\delta g_{\mu\nu}}|_\beta = 0$. This extremum of the effective action is reached on regular manifolds without conical singularities and the equilibrium metric is function of the temperature $\beta$ and parameters fixing the macro-state of the system (mass $M$, charge $Q$, etc.)\(^4\). Now the equilibrium free energy $\beta F = W(\beta, g_{\mu\nu}(\beta))$ gives us the thermodynamical entropy $S^{TD} = (\beta d\beta - 1)W(\beta, g_{\mu\nu}(\beta))$. Note that for equilibrium states the total derivative $d_\beta W = \partial_\beta W + \frac{\delta W(\beta, g_{\mu\nu})}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta \beta}$ coincides with the partial. Then we obtain that two entropies indeed coincide, $S^{TD} = S_{\text{ent}}$.

However, in order to calculate $S^{TD}$ we must know exactly the form of the quantum-corrected configuration $g_{\mu\nu}(\beta)$ that is normally out of our knowledge. On the other hand the calculation of $S_{\text{ent}}$ does not require such an information and we can obtain exactly the entropy (off-shell) as a function of metric and its derivatives on the horizon $\Sigma$. It should be emphasized that there is no contribution to $W(\beta, g_{\mu\nu}(\beta))$ due to the conical singularity and we deal with the standard ultraviolet divergences coming from the bulk terms in the effective action. They result in the corresponding divergences of the entropy which are obviously regularized by the standard renormalization of the gravitational couplings. So in terms of the thermodynamical entropy our main statement holds automatically.

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\(^3\)I wish to thank V.P.Frolov for discussing this point.

\(^4\)Really the minimization of the functional $W(\beta, g_{\mu\nu})$ under $\beta$ fixed includes also variations in the space of macro-parameters. Therefore, the equilibrium state lies on the constraint $\beta = \beta(M, Q)$. 

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Appendix A: Curvature tensors on $E_\alpha$ [22].

Consider space $E_\alpha$ which is $\alpha$-fold covering of a smooth manifold $E$ along the Killing vector $\partial_\varphi$ generating abelian isometry. Let surface $\Sigma$ be a stationary point of this isometry and near $\Sigma$ space $E_\alpha$ looks as direct product $\Sigma \times C_{\alpha}$ of the surface $\Sigma$ and two-dimensional cone $C_{\alpha}$ with angle deficit $\delta = 2\pi (1-\alpha)$. Outside the singular surface $\Sigma$ the space $E_\alpha$ has the same geometry as smooth manifold $E$. In particular, their curvature tensors coincide. However, at the surface $\Sigma$ there exists a conical singularity which results in singular (delta-function like) contribution to the curvatures. To extract this contribution exactly one can use some regularization procedure replacing the singular space $E_\alpha$ by a sequence of regular manifolds $\tilde{E}_\alpha$. In the limit $\tilde{E}_\alpha \to E_\alpha$ we obtain the following result [22]:

\[
\begin{align*}
R^{\mu\nu}_{\alpha\beta} &= \tilde{R}^{\mu\nu}_{\alpha\beta} + 2\pi (1-\alpha) \left( (n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha) \right) \delta_\Sigma \\
R^{\mu}_{\nu} &= \tilde{R}^{\mu}_{\nu} + 2\pi (1-\alpha)(n^\mu n_\nu)\delta_\Sigma \\
R &= \tilde{R} + 4\pi (1-\alpha)\delta_\Sigma
\end{align*}
\]

(A.1)

where $\delta_\Sigma$ is the delta-function: $\int_M f \delta_\Sigma = \int_\Sigma f ; \; n^k = n^k_\mu dx^\mu$ are two orthonormal vectors orthogonal to $\Sigma$, $(n^\mu n_\nu) = \sum_{k=1}^2 n^k_\mu n^k_\nu$ and the quantities $\tilde{R}^{\mu\nu}_{\alpha\beta}, \tilde{R}^{\mu}_{\nu}$ and $\tilde{R}$ are computed in the regular points $E_\alpha/\Sigma$ by the standard method.

These formulas can be applied to define the integral expressions:

\[
\begin{align*}
\int_{E_\alpha} R &= \alpha \int_E \tilde{R} + 4\pi (1-\alpha) \int_\Sigma , \\
\int_{E_\alpha} R^2 &= \alpha \int_E \tilde{R}^2 + 8\pi (1-\alpha) \int_\Sigma \tilde{R} + \mathcal{O}(1-\alpha)^2 , \\
\int_{E_\alpha} R^{\mu\nu} R^{\mu\nu} &= \alpha \int_E \tilde{R}^{\mu\nu} \tilde{R}^{\mu\nu} + 4\pi (1-\alpha) \int_\Sigma \tilde{R}_{i\bar{i}} + \mathcal{O}(1-\alpha)^2 , \\
\int_{E_\alpha} R^{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} &= \alpha \int_E \tilde{R}^{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu\lambda\rho} + 8\pi (1-\alpha) \int_\Sigma \tilde{R}_{ij\bar{i}\bar{j}} + \mathcal{O}(1-\alpha)^2 ,
\end{align*}
\]

where $\tilde{R}_{i\bar{i}} = \tilde{R} n_i^\mu n_\mu^\nu$ and $\tilde{R}_{ij\bar{i}\bar{j}} = \tilde{R} n_i^\mu n_j^\lambda n_\lambda^\nu n_\rho^\rho$. The first integrals in right part of (A.2)-(A.5) are defined on the smooth space $E$; they are proportional to $\alpha$. The terms $\mathcal{O}(1-\alpha)^2$ in (A.3)-(A.5) are really something like $\delta_\Sigma^2$. They are ill-defined and turn to be dependent on the regularization prescription.
and singular in the limit $E_\beta \rightarrow \mathcal{M}_\beta$. But these terms are not important, for example, in calculation of entropy.

**Appendix B: The heat kernel expansion of operator $(- + \xi \bar{R})$ on $E_\alpha [20]$.**

Consider on space $E_\alpha$, possessing an abelian isometry, the operator $-\xi = - + \xi \bar{R}$, where $\bar{R}$ is regular part of the scalar curvature $R$ on $E_\alpha$ (see (A.1)). Then we have the following heat kernel expansion

$$\ln \det (-\xi) = \int_{\epsilon}^{\infty} \frac{ds}{s} Tr K_{E_\alpha}(s),$$

$$K_{E_\alpha}(s) = e^{-s\xi} = \frac{1}{(4\pi s)^d} \sum_n \bar{a}_n s^n, \quad s \rightarrow 0 \quad (B.1)$$

where the $\bar{a}_n(x, x)$

$$\bar{a}_n(x, x) = \bar{a}_{n_0}^x(x, x) + \bar{a}_{n,\alpha}(x, x) \delta_\Sigma \quad (B.2)$$

is sum of the standard coefficient $\bar{a}_{n_0}^x(x, x)$ for smooth manifold $E [26]$:

$$a_{n_0}^x = 1, \quad a_{n_0}^x = (\frac{1}{6} - \xi) \bar{R}$$

$$a_{n_0}^x = \frac{1}{180} \bar{R}^2_{\mu'\nu'\alpha\beta} - \frac{1}{180} \bar{R}^2_{\mu\nu} - \frac{1}{6} \frac{1}{5} - \xi \bar{R} + \frac{1}{2} \frac{1}{6} - \xi^2 \bar{R}^2 \quad (B.3)$$

and a part coming from the singular surface $\Sigma$ (stationary point of the isometry):

$$a_{0,\alpha} = 0; \quad a_{1,\alpha} = \frac{\pi (1 - \alpha)(1 + \alpha)}{3 \alpha} \int_\Sigma \sqrt{\gamma} d^2\theta;$$

$$a_{2,\alpha} = \frac{\pi (1 - \alpha)(1 + \alpha)}{3 \alpha} \int_\Sigma (\frac{1}{6} - \xi) \bar{R} \sqrt{\gamma} d^2\theta$$

$$- \frac{\pi (1 - \alpha)(1 + \alpha)(1 + \alpha^2)}{180 \alpha^3} \int_\Sigma \bar{R}_{\mu\nu} n^\mu_i n^\nu_j - 2 \bar{R}_{\mu'\nu'\alpha\beta} n^\mu_i n^\nu_j n^\alpha_i n^\beta_j \sqrt{\gamma} d^2\theta \quad (B.4)$$

where $n^i$ are two vectors orthogonal to surface $\Sigma$ ($n^i_i n^j_j g_{\mu\nu} = \delta_{ij}$) and $\gamma$ is metric on the surface $\Sigma$. 

14
References


