ONE-LOOP AMPLITUDES IN EUCLIDEAN QUANTUM GRAVITY

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This paper studies the linearized gravitational field in the presence of boundaries. For this purpose, ζ-function regularization is used to perform the mode-by-mode evaluation of BRST-invariant Faddeev-Popov amplitudes in the case of flat Euclidean four-space bounded by a three-sphere. On choosing the de Donder gauge-averaging term, the resulting ζ(0) value is found to agree with the space-time covariant calculation of the same amplitudes, which relies on the recently corrected geometric formulas for the asymptotic heat kernel in the case of mixed boundary conditions. Two sets of mixed boundary conditions for Euclidean quantum gravity are then compared in detail. The analysis proves that one cannot restrict the path-integral measure to transverse-traceless perturbations. By contrast, gauge-invariant amplitudes are only obtained on considering from the beginning all perturbative modes of the gravitational field, jointly with ghost modes.

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I. INTRODUCTION

In the Euclidean functional integral approach to quantum gravity [1], one deals with amplitudes written formally as path integrals over all Riemannian four-geometries matching the boundary data on (compact) Riemannian three-geometries $(\Sigma_1, h_1)$ and $(\Sigma_2, h_2)$. To take into account the gauge freedom of the theory, the path-integral measure also includes suitable ghost fields, described geometrically by a one-form, hereafter denoted by $\varphi = \varphi_\mu dx^\mu$ (see appendix), subject to boundary conditions at $(\Sigma_1, h_1)$ and $(\Sigma_2, h_2)$. Although a rigorous definition of the Feynman sum [2] over all Riemannian four-geometries with their topologies does not yet exist, the choice of boundary conditions still plays a key role to obtain a well-defined elliptic boundary-value problem, which may be applied to the semiclassical analysis of the quantum theory.

In quantum cosmology, it was proposed in Refs. [3,4] that no boundary conditions should be imposed at the three-geometry $(\Sigma_1, h_1)$, since this might shrink to a point in the case of the quantum state of the universe. One would then have to impose suitable boundary conditions only at $(\Sigma_2, h_2)$, by describing the quantum state of the universe in terms of a Euclidean path integral over all compact Riemannian four-geometries matching the boundary data at $(\Sigma_2, h_2)$. Although this approach to quantum cosmology still involves a number of formal definitions, the semiclassical evaluation of the corresponding wave function may be put on solid grounds. The one-loop analysis is related to mathematical and physical subjects such as cobordism theory (i.e. under which conditions a
compact manifold is the boundary of another compact manifold), the geometry of compact
Riemannian four-manifolds, the asymptotic heat kernel, the one-loop effective action and
the use of mixed boundary conditions in quantum field theory (see below).

In particular, over the last ten years many efforts have been produced to evaluate
one-loop quantum amplitudes for gauge fields and the gravitational field in the presence
of boundaries, either by using the space-time covariant Schwinger-DeWitt method [5] or
the mode-by-mode analysis which relies on $\zeta$-function regularization [6,7].

The main motivations were the need to understand the relation between different ap-
proaches to quantum field theories in the presence of boundaries, and the quantization
of closed cosmologies. Indeed, boundaries play an important role in the Feynman path-
integral approach to quantum gravity [8] as we just said, in choosing BRST-covariant and
gauge-invariant boundary conditions for quantum cosmology [9,10] and in studying dif-
ferent quantizazion and regularization techniques in field theory. In particular, for the latter
problem, discrepancies were found in the semiclassical evaluation of quantum amplitudes
by using space-time covariant methods, where the scaling factor of one-loop quantum am-
plitudes coincides with the Schwinger-DeWitt $A_2$ coefficient in the heat-kernel expansion,
or noncovariant methods, where the same factor can be computed within the framework
of $\zeta$-function regularization, and it is expressed through the $\zeta(0)$ value.

If one reduces a field theory with first-class constraints to its physical degrees of
freedom before quantization [11-14], one of the main problems is whether the resulting
quantum theory is equivalent to the theories relying on the Faddeev-Popov gauge-averaging
method or on the extended-phase-space Hamiltonian path integral of Batalin, Fradkin and
Vilkovisky, where one takes into account ghost and gauge modes [13-14]. We will see that, in a manifestly gauge-invariant formulation of such theories, there seem to be no unphysical modes, in that there are no subsets of the set of all perturbative modes whose effects cancel exactly the ones of ghost modes. This lack of cancellation turns out to be essential to achieve agreement between different techniques (see Sec. IV).

In Ref. [11], the \(\zeta(0)\) calculation was performed for gravitons by restricting the path-integral measure to transverse-traceless perturbations in the case of flat Euclidean four-space bounded by a three-sphere. In Refs. [15-17], this result was generalized to the part of the Riemannian de Sitter four-sphere bounded by a three-sphere. Both results did not coincide with those obtained by a space-time covariant method [9]. Hence the natural hypothesis arises that the possible non-cancellation of the contributions of gauge and ghost modes can be the cause of the discrepancy. In Refs. [18-21] such a suggestion was checked for the electromagnetic field on different manifolds and in different gauges.

In Ref. [22] the asymptotic heat kernel for second-order elliptic operators was obtained in the case of pure and mixed boundary conditions in real Riemannian four-manifolds, and in Ref. [23] this analysis has been improved. In the light of these results, the conformal anomalies on Einstein spaces with boundaries have been re-calculated in Ref. [24].

In Ref. [25], we calculated the \(\zeta(0)\) value for gravitons in the de Donder gauge on the part of flat four-dimensional Euclidean space bounded by two concentric three-spheres, taking into account the contribution of gauge modes and ghosts. The result was in agreement with the space-time covariant calculation. Hence we here investigate the linearized gravitational field in the geometric framework of Ref. [11] (i.e. flat Euclidean space bounded
by a three-sphere), and we compare the resulting $\zeta(0)$ value with the space-time covariant calculation of the same Faddeev-Popov amplitudes, by using the recently corrected geometric formulas for the asymptotic heat kernel in the case of mixed boundary conditions [24]. For our purposes, we use the version of the $\zeta$-function technique [6] elaborated in Refs. [15-17]. Hence we write $f_n(M^2)$ for the function occurring in the equation obeyed by the eigenvalues by virtue of boundary conditions, and $d(n)$ for the degeneracy of the eigenvalues parametrized by the integer $n$. One then defines the function

$$I(M^2, s) \equiv \sum_{n=n_0}^{\infty} d(n) n^{-2s} \ln f_n(M^2).$$

(1.1)

Such a function has an analytic continuation to the whole complex-$s$ plane as a meromorphic function, i.e.

$$"I(M^2, s)" = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s).$$

(1.2)

Then the $\zeta(0)$ value is

$$\zeta(0) = I_{\log} + I_{\text{pole}}(\infty) - I_{\text{pole}}(0),$$

(1.3)

where $I_{\log}$ is the coefficient of $\log M$ from $I(M^2, s)$ as $M \rightarrow \infty$, and $I_{\text{pole}}(M^2)$ is the residue at $s = 0$. The uniform asymptotic expansions of basis functions as both their order and $M$ tend to $\infty$, yield $I_{\log}$ and $I_{\text{pole}}(\infty)$, while the limiting form of basis functions as $M \rightarrow 0$ and $n \rightarrow \infty$ yields $I_{\text{pole}}(0)$ [15-17,19-21,25].

In Sec. II we consider the mixed boundary conditions of Refs. [9,10] and compute the $\zeta(0)$ value for the linearized gravitational field on the part of flat four-dimensional Euclidean space bounded by a three-sphere, taking into account the contributions of gauge
modes and ghosts. Sec. III studies instead the mixed boundary conditions first proposed in Ref. [26]. Sec. IV compares the results of Sec. II with the ones deriving from a geometric analysis of the asymptotic heat kernel. Concluding remarks are presented in Sec. V, and relevant details are given in the appendix.

II. LUCKOCK-MOSS-POLETTI
BOUNDARY CONDITIONS

In this section we evaluate \( \zeta(0) \) for the linearized gravitational field in the de Donder gauge [25]

\[
\Phi_{\nu}^{dD}(h) \equiv \nabla^\mu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \right),
\]

where \( h \) is the perturbation of the background four-metric \( g \), and \( \nabla \) is the Levi-Civita connection compatible with \( g \), i.e. \( \nabla g = 0 \). This gauge-averaging functional leads to the familiar form of the elliptic operator acting on metric perturbations. For example, in the case of a flat Euclidean background, this operator reduces to \(- \Box \equiv -\nabla^\mu \nabla_\mu \) [25].

Our background four-geometry is indeed flat Euclidean space bounded by a three-sphere, studied also in Ref. [11], and we take into account the contributions of gauge modes and ghosts.

In Refs. [9,10] mixed boundary conditions have been introduced for quantum gravity and quantum cosmology. They are motivated by the need to obtain BRST-covariant boundary conditions which lead to gauge-invariant quantum amplitudes (see, however, Sec. V). This request leads to Dirichlet boundary conditions on the normal component
The Luckock-Moss-Poletti (also referred to as LMP) boundary conditions read (n being the normal to the boundary) [9,10,25]

\[ \left[ \varphi_0 \right]_{\partial M} = 0, \] (2.5)

\[ \left[ (2K^\sigma_{\sigma} + n^\sigma \nabla_{\sigma}) n^\mu n^\nu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} h_{\alpha\beta} \right) \right]_{\partial M} = 0, \] (2.4)

\[ \left[ h_{0i} \right]_{\partial M} = 0, \] (2.3)

\[ \left[ h_{ij} \right]_{\partial M} = 0, \] (2.2)

where \( K_{\mu\nu} \) and \( P^\nu_{\mu} \equiv \delta^\nu_{\mu} - n^\mu n^\nu \) are the extrinsic-curvature tensor of the boundary and the tangential projection operator respectively.

The mode-by-mode form of the LMP boundary conditions appears already in Ref. [25], and hence we here re-write them only for the spatial components \( \varphi_i \) of the ghost, i.e.

\[ \left[ \partial \varphi_i / \partial \tau - \frac{2}{\tau} \varphi_i \right]_{\partial M} = 0, \] (2.7)

to correct an unfortunate mistake in Ref. [25], where the numerical coefficient of \( 1/\tau \) is incorrectly given (cf. Eqs. (3.3), (3.12)-(3.13) therein). That mistake did not affect the
$\zeta(0)$ value for ghosts, but it would affect the one-boundary calculations we are studying.

On inserting the mode-by-mode expansion of the components $\varphi_i(x, \tau)$ of the ghost one-form (see the appendix and Ref. [25]) into the boundary conditions (2.7) one obtains

$$\frac{d m_n}{d\tau} - \frac{2}{\tau} m_n = 0 \quad \text{at} \quad \partial M, \quad (2.8)$$

$$\frac{d p_n}{d\tau} - \frac{2}{\tau} p_n = 0 \quad \text{at} \quad \partial M. \quad (2.9)$$

Following the notation of the appendix and the technique used in Ref. [25], one finds the following contributions to $\zeta(0)$:

$$\zeta(0)_{\text{transverse-traceless modes}} = -\frac{278}{45}, \quad (2.10)$$

$$\zeta(0)_{\text{scalar modes}} = 18 - \frac{1}{60} + \frac{5}{2} - 9 - \frac{1}{6} - \frac{1}{180} = \frac{509}{45}, \quad (2.11)$$

$$\zeta(0)_{\text{partially decoupled modes}} = -2 - 17 = -19, \quad (2.12)$$

$$\zeta(0)_{\text{vector modes}} = 15 - \frac{41}{60} - \frac{2}{3} - \frac{31}{180} = \frac{1213}{90}, \quad (2.13)$$

$$\zeta(0)_{\text{decoupled vector mode}} = -\frac{21}{2}, \quad (2.14)$$

$$\zeta(0)_{\text{scalar ghost modes}} = -2 \left( \frac{119}{120} + \frac{899}{360} \right) = -\frac{314}{45}, \quad (2.15)$$

$$\zeta(0)_{\text{vector ghost modes}} = -2 \left( \frac{19}{120} + \frac{209}{360} \right) = -\frac{133}{90}, \quad (2.16)$$

$$\zeta(0)_{\text{decoupled ghost mode}} = -\frac{5}{2}. \quad (2.17)$$
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For example, in (2.11) we evaluate $I_{\log}$ by using the uniform asymptotic expansions of modified Bessel functions. After eliminating fake roots of order $(4n - 1)$ as $M \to 0$ one finds that the coefficient of $\ln M$ is $-(4n + 1)$. Hence one obtains

$$I_{\log} = \sum_{n=3}^{\infty} \frac{n^2}{2} (1 - 4n) = -2 \zeta_R(-3) + 18 + \frac{5}{2} = -\frac{1}{60} + 18 + \frac{5}{2} .$$ (2.18)

To evaluate $I_{\text{pole}}(\infty)$, the structure of the resulting $n$-dependent coefficient in the eigenvalue condition is

$$\sigma_\infty(n) = 12n \frac{(n^2 - 1)}{(n^2 - 4)} .$$ (2.19)

and it is easy to see that the term $\frac{n^2}{2} \ln(\sigma_\infty(n))$ does not contribute as $n \to \infty$ (i.e. it has no coefficient of $\frac{1}{n}$), and hence $I_{\text{pole}}(\infty) = 0$.

Last, as $M \to 0$ and $n \to \infty$, the $n$-dependent coefficient which contributes to $I_{\text{pole}}(0)$ is

$$\sigma_0(n) = 12 \Gamma^{-4}(n) \frac{(n + 1)(n + 4)(n - 1)^2}{n^3(n - 2)(n + 2)^2} .$$ (2.20)

Thus, using Stirling’s asymptotic expansion of the $\Gamma$-function [27], and after taking the coefficient of $\frac{1}{n}$ as $n \to \infty$ in $\frac{n^2}{2} \ln(\sigma_0(n))$, one finds

$$I_{\text{pole}}(0) = 9 + \frac{1}{6} + \frac{1}{180} .$$ (2.21)

Finally, by virtue of (2.10)-(2.17) one gets the full $\zeta(0)$ value for gravitons

$$\zeta(0) = -\frac{758}{45} .$$ (2.22)
Hence the contributions of gauge and ghost modes do not cancel each other (cf. Ref. [11]) and our result, as we will show in Sec. IV, coincides with that obtained by using the covariant Schwinger-DeWitt technique on the part of flat Euclidean four-space bounded by a three-sphere.

### III. BARVINSKY BOUNDARY CONDITIONS

The boundary conditions studied in Sec. II are not the only possible set of mixed boundary conditions for quantum gravity. As shown in Ref. [26], one can also set to zero at the boundary the gauge-averaging functional, the whole ghost one-form, and the perturbation of the induced three-metric. With the notation of Sec. II, after making a gauge transformation of the metric perturbation $h_{\mu\nu}$ according to the law [11]

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} + \nabla(\mu \varphi_\nu), \quad (3.1)$$

one finds in the de Donder gauge (denoting by $\lambda$ the eigenvalues of the elliptic operator $-g_{\mu\nu} \square - R_{\mu\nu}$)

$$\Phi^dD(\hat{h}) - \Phi^dD(h) = -\frac{1}{2} \left( g_{\mu\nu} \square + R_{\mu\nu} \right) \varphi^\mu = \frac{\lambda}{2} \varphi_\nu, \quad (3.2)$$

for any background with Ricci tensor $R_{\mu\nu}$. In our flat Euclidean background, the Ricci tensor vanishes, and on making a 3+1 split of the de Donder functional $\Phi^dD$ and of the ghost one-form $\varphi_\mu$, the boundary conditions proposed in Ref. [26] read

$$\left[ h_{ij} \right]_{\partial M} = \left[ \hat{h}_{ij} \right]_{\partial M} = 0 , \quad (3.3)$$

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\[
[\Phi_0^{dD}(h)]_{\partial M} = [\Phi_0^{dD}(\hat{h})]_{\partial M} = 0 ,
\]

\[
[\Phi_i^{dD}(h)]_{\partial M} = [\Phi_i^{dD}(\hat{h})]_{\partial M} = 0 ,
\]

\[
[\varphi_0]_{\partial M} = 0 ,
\]

\[
[\varphi_i]_{\partial M} = 0 .
\]

Note that the vanishing of the whole ghost one-form at the boundary ensures the invariance of the boundary conditions (3.3) under the transformations (3.1) (see also Eq. (5.1)). At that stage, the only remaining set of boundary conditions on metric perturbations, whose invariance under (3.1) is again guaranteed by (3.6)-(3.7), is given by (3.4)-(3.5) by virtue of (3.2). In this respect, these boundary conditions are the natural generalization of magnetic boundary conditions for Euclidean Maxwell theory, where one sets to zero at the boundary the tangential components of the potential, the gauge-averaging functional, and hence the ghost zero-form [18-21]. The boundary conditions (3.3)-(3.7) were considered in Ref. [26] as part of the effort to understand the relation between the wave function of the universe and the effective action in quantum field theory. The loop expansion in quantum cosmology was then obtained after a thorough study of boundary conditions for the propagator [26].

In the light of (3.3), the boundary conditions (3.4)-(3.5) lead to mixed boundary conditions on the metric perturbations which take the form

\[
\left[ \frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} \left( g^{ij} h_{ij} \right) + \frac{2}{\tau^2} h_{0i} \right]_{\partial M} = 0 ,
\]

\[
\left[ \frac{\partial h_{0i}}{\partial \tau} + \frac{3}{\tau} h_{0i} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \right]_{\partial M} = 0 .
\]
To evaluate the scaling behaviour of the corresponding one-loop amplitudes, it is necessary to write down the mode-by-mode form of the boundary conditions (3.8)-(3.9), (3.3) and (3.6)-(3.7). They lead to (see Eqs. (A1)-(A13) of the appendix)

\[
\frac{da_n}{d\tau} + \frac{6}{\tau} a_n - \frac{1}{\tau^2} \frac{dc_n}{d\tau} - \frac{2}{\tau^2} b_n = 0 \quad \text{at} \quad \partial M , \tag{3.10}
\]

\[
\frac{db_n}{d\tau} + \frac{3}{\tau} b_n - \left(\frac{n^2 - 1}{2}\right) a_n = 0 \quad \text{at} \quad \partial M , \tag{3.11}
\]

\[
\frac{dc_n}{d\tau} + \frac{3}{\tau} c_n = 0 \quad \text{at} \quad \partial M , \tag{3.12}
\]

\[
d_n = 0 \quad \text{at} \quad \partial M , \tag{3.13}
\]

\[
e_n = 0 \quad \text{at} \quad \partial M , \tag{3.14}
\]

\[
f_n = 0 \quad \text{at} \quad \partial M , \tag{3.15}
\]

\[
k_n = 0 \quad \text{at} \quad \partial M , \tag{3.16}
\]

\[
l_n = 0 \quad \text{at} \quad \partial M , \tag{3.17}
\]

\[
m_n = 0 \quad \text{at} \quad \partial M , \tag{3.18}
\]

\[
p_n = 0 \quad \text{at} \quad \partial M . \tag{3.19}
\]

On using the technique outlined in the introduction and applied also in Sec. II and in Ref. [25], the corresponding contributions to \(\zeta(0)\) are found to be (see the appendix for details)

\[
\zeta(0)_{\text{transverse-traceless modes}} = -\frac{278}{45} , \tag{3.20}
\]

\[
\zeta(0)_{\text{scalar modes}} = 18 - \frac{1}{60} - 1 - \frac{1}{180} = \frac{764}{45} , \tag{3.21}
\]
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\[ \zeta(0)_{\text{partially decoupled modes}} = -2 - 15 = -17, \]  
\[ \zeta(0)_{\text{vector modes}} = 12 - \frac{11}{60} - \frac{2}{3} - \frac{31}{180} = \frac{494}{45}, \]  
\[ \zeta(0)_{\text{decoupled vector mode}} = -\frac{15}{2}, \]  
\[ \zeta(0)_{\text{scalar ghost modes}} = -2 \left( \frac{179}{120} + \frac{59}{360} \right) = -\frac{149}{45}, \]  
\[ \zeta(0)_{\text{vector ghost modes}} = -2 \left( -\frac{41}{120} - \frac{31}{360} \right) = \frac{77}{90}, \]  
\[ \zeta(0)_{\text{decoupled ghost mode}} = \frac{5}{2}. \]  

Hence the full \( \zeta(0) \) for linearized gravity subject to the Barvinsky boundary conditions [26] is found to be

\[ \zeta(0) = -\frac{241}{90}. \]  

Note that the result (3.28) differs from the one obtained in the previous section: \( \zeta(0) = -\frac{758}{45} \). As far as we can see, this property reflects two different physical situations described by two different sets of mixed boundary conditions (see also the comments in Sec. V). In the following section, the result (2.22) is checked by using geometric formulas for the asymptotic heat kernel.
IV. GEOMETRIC RESULTS ON THE HEAT KERNEL

In this section we calculate by means of a geometric method the Schwinger-DeWitt coefficient $A_2$ for the case of Luckock-Moss-Poletti boundary conditions. The result obtained coincides with the one found in Sec. II by a mode-by-mode analysis. The space-time covariant formulas for a second-order elliptic operator on a manifold with boundaries in the case of pure and mixed boundary conditions were obtained in Ref. [22] and corrected in Ref. [23]. We use them in the form presented in Ref. [24].

The Schwinger-DeWitt coefficient $A_2$ for the elliptic operator

$$-D_\mu D^\mu + X,$$

where $D_\mu$ is a gauge derivative with curvature $F_{\mu\nu}$, can be written as

$$16\pi^2 A_2 = \int_\mathcal{M} b_2 d\mu + \int_{\partial\mathcal{M}} c_2 d\mu .$$

(4.1)

The volume coefficient $b_2$ is well-known [5], while surface terms depend upon the choice of boundary conditions. We use mixtures of Dirichlet and Robin boundary conditions,

$$P_- \phi = 0 , \quad (\psi + n^\sigma \nabla_\sigma)P_+ \phi = 0 ,$$

(4.2)

where $P_\pm$ are projection operators [9,10,24].

The results can be expressed in terms of polynomials in the curvature tensor $R_{\mu\nu\alpha\beta}$ of the background four-manifold and in terms of the extrinsic-curvature tensor of the
boundary (hereafter $R$ is the trace of the Ricci tensor, and $K$ is the trace of $K_{\mu\nu}$), i.e.

$$q \equiv \frac{8}{3} K^3 + \frac{16}{3} K_{\mu}^{\alpha} K_{\nu}^{\alpha} K_{\mu}^{\nu} - 8 K K_{\mu\nu} K_{\mu\nu} + 4 K R - 8 R_{\mu\nu}(K n^{\mu} n^{\nu} + K^{\mu\nu}) + 8 R_{\mu\nu\alpha\beta} K^{\mu\alpha} n^{\nu} n^{\beta},$$  

(4.3)

and

$$p \equiv K_{\mu}^{\nu} K_{\nu}^{\alpha} K_{\mu}^{\mu} - K K_{\mu\nu} K_{\mu\nu} + \frac{2}{9} K^3.$$

(4.4)

For Dirichlet boundary conditions [24]

$$c_2^D = \text{Tr} \left[ -\frac{1}{360} q + \frac{2}{35} p - \frac{1}{3} \left( X - \frac{1}{6} R \right) K - \frac{1}{2} n^\sigma \nabla_\sigma \left( X - \frac{1}{6} R \right) + \frac{1}{15} C_{\mu\nu\alpha\beta} K^{\mu\alpha} n^{\nu} n^{\beta} \right],$$  

(4.5)

while for Robin boundary conditions [24]

$$c_2^R = \text{Tr} \left[ -\frac{1}{360} q + \frac{2}{45} p - \frac{1}{3} \left( X - \frac{1}{6} R \right) K + \frac{1}{2} n^\sigma \nabla_\sigma \left( X - \frac{1}{6} R \right) - \frac{4}{3} \left( \psi - \frac{1}{3} K \right)^3 + 2 \left( X - \frac{1}{6} R \right) \psi \left( \psi - \frac{1}{3} K \right) \left( \frac{2}{45} K^2 - \frac{2}{15} K_{\mu\nu} K_{\mu\nu} \right) + \frac{1}{15} C_{\mu\nu\alpha\beta} K^{\mu\alpha} n^{\nu} n^{\beta} \right].$$  

(4.6)

For mixed boundary conditions [23,24]

$$c_2 = \text{Tr} \left[ P_+ c_2^R + P_- c_2^D - \frac{2}{15} (P_+ |\psi)(P_+ |\psi) K - \frac{4}{15} (P_+ |\psi)(P_+ |\psi) K^{ij} + \frac{4}{3} (P_+ |\psi)(P_+ |\psi) P_+ \psi - \frac{2}{3} P_+(P_+ |\psi) n^\mu F_{\mu\rho} \right],$$  

(4.7)

where Latin indices run from 1 to 3, Greek indices run from 0 to 3, and the stroke denotes (as in the rest of our paper) covariant differentiation tangentially with respect to the
three-dimensional Levi-Civita connection of the boundary [22]. The Luckock-Moss-Poletti boundary conditions for gravitons imply that [9]

\[ \left( P_{+\text{gravitons}} \right)^{\alpha\beta}_{\mu\nu} \equiv n_\mu n_\nu \left( 2n^\alpha n^\beta - g^\alpha\beta \right), \]  

(4.8)

and

\[ \psi_{\text{gravitons}} = 2K. \]  

(4.9)

Similarly, for ghosts one has

\[ \left( P_{+\text{ghosts}} \right)^{\nu}_{\mu} \equiv \delta^{\nu}_{\mu} - n_\mu n^\nu, \]  

(4.10)

and

\[ \psi_{\text{ghosts}} = -\frac{K}{3}. \]  

(4.11)

Note that, on taking traces for gravitons in Eq. (4.7), one has to use the generalized Kronecker symbol

\[ \delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2} \left( \delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu \right). \]

Now inserting (4.8) and (4.9) into Eqs. (4.1), (4.6)-(4.7) one obtains for gravitons

\[ A_{2\text{gravitons}} = -\frac{98}{9}. \]  

(4.12)

Analogously, using the expressions (4.10)-(4.11) one finds for ghosts

\[ A_{2\text{ghosts}} = -\frac{268}{45}. \]  

(4.13)
These results are in full agreement with those in Sec. II.

V. CONCLUSIONS

Our paper has studied one-loop quantum gravity with boundaries in the limiting case of small three-geometries, which is relevant for quantum cosmology [11]. This leads to the analysis of flat Euclidean four-space bounded by a three-sphere, with mixed boundary conditions on metric perturbations. The results of our investigation are as follows.

First, we have completed and improved the analysis of Luckock-Moss-Poletti mixed boundary conditions [9,10] considered in our earlier work [25]. These are motivated by the analysis of BRST transformations at the boundary, and are more relevant for supersymmetric theories of gravitation [7,10]. The eight contributions to the full $\zeta(0)$ for Faddeev-Popov amplitudes have been calculated in detail by means of a mode-by-mode analysis, and the resulting $\zeta(0)$ value has been found to agree with the geometric theory of the asymptotic heat kernel. The latter relies on the results appearing in Refs. [22-24].

Second, we have performed a detailed analysis of the mixed boundary conditions for quantum gravity proposed by Barvinsky in Ref. [26]. Their main merit is the invariance under the transformations (3.1) on metric perturbations. The technique of $\zeta$-function regularization, jointly with a mode-by-mode analysis, has made it possible to obtain the full $\zeta(0)$ value as in Eq. (3.28). Indeed, this result differs from the one obtained in Sec. II. Thus, different boundary conditions lead to different semiclassical wavefunctions, with the exception of Euclidean Maxwell theory, where magnetic or electric boundary conditions
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correspond to the same semiclassical amplitudes in the Lorentz gauge [20-21]. It should be emphasized that, in Sec. II, the invariance of the boundary conditions (2.3) under (3.1) leads to the Robin boundary conditions (2.6) on the ghost one-form. However, this implies that the boundary conditions (2.2) are not invariant under (3.1), since

\[ \hat{h}_{ij} - h_{ij} = \varphi_{(i|j)} + K_{ij} \varphi_0. \]  

(5.1)

In other words, the right-hand side of (5.1) does not vanish at \( \partial M \) if (2.6) holds, because it reduces to \[ \left[ \varphi_{(i|j)} \right]_{\partial M} \] by virtue of (2.5). Indeed, in Ref. [9] the authors acknowledge that the boundary conditions (2.2)-(2.6) are not entirely BRST invariant, but they say, without explicit proof, that this does not affect the gauge invariance of the resulting quantum amplitudes. Hence we found it appropriate to consider also the boundary conditions of Sec. III, which are instead completely invariant under the transformations (3.1). On the other hand, the boundary conditions of Sec. II are motivated by self-adjointness theory [9] and are in agreement with the results on boundary conditions for one-forms, which should be mixed for gauge fields [23,28]. Hence the boundary conditions of Sec. III are less natural in this respect.

Third, our analysis proves that, also in the one-boundary problem, one cannot restrict the path-integral measure to transverse-traceless perturbations. By contrast, ghost modes and the whole set of perturbative modes of the gravitational field are all necessary to obtain gauge-invariant (one-loop) amplitudes.

It now remains to be seen what happens on considering arbitrary relativistic gauges and curved backgrounds. As one already knows from Euclidean Maxwell theory, it is not
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possible to diagonalize the operator matrix of the problem for arbitrary gauge conditions [21]. In this respect, it appears interesting to characterize all relativistic gauge conditions which enable one to express the perturbative modes as linear combinations of Bessel functions [20,21,25]. Moreover, it is necessary to obtain geometric formulas for \( \zeta(0) \) in the case of Barvinsky boundary conditions studied in Sec. III. These boundary conditions are not naturally expressed in terms of projection operators, and hence the method of Refs. [22-24] cannot be applied. Nevertheless, work is in progress on this problem by the author of Refs. [23,28] and his collaborators. Last, but not least, the analysis of non-relativistic gauges for Euclidean quantum gravity within the Faddeev-Popov formalism, and the non-local nature of the one-loop effective action, deserve careful consideration.

If one wants to relate our analysis to the Lorentzian theory, there is also the problem of interpreting the quantum state corresponding to the boundary conditions of Secs. II and III. Moreover, the impossibility to restrict the measure of the Euclidean path integral to transverse-traceless perturbations raises further interpretive issues for the Lorentzian theory, where such a reduction to physical degrees of freedom is instead quite natural. Thus, although the path-integral formulation of Euclidean quantum gravity [29] does not provide a mathematically consistent theory of the quantized gravitational field, the detailed calculations and the open problems presented in our paper seem to add evidence in favour of quantum cosmology having a deep influence on modern quantum field theory in four-dimensions [7,30].
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APPENDIX

Following Ref. [25], the metric perturbations are expanded on a family of three-spheres centred on the origin as

\[ h_{00}(x, \tau) = \sum_{n=1}^{\infty} a_n(\tau) Q^{(n)}(x), \]

\[ h_{0i}(x, \tau) = \sum_{n=2}^{\infty} \left[ b_n(\tau) \frac{Q^{(n)}_i(x)}{(n^2 - 1)} + c_n(\tau) S^{(n)}_i(x) \right], \]

\[ h_{ij}(x, \tau) = \sum_{n=3}^{\infty} \left[ d_n(\tau) \left( \frac{Q^{(n)}_{ij}(x)}{(n^2 - 1)} + \frac{c_{ij}}{3} Q^{(n)}(x) \right) \right] + \sum_{n=1}^{\infty} \frac{e_n(\tau)}{3} c_{ij} Q^{(n)}(x) \]
where $Q^{(n)}(x), S_i^{(n)}(x), G_{ij}^{(n)}(x)$ are scalar, transverse vector, and transverse-traceless tensor hyperspherical harmonics, respectively, on a unit three-sphere with metric $c_{ij}$. The components $\varphi_0(x, \tau)$ and $\varphi_1(x, \tau)$ of the ghost one-form are expanded as in (A1)-(A2), providing one replaces $a_n(\tau), b_n(\tau), c_n(\tau)$ by the modes $l_n(\tau), m_n(\tau)$ and $p_n(\tau)$ respectively [25]. Note that, strictly, our ghost one-form corresponds to a ghost $\eta_\mu$ and an anti-ghost $\bar{\eta}_\mu$. They obey the same boundary conditions introduced for $\varphi_\mu$ [10], and their contribution to $\zeta(0)$ is obtained by using the multiplicative factor -2, as in Secs. II-IV.

In the one-boundary problems studied in our paper, regular perturbative modes are only obtained by setting to zero the coefficients multiplying the modified Bessel functions $K_n$, since such functions are singular at the origin of flat Euclidean four-space. Hence one finds for scalar-type gravitational perturbations (cf. Ref. [25])

$$a_n(\tau) = \frac{1}{\tau} \left[ \gamma_1 I_n(M\tau) + \gamma_3 I_{n-2}(M\tau) + \gamma_4 I_{n+2}(M\tau) \right],$$

$$b_n(\tau) = \gamma_2 I_n(M\tau) + (n + 1) \gamma_3 I_{n-2}(M\tau) - (n - 1) \gamma_4 I_{n+2}(M\tau),$$

$$d_n(\tau) = \tau \left[ -\gamma_2 I_n(M\tau) + \frac{(n + 1)}{(n - 2)} \gamma_3 I_{n-2}(M\tau) + \frac{(n - 1)}{(n + 2)} \gamma_4 I_{n+2}(M\tau) \right],$$

$$c_n(\tau) = \tau \left[ 3\gamma_1 I_n(M\tau) - 2\gamma_2 I_n(M\tau) - \gamma_3 I_{n-2}(M\tau) - \gamma_4 I_{n+2}(M\tau) \right].$$
The basis functions for vector-like gravitational perturbations are

\begin{equation}
\epsilon_n(\tau) = \tilde{\epsilon}_1 I_{n+1}(M\tau) + \tilde{\epsilon}_2 I_{n-1}(M\tau),
\end{equation}

\begin{equation}
f_n(\tau) = \tau \left[ -\frac{1}{n+2} \tilde{\epsilon}_1 I_{n+1}(M\tau) + \frac{1}{n-2} \tilde{\epsilon}_2 I_{n-1}(M\tau) \right],
\end{equation}

and the basis function for transverse-traceless symmetric tensor harmonics reads

\begin{equation}
k_n = a_1 \tau I_n(M\tau).
\end{equation}

Finally the basis functions for ghosts are

\begin{equation}
l_n(\tau) = \frac{1}{\tau} \left[ \kappa_1 I_{n+1}(M\tau) + \kappa_2 I_{n-1}(M\tau) \right],
\end{equation}

\begin{equation}
m_n(\tau) = -(n-1)\kappa_1 I_{n+1}(M\tau) + (n+1)\kappa_2 I_{n-1}(M\tau),
\end{equation}

\begin{equation}
p_n(\tau) = \partial I_n(M\tau).
\end{equation}

The equations (A4)-(A13) hold both in Sec. II and in Sec. III, and we here focus on the latter application, since the former was treated in detail in Ref. [25].

On inserting the form of \(a_n(\tau), b_n(\tau), d_n(\tau), e_n(\tau)\) into the boundary conditions (3.10)-(3.11) and (3.13)-(3.14), one gets an eigenvalue condition for coupled scalar modes given by the vanishing of the determinant of a 4 × 4 matrix. As \(M \to 0\), fake roots of order \(4n\) are found to arise. In the calculation of \(I_{\log}\), the factors \(\sqrt{M}\) and \(\frac{1}{\sqrt{M}}\) compensate each other as \(M \to \infty\), and hence \(I_{\log}\) is found to be

\begin{equation}
I_{\log} = \sum_{n=3}^{\infty} \frac{n^2}{2} (-4n) = -2\zeta_6(-3) + 18 = -\frac{1}{60} + 18.
\end{equation}
Moreover, as $M \to \infty$ and $n \to \infty$, the $n$-dependent coefficient in the eigenvalue condition takes the form (cf. Eq. (2.19))

$$\rho_\infty(n) = 12n \frac{(n^2 - 1)}{(n^2 - 4)}.$$  \hspace{1cm} (A15)

Since $\frac{\rho_\infty(n)}{n}$ is an even function of $n$, the term $\frac{n^2}{2}\ln(\rho_\infty(n))$ has no coefficient of $\frac{1}{n}$ as $n \to \infty$, and hence $I_{\text{pole}}(\infty) = 0$.

By contrast, as $M \to 0$ and $n \to \infty$, the $n$-dependent coefficient in the eigenvalue condition takes the form (cf. Eq. (2.20))

$$\rho_0(n) = \Gamma^{-4}(n) \left(1 - \frac{1}{n}\right) \frac{48}{(n + 1)(n - 2)}.$$ \hspace{1cm} (A16)

Thus, by virtue of Stirling’s asymptotic expansion [27]

$$\ln \Gamma(n) \sim \left(n - \frac{1}{2}\right) \ln(n) - n + \frac{1}{2} \ln(2\pi) + \frac{1}{12} \frac{1}{n} - \frac{1}{360} \frac{1}{n^3} + O(n^{-5}),$$ \hspace{1cm} (A17)

the $I_{\text{pole}}(0)$ value, which is the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \to \infty$ of $\frac{n^2}{2}\ln(\rho_0(n))$, turns out to be

$$I_{\text{pole}}(0) = \frac{1}{180} - \frac{1}{6} + \frac{4}{3} - \frac{1}{6} = \frac{181}{180}.$$ \hspace{1cm} (A18)

The same technique yields the contributions (3.22)-(3.27) to the full $\zeta(0)$, bearing in mind that, for decoupled modes, the contribution to $\zeta(0)$ is only given by the $I_{\log}$ coefficient [15-17].


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