A SELECTION OF FORMULAE AND DATA USEFUL
FOR THE DESIGN OF A.G. SYNCHROTRONS

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EDITOR'S NOTE

The present collection of formulae and data was first written up as a CERN internal report (MPS-SI/Int. DL/68-3). In view of the friendly reception (and the rapid "sale" of 600 copies) it has now been newly edited and distributed more widely.

The changes consisted in eliminating about half a dozen errors, improving the clarity of text and notation, and adding some formulae and a more complete table of RF "bucket" parameters (Appendix C).


We apologize to those who prefer the larger (A4) size, but there was a (slight) majority in favour of the smaller size, easier to handle and to ship (2000 copies printed).

K.H. Reich

Meyrin, 23.3.1970

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x) The present preprints of size A4 are strictly limited in number and will not be reproduced.
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### LIST OF FREQUENTLY OCCURRING SYMBOLS, THEIR MEANINGS AND UNITS

<table>
<thead>
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<th>Description</th>
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<tr>
<td>A</td>
<td>RF &quot;bucket&quot; area (in longitudinal phase plane **)) [see page 31]</td>
</tr>
<tr>
<td>$A_H$</td>
<td>acceptance in horizontal phase plane ** [see page 18] (= area of largest</td>
</tr>
<tr>
<td>$A_V$</td>
<td>acceptance in vertical phase plane ** [see page 18] acceptable ellipse/π)</td>
</tr>
<tr>
<td>$B, B_0$</td>
<td>magnetic flux density, in teslas[T], nominal value</td>
</tr>
<tr>
<td>$C, C_0$</td>
<td>length of orbit [m], nominal value</td>
</tr>
<tr>
<td>$c$</td>
<td>velocity of light [m/s]</td>
</tr>
<tr>
<td>$e$</td>
<td>electronic charge [C]</td>
</tr>
<tr>
<td>eV</td>
<td>maximum energy gain per turn [keV] unless units</td>
</tr>
<tr>
<td>$E$</td>
<td>total energy of particle [GeV]</td>
</tr>
<tr>
<td>$E_0$</td>
<td>rest energy of particle [GeV]</td>
</tr>
<tr>
<td>$f_a$</td>
<td>accelerating frequency [Hz]</td>
</tr>
<tr>
<td>$f$</td>
<td>revolution frequency [Hz]</td>
</tr>
<tr>
<td>$f_\infty$</td>
<td>asymptotic value of $f_a$ reached at $\beta = 1$</td>
</tr>
<tr>
<td>$g$</td>
<td>gradient of magnetic field, in teslas per metre [Tm⁻¹]</td>
</tr>
<tr>
<td>$h$</td>
<td>harmonic number = $f_a/f$</td>
</tr>
<tr>
<td>$K$</td>
<td>focal constant [m⁻²]</td>
</tr>
<tr>
<td>$m$</td>
<td>mass of particle [GeV/c²]</td>
</tr>
<tr>
<td>$m_p$</td>
<td>mass of proton [GeV/c²]</td>
</tr>
<tr>
<td>$n$</td>
<td>field index = $(-\rho_0/B_0)(\partial B/\partial x)$</td>
</tr>
<tr>
<td>$p, p_0$</td>
<td>momentum of particle [GeV/c], nominal value</td>
</tr>
<tr>
<td>$Q$</td>
<td>number of betatron oscillations per revolution</td>
</tr>
<tr>
<td>$R, R_0$</td>
<td>mean orbit radius (= $C/2\pi$), nominal value, [m] unless stated otherwise</td>
</tr>
<tr>
<td>$T$</td>
<td>kinetic energy of particle [GeV]</td>
</tr>
<tr>
<td>$V$</td>
<td>peak accelerating voltage per turn [kV]</td>
</tr>
</tbody>
</table>

*) In square brackets

**) With these definitions the available six-dimensional hypervolume is $A_6 = \pi^2 A_H A_V R A$ where $A$ is in $(\Delta p/m_0 c) - \psi$ coordinates.
$a_p$ momentum compaction factor

$\alpha(s)$ Twiss parameter

$\beta$ ratio of particle velocity to that of light ($= v/c$)

$\beta(s), \beta_{H,V}$ betatron amplitude function

$\Gamma = \sin \phi_s$ where $\phi_s$ refers to the synchronous particle

$\gamma(s)$ Twiss parameter

$\Theta$ deflection angle

$\epsilon$ emittance in transverse plane [see page 18] ($= \text{area of ellipse}/\pi$

$\epsilon_H$ horizontal beam emittance* [see page 18] occupied by beam in respective plane

$\epsilon_V$ vertical beam emittance* [see page 18]

$\phi$ azimuthal angle

$\mu$ phase shift of betatron oscillation for one focusing period

$\rho$ bending radius [m], positive from centre towards outside

$\phi$ "phase angle" between particle and zero crossing of RF voltage

$\phi_s$ "phase angle" for synchronous (phase stationary) particle

$\phi(s)$ phase advance of the betatron oscillation

Other symbols are defined as they occur.

Coordinate system of particle: (Definitions of $s, x, y, z$ as in Courant and Snyder[1,3**])

$s$ distance along beam axis

$x$ horizontal transverse coordinate, same sign as $\rho$

$z$ vertical transverse coordinate, positive towards sky

$y$ general transverse coordinate

$<x>$ arithmetic mean of $x$

$\text{r.m.s. value of } x$

A prime denotes differentiation with respect to $s$.

A dot denotes differentiation with respect to time.

*) With these definitions the six-dimensional invariant hypervolume occupied by the beam is $V_e = (\pi \beta \gamma)^2 \epsilon_H \epsilon_V R S_\phi$ where $S_\phi$ is the area [in $(\Delta p/m c) - \phi$ coordinates] occupied by one bunch in the longitudinal phase plane

**) See page 43 for references
PART I

BASIC RELATIONS
1. PARTICLE VELOCITY, MOMENTUM AND ENERGY

1.1 Relations between $\beta$, $cp$, $E_0$, $T$, $E$, $\gamma$

<table>
<thead>
<tr>
<th>In terms of Wanted</th>
<th>$\beta$</th>
<th>$cp$</th>
<th>$T$</th>
<th>$E$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = \beta$</td>
<td>$[(E_0/cp)^2 + 1]^{-\gamma_2}$</td>
<td>$[1 - (1 + T/E_0)^{-2}]^{\gamma_2}$</td>
<td>$[1 - (E_0/E)^2]^{\gamma_2}$</td>
<td>$(1 - \gamma^{-2})^{\gamma_2}$</td>
<td></td>
</tr>
<tr>
<td>$cp = E_0(\beta^{-2} - 1)^{-\gamma_2}$</td>
<td>$cp/E$</td>
<td>$[T(2E_0 + T)]^{\gamma_2}$</td>
<td>$(E^2 - E_0^2)^{\gamma_2}$</td>
<td>$E_0(\gamma^2 - 1)^{\gamma_2}$</td>
<td></td>
</tr>
<tr>
<td>$E_0 = cp/\beta^{\gamma_2}$</td>
<td>$cp(\gamma^2 - 1)^{-\gamma_2}$</td>
<td>$T/(\gamma - 1)$</td>
<td>$(E^2 - c^2p^2)^{\gamma_2}$</td>
<td>$E/\gamma$</td>
<td></td>
</tr>
<tr>
<td>$T = [(1 - \beta^2)^{-\gamma_2} - 1]E_0$</td>
<td>$[E_0^5 + c^2p^2]^{\gamma_2} - E_0$</td>
<td>$T$</td>
<td>$E - E_0$</td>
<td>$E_0(\gamma - 1)$</td>
<td></td>
</tr>
<tr>
<td>$\gamma = (1 - \beta^2)^{-\gamma_2}$</td>
<td>$cp/E_0\beta$</td>
<td>$[1 - (cp/E_0)^2]^{\gamma_2}$</td>
<td>$1 + T/E_0$</td>
<td>$E/E_0$</td>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

In a synchrotron:

$\beta = 2\pi f/c$ \hspace{1cm} ($f = f_a/h$)

and $p[GeV/c] = 0.2997925 B\rho \text{ [Tm]}$, or $p \text{ [VAS}^2\text{s}^{-1}] = eB\rho \text{ [As Tm]}$. 
1.2 First Derivatives

<table>
<thead>
<tr>
<th>In terms of Wanted</th>
<th>dβ</th>
<th>d(cp)</th>
<th>dy = dE/E₀ = dT/E₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>dβ =</td>
<td>dβ</td>
<td>[1 + (cp/E₀)^2]^-γ/2 d(cp)/E₀</td>
<td>γ^-2(y^2 - 1)^-γ/2 dy</td>
</tr>
<tr>
<td>d(cp) =</td>
<td>E₀(1 - β^2)^-γ/2 dβ</td>
<td>d(cp)</td>
<td>E₀γ(y^2 - 1)^-γ/2 dy</td>
</tr>
<tr>
<td>dy =</td>
<td>β(1 - β^2)^-γ/2 dβ</td>
<td>[1 + (E₀/cp)^2]^-γ/2 d(cp)/E₀</td>
<td>β^-1 y^-3 dy</td>
</tr>
<tr>
<td>= dE/E₀ =</td>
<td>βγ^3 dβ</td>
<td>β d(cp)/E₀</td>
<td></td>
</tr>
<tr>
<td>= dT/E₀ =</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.3 Logarithmic first derivatives

<table>
<thead>
<tr>
<th>In terms of Wanted</th>
<th>dβ/β</th>
<th>dp/p</th>
<th>dT/T</th>
<th>dE/E = dy/y</th>
</tr>
</thead>
<tbody>
<tr>
<td>dβ/β =</td>
<td>dβ/β</td>
<td>γ^-2 dp/p</td>
<td>[γ(γ + 1)]^-1 dT/T</td>
<td>(γ^2 - 1)^-1 dy/y</td>
</tr>
<tr>
<td>= dp/p =</td>
<td>γ^2 dβ/β</td>
<td>dp/p</td>
<td>[γ/(γ + 1)] dT/T</td>
<td>β^-2 dy/y</td>
</tr>
<tr>
<td>= dT/T =</td>
<td>γ(γ + 1) dβ/β</td>
<td>(1 + γ^-1) dp/p</td>
<td>dT/T</td>
<td>γ(γ - 1)^-1 dy/y</td>
</tr>
<tr>
<td>= dE/E =</td>
<td>(βγ)^2 dβ/β</td>
<td>β^2 dp/p</td>
<td>(1 - γ^-1) dT/T</td>
<td>dy/y</td>
</tr>
<tr>
<td>= dy/y =</td>
<td>(γ^2 - 1) dβ/β</td>
<td>dp/p - dβ/β</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

See page 1 for meaning of symbols.
2. ENERGY AVAILABLE IN COLLISION BETWEEN TWO PARTICLES

β, γ, and θc measured in the laboratory frame.

2.1 General two-body collision along the same line

\[ E_{\text{c.m.}} = \left[ m_1^2 + m_2^2 + 2m_1m_2Y_1Y_2 (1 - \beta_1\beta_2) \right]^{1/2}, \]

where β is counted algebraically, and \( E_{\text{c.m.}} \) is the total energy in the centre-of-mass frame, i.e. the maximum available energy.

2.2 Two identical particles

i) One particle at rest: \( \beta_1 = 0, \ Y_1 = 1 \)

\[ E_{\text{c.m.}} = m (2 + 2Y_2)^{1/2} \approx m (2Y_2)^{1/2} \quad \text{for} \ Y_2 \gg 1. \]

ii) Two particles having velocities of the same magnitude but of opposite sign: \( Y_1 = Y_2 = Y; \ \beta_1 = -\beta_2; \)

\[ E_{\text{c.m.}} = 2E = 2mY. \]

If a proton colliding with another proton at rest can liberate the same energy as a collision between two protons with opposite velocities, its energy is defined by

\[ Y_{\text{eq}} = 2Y^2 - 1 \approx 2Y^2 \quad \text{for} \ Y \gg 1. \]

iii) Two particles having velocities of the same magnitude but making a small angle \( \theta_c; \)

\[ E_{\text{c.m.}} = 2E \left( 1 - \beta^2 \sin^2 (\theta_c/2) \right)^{1/2} \]

\[ \approx 2E \cos (\theta_c/2) \quad \text{for} \ \beta \approx 1. \]

\[ Y_{\text{eq}} = 2Y^2 \cos^2 (\theta_c/2) - 1 \]

\[ \approx 2Y^2 \cos^2 (\theta_c/2) \quad \text{for} \ Y \gg 1. \]

See page 1 for meaning of symbols
3. MAGNETIC AND ELECTRIC DEFLECTION

3.1 Magnetic deflection

a) Deflection angle $\theta \ [\text{rad}] = BL_1/(Bp) = 0.2997925 \ BL_1/p \ [\text{Tm}/\text{GeV}/\text{c}]$

b) Beam rigidity (magnetic bending radius)

\[ Bp \ [\text{Tm}] = 3.3356 \ p \ [\text{GeV}/\text{c}] \]
\[ = 3.1297 \ \beta_Y \] for protons (refer to Table 1.1 for other expressions)

c) Sagitta

\[ d_1 = \frac{1}{2} l_1 \tan \left( \frac{\theta}{4} \right) = 2\rho \sin^2 \left( \frac{\theta}{4} \right) \approx \frac{\rho \theta^2}{8} = \frac{l_1^2}{8 \rho} \]

\[ d_2 = l_2 \tan \left( \frac{\theta}{2} \right) = \rho (1 - \cos \theta) \approx \frac{\rho \theta^2}{2} = \frac{l_2^2}{2 \rho} \]

3.2 Electric deflection

a) Deflection angle

\[ \theta \ [\text{rad}] = \arctan \left( E\ell /\beta \right) \ [10^9 \ \text{V}/(\text{GeV}/\text{c})] \]

b) Sagitta

\[ d [\text{m}] = E\ell^2 /2\beta p \ [10^9 \ \text{V}m/(\text{GeV}/\text{c})] \]
3.3 **Comparison of electric and magnetic deflection**

For small $\vartheta$, 

$$B[T] \approx E/(300 \beta) [\text{MV/m}]$$

for the same deflection.

Equivalent deflection for high fields, $B = 2T$, $E = 10 \text{ MV/m}$ corresponds to $\beta = \tfrac{\gamma_0}{2}$; and, for protons, $p = 16 \text{ MeV/c}$, $T = 0.13 \text{ MeV}$.

4. **SOME FORMULAE FOR QUANTITIES RELATED TO SYNCHROTRONS**

4.1 **Mean machine radius**

$$R_0 = C_0/2\pi = (1 + k)\rho_0,$$

where $k$ is the circumference factor.

4.2 **Relations between $p$, $R$, $B$, $f$, $\beta$ and their derivatives**

4.2.1 $p$, $R$, $B$*)

a) **Fundamental equation**

$$p = e\rho_0 \left( \frac{R}{R_0} \right)^{1/\alpha_p} B$$

b) **Definition of $\alpha_p$**

$$\alpha_p = \frac{P}{R} \left( \frac{\partial R}{\partial p} \right)_B \left( \approx \frac{1}{Q^2} \right)$$

4.2.2 $f$, $\beta$, $R$

$$f = \beta c/2\pi R.$$

4.2.3 **Definition of transition energy $E_{tr} = \gamma_{tr} E_0$**

$$\frac{P}{f} \left( \frac{\partial f}{\partial p} \right)_B = \frac{1}{\gamma_{tr}^2} - \alpha_p = 0$$

$$\gamma_{tr} = 1/\sqrt{\alpha_p} \left( \approx Q \right).$$

*) $B$ is defined on the nominal orbit $C_0$. 
4.2.4 Differential relations

<table>
<thead>
<tr>
<th>Variables</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>B, p, R</td>
<td>( \frac{dp}{p} = \gamma_{tr}^2 \frac{dR}{R} + \frac{dB}{B} )</td>
</tr>
<tr>
<td>f, p, R</td>
<td>( \frac{dp}{p} = \gamma^2 \frac{df}{f} + \gamma^2 \frac{dR}{R} )</td>
</tr>
<tr>
<td>B, f, p</td>
<td>( \frac{dB}{B} = \gamma_{tr}^2 \frac{df}{f} + \frac{\gamma^2 - \gamma_{tr}^2}{\gamma^2} \frac{dp}{p} )</td>
</tr>
<tr>
<td>B, f, R</td>
<td>( \frac{dB}{B} = \gamma^2 \frac{df}{f} + (\gamma^2 - \gamma_{tr}^2) \frac{dR}{R} )</td>
</tr>
</tbody>
</table>

4.3 Relation between currents and number of particles

a) Number of injected particles in terms of linac current:

In the case of multiturn injection

\[
N = 1.5082 \times 10^{11} n_t \varepsilon_a \varepsilon_{tp} R I_L / \beta \quad \text{[Am]}
\]

where

- \( N \) is the number of trapped particles in the synchrotron;
- \( n_t \) is the number of injected turns;
- \( I_L \) is the linac current [A];
- \( \varepsilon_a \) is the mean transverse phase space injection efficiency;
- \( \varepsilon_{tp} \) is the longitudinal trapping efficiency.

b) Circulating current:

\[
I \quad \text{[A]} = \frac{(ec/2\pi)(N\beta/R)}{7.6441 \times 10^{-12} N\beta/R \quad \text{[m^{-1}]}},
\]

Number of charges passing per microsecond = \( 6.2418 \times 10^9 I \quad \text{[mA]} \),
or \( I \quad \text{[mA]} = 1.6021 \times 10^{-10} \times \text{number of charges passing per microsecond} \).
ADDITIONAL FORMULAE
PART II

TRANSVERSE PHASE SPACE
1. MATRIX FORMULATION OF BEAM DYNAMICS

1.1 General form of matrices with dispersive terms

The general matrix $M$ is defined by

$$\begin{pmatrix} y \\ y' \\ \Delta p/p_B \end{pmatrix} = M(\beta | A) \begin{pmatrix} y \\ y' \\ \Delta p/p_A \end{pmatrix}$$

1.2 Drift length $\ell$

$$M_\ell = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.3 Dipole magnet

a) Notation and definitions

---

* See page 43 for references
Fringe field, linear approximation:

b) Pure sector magnet
\((\varepsilon_1 = \varepsilon_2 = 0, b = 0)\)

\[
\mathbf{M}_H^S = \begin{pmatrix}
\cos \theta & \rho \sin \theta & \rho(1 - \cos \theta) \\
-\sin \theta & \cos \theta & \sin \theta \\
\rho & 0 & 1
\end{pmatrix}
\] in horizontal plane (plane of deflection)

\[
\mathbf{M}_V^S = \begin{pmatrix}
1 & \rho \rho & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] in vertical plane

c) Edge effect with linear fringe field

\([4,100]*\)

\[
\mathbf{M}_H^E = \begin{pmatrix}
1 & 0 & 0 \\
\frac{\tan \varepsilon}{\rho} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] \(\varepsilon = \varepsilon_1\) for entrance
\(\varepsilon = \varepsilon_2\) for exit

\[
\mathbf{M}_V^E = \begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{\rho} \left(\frac{b}{6\rho \cos \varepsilon} - \tan \varepsilon\right) & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] \(\varepsilon = \varepsilon_1\) for entrance
\(\varepsilon = \varepsilon_2\) for exit

*) See page 43 for references
14 Gradient sector magnet

a) Equation of profile
\[
\left( \frac{\rho_0}{n} - x \right) z = \frac{\rho_0}{n} z_0
\]
z_0 is the half-aperture where \( \rho = \rho_0 \).
Also \( g[Tm^{-1}] = -nB_0/\rho_0 \)
\[ [Tm^{-1}] = -3.356 \, n\rho_0/\rho_0^2 \]
\[ [(GeV/c)m^{-2}] \]

b) Focusing plane

\[
M_F^* = \begin{pmatrix}
\cos \zeta & \frac{1}{\sqrt{K}} \sin \zeta & \frac{1}{\rho K} (1 - \cos \zeta) \\
-\sqrt{K} \sin \zeta & \cos \zeta & \frac{1}{\rho \sqrt{K}} \sin \zeta \\
0 & 0 & 1
\end{pmatrix}
\]
where \( K[m^{-2}] = (|n| + 1)/\rho_0^2 \) (horizontal plane)
\( = |n|/\rho_0^2 \) (vertical plane)
\( \zeta = \ell_m \sqrt{K} \), \( \ell_m \) is the magnetic length.

c) Defocusing plane

\[
M_D^* = \begin{pmatrix}
\cosh \zeta & \frac{1}{\sqrt{K}} \sinh \zeta & \frac{1}{\rho K} (\cosh \zeta - 1) \\
\sqrt{K} \sinh \zeta & \cosh \zeta & \frac{1}{\rho \sqrt{K}} \sinh \zeta \\
0 & 0 & 1
\end{pmatrix}
\]
where \( K[m^{-2}] = (|n| - 1)/\rho_0^2 \) (horizontal plane)
\( = |n|/\rho_0^2 \) (vertical plane)
\( \zeta = \ell_m \sqrt{K} \), \( \ell_m \) is the magnetic length.

d) Pure quadrupole lens
The same matrices as in points (b) and (c) are used, with \( 1/\rho = 0 \)
and \( K[m^{-2}] = 0.2997925 \, g/\rho_0 \)
\[ [Tm^{-1}/(GeV/c)] \]
\( = -n/\rho_0^2 = g/(B\rho) \)
\[ [(m^{-2})] \]
\( = 0.31952 \, g/\beta \gamma \)
\[ [Tm^{-1}] \] for protons.

*) Valid for coordinate system defined on page 2.
2. DESCRIPTION OF SINGLE PARTICLE MOTION IN A SYNCHROTRON

2.1 Equation of motion

\[ d^2x/ds^2 + K_x(s)x = (\Delta \rho/\rho(s)) \]
\[ d^2z/ds^2 + K_z(s)z = 0. \]

2.2 Solution of the equation of motion

\[ y(s) = \sqrt{\epsilon(\beta(s))}\cos[\psi(s) + \delta] \]
\[ y'(s) = \sqrt{\epsilon(\beta(s))}[\alpha(s)\cos[\psi(s) + \delta] + \sin[\psi(s) + \delta]] \]
\[ \psi(s) = \int ds/\beta(s) \]
\[ \beta(s)y(s) = 1 + \alpha^2(s) \]
\[ \beta'(s) = -2\alpha(s) \]
\[ \tan[\psi(s) - \chi(s)] = 1/\alpha(s) \]
\[ \sin[\psi(s) - \chi(s)] = -[\beta(s)y(s)]^{-1/2} \cdot \]

Envelope equation:

\[ \sqrt{\beta''} + K(s)\sqrt{\beta} - \beta^{3/2} = 0 \]

Initial conditions:

\[ e = \gamma(0)y^2(0) + \beta(0)y'^2(0) + 2\alpha(0)y(0)y'(0) \]
\[ \cos \delta = y(0)/\sqrt{\epsilon(\beta(0))} \]
\[ \tan \delta = -[\alpha(0) + \beta(0)y'(0)/y(0)] \cdot \]

For \( y(0) = 0 \):

\[ y(s) = y'(0)\sqrt{\beta(0)}\beta(s)\sin \psi(s) \]
\[ y'(s) = -y'(0)\sqrt{\beta(0)}\beta(s)[\alpha(s)\sin \psi(s) - \cos \psi(s)] \cdot \]
\[ Q = \psi(c)/(2\pi) \]

Betatron wavelength \( \lambda = 2\pi t/Q \)

Form factor \( F = \beta_{\text{max}}^{\sqrt{Q/R}} \cdot \)

2.3 Sinusoidal approximation

\[ \psi(s) \approx 2ms/\lambda = Q 2\pi s/C ; \quad \beta(s) \approx R/Q \]
\[ y(s) \approx \sqrt{\epsilon R/Q} \cos(Qs/R + \delta) \cdot \]
2.4 Motion through one period (cell) of length $L_p$

$\xi(s)$, $\alpha(s)$, $\beta(s)$, $\gamma(s)$ are periodic with period $L_p$,
and $\psi(s + L_p) = \psi(s) + \mu$, $\chi(s + L_p) = \chi(s) + \mu$.

The transfer matrix through one period may be written as

$$M_{a + L_p | s} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos \mu + a(s) \sin \mu & \beta(s) \sin \mu \\ -\gamma(s) \sin \mu & \cos \mu - a(s) \sin \mu \end{bmatrix}$$

$\cos \mu = \gamma^2 (a_{11} + a_{22})$
$\gamma(s) = -\frac{a_{21}}{\sin \mu}$
$\beta(s) = a_{12}/\sin \mu$
$a(s) = \gamma^2 (a_{11} - a_{22})/\sin \mu$.

2.5 Transfer matrix through any section

a) If the Twiss parameters at points $s_1$, $s_2$ are $(\beta_1$, $\alpha_1$, $\gamma_1)$ and
$(\beta_2$, $\alpha_2$, $\gamma_2)$, respectively, the $2 \times 2$ transfer matrix from $s_1$ to $s_2$
can be written as

$$M(s_2 | s_1) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \beta_2 / \beta_1 (\cos \Delta \psi + \alpha_1 \sin \Delta \psi) & \sqrt{\beta_1 \beta_2} \sin \Delta \psi \\ \sqrt{\beta_1 \beta_2} \sin \Delta \psi & \beta_1 / \beta_2 (\cos \Delta \psi - \alpha_2 \sin \Delta \psi) \end{pmatrix}$$

where $\Delta \psi = \psi(s_2) - \psi(s_1)$.

b) Transformation of the Twiss parameters through a beam transfer section:

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{21}m_{11} & 1 + 2m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

$$\tan \Delta \psi = m_{22}/[m_{11} \beta(s_1) - m_{12} \alpha(s_1)]$$

Example: drift length $l$,

$$\begin{align*}
\beta(s_2) &= \beta(s_1) - 2a(s_1)l + \gamma(s_1)l^2 \\
\alpha(s_2) &= \alpha(s_1) - \gamma(s_1)l \\
\gamma(s_2) &= \gamma(s_1),
\end{align*}$$

and $\psi(s_2) = \psi(s_1) + \arctan \{l/[\beta(s_1) - \alpha(s_1)l]\}$. 
2.6 Normalization

After a normalisation transformation \((\eta', \gamma') = M(\eta, \gamma)\) (with \(\eta' = \frac{d\eta}{d\phi}\)),
the transverse phase space trajectories have the form of circles on which
a phase advance \(\phi(s)\) produces simply a rotation by \(\phi\). Possible trans-
formations matrices:

\[
M = \begin{pmatrix}
\frac{1}{\sqrt{\beta(s)}} & 0 \\
\frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)}
\end{pmatrix}; \text{ or, when } \alpha = 0, \quad M = \begin{pmatrix}
1 & 0 \\
0 & \beta(s)
\end{pmatrix}
\]

3. ELLIPSE REPRESENTATION IN TRANSVERSE PHASE SPACE (see page 18 for units)

3.1 The Courant-Snyder invariant

\[
\gamma(s)y^2 + 2a(s)yy' + \beta(s)y'^2 = \epsilon = \frac{\text{area}}{\pi}.
\]

The largest area contained in the synchrotron is given by the acceptance
\(A_{H,V} = r^2/\beta_{\text{max}}\), where \(r\) is the half-aperture of the vacuum chamber at \(\beta_{\text{max}}\).

3.2 Ellipse parameters

An ellipse centred at the origin of the phase plane is determined by
three independent parameters. According to the problem involved, one can
choose one of the following sets:

i) Twiss parameters \(a, \beta, \gamma\) and \(\epsilon\) giving the emittance (see Section 3.1).

These parameters transform with the matrix given in Section 2.5 (b);
ii) the elements \(c_i\) of a 2 \times 2 matrix which transforms by multiplication
with \(M(s_2 | s_1)\); \((c_3y - c_1y')^2 + (c_4y - c_2y')^2 = \epsilon^2\)
iii) \(L, S, \) and \(\epsilon\) where \(L\) is the ratio of the ellipse axes \(a/b\) at the waist,

\(S\) is the distance of the waist along the beam (\(>0\) if waist upstream).

The optical transformations from \(s_1\) to \(s_2\) are:

\[
L(s_2) = \frac{L(s_1)}{[m_{z1}L(s_1)]^2 + [m_{z1}S(s_1) + m_{z2}^2]^2}
\]

\[
S(s_2) = \frac{m_{z1}L^2(s_1) + [m_{z1}S(s_1) + m_{z2}][m_{z1}S(s_1) + m_{z2}]}{[m_{z1}L(s_1)]^2 + [m_{z1}S(s_1) + m_{z2}]^2}
\]

Conversion from one set to the other is given in Table 3.3.
### 3.3 Conversion of ellipse parameters

<table>
<thead>
<tr>
<th>given</th>
<th>( \alpha, \beta, \gamma, \epsilon )</th>
<th>( \begin{pmatrix} c_1 &amp; c_2 \ c_3 &amp; c_4 \end{pmatrix} )</th>
<th>( L, S, \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta \gamma - \alpha^2 = 1 )</td>
<td>( \epsilon = c_1 c_4 - c_2 c_3 )</td>
<td>( \begin{pmatrix} \alpha &amp; \beta \ \gamma &amp; \epsilon \end{pmatrix} )</td>
<td>( -S/L )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( -\frac{(c_1 c_3 + c_2 c_4)}{\epsilon} )</td>
<td>( L + S^2/L )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( \beta )</td>
<td>( \frac{(c_1^2 + c_2^2)}{\epsilon} )</td>
<td>( \frac{1}{L} )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( \gamma )</td>
<td>( \frac{(c_3^2 + c_4^2)}{\epsilon} )</td>
<td>( \frac{1}{L} )</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>( \sqrt{\epsilon \beta} )</td>
<td>( c_1 )</td>
<td>( \sqrt{\frac{L \epsilon}{\gamma}} )</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>( 0 )</td>
<td>( c_2 )</td>
<td>( S \sqrt{\frac{\epsilon}{\gamma}} )</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>( -\alpha \sqrt{\frac{\epsilon}{\beta}} )</td>
<td>( c_3 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>( \frac{\epsilon}{\beta} )</td>
<td>( c_4 )</td>
<td>( \frac{\epsilon}{L} )</td>
</tr>
<tr>
<td>( L )</td>
<td>( \frac{1}{\gamma} )</td>
<td>( \frac{\epsilon}{(c_3^2 + c_4^2)} )</td>
<td>( L )</td>
</tr>
<tr>
<td>( S )</td>
<td>( -\frac{\alpha}{\gamma} )</td>
<td>( \frac{(c_1 c_3 + c_2 c_4)}{(c_3^2 + c_4^2)} )</td>
<td>( S )</td>
</tr>
</tbody>
</table>

**Comments on units:**

The equations of Sections 3.1 to 3.4 are valid for either of the two following sets of units:

i) All lengths in metres, all angles in radians, the emittances in rad m, \( L = \frac{y_1}{y_3'} \) in m/rad, \( S \) in m/rad.

ii) All phase plane dimensions in mm and mrad, emittances in mrad mm, \( \beta = \frac{[\text{mm/mrad}]}{[\text{m/}\text{rad}]} \) if defined as \( y_2/y_4' \) or \( \beta = [\text{m}] \) if defined as reduced betatron wavelength(similarly \( \gamma = [\text{mm/mrad}], \) or \( [\text{m}^{-1}] \), \( L = [\text{mm/mrad}], \) \( S = [\text{mm/mrad}] \) if defined as \( y_3/y_3' \) or \( S = [\text{m}] \) if defined as distance from waist.

**N.B.:** the values of \( \alpha, \beta, \gamma, L \) and \( S \) do not depend on the choice between i) and ii).
### Geometrical properties of the ellipse

<table>
<thead>
<tr>
<th>$\alpha$, $\beta$, $\gamma$, $\epsilon$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$L$, $S$, $\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta Y - \alpha^2 = 1$</td>
<td>$\epsilon = c_1 c_4 - c_2 c_3$</td>
<td>$\frac{1}{2} (c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$</td>
<td>$H = \frac{1}{2L} (L^2 + S^2 + 1)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| $H = \frac{1}{2} (\beta + \gamma)$ | $\frac{\epsilon}{\sqrt{c_3^2 + c_4^2}}$ | $\sqrt{c_1^2 + c_2^2}$ | $\sqrt{\epsilon L}$ |}

| $y_1$ | $\sqrt{\epsilon / \gamma}$ | $\sqrt{\epsilon / \beta}$ | $\sqrt{\epsilon / \gamma}$ | $\sqrt{\epsilon / \beta}$ |
| $y_2$ | $\frac{\sqrt{\epsilon / \beta}}{\sqrt{\gamma}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ |
| $y_3$ | $\frac{\sqrt{\epsilon / \beta}}{\sqrt{\gamma}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ |
| $y_4$ | $\frac{\sqrt{\epsilon / \beta}}{\sqrt{\gamma}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ | $\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$ |

\[
\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}} = \frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}
\]

### Other properties

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = \frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$</td>
<td>$b = \frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$</td>
<td>$\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}$</td>
</tr>
</tbody>
</table>

\[
\frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}} = \frac{\sqrt{\epsilon / \gamma}}{\sqrt{\beta}}
\]

### Additional expressions

\[
\tan \xi = \frac{- \alpha (H + \sqrt{H^2 - 1})}{\beta (H + \sqrt{H^2 - 1}) - 1}
\]

\[
\sin 2\xi = \frac{- \alpha \sqrt{H^2 - 1}}{1 - \alpha^2 (H^2 - 1)}
\]

\[
\cos 2\xi = \frac{(\beta - \gamma) / \sqrt{H^2 - 1}}{1 - \alpha^2 (H^2 - 1)}
\]

\[
\tan 2\xi = \frac{- 2 \alpha (\beta - \gamma)}{1 - \alpha^2 (H^2 - 1)}
\]

\[
S / [L \sqrt{H^2 - 1} - 1]
\]

\[
L / \sqrt{H^2 - 1}
\]

\[
(L^2 + S^2 - 1) / 2L \sqrt{H^2 - 1}
\]

\[
2S / (L^2 + S^2 - 1)
\]
3.4.4 Two ellipses $S_1, S_2$ with same area $S$ and centre

Common area $S_c$ is given by

$$\frac{S_c}{S} = \frac{4}{\pi} \arctan \left[ D - \sqrt{D^2 - 1} \right]^{1/2}$$

where

$$D = \frac{1}{2} (\beta_2 \gamma_1 + \gamma_2 \beta_1 - 2 \alpha_1 \alpha_2)$$

$$= 1 + \frac{(L_2 - L_1)^2 + (S_2 - S_1)^2}{2L_1L_2}.$$ 

For meaning of $\alpha, \beta, \gamma, L, S$ see 3.2.

3.4.5 Three ellipses

a) Area of ellipse $S_3$, similar to $S_2$, such that $S_3$ circumscribes $S_1$: (area $S_1 = area S_2 = S$)

b) Area of ellipse $S_3$ circumscribing two ellipses $S_1, S_2$ of same area $S$:

$$\frac{S_3}{S} = (D + \sqrt{D^2 - 1})^{1/2}$$

See 3.4.4 for meaning of $D$. 

\[ \text{Diagram} \]
3.5 Relations between areas of rectangles, ellipses and circles

3.5.1 Rectangle and ellipse

\[
\frac{S_{\text{min ellipse}}}{S_{\text{inscribed rect.}}} = \frac{\pi}{2}
\]

\[
\frac{S_{\text{circumscribed rect.}}}{S_{\text{ellipse}}} = \frac{4}{\pi}
\]

3.5.2 Semi-circle and ellipse

\[
\frac{S_{\text{min ellipse}}}{S_{\text{semi-circle}}} = \frac{8\sqrt{3}}{9} \approx 1.54
\]

3.5.3 Two maximum ellipses circumscribed by a circle

For a non-zero septum, see [9, Fig. 9]

\[
\frac{S_{\text{circle}}}{S_{\text{max ellipse}}} = \frac{3\sqrt{3}}{2} \approx 2.60
\]
4. CLOSED ORBIT

4.1 Closed orbit for a momentum deviation \( \Delta p/p \)

The closed orbit for a given \( \Delta p/p \) is given by its phase space coordinates:

\[
\begin{pmatrix}
    e(s) \\
    e'(s)
\end{pmatrix} = \frac{\Delta p/p}{2(1 - \cos \mu)} \begin{pmatrix}
    m_1 & m_2 & m_3 \\
    m_2 & m_3 & m_4 \\
    m_3 & m_4 & m_5
\end{pmatrix},
\]

the \( m_{ij} \) being the elements of the 3x3 transfer matrix through one period.

The transformation of the closed orbit vector through the machine is as in Section II.1.1., page 12.

4.2 Orbit deformations *)

4.2.1 One and two dipoles

- a) One dipole producing a deflection angle \( \vartheta \)

\[
y(s_1) = 0.5 \vartheta(s_1) \varphi(s_1) \cot(\pi Q)
\]

\[
y'(s_1) = 0.5 \vartheta(s_1) [1 - \alpha(s_1) \cot(\pi Q)]
\]

The maximum orbit deviations are

\[
\hat{y}(s) = 0.5 \vartheta(s_1) [\beta(s_1) \beta(s)]^{1/2}/\sin(\pi Q)
\]

and occur approximately at distances (in betatron oscillation phase) of

\[
\Delta \psi_i \approx \pm \pi(\psi - m), \quad m = 1, 2, 3, \ldots < Q.
\]

*) Complete decoupling between betatron and synchrotron oscillations,
b) Reduction of the deformation by a second dipole

For best reduction between $s_2$ and $s_1$ of a deformation described in a), a second dipole positioned at $s_2$ and spaced by $\Delta \psi$ should provide a deflection

$$\vartheta(s_2) = -\cos \Delta \psi [\beta(s_1) \beta(s_2)]^{1/2} \vartheta(s_1).$$

The remaining relative deformations of the corrected orbit are:

between $s_2$ and $s_1$ ("outside" dipoles):

$$\frac{\vartheta_c}{\vartheta} = \sin \Delta \psi$$

between $s_1$ and $s_2$ ("between" dipoles):

$$\frac{\vartheta_c}{\vartheta} = [\cos \Delta \psi + 2 \sin^2 \pi Q - \cos 2(\pi Q - \Delta \psi)]^{1/2}.$$

4.2.2 Distortions due to random errors

The number of magnet units $m$ is assumed to be $m > 3Q$.

a) r.m.s. value of $y(s)$

$$\langle y \rangle_{\text{rms}} = \sqrt{\frac{\pi}{2}} \frac{R}{\rho} |n| \frac{\langle \delta \rangle_{\text{rms}}}{\sqrt{m}},$$

where $\delta$ is the position error of one of the $m$ gradient magnets.

More generally, one has

$$\langle y \rangle_{\text{rms}} = \frac{1}{2\sqrt{2}} |\sin \pi Q| \sqrt{\bar{\beta}} \sqrt{\sum_{i=1}^{n} \beta_i} \psi_i^{2},$$

where $\bar{\beta} = (1/\bar{c}) \int_{0}^{\bar{c}} \beta(s) ds$ and the equivalent kicks $\psi_i$ are for the various cases of interest:

*) See page 43 for references
<table>
<thead>
<tr>
<th>Type of element</th>
<th>Source of kick</th>
<th>r.m.s. value</th>
<th>$\psi_i$</th>
<th>Directions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient element</td>
<td>Displacement</td>
<td>$&lt;\Delta y&gt;$</td>
<td>$K_i \ell_i &lt;\Delta y&gt;$</td>
<td>x and z</td>
</tr>
<tr>
<td>Bending elements</td>
<td>Tilt</td>
<td>$&lt;\Delta \theta_e&gt;$</td>
<td>$\theta_i &lt;\Delta \theta_e&gt;$</td>
<td>z only</td>
</tr>
<tr>
<td>Bending elements</td>
<td>Field error</td>
<td>$&lt;\Delta B/B&gt;$</td>
<td>$\theta_i &lt;\Delta B/B&gt;$</td>
<td>x only</td>
</tr>
<tr>
<td>Straight sections</td>
<td>Stray field</td>
<td>$&lt;\Delta B_s&gt;$</td>
<td>$\ell_i &lt;\Delta B_s&gt;/\rho B_{\text{inj}}$</td>
<td>x and z</td>
</tr>
<tr>
<td>Gradient and</td>
<td>Displacement</td>
<td>$&lt;\Delta y&gt;$</td>
<td>$K_i \ell_i [(&lt;\Delta y^2&gt; + \rho^2/\pi^2 (&lt;\Delta B/B)^2)]^{1/2}$</td>
<td>x and z</td>
</tr>
<tr>
<td>bending elements</td>
<td>and field error</td>
<td>$&lt;\Delta B/B&gt;$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Value $\hat{y}$ not exceeded with a probability $P$

$$\hat{y}_P(s) = k(P) \left[ 1 + \frac{|\sin \pi Q|}{3} \right] \sqrt{\frac{\beta(s)}{\beta}} \sqrt{2} <y>$$

with $k(P)$ given by

![Diagram](image)

<table>
<thead>
<tr>
<th>k</th>
<th>P 50%</th>
<th>P 75%</th>
<th>P 90%</th>
<th>P 98%</th>
</tr>
</thead>
<tbody>
<tr>
<td>rectangular vacuum chamber</td>
<td>1.11</td>
<td>1.41</td>
<td>1.72</td>
<td>2.14</td>
</tr>
<tr>
<td>elliptical vacuum chamber</td>
<td>1.28</td>
<td>1.63</td>
<td>1.95</td>
<td>2.40</td>
</tr>
</tbody>
</table>

*) This bracket takes into account the mean influence of higher harmonics.
5. EFFECTS OF VARIOUS FOCUSING PERTURBATIONS ON THE FREQUENCY 
AND AMPLITUDE OF BETATRON OSCILLATIONS

5.1 Change in frequency due to tuning of quadrupole lenses or gradient errors

The frequency shift is given by

\[ \Delta \Omega = \frac{1}{4\pi} \int_0^C \beta(s)k(s)ds \]

where \( 2\pi Q_0 \) is the unperturbed phase shift around the orbit of length \( C \) and \( k(s) \) the focal constant of the perturbation(s).

In the case of small frequency shifts this becomes

\[ \Delta \Omega \approx \frac{1}{8\pi^2} \int_0^C \beta(s)k(s)ds \]

For \( m \gg Q \), random errors in \( m \) elements produce a shift

\[ \Delta \Omega = \frac{1}{4\pi} \sqrt{\sum_i \left( \beta_i K_i \xi_i \right)^2} \Delta K \]< \( \frac{\Delta K}{K} \) _rms

where the symbols are the same as in Section 4.2.2, pages 23 and 24.

5.2 Tuning of momentum dependent frequency shifts by means of sextupoles

To compensate a shift caused by \( k(s) = -K(s)/\mu/p \) one needs (in the case of an ideal closed orbit) a sextupole field \( \delta^2 B_z/\delta x^2 \) such that

\[ \int_0^C \beta(s) \left[ \frac{\delta^2 B_z}{\delta x^2} (s)e(s) - Bp k(s) \right] ds = 0 \]

where \( e(s) \) is defined in Section 4.1 on page 22.

5.3 "Beating" of amplitudes

The beat factor characterising the amplitude function modified by gradient errors is

\[ G = [\beta(\text{actual})/\beta(\text{ideal})]_{\text{max}} \]

*) See page 43 for references
In practice one is more interested in $(\Delta\tilde{y}/\tilde{y})_p = 0.5(0 - 1)$. Similarly as for the orbit (Section 4.2.2, page 23) one has:

$$
(\Delta\tilde{y}/\tilde{y})_p = \frac{k(p)}{4} \left[ \frac{1}{3} + \left[ \frac{1}{\sin 2\pi q} \right] \right] \sqrt{\sum_{i} m_i (\beta_i K_i \epsilon_i)^2 < \frac{\Delta k}{k} >_{\text{rms}}}
$$

5.4 Stopbands due to random gradient errors

The total width of the stopband is

$$
\delta Q = 2\Delta Q
$$

where the $\Delta Q$ is given in Section 5.1 on page 25.

6. SPACE CHARGE LIMIT

Symbols:

- $N$: limit of the number of particles in the synchrotron
- $B_f$: bunching factor (<1)
- $b[m]$: mean semi-minor beam axis (vertical)
- $a[m]$: mean semi-major beam axis (horizontal)
- $\Delta(Q^2)$: $Q_0^2 - Q^2 \approx 2Q_0 \Delta Q$
- $r$: classical particle radius ($= e/(4\pi \epsilon_0 m e^2 [eV])$, see p. 45)
- $2h[m]$: vertical aperture of the vacuum chamber
- $2w[m]$: horizontal aperture of the vacuum chamber
- $2v[m]$: height of the magnet gap.

a) Individual particle limit (without neutralization)

$$
N_{\text{ind}} = -0.5 \pi b(a + b)(R r F)^{-1} \beta^2 \gamma^3 B_f \Delta(Q_i^2)
$$

$$
\approx - (\pi \epsilon_V \beta \gamma)(1 + \sqrt{\epsilon_H/\epsilon_V})(rF)^{-1} \beta \gamma^2 B_f \Delta Q_i
$$

where

$$
F = 1 + [b(a + b)/h^2]\left[ \epsilon_1 [1 + B_f(\gamma^2 - 1) + \epsilon_2 B_f(\gamma^2 - 1)(h^2/\gamma^2)] \right]
$$

with $\epsilon_1, \epsilon_2$, the image force coefficients given on page 27, and $\epsilon_H, \epsilon_V$ in rad m.

*) See page 43 for references
For parallel straight pole pieces, and to good approximation for wedged-shaped poles, the magnetostatic image coefficients have the values $\varepsilon_2 = 0.411, \xi_2 = 0.617$.

b) Coherent particle limit (without neutralization)

\[ N_{coh} = -\pi Q_0 \hbar^2 (R \tau F)^{-1} \beta^2 \gamma^3 B_f \Delta Q_c \]

where near an integral resonance

\[ F = \xi_1 [1 + B_f (\gamma^2 - 1)] + \xi_2 B_f (\gamma^2 - 1) \hbar^2 / \nu^2 \]

and near a half-integral resonance

\[ F = \xi_1 + \varepsilon_1 B_f (\gamma^2 - 1) + \varepsilon_2 B_f (\gamma^2 - 1) \hbar^2 / \nu^2 \]

with $\varepsilon_1$, $\varepsilon_2$, $\xi_1$ and $\xi_2$ as given above.

For $B_f \gamma^2 >> 1$, one has near an integral resonance

\[ N_{coh} \approx -\pi Q_0 \hbar^2 [R \tau (\xi_1 + \xi_2 \hbar^2 / \nu^2)]^{-1} \gamma \Delta Q_c \]

*) See page 43 for references
ADDITIONAL FORMULAE
PART III

LON GITUDINAL PHASE SPACE
1. **ACCELERATING VOLTAGE**

\[ V \sin \varphi_s [\text{kV}] = 10^{-3} \left[ C (\dot{\rho} \dot{B} + B \ddot{\rho} - B \dot{S}_F) \right] \left[ \text{m}^2 \text{T/s} \right] \]

where the index \( s \) refers to the synchronous particle and \( S_F \) is an equivalent area such that \( B S_F \) is the total flux enclosed by \( C \).

For \( \dot{\rho} = S_F = 0 \), one has

\[ V \sin \varphi_s [\text{kV}] = 0.020958 \, R \, \dot{p} \left[ \text{m} \text{(GeV/c)}/\text{s} \right] \]

(see Table 1.1.1 for other expressions).

2. **ACCELERATING FREQUENCY**

\[ f_a [\text{Hz}] = \frac{h \, f}{c} = \frac{h \, c \, \beta / C}{[s^{-1}]} \]

\[ = \frac{h \, c}{C^2} \left[ B^2 [\text{T}^2] \left( \frac{E_0 [\text{MeV}]}{c \rho [\text{km}^2/\text{s}]} \right)^2 \right]^{-\gamma/2} \left[ \text{m}^2 \text{T/s} \right]. \]

(see Table 1.1.1 for other expressions).

See Section 1.4.2.4, page 9, for differential relations.

3. **SYNCHROTRON OSCILLATIONS**

3.1 **Equation of motion**

(Above transition \( \varphi \) should be replaced by \( \varphi + \pi - 2 \varphi_s \); for transition see Section 1.4.2.3, page 8.)

\[ \frac{\dot{\varphi}}{\gamma^2_s} = (E_s / \eta_s \, \gamma^2_s) \, \dot{\varphi} = (h \, e \nu / 2 \pi) (\sin \varphi - \sin \varphi_s) \]

where \( E_s \) is in keV, \( \eta = \gamma_{tr}^2 - \gamma^{-2} \), and \( \Omega \) is the particle angular velocity on the orbit of length \( C \). To transform to other co-ordinates one has in the absence of perturbations

\[ \varphi = h \left( \Omega_s - \Omega \right) \]

\[ = (h \, \Omega_s / \eta_s \, \gamma_{tr}^2 / R) \Delta R \]

\[ = (h \, \Omega_s / \eta_s / \beta_s \, \gamma_s) (\Delta p / m_0 c) \]

\[ = (h \, \Omega_s / \eta_s / \beta_s^2 \, E_s) \Delta E \]

See page 1 for meaning of symbols.
3.2 Bucket size

3.2.1 Without space charge effects

a) Bucket area, (half) height and coordinates

<table>
<thead>
<tr>
<th>Bucket area</th>
<th>Bucket (half) height</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\text{heV})^{1/2} a(\Gamma) (16\gamma/h) (2\pi E</td>
<td>\eta</td>
<td>)^{-1/2})</td>
</tr>
<tr>
<td>((\text{heV})^{1/2} a(\Gamma) (16\beta/h) (E/(2\pi</td>
<td>\eta</td>
<td>))^1/2)</td>
</tr>
<tr>
<td>((\text{heV})^{1/2} a(\Gamma)[16\alpha/(h\beta)](E/(2\pi</td>
<td>\eta</td>
<td>)^{1/2})</td>
</tr>
<tr>
<td>((\text{heV})^{1/2} a(\Gamma)[16\alpha/(h^2 \Omega)](E/(2\pi</td>
<td>\eta</td>
<td>))</td>
</tr>
</tbody>
</table>

For \(\alpha(\Gamma)\) see below and Appendix C; for \(\eta\) see Section 3.1 on preceding page.

\[ Y = Y(\Gamma) = \frac{\dot{\varphi}_{\text{max}}}{\sqrt{2}} \frac{2\pi \nu_0}{\varphi_B = 0} = \frac{\dot{\varphi}_{\text{max}} (\text{heV})^{1/2} (\beta/\Omega)(\pi E|\eta|)^{1/2}}{2} \text{[E and eV in keV}\]

\(\varphi\) in rad, \(\Delta R\) in cm.

Ideal adiabatic trapping of a linac beam with \(\pm \Delta E_L\) leads to a minimum bucket (half) height \(\Delta E = (\pi/2)\Delta E_L\)

b) Bucket width, normalised (half) height and area (see Appendix C for numbers)
3.2.2 Reduction of bucket area due to space charge effects (below transition)

This reduction can be obtained from Fig. III.3.2.2, where

\[ A_{\text{sp.c.}} = 4\pi h g_c E_0 r_p N/(\text{ReV} \gamma^2) \]

with

- \( N \) = number of accelerated particles
- \( g_c = 1 + 2 \ln (\text{vacuum chamber diameter}/\text{beam diameter}) \)
- \( r_p \) = classical proton radius

and \( E_0 \) and \( eV \) are in the same units (as are \( r_p \) and \( R \)).

Fig. III.3.2.2 \( (A - A_{\text{sp.c.}})/A = f(A_{\text{sp.c.}}) \) (for constant density in phase space)

For \( \varphi_S = 0^\circ \) (and a \( \cos^2 \) distribution in real space) one has

\[ A_{\text{sp.c.}}/A = [1 - g_c e h N/(4\pi e_0 \gamma^2 R V)]^{1/2}, \]

where \( V \) is in volts.
3.3 Frequency of synchrotron oscillations

In the case of small phase oscillation amplitudes around the stable phase $\varphi_s$, the equation of motion is

$$ (\Delta \varphi) = (\delta^2 - 1)\delta^{-3} C'[\sin(\varphi_s + \Delta \varphi) - \sin \varphi_s] = (\delta^2 - 1)\delta^{-3} C' \cos \varphi_s \Delta \varphi $$

where $\delta^2 = \eta \gamma^2 + 1 = \alpha\gamma^2$, $C' = 2\pi \int_0^\infty \gamma_{\text{tr}}^{-3} eV/(hE_0)$ and $eV$ and $E_0$ are in the same units. (See Section 1.4.2.3, page 8, for transition.)

The frequency of these oscillations is \([eV \text{ and } E_0 \text{ in same units}]\)

$$ \nu_0 = \left[ (1 - \delta^2)\delta^{-3} C' \cos \varphi_s \right]^{1/2}/(2\pi) = \left[ \int_0^\infty |\eta| eV \cos \varphi_s/(2\pi E_0) \right]^{1/2}. $$

In terms of the bucket area $A$ \([\Delta \varphi/(m_e c) - \varphi \text{ coordinates}]\):

$$ \nu_0 = [A E_0 \int_0^\infty |\eta|/(16E_0 \alpha(\Gamma))]|\cos \varphi_s|^{1/2} $$

or

$$ \nu_0 = [AeB|\eta|/(32\pi R \gamma_\varphi(\Gamma))]|\cos \varphi_s|^{1/2}. $$

Alternatively

$$ \nu_0 = \left[ \cos \varphi_s/(4\pi^2) \right] \left[ c^2/(2\pi R^2 E_0) \right] (h eV) \left( 1 - \gamma_{\text{tr}}^{-2} \gamma^2 \right) \left( \gamma^3 \right)^{1/2}. $$

3.4 Adiabatic damping of small-amplitude oscillations

$$ \gamma(t) = \left[ 1 + [B(t)/B(t_0)]^2 \right]^{1/2}. $$

3.4.1 Phase amplitude \([eV \text{ and } E_0 \text{ in same units}]\)

$$ \Delta \varphi(t) = D [eV(t) \cos \varphi_s]^{-1/4} \left[ 1 - \gamma_{\text{tr}}^{-2} \gamma^2(t) / \gamma^3(t) \right]^{1/4} $$

where

$$ D = \text{constant} = \Delta \varphi_i \left[ 2\pi R^2 E_0 / (h c^2) \right]^{1/4} $$

with $i$ denoting the initial values.

$$ \Delta \varphi(t)/\Delta \varphi_i = [Vi/V(t)]^{1/4} \left[ 1 - \gamma_{\text{tr}}^{-2} \gamma^2(t) / \gamma^3(t) \right]^{1/4} \left[ 1 - \gamma_{\text{tr}}^{-2} \gamma_i^2 / \gamma_i^3 \right]^{-1/4}. $$

* ) See page 43 for references
3.4.2 **Energy amplitude**

\[ \Delta \varepsilon(t) = G[e^\varepsilon(t) \cos \phi_s]^{\frac{1}{4}} \beta(t)[\gamma^3(t) / |1 - \gamma_{tr}^{-2} \gamma^2(t)|]^{\frac{1}{4}} \]

where

\[ G = \text{constant} = \lambda \phi_\gamma \left[ E_0 R / (hc) \right] \left[ h c^2 / (2\pi R^2 E_0) \right]^{\frac{1}{4}} \]

\[ \Delta \varepsilon(t) / \Delta \varepsilon_i = [\psi(t)/\psi_i]^{\frac{1}{4}} [\beta(t)/\beta_i] [\gamma(t) / |1 - \gamma_{tr}^{-2} \gamma^2(t)|]^{\frac{1}{4}} [1 - \gamma_{tr}^{-2} \gamma^2(t) / \gamma_i^2]^{\frac{1}{4}} \]

3.4.3 **Radial amplitude**

\[ \Delta R(t) = H[e^\varepsilon(t) \cos \phi_s]^{\frac{1}{4}} \beta(t)[\gamma^3(t) / |1 - \gamma_{tr}^{-2} \gamma^2(t)|]^{\frac{1}{4}} \]

where

\[ H = \lambda \phi_\gamma \left[ \gamma_{tr}^{-2} R^2 / (hc) \right] \left[ h c^2 / (2\pi R^2 E_0) \right]^{\frac{1}{4}} \]

\[ \Delta R / \Delta R_i = [\beta(t) / \beta_i][\gamma(i) / \gamma(t)] [\Delta \theta_i / \Delta \theta(t)] . \]

4. **DEBUNCHING**

4.1 **Debunching time**

In the absence of RF fields the beam "debunches" *) itself in the synchrotron (i.e. front end of one bunch reaches the tail of the next bunch ahead) after a time **

\[ t_{db} \approx (\pi - \Delta \phi)[2\pi f_a |\gamma^{-2} - \gamma_{tr}^{-2} | \Delta \psi / p]^{-1} \]

where \( 2\Delta \phi \) and \( 2\Delta p \) are the total phase and momentum spreads.

Special cases:

a) Low energy

\[ \gamma^2 << \gamma_{tr}^2 \quad \Delta \phi << \pi \text{ (strong damping in linac)} \]

\[ t_{db} = \gamma^2 (2f_a \Delta \psi / p)^{-1} = \gamma(\gamma + 1)(2f_a \Delta \psi / p)^{-1} . \]

*) This azimuthal spreading does not involve any reduction of \( \Delta \psi / p \).

**) Strictly valid for "rectangular" bunches. For "oval" bunches more time may be required in practice.
b) Well above transition

\[ \gamma^2 \gg \gamma_{tr}^2 \quad \Delta \phi \ll \pi \text{ (damping in synchrotron)} \]

\[ t_{db} = \gamma_{tr}^2 \left( 2f_a \Delta p/p \right)^{-1} \]

Complete overlapping (front end reaches centre of next bunch ahead) requires \( 2t_{db} \).

4.2 Travelling distance required for debunching

\[ s_{db} = c \beta t_{db} \]

For \( \gamma^2 \ll \gamma_{tr}^2 \) and \( \Delta \phi \ll \pi \), this becomes with \( 2D = \text{distance between centre of bunches} \)

\[ s_{db} = D/(\Delta \beta/\beta) \]

See Table I.1.1 for other expressions.
ADDITIONAL FORMULAE
PART IV

ROUGH EVALUATION OF MAJOR ACCELERATOR SYSTEMS
1. MAGNETS (non-saturated)

1.1 Bending magnet

   a) Excitation current

   \[ N_B I_{(\text{ampere-turns})} = B h_B / \mu_0 (\text{mT}/(\text{H m}^{-1})) \]

   where \( h_B \) is the (mean) gap height

   \[ N_B I/(B h_B) \approx 800 \text{ ampere-turns}/(0.1 \text{T} \times 0.01 \text{ m gap height}) \]

   b) Inductance

   \[ L_B[H] \approx N_B^2 \mu_0 w f_B / h_B \]

   \[ w = w_a + \frac{2}{3} w_c \quad (\text{for window frame magnet}) \]

   \[ w = w_p + \frac{1}{2} h_B \quad (\text{for magnet with poles}) \]

   where

   \( w_a \) = aperture between coils, \( w_c \) = coil width

   \( w_p \) = pole width

   \( \ell_B \) = the total magnetic length

   c) Stored energy

   \[ W_B[W_s] \approx B^2 h_B w \ell_B / (2\mu_0) \quad [\text{T}^2 \text{ m}^3/(\text{H m}^{-1})] \]

   where \( w \) is as in b).

1.2 Quadrupole lens

   a) Excitation current per pole

   \[ N_Q I_{(\text{ampere-turns})} = g r_Q^2 / (2\mu_0) \quad [\text{Tm}/\text{H m}^{-1}] \]

   \[ N_Q I/(g r_Q^2) \approx 400 \text{ ampere-turns}/[10 \text{ T/m}(0.01 \text{ m bore radius})^2] \]

   b) Inductance

   \[ L_Q[H] \approx 8\mu_0 N_Q y_{\text{max}}(y_{\text{max}} + 2/3 w_c) f_Q / r_Q^2 \]

   where \( y_{\text{max}} \) is the distance from the lens centre to the coil face

   and \( f_Q \) is the total magnetic length.
c) Stored energy

\[ W_Q[W_s] \approx g^2 r_Q^2 y_{\text{max}} \left( y_{\text{max}} + 2/3 w_c \right) e^Q/\mu_0. \]

1.3 Bending magnet and quadrupole lens excited in series

\[ B = N_B \beta r_Q^2 \gamma K/(0.639 h_B N_Q), \quad \text{(for protons)} \]

or

\[ K = (0.6/p)(N_Q/N_B)(h_B/r_Q^2) B \]

where \( r_Q \) and \( h_B \) in m.

See Section II.1.4.d) for other expressions.

1.4 Cooling water requirements

To cool a conductor heated by a power loss \( N[kW] \), one needs a water flow of

\[ G_w[\ell/s] \approx N/(4.2 \Delta t) = 10^{-3} v A_F = 10^{-3} v F_s d_h^2, \]

where

\[ \Delta t[^\circ C] \quad \text{is the allowed temperature increase} \]
\[ v[\text{m/s}] \quad \text{is the velocity of the cooling water} \]
\[ A_F[\text{mm}^2] \quad \text{is the flow area} \]
\[ F_s = A_F/d_h^2 \quad \text{is the shape factor (= \( \pi/4 \) for round holes)} \]
\[ d_h[\text{mm}] = 4A_F/\text{perimeter} \quad \text{is the hydraulic diameter.} \]

For turbulent flow the required pressure drop may be obtained from

\[ \Delta P_w[\text{kg/cm}^2] = 0.18 L_c v^{1.75}/(F_s d_h^{1.25}) \]

where \( L_c[\text{m}] = \text{length of conductor, and it is noted that } 0.18(\pi/4)^{-1.75} = 0.28. \)

If \( d_h \) is in metres, this becomes for round holes

\[ \Delta P_w[\text{kg/cm}^2] = 5 \times 10^{-5} L v^{1.75}/d_h^{1.25}. \]

Alternatively, one has (with somewhat more pessimistic assumptions about the pressure loss)

\[ G_w[\ell/s] = 1.25 10^{-3}(1 + 0.009 t_w) F_s d_h^{2.71}(\Delta P_w/L_c)^{0.57} \]

where \( t_w[^\circ C] \) is the water temperature.

*) See page 43 for references
2. FERRITE-LOADED RF ACCELERATING CAVITY

2.1 Basic quantities

\[ Z = \frac{1}{C_c} \]

Equivalent transmission line

Cavity cross section

\[ \text{Short circuit} \]

\[ Z = (L_d/C_d)^{1/2} = 60(\mu_{eff}/\epsilon_{eff})^{1/2} \ln(R_3/R_1) \]

\[ L_d = \left[ \frac{\epsilon_{eff}}{\mu_{eff}} \frac{\mu_0 \epsilon_c \ln(R_3/R_1)}{2\pi} \right] \]

\[ C_d = 2\pi \epsilon_{eff} \epsilon_0 \frac{\epsilon_c}{\ln(R_3/R_1)} \]

\[ Z = (L_d/C_d)^{1/2} = 60(\mu_{eff}/\epsilon_{eff})^{1/2} \ln(R_3/R_1) \]

\[ \lambda = \frac{\nu}{f_a} = \frac{\epsilon}{[f_a (\mu_{eff} \epsilon_{eff})^{1/2}] = \epsilon_c/[f_a (L_d C_d)^{1/2}] \]

2.2 Length of cavity

For resonance (assuming negligible losses)

\[ \tan(2\pi \epsilon_c / \lambda) = (\omega_0 C_c Z)^{-1} \]

i.e.

\[ \tan(2\pi \epsilon_c / \lambda) = (\omega_0 C_c Z)^{-1} \]
which becomes for small arguments with \( \omega_0 \approx (L_d C_e)^{-1/2} \)

\[
\varepsilon_c \approx \lambda \left( \frac{C_d}{C_e} \right)^{1/2} (2\pi).
\]

2.3 Equivalent resonant circuit

If one knows the complex cavity admittance

\[ Y = G + jB = (G^2 + B^2)^{1/2} e^{j\phi} \]

as a function of \( \omega \), one can find \( \varepsilon_c \) from

\[
\varepsilon_c = \frac{(G_e \tan \phi)}{(2\Delta \omega)}
\]

where the phase angle \( \phi \) pertains to the frequency \( \omega = \omega_0 \pm \Delta \omega \), and hence

\[
L_e = \left( \omega_0^2 C_e \right)^{-1}.
\]

The relation between the equivalent and the "real" quantities is as follows (on the basis of \( B = B_e = 0; dB/d\omega = dB_e/d\omega \) for \( \omega = \omega_0 \))

\[
\varepsilon_c = 0.5 C_c + 0.5 C_d [\sin^2(2\pi \varepsilon_c /\lambda)]^{-1}
\]

\[
L_e = 2L_d \left( \frac{2\pi \varepsilon_c /\lambda}{(C_c/C_d)} + [\sin^2(2\pi \varepsilon_c /\lambda)]^{-1} \right)^{-1}.
\]

2.4 Longitudinal variation of power loss

\[
P(\varepsilon) = P_{\text{max}} \cos^2(2\pi \varepsilon_c /\lambda)
\]

\[
\bar{P} = P_{\text{max}} \left[ 0.5 + \left[ 0.25 \sin(4\pi \varepsilon_c /\lambda) \right]/(2\pi \varepsilon_c /\lambda) \right]
\]

where \( P_{\text{max}} \) is the maximum loss (occurring at the short circuit) and the power loss per unit volume is assumed to be constant.
3. VACUUM PRESSURE REQUIRED

The natural growth of the beam emittance in either transverse plane is given by

\[ \Delta(\epsilon \beta Y) [\text{mrad mm}] = 0.32 \lambda P \log(1 - \eta) \int_{t_o}^{t_1} \beta^{-2} \gamma^{-1} dt \text{ [m Torr s]} \]

where \( \lambda = R/Q \), \( P \) = nitrogen equivalent pressure and \( \eta \) is the fraction of the particles contained in the emittance

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>0.5</th>
<th>0.8</th>
<th>0.9</th>
<th>0.95</th>
<th>0.97</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log(1 - \eta) )</td>
<td>0.694</td>
<td>1.67</td>
<td>2.3</td>
<td>3.0</td>
<td>3.5</td>
</tr>
</tbody>
</table>

\( a) \) For \( (\dot{\beta} Y) = \text{const} \), one has

\[ \int_{t_0}^{t_1} \beta^{-2} \gamma^{-1} dt = (\dot{\beta} Y)^{-1} \left( 1/\beta(t_0) - 1/\beta(t_1) \right) + \log(\gamma(t_1)[1 + \beta(t_1)]/\gamma(t_0)[1 + \beta(t_0)]) \]

\( b) \) In the case of sinusoidal excitation of the magnet field

\[ \beta(t)Y(t) = (\beta Y)_{d.c.} - (\beta Y)_{a.c.} \cos \omega m t \]

\[ = 0.5 \left\{ (\dot{\beta} Y) + (\ddot{\beta} Y) - [(\dot{\beta} Y) - (\ddot{\beta} Y)] \cos \omega m t \right\} \]

one has to good approximation (for \( \dot{\beta} \approx 1 \) and \( \ddot{\beta} \leq 1 \))

\[ \int_{t_0}^{t_1} \beta^{-2} \gamma^{-1} dt \approx \pi(1/\dot{\beta} + 1/\ddot{\beta})/\{2\omega m [(\dot{\beta} Y)(\ddot{\beta} Y)]^{1/2} \}. \]
REFERENCES ([1,3] means page 3 of reference 1)

18. U. Bigliani, "Système HF du Booster : Capture dans l'espace de phase longitudinal", CERN SI/Int. EL 68-2, Appendice IV.
### Table of Constants

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Meaning</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>velocity of light</td>
<td>$2.997925 \times 10^8$</td>
<td>m/s</td>
</tr>
<tr>
<td>e</td>
<td>electronic charge</td>
<td>$1.6021 \times 10^{-19}$</td>
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<td>h</td>
<td>Planck's constant</td>
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</tr>
<tr>
<td>$h/e$</td>
<td>quantum charge ratio</td>
<td>$4.1355 \times 10^{-15}$</td>
<td>Js C$^{-1}$</td>
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<td>Boltzmann's constant</td>
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<td>$m_d$</td>
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<td>m</td>
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These values are taken from the 48th edition of the CRC Handbook of Chemistry and Physics. (Still within the limits of errors of the values given in the 50th edition.)
Definitions: \( f(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2/2); \quad u^2 = 1 \)

\[
P(u) = \int_{-u}^{u} f(u) \, du
\]

One-dimensional density distribution:

\[
\rho(x) = (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp(-x^2/(2\sigma^2)) = \sigma^{-1} f(x/\sigma)
\]

Normalization:

\[
\int_{-\infty}^{\infty} \rho(x) \, dx = 1; \quad \int_{-\infty}^{\infty} \rho(x) \, dx = P(x/\sigma)
\]

Variance:

\[
\overline{x^2} = \int_{-\infty}^{\infty} x^2 \rho(x) \, dx = \sigma^2
\]

Standard deviation:

\[
\overline{x^2} = \sqrt{\overline{x^2}} = \sigma
\]

Two-dimensional density distribution (x, z uncorrelated):

\[
\rho(x, z) = (2\pi \sigma_1 \sigma_2)^{-1} \exp(-x^2/(2\sigma_1^2) - z^2/(2\sigma_2^2)) = (\sigma_1 \sigma_2)^{-1} f(x/\sigma_1)f(z/\sigma_2)
\]

Normalization: over the whole plane \( \iint \rho \, dx \, dz = 1 \)

Variances:

\[
\overline{x^2} = \sigma_1^2 \text{ for any } z \text{ or for all } z; \quad \overline{z^2} = \sigma_2^2 \text{ similarly}
\]

The ellipses \( x^2/\sigma_1^2 + z^2/\sigma_2^2 = \nu^2 \) are lines of constant \( \rho \) with

\[
\rho = (2\pi \sigma_1 \sigma_2)^{-1} \exp(-\nu^2/2); \quad \nu^2 = 2.
\]

\begin{table}
\begin{tabular}{cccccc}
| \( u \) & \( v \) & \( f(u) \) & \( P(u) \) & \( 1 - \exp(-\nu^2/2) \) & \( \nu\sqrt{2} \) & \( \nu^2/2 \) \\
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$E_p = 938.26$ MeV

APPENDIX B

PROTON KINEMATICAL TABLE

- 47 -
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\[ \alpha = \frac{A}{(\beta + 1 \cdot \gamma + 1 \cdot \delta + 1 \cdot \eta)} \]