Supersymmetric sigma models and the ’t Hooft instantons

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Abstract

Witten’s linear sigma model for ADHM instantons possesses a natural (0, 4) supersymmetry. We study generalizations of the infrared limit of the model that are invariant under (4, 4) supersymmetry. In the case of four space-time dimensions a background with a conformally flat metric and torsion is required. The geometry is specified by a single real scalar function satisfying Laplace’s equation. It gives rise to ’t Hooft instantons for the gauge group $SU(2)$, instead of the general ADHM instantons for an $SO(n)$ gauge group in the case (0, 4).
1 Introduction

Recently, Witten [1] constructed a (0, 4) supersymmetric linear sigma model in two dimensions with a potential term. Remarkably, under simple assumptions about the structure of the (0, 4) multiplets involved, the Yukawa couplings of the model satisfy the ADHM equations of the instantons construction [2]. In the infrared limit the massless chiral fermions in the model are coupled to an instanton gauge field, obtained according to the ADHM prescription [2]. The fact that (0, 4) supersymmetry is related to self-duality of the target-space gauge fields is not new (see, for instance, [3]). However, it is a very unusual feature of Witten’s model that it restricts the general self-dual gauge fields to those with finite action, i.e., instantons. In [4] we gave a manifestly supersymmetric and off-shell formulation of the ADHM sigma model in harmonic (0, 4) superspace.

In the present paper we study generalizations of Witten’s model to (4, 4) supersymmetry. More precisely, we examine the conditions for the model to have off-shell (4, 4) supersymmetry in the infrared limit. A necessary condition is that the number of massless chiral fermions be tied to the dimension of the target space-time. In this case the linearized action of the massless sector possesses an off-shell (4, 4) supersymmetry. We find the most general action of the massless fields consistent with this supersymmetry. It necessarily involves nontrivial space-time geometry with torsion. In four space-time dimensions supersymmetry restricts the Yang-Mills gauge group to $SU(2)$ and the metric to be conformally flat. The conformal factor satisfies Laplace’s equation. In fact, it is identified with the scalar field of the ’t Hooft construction of the $5k'$ family of $SU(2)$ instantons [5]. We also find analogs of the ’t Hooft instantons in higher dimensions.

In section 2 we briefly review the harmonic superspace construction of the ADHM sigma model [4]. In section 3 we concentrate on its infrared limit and generalize it to include a nontrivial space-time background. Then we study the conditions for the massless theory to possess off-shell (4, 4) supersymmetry. In section 4 we derive the component action and discuss further problems. In the concluding section we compare our results with various facts about (4,4) sigma models known in the literature.

2 ADHM sigma model in (0, 4) harmonic superspace

In this section we shall review the construction of the ADHM sigma model with manifest (0, 4) supersymmetry. More details can be found in [4]. The supersymmetry algebra is given by

$$\{Q_{-}^{AA'}, Q_{-}^{-BB'}\} = 2\epsilon^{AB} e^{A'B'} P_{-}$$

(1)

Here $\pm$ indicate the Lorentz ($SO(1,1)$) weights, and we always write the weights as lower indices. $A$ and $A'$ are doublet indices of the $SO(4) \sim SU(2) \times SU(2)'$ automorphism group of (1). The supercharges satisfy the reality condition $Q_{-}^{AA'} = e_{AB} e_{A'B'} Q_{-}^{-BB'}$. The algebra (1) can be realized on the super-world sheet with coordinates $x_{++}, x_{--}, x_{+-}, x_{-+}, \theta_{\pm}^{AA'}$:

$$\delta x_{++} = -i e_{+}^{AA'} \theta_{+}^{AA'}, \quad \delta x_{--} = 0, \quad \delta \theta_{+}^{AA'} = e_{+}^{AA'}.$$

(2)

Following the method of $SU(2)$ harmonic superspace [6], we extend the world sheet by means of the harmonic variables $u^{\pm A} \in SU(2)/U(1) : u^{+A} u^{-A} = 1, \quad u^{+A} = u^{-A}$.
These variables are inert under supersymmetry, transform as doublets of one of the $SU(2)$ subgroups of automorphisms, and are defined modulo the $U(1)$ subgroup of $SU(2)$. In essence, harmonics parametrize the two-sphere $S^2$ of all possible choices of the three complex structures characteristic for $(0,4)$ supersymmetry. The extended superspace $x_+, x_-, \theta^A_A$, $u^\pm_A$ possesses an invariant subspace of lower Grassmann dimension:

\begin{align}
\dot{x}^+_+ &= x^+_+ + i \theta^A_A \theta^B_A u^+_A u^+_B \, x^-- , \quad \theta^{+-}_+ = u^+_A \theta^A_A , \quad u^+_A ; \\
\delta \dot{x}^+_+ &= -2 i \epsilon^A_A u^+_A \theta^+_A , \quad \delta x^-- = 0 , \quad \delta \theta^+_+ = \epsilon^A_A u^+_A .
\end{align}

Note that in the notation $\theta^\pm_+$ the upper index $\pm$ is a $U(1)$ and the lower one is Lorentzian. It is this analytic superspace that is most adequate for describing multiplets of $(0,4)$ supersymmetry. The analytic superfields $\Phi^A(x, \theta^+)$ carry in general a $U(1)$ harmonic charge and an $SO(1,1)$ weight. They have a very short Grassmann expansion, e.g.,

$$\Phi^A(x, \theta^+, u) = \phi^A(x, u) + \theta^+ A \epsilon^{-1}_A(x, u) + (\theta^+_+)^2 f^{22}(x, u) ,$$

where $(\theta^+_+)^2 \equiv \theta^+ A \theta^+_A \epsilon^A_A$. The coefficients in (5) are harmonic-dependent fields; they can be expanded in terms of spherical harmonics on $S^2$. For certain values of the $U(1)$ charge $q$ such superfields can be made real in the sense of a special conjugation on $S^2$ (for more details about the harmonic formalism see [6], [4]).

Finally, to complete the formalism we need the expression for the covariant harmonic derivative in the analytic superspace (3):

$$D^{++} = u^+ A \frac{\partial}{\partial u^+ A} + i (\theta^+_+)^2 \frac{\partial}{\partial \dot{x}^+_+} .$$

It will help us to write down irreducibility conditions and equations of motion for harmonic superfields.

The ADHM sigma model of Witten exploits three types of $(0,4)$ multiplets: chiral (right-handed) fermions and two different scalar multiplets that contain both bosons and left-handed fermions. The chiral fermions are described in harmonic superspace by the following anticommuting and real (in the sense mentioned above) superfields

$$\Lambda^a_+(x, \theta^+, u) = \lambda^a_+(x, u) + \theta^{+A}_+ S^a_4(x, u) + i (\theta^+_+)^2 \sigma^{-a}_-(x, u)$$

with the free action

$$S_A = \frac{1}{2} \int d^2 x \, d^2 u \theta^+_+ \Lambda^a_+ \, D^{++} \Lambda^a_+ .$$

The external index $a = 1, \ldots, n+4k'$ is an $SO(n+4k')$ one. The free equation of motion $D^{++} \Lambda^a_+ = 0$ shows that all the components of $\Lambda^a_+$ are auxiliary, except for the lowest component in the harmonic expansion of $\lambda^a_+ (x, u)$, i.e., $\lambda^a_+(x, u) = \lambda^a_+(x)$. These are the physical chiral fermions satisfying the free equation $\partial_{--} \lambda^a_+(x) = 0$ and forming a trivial on-shell representation of $(0,4)$ supersymmetry. However, as shown in [4], its off-shell version requires the infinite sets of auxiliary fields contained in (7), except if the number

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1To simplify the notation we shall not write explicitly $\dot{x}^+_+$ when it is clear that we work in the analytic superspace (3).
of chiral fermions is a multiple of four (these “short” multiplets were discussed in the second ref. [3], see also [7]).

One of the scalar multiplets (we shall call it non-twisted) involves the coordinates $X^{AY}$ of the Euclidean target space $R^4$, in which the Yang-Mills fields will be defined. 2 It is described by the real analytic superfield $X^{+Y}(x, \theta^+, u)$ ($Y = 1, 2$ is an $Sp(1) \sim SU(2)$ index), which satisfies the following harmonic irreducibility condition

$$D^{++} X^{+Y} = 0.$$  

(9)

The solution to it

$$X^{+Y}(x, \theta^+, u) = X^{AY}(x) u^+_A + i \theta^+_A \psi^+_{-A'}(x) - i (\theta^+_A)^2 \partial_- X^{AY}(x) u^-_A$$  

(10)

contains 4 bosonic and 4 fermionic real off-shell fields. The free action for the this multiplet is again given as an integral over the analytic superspace:

$$S_X = i \int d^2x d^2\theta^+_A X^{+Y} \partial^{++} X^{+Y}.$$  

(11)

Finally, the last multiplet is the so-called “twisted” scalar multiplet, in which the $SU(2)$ indices carried by the bosons and fermions are interchanged (as compared to the non-twisted multiplet). Its superspace description requires a set of *anticommuting abelian gauge* superfields $\Phi^{+Y'}_+(x, \theta^+, u)$ ($Y' = 1, 2, \ldots, 2k'$ is an $Sp(k')$ index). The gauge transformations have the form

$$\delta \Phi^{+Y'}_+ = D^{++} \omega^{+Y'}_+$$  

(12)

with analytic parameters $\omega^{+Y'}_+(x, \theta^+, u)$. Using these parameters one can choose the “harmonically short” and non-manifestly-supersymmetric *Wess-Zumino-type gauge*

$$\Phi^{+Y'}_+(x, \theta^+, u) = \theta^{+A'}_+ \phi^{Y'}_{-A'}(x) + i (\theta^+_A)^2 u^-_A \chi^{Y'A}(x).$$  

(13)

in which only the $4k'$ real physical bosons $\phi^{Y'}_A$ and fermions $\chi^{Y'A}$ are left. This multiplet is off shell. The gauge invariant free action for the superfield $\Phi^+_+$ has been given in [4] and we shall not need it here.

Now we turn to the discussion of the potential-type coupling of the above three $(0,4)$ multiplets. Such a coupling is severely restricted by dimension, $SO(1,1)$ and $U(1)$ invariance, as well as Grassmann analyticity. An important additional assumption made by Witten in [1] is that the part $SU(2)'$ of the $(0,4)$ supersymmetry automorphism group is preserved (this requirement is motivated by the desire to obtain a CFT in the infrared limit). As shown in [4], then the only possible coupling term is

$$S_{int} = m \int d^2x d^2\theta^+_A \Phi^{+Y'}_+ \psi^{+Y'}_+(X^+, u) \Lambda^{+a}_+,.$$  

(14)

It is invariant under the gauge transformation (12) (together with the kinetic term (8) for $\Lambda^+_+$) provided the chiral fermions transform as

$$\delta \Lambda^{+a}_+ = m \psi^{+a}_+(X^+, u) \omega^{-Y'}_+. $$  

(15)

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2 We shall discuss the generalization to the case of $R^{4k}$ at the end of section 3.
and the matrix \( v^+_{Y^i}(X^+, u) \) satisfies the following two conditions
\[
v^+_{Y^i}, v^+_{Z^i} = 0, \\
D^{++} v^+_{Y^i}(X^+, u) = 0.
\] (16) (17)

The general solution to (17) is (recall (9))
\[
v^+_{Y^i}(X^+, u) = u^{+A} \alpha^a_{AY^i} + \beta^a_{Y^i} X^Y, \\
v^+_{Z^i}(X^+, u) = u^{+A} \gamma^a_{AZ^i} + \delta^a_{Z^i} X^Z,
\] (18)

where the matrices \( \alpha, \beta \) are constant. At \( \theta^+=0 \), the matrix \( v^+_{Y^i}(X^+, u) \) reduces to
\[
v^+_{Y^i}(X^+, u)|_{\theta=0} = u^{+A} (\alpha^a_{AY^i} + \beta^a_{Y^i} X^Y) \equiv u^{+A} \Delta^a_{AY^i},
\] (19)

and then the other condition (16) implies for the matrix \( \Delta^a_{AY^i} \)
\[
\Delta^a_{AY^i}, \Delta^a_{BZ^j} + (A \leftrightarrow B) = 0.
\] (20)

The matrix \( \Delta^a_{AY^i} \) linear in \( X \) and satisfying (20) is the starting point in the ADHM construction for instantons [2].

Now we discuss the infrared limit of the theory. To this end one has to separate the massless and massive modes. Among the \( n + 4k' \) left-handed fermions \( \lambda^a_+ \) contained in \( \Lambda^a_+ \) there is a subset of \( 4k' \) which are paired with the right-handed fermions in \( \Phi \) and become massive (together with the bosons from \( \Phi \)). The remaining chiral fermions stay massless. To diagonalize the action, we complete the \( 2k' \times (n + 4k') \) matrix \( v^+_{Y^i}(X^+, u) \) to a full orthogonal matrix \( v^{\tilde{a}}(X^+, u) \), where the \( n + 4k' \) dimensional index \( \tilde{a} = (+Y^i, -Y^i, i) \) and \( i = 1, \ldots, n \) is a vector index of the group \( SO(n) \). Orthogonality means
\[
v^{\tilde{a}}v^{\tilde{b}} = \delta^{\tilde{a}\tilde{b}},
\] (21)

where \( \delta^{+Y^i,-Z^i} = -\delta^{-Y^i,+Z^i} = \delta^{Y^i,+Z^i} = \delta^{-Y^i,-Z^i} = \delta^{\pm Y^i,i} = 0 \). Since \( v^{\tilde{a}} \) is a function of \( X^+ \) and \( u^+ \), we take the other blocks of \( v^{\tilde{a}} \), namely \( v^{-Y^i} \) and \( v^i \) to be such functions too.

With the help of the matrix \( v^{\tilde{a}} \) we can make a change of variables from the superfield \( \Lambda^a_+ \) to \( \Lambda^a_{\tilde{A}} = v^{\tilde{a}} \Lambda^a_+ \). Then the gauge transformation (15) gets the form
\[
\delta \Lambda^a_{\tilde{A}} = m \omega^a_{\tilde{A}}, \quad \delta \Lambda^a_{\tilde{Y}'} = \delta \Lambda^a_{\tilde{Y}'} = 0,
\] (22)

hence the superfields \( \Lambda^a_{\tilde{Y}'} \) can be completely gauged away. Further, the superfields \( \Lambda^a_{\tilde{Y}'} \) enter the action without derivatives; their elimination results in the following Lagrangian for the chiral fermions
\[
\mathcal{L}^+_{\tilde{A}}(\Lambda) = \frac{1}{2} \Lambda^a_+ [\tilde{g}^{ij} D^{++} + (V^{++})^{ij}] \Lambda^a_+ \\
= \frac{1}{2} [(V^{++})_{Y^i} \Lambda^a_+ + m \Phi^a_{Y^i}] [(V^{++} - 1)_{Y^i} \Lambda^a_+ + m \Phi^a_{Y^i}].
\] (23)

Here we used the notation
\[
(V^{++})^{\tilde{a}\tilde{b}} = v^{\tilde{a}} D^{++} v^{\tilde{b}} = v^{\tilde{a}} D^{++} v^{\tilde{a}} = (V^{++})_{Y^i} Z^i = (V^{++})_{Y^i} Z^i.
\] (24)
In the infrared limit \( m \to \infty \) the kinetic term for \( \Phi_+^{\pm} \) is suppressed, so the second line of (23) becomes auxiliary and can be dropped. The final result for the massless sigma model is:

\[
S_{m-\infty} = \int d^2x dud\theta_+^{\pm} \left[ iX^+ Y \partial_{++} X_+^+ + iP_{++}^Y D^{++} X_+^+ + \frac{1}{2} \Lambda^i_+ (\delta^{ij} D^{++} + (V^{++})^{ij}) \Lambda^j_+ \right].
\] (25)

Here we have added the kinetic term (11) for \( X^{+Y} \) and have introduced the harmonic irreducibility condition (9) into the action with the Lagrange multiplier \( P_{++}^Y \). The object

\[
(V^{++})^{ij}(X^+, u) = v^i(X^+, u)D^{++} v^j(X^+, u).
\] (26)

is the twistor transform of the ADHM \( SO(n) \) gauge field (or, we should rather say, the harmonic version [8] of Ward’s [9] instanton construction).

The alternative to the manifestly supersymmetric gauge \( \Lambda^{++Y} = 0 \) above is the Wess-Zumino gauge (13). In it, after a suitable diagonalization and again in the infrared limit one finds an ADHM gauge field coupled to the massless subset of the chiral fermions \( \lambda^a_+ \) [1].

This completes our review of the superfield construction for the ADHM sigma model.

### 3 Searching for (4,4) supersymmetry

The procedure of the previous section lead to the massless action (25) for the superfields \( X^{+Y} \) and \( \Lambda^i_+ \). It involves four real bosons \( X^{AY} \) and four real left-handed fermions \( \psi^{A\bar{Y}} \) coming from the superfield \( X^{+Y} \), as well as \( n \) real right-handed fermions \( \lambda^i_+ \) from the matter superfields \( \Lambda^i_+ \). If we want to form a (4,4) multiplet out of them, the first necessary condition is to match the numbers of left- and right-handed fermions. Consequently, we have to choose \( n = 4 \) and restrict the gauge group to (at most) \( SO(4) \sim SU(2) \times SU(2) \).

The search for further (4,4) supersymmetry is based on an examination of the flat action obtained by putting \( V^{++} = 0 \) in (25):

\[
S_{\text{free}} = \int d^2x dud\theta_+^{\pm} \left( iX^{+Y} \partial_{++} X_+^+ + iP_{++}^Y D^{++} X_+^+ + \frac{1}{2} \Lambda^i_+ D^{++} \Lambda^i_+ \right).
\] (27)

It is not hard to check that this free action has two different off-shell (4,4) supersymmetries, depending on how the \( SU(2) \) indices are involved in the transformation laws. The first possibility is obtained by replacing the \( SO(4) \) vector index \( i \) by the \( SU(2) \times SU(2) \) pair \( A\bar{Y} \) (\( \bar{A} \) is an \( SU(2) \) index of a new type):

\[
\delta X^{+Y} = i\varepsilon_{-A}^{+} \Lambda^A_+ Y,
\]

\[
\delta \Lambda^A_+ = -2\varepsilon^{A}_{-} \partial_{++} X^{+Y} + \varepsilon^{A}_{+} P_{++}^Y,
\]

\[
\delta P_{++}^Y = -2i\varepsilon^{A}_{-} \partial_{++} \Lambda^A_+ Y,
\] (28)

where \( \varepsilon_{\pm A}^{\pm} = u_{\pm A}^{\pm} \varepsilon_{\pm}^{A} \). Actually, these transformation laws originate from the \( \theta_+^{\pm} \) expansion of the (4,4) superfield \( q^{+Y}(\theta_+^{\pm}, \theta_+^{\mp}) \) obtained by dimensional reduction from the \( N = 2 \) \( D = 4 \) hypermultiplet [6]:

\[
q^{+Y} = X^{+Y} + i\theta_+^{+A} \Lambda^A_+ Y - \frac{i}{2} (\theta_+^{+})^2 P_{++}^Y.
\] (29)
Moreover, in this case the free action (27) itself can be derived from the hypermultiplet action
\[ S = \int d^4 x \, d^4 \theta^+ \, q^{+Y} D^{++} \phi^+_Y. \] (30)

The second possibility is obtained by writing \( i \) as \( A \hat{Y} \) (now \( \hat{Y} \) is another type of \( SU(2) \) index) and then decomposing \( \Lambda^{+Y}_+ \) into harmonic \( U(1) \) projections \( \Lambda_+^{+Y} = u^\pm_+ A^{+Y}_+ (\Lambda^{+Y}_+ \) should not be confused with \( \Lambda^{+Y} \) from section 2):
\[
\delta X^{+Y} = i \varepsilon^{+Y}_- \Lambda^{+Y}_+, \\
\delta \Lambda^{+Y}_+ = -2 \varepsilon^{+Y}_- \partial_{++} X^{+Y}_+, \\
\delta \Lambda^{-Y}_+ = \varepsilon^{+Y}_- P^{+Y}_+, \\
\delta P^{+Y}_+ = -2 i \varepsilon^{+Y}_- \partial_{++} \Lambda^{-Y}_+. \] (31)

It is not hard to verify that in both cases the algebra of the supersymmetry transformations closes off shell. As one can see from (28) and (31), the main difference between the two types of supersymmetry amounts to interchanging different types of \( SU(2) \) indices (“twist”). Therefore we shall refer to the transformations (28) as non-twisted and to (31) as twisted (4,4) supersymmetry. It is a well-known fact that dimensional reduction from \( N = 2D = 4 \) gives rise to the former, whereas the latter is specific to two dimensions [10], [11].

The main question now is whether we can turn on a background in the free action (27) compatible with any of the above supersymmetries. The advantage of dealing with off-shell supersymmetry is that we do not have to adjust the transformation laws to the interaction. As will be clear from the end results, a simple self-dual Yang-Mills background like in (25) cannot be compatible with (4,4) supersymmetry. One is forced to introduce an additional “curved” deformation of the free action (this point has already been made clear in [12]). In order not to miss any possibility, we shall examine the most general background for the action (27) allowed by the Lorentz and \( U(1) \) properties and dimensions of the superfields \( X, P, \Lambda \) (note that \( X \) is dimensionless and \( [P] = 1, [\Lambda] = 1/2 \)). So, we write down
\[ S = \int d^2 x \, d^2 \theta^+ \left[ i \mathcal{L}^{+Y} \partial_{++} X^{+Y}_- + i P^{+Y} \left( D^{++} X^{+Y}_- + \mathcal{L}^{+3Y}_- \right) + \frac{1}{2} \Lambda^{+Y}_+ (\delta^{ij} D^{++} + (V^{++})^{ij}_+) \Lambda^{+Y}_+ \right]. \] (32)

Here \( \mathcal{L}^{+Y}(X^+, u), \mathcal{L}^{+3Y}(X^+, u) \) and \((V^{++})^{ij}(X^+, u)\) are for the time being arbitrary functions of \( X^{+Y} \) and \( u^{+Y}_+ \).

The next step is to vary the action (32) under either (28) or (31), derive the corresponding restrictions on the potentials \( \mathcal{L}^{+Y}, \mathcal{L}^{+3Y}, (V^{++})^{ij} \) and solve them in terms of unconstrained prepotentials. The computations are straightforward, therefore here we shall only give the final answers. Up to insignificant field redefinitions those are:

### 3.1 Non-twisted case

Here the potential \( \mathcal{L}^{+Y} \) takes the form (after field redefinitions) \( \mathcal{L}^{+Y} = X^{+Y} \). The other two potentials are expressed in terms of a single scalar prepotential with \( U(1) \) charge +4

\[ \mathcal{L}^{+3Y} = \frac{1}{2} \mathcal{L}^{+Y} \mathcal{L}^{+Y} - u^{+Y}_+ X^{+Y}_- \mathcal{L}^{+Y} \]

\[ (V^{++})^{ij} = \frac{1}{2} \mathcal{L}^{+Y} \delta^{ij} - \frac{1}{2} \mathcal{L}^{+Y} \partial_{++} \mathcal{L}^{+Y} \delta^{ij} - \frac{1}{2} \mathcal{L}^{+Y} \partial_{++} \mathcal{L}^{+Y} \partial_{++} \mathcal{L}^{+Y} \delta^{ij} \]
\[ \mathcal{L}^+(X^+, u) \] as follows

\[ \mathcal{L}^+_{V^3} = \partial_V^+ \mathcal{L}^+ = (V^{++})_{AB}|_{\bar{B}Z} = -\epsilon_{\bar{A}B} \partial_V^+ \partial_Z^+ \mathcal{L}^+ . \]  \hspace{1cm} (33)

Once again, one realizes that this form of the action originates from the dimensional reduction of the general \( N = 2 \) \( D = 4 \) hypermultiplet action [14]

\[ S = \int d^6 x d\theta^+ \left[ \theta^+ \mathcal{L}^+ + \mathcal{L}^+ + \mathcal{L}^+(q^+, u) \right] . \]  \hspace{1cm} (34)

The prepotential \( \mathcal{L}^+ \) in (34) has been shown in [14] to generate the most general hyper-Kähler manifolds. Such manifolds are torsion-free and the connection term \( (V^{++})_{AB}|_{\bar{B}Z} \) in (33) is in fact the Christoffel connection.

3.2 **Twisted case**

The requirement of (4,4) supersymmetry of the type (31) leads to the following restrictions on the potentials in (32):

\[ \mathcal{L}^+_{V^3} = 0 , \quad (V^{++})_{AB}|_{\bar{B}Z} = \epsilon_{\bar{A}B} u_{\bar{A}}^a u_B^a [1 - V(X^+, u)] , \quad \partial_V^+ \mathcal{L}^+ - \partial_Z^+ \mathcal{L}^+ = -2\epsilon_{\bar{A}B} V(X^+, u) . \]  \hspace{1cm} (35)

Once more we have a single scalar prepotential \( V(X^+, u) \), but this time it carries no \( U(1) \) charge. Note that eq. (35) determines the potential \( \mathcal{L}^+ \) up to a gradient \( \partial_V^+ \mathcal{L}^+ \), but one can easily see that this is a gauge invariance of the action (32). Written down in terms of the restricted potentials (35), the action compatible with twisted supersymmetry takes the form

\[ S = \int d^6 x d\theta^+ \left( i \mathcal{L}^+(X^+, u) \partial_{++} X^+_Y + i P^{++}_{++} D^{++} X^+_Y 
- \Lambda^{+Y}_{++} \Lambda^{+Y}_{++} - \Lambda^{+Y}_{++} \Lambda^{+Y}_{++} \right) . \]  \hspace{1cm} (36)

We see that in fact this action is very similar to the one obtained in section 2. The difference is that in (36) the gauge group is reduced to \( SU(2) \) (the gauge superfield \( V^{++} \) being given in (35)) and the kinetic term for \( X^+ \) is deformed by the potential \( \mathcal{L}^{++} \). Note that the presence of this potential does not affect in any way the arguments leading to the ADHM-type interaction in section 2; there we have never used the kinetic term for \( X^+ \).

Finally, we briefly mention the generalization of the above results to the case of \( 4k \) target space dimensions. In fact, the ADHM sigma model of Witten [1] reviewed in section 2 can equally well accommodate \( R^{4k} \) as its target space, just replacing the \( Sp(1) \) spinor index \( Y \) of \( X^{++} \) by an \( Sp(k) \) index. For our study of (4,4) supersymmetry it is convenient to adapt the notation as follows. Instead of \( X^{++} \) we write \( X^{++}_\alpha \), where \( Y \) is still an \( Sp(1) \) spinor index and we have added a new vector index \( \alpha \) of \( SO(k) \) (thus \( Y \alpha \) forms a decomposition of an \( Sp(k) \) spinor index under \( Sp(1) \times SO(k) \)). The same index \( \alpha \) will be attached to the chiral fermions, e.g., \( \Lambda^{++}_{4Y\alpha} \). This amounts to an obvious modification of the (4,4) supersymmetry transformation rules (28) and (31) and of the
general interacting action (32). In the non-twisted case we find once again a hyper-Kähler background, hence no gauge field. In the twisted case we obtain the analog of (36)

\[ S = \int d^2x dud\theta^+_\alpha \left( i \mathcal{L}^{+Y}_\alpha(X^+,u)\partial_{++}X^+_Y + i \mathcal{L}^{-Y}_\alpha(X^+,u)\partial_{--}X^-_Y \right. \]

\[ \left. - \Lambda^+_{++} D^{++} \Lambda^+_{--} - \frac{1}{2} \Lambda^+_{++} \Lambda^+_{--} \right). \]  

(37)

Here the potentials \( \mathcal{L}^{+Y}_\alpha(X^+,u) \) and \( V_\alpha(X^+,u) = V_\alpha(X^+,u) \) are related by the constraint

\[ \partial_{++}^\alpha \mathcal{L}^+_{Z\beta} - \partial_{--}^\beta \mathcal{L}^-_{Y\alpha} = -2\epsilon_{YZ} V_\alpha \beta. \]  

(38)

We see that the gauge connection \( V^{++} \) from (32) takes the form \( (V^{++})_{AY\alpha | B\beta} = \epsilon_{YZ} u^+_A u^+_B [\delta_{\alpha \beta} - V_\alpha \beta] \), so it corresponds to the gauge group \( Sp(k) \) instead of \( SU(2) \sim Sp(1) \) in the four-dimensional case.

### 4 Components of the twisted sigma model

In order to better understand the content of the twisted sigma model (36) we shall present its component expansion (in the case of 4 space-time dimensions only). This is made very easy by the fact that the Lagrange multipliers \( P^{+Y}_{++} \) and \( \Lambda^{-Y}_{++} \) in (36) force both superfields \( X^+_Y \) and \( \Lambda^+_{++} \) to depend trivially on the harmonics, therefore they contain only finite numbers of fields:

\[ X^{+Y} = X^{AY}(x) u^+_A + i \theta^+_A Y^A \psi^Y_A(x) - i (\theta^+_A)^2 \partial_{--} X^{AY}(x) u^-_A, \]

\[ \Lambda^+_{++} = \Lambda^{AY}_+(x) u^+_A + \theta^+_A \psi^T_A(x) - i (\theta^+_A)^2 \partial_{--} \Lambda^+_{++}(x) u^-_A. \]  

(39)

We then insert these expansions in the action and do the Grassmann integral. The field \( s \) is easily seen to be auxiliary, so we eliminate it and in the end obtain the following sigma model for the fields \( X^{AY}, \psi^Y_A, \Lambda^+_{++} \):

\[ S = \int d^2x \left[ - (\epsilon_{AB} g(X) + b_{AB}(X)) \right. \partial_{++} X^{AY}(x) \partial_{--} X^{BY}(x) + \frac{i}{2} g(X) \psi^{AY}_A \left. \partial_{++} \psi_{-AY}(x) \right] \]

\[ \left. + \frac{i}{2} g(X) \Lambda^+_{++} \partial_{--} \Lambda^+_{++}(x) + \frac{i}{2} \psi^{AY}_A \psi_{-AY}(x) \left. \partial_{++} X^{CZ}(x) \Omega_{CZ} \psi^{T} \Omega_{CZ} \psi^{T}(x) \right] \]

\[ \left. + \frac{i}{2} \lambda^{AY}_A \lambda^B_{++} \partial_{--} X^{CZ} \Omega_{CZ} \Lambda^+_{++}(x) + \lambda^{AY}_A \lambda^B_{++} \psi^{AT}_A \psi^{T}_A \left. \partial_{--} R_{AB}(x) \right]. \]  

(40)

Here the sigma model metric is given by

\[ g_{AYBZ}(X) = \epsilon_{ABCV} g(X) \]  

and the two-form (the torsion potential) is

\[ b_{AYBZ}(X) = \epsilon_{YZ} b_{AB} = 2 \int du [u^+_A u^-_B] V(X^+, u) \]  

the spin connections are

\[ \Omega_{CZ} \psi^{T}(X) = \epsilon_{T} (\partial_{CV}) g, \quad \Omega_{CZ} \Lambda^+_{++}(X) = - \epsilon_{C(B \partial_A)Z} g. \]  

(43)
Finally, the curvature $R$ in the four-fermion term is constructed in the usual way.

The action (40) has two remarkable properties. Firstly, we see that the geometric objects - metric and torsion (but not the two-form itself) are expressed in terms of a single real scalar function $g(X^{A^Y})$. In particular, this means that the sigma model metric is conformally flat. Secondly, the function $g(X^{A^Y})$ satisfies Laplace's equation. This is obvious from the definition (41):

$$g(X) = \int du \partial^{-Y} \partial_Y^X V(X^+, u) = 0$$

(44)

because of the holomorphic dependence of the potential $V(X^+, u)$ on $X^{+Y}$. In fact, what we see here is an example of Penrose's transform [15], where the solutions to Laplace's equation are parametrized by an unconstrained holomorphic function in twistor space (the role of twistor variables here is played by the $SU(2)$ harmonics). The spin connections in (43) consist of two parts - of the Riemannian connection and of the torsion. As explained in [12], one of the two spin connections can also be viewed as an $SU(2)$ gauge field. As a consequence of (4,4) off-shell supersymmetry this gauge field has the typical form of a 't Hooft instanton solution [5]. Remembering that we started from the action (25), in which the gauge field was by construction of the instanton (ADHM) type, we see that indeed we deal with a sigma model in which 't Hooft's instantons appear in a natural way.

5 Conclusions

In this paper we have studied the conditions under which the massless (0,4) sigma model involving chiral fermions coupled to an ADHM gauge field and obtained by Witten's procedure can have a larger (4,4) supersymmetry. The main assumption we have made is that the interacting theory should preserve the off-shell (4,4) supersymmetry of the free theory. We have seen that starting from the non-twisted free supersymmetry we could only obtain a hyper-Kähler sigma model without torsion and, consequently, without a self-dual gauge field. If we choose to preserve the other, twisted supersymmetry of the free theory, we obtain strong restrictions on the possible background: it must have a conformally flat metric and torsion expressed in terms of a single real scalar function $g(X)$. The latter satisfies Laplace's equation and thus gives rise to an $SU(2)$ instanton gauge field of the 't Hooft type.

We should point out that one could reach the same conclusions by applying the general results of [10] to the special case of a four-dimensional target space. In [10] the analysis of the conditions for (4,4) supersymmetry in a sigma model is carried out in terms of (2,2) superfields (chiral and twisted chiral). This implies choosing holomorphic coordinates in the target space. The sigma model Lagrangian is given by a Kähler potential. Our scalar function $g(X)$ appears in [10] as a second-order derivative of the Kähler potential. The main difference between this approach and ours (which makes use of (0,4) superfields) is in the treatment of the $SU(2)$ symmetry inherent to the problem we address here. Working in a holomorphic basis inevitably leads to loosing manifest $SU(2)$ invariance.

Another study of (4,4) twisted sigma models has recently been presented in [16], this time using a double-harmonic (4,4) superspace formalism. The results obtained
there agree with ours. We believe that the (0,4) approach may prove more efficient in investigating the possible linear (4,4) sigma models with potential terms.

We would also like to note that the relevance of ’t Hooft instantons in string-inspired sigma models has been shown very clearly in [12]. There one deals with two distinct cases: first with (0,4) and then with (4,4) supersymmetry. In the context of (0,4) supersymmetry the coupling to a ’t Hooft instanton gauge field (as opposed to a general, ADHM type one) appears as an Ansatz, which fits nicely with string theory. Then, for reasons having to do with the quantum behaviour of the model, the authors of [12] want to extend the (0,4) supersymmetry to full (4,4) and realize that the self-dual gauge field must necessarily be identified with the spin connection with torsion. Thus, a (4,4) model incorporating a self-dual gauge field cannot have a flat geometry. A point which is missing in [12] is the observation that in the context of (4,4) supersymmetry ’t Hooft instantons are not an Ansatz any more, but are the only possibility. We also remark that some conclusions reached in [12] about the general conditions on the background for (0,4) supersymmetry have been later on corrected in [17].

In conclusion we may say that the study of instantons in the context of (4,4) sigma models in this paper should be considered as a first step only. The more interesting question is whether there exists a linear (4,4) sigma model which automatically gives, in the infrared limit, the model with ’t Hooft instantons discussed here. An encouraging sign is the fact that the free action of section 2, in which we only keep the mass term in (14) (but drop the Yukawa couplings), does indeed have a (4,4) supersymmetry. However, the massive (4,4) multiplet has central charge and is on shell, which makes the analysis of the general potential self-interaction more difficult. We hope to come back to this problem in the near future.

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