Analytic approach to the polarization of the cosmic microwave background in flat and open universes

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Abstract

We develop an analytic method and approximations to compute the polarization induced in the cosmic microwave background radiation on a wide range of angular scales by anisotropic Thomson scattering in presence of adiabatic scalar (energy-density) linear fluctuations. The formalism is an extension to the polarized case of the analytic approach recently developed by Hu and Sugiyama to evaluate the (unpolarized) temperature correlation function. The analytic approach helps to highlight the dependence of potentially measurable polarization properties of the cosmic microwave background radiation upon various parameters of the cosmological model. We show, for instance, that the ratio between the multipoles of the temperature and polarization correlation functions depends very sensitively upon the value of $\Omega_0$, the matter density in units of the critical, in an open universe.

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I. INTRODUCTION

The cosmic microwave background radiation (CMB) is a seemingly unlimited source of information about the history and evolution of the Universe. Its existence is one of the most powerful arguments for the standard “hot” big-bang cosmology. Its high degree of isotropy is an indication of the Universe large scale homogeneity. Its precise black-body spectrum places stringent bounds on alternative cosmologies. Its large scale anisotropies uncover tiny fluctuations in the gravitational potential, most likely the same that once upon a time led to the formation of galaxies and other large scale structure in the Universe.

The polarization properties of the cosmic microwave background radiation constitute yet another set of observables whose eventual measurement is still to bear fruit. A positive measurement of a degree of polarization would provide much further insight into the Universe history and evolution. The current bound on the degree of linear polarization of the CMB on large angular scales, $P < 6 \times 10^{-5}$ [1], already serves to set limits and constraints upon alternative cosmological scenarios.

The fact that anisotropic Thomson scattering of photons and electrons around the time of decoupling induces a degree of linear polarization in the cosmic microwave background [2] was pointed out shortly after its discovery in 1965. Several estimates were made of the degree of polarization expected. The task is to evaluate the Stokes parameters of the photon distribution function, which satisfies a Boltzmann equation with a Thomson scattering collisional term [3]. The first computations were performed in homogeneous but anisotropically expanding universes [2,4–8]. Energy-density fluctuations [9–14] and long wavelength gravitational waves [15,16,12,13,17] were also considered as the source of the anisotropy that leads to polarization. Typically, at least within standard recombination histories, the degree of linear polarization turns out to be almost two orders of magnitude smaller than the large scale temperature anisotropy. This is still well beyond current detection capabilities. Nevertheless, with large scale anisotropies in the CMB now positively measured by COBE-DMR [18] and other experiments, it is worthwhile to further analyse the predictions for the CMB polarization properties within alternative realistic cosmological models, and its dependence upon various parameters.

Very detailed numerical computations of the CMB temperature fluctuations and polarization correlation functions on all angular scales were performed by Bond and Efstathiou [11] for standard $\Omega_0 = 1$ cold dark matter dominated universes, with scale invariant adiabatic or isocurvature scalar fluctuations. It is one of the purposes of the present paper to reproduce the most significant features of the multipole expansion of the polarization correlation function on all angular scales in this cosmological model from an analytic approach, to better highlight its dependence upon various parameters. To achieve that goal, in Section II we develop a method and perform approximations that very closely parallel the recent analytic approach introduced by Hu and Sugiyama to evaluate the temperature anisotropies on all angular scales for the unpolarized case [19]. In Section III the multipole expansion of the polarization correlation function is performed, and an expression for the total degree of linear polarization is derived. In Section IV we specialize to a specific cosmological model,
namely a spatially-flat, $\Omega_0 = 1$ cold dark matter cosmology with scale invariant scalar fluctuations normalized to the large angle anisotropy measured by COBE, under the assumption that the baryonic energy density is much smaller than the radiation energy density at the time of decoupling. In Section V we find the functional dependence of the ratio between the multipoles of the temperature fluctuation and of the polarization correlation functions upon $\Omega_0$, the mass-density in units of the critical, both in open universes as well as in spatially-flat models with a non-vanishing cosmological constant, with scale-invariant scalar fluctuations as the source of the anisotropy. Section VI rounds up the conclusions.

II. ANALYTIC APPROACH

A. Tight Coupling approximation

The Boltzmann equations for the photon distribution function in realistic cold dark matter cosmological models with energy-density linear fluctuations have been numerically solved, and the predictions for the CMB temperature correlation function on all angular scales have been analysed in great detail [9,20,11]. The Boltzmann code was also used to study the dependence of the microwave background anisotropies upon various cosmological parameters [21]. Notwithstanding the thoroughness of these numerical studies, it is very useful to have some analytic approximation to the exact solution, to gain even further insight into the nature and origin of the CMB anisotropies. Some analytic methods and approximations to compute with reasonable accuracy the CMB anisotropies on all angular scales were recently developed [19,22,23], improving upon the original work of Sachs and Wolfe [24].

Here we extend to the polarized case the analytic approach that Hu and Sugiyama introduced to compute CMB anisotropies [19]. The method is in turn an extension of the standard tight coupling analysis [25,26] to include realistic time dependent gravitational potentials, described in a gauge invariant formalism [27,28]. It is based upon an expansion of the CMB temperature fluctuation in inverse powers of the differential optical depth. In the tight coupling regime, when the effectiveness of Thomson scattering of CMB photons with free electrons makes the differential optical depth high, a perturbative expansion to first order constitutes a very good approximation to the exact result. While photons and baryons are tightly coupled, all higher multipoles of the temperature fluctuation can be evaluated in terms of the monopole, which in turn obeys the equation of a forced and damped oscillator [19]. We will see now how to include the polarization dependence of Thomson scattering into this formalism.

The CMB gauge-invariant temperature fluctuation for a given direction of observation, described in terms of polar angles $\theta, \phi$, is the relative temperature fluctuation around the mean evaluated in the shear-free (Newtonian) gauge: $\Delta_T(\theta, \phi) \equiv \Delta T(\theta, \phi)/T$. We compute first the temperature fluctuations induced by only one Fourier mode of the scalar fluctuations in the gravitational potential, with wave-vector $\vec{k}$. Then it is convenient to use a reference
frame such that \( \hat{z} \parallel \vec{k} \), since there is axial symmetry around the direction of \( \vec{k} \). The degree of linear polarization, \( \Delta P \), is defined in terms of the Stokes parameters \( Q \) and \( U \) [3] of the CMB radiation. We choose the two orthogonal directions into which the intensity is projected to define the Stokes parameters as \( \theta \) and \( \phi \) of spherical coordinates. The advantage is that, in this basis, \( U = 0 \), and the only non-vanishing Stokes parameter is \( Q = I_{\theta} - I_{\phi}/I_{\theta} + I_{\phi} = \Delta P \) [29].

Given the axial symmetry around the direction of \( \vec{k} \parallel \hat{z} \), the multipole expansion of either the temperature fluctuation or the polarization in a given direction of observation \( \hat{n} \) can be written as

\[
\Delta(\hat{n}, \vec{k}) = \sum_{l} (2l + 1) \Delta_l P_{l}(\mu)
\]

with \( P_{l} \) the Legendre polynomials and \( \mu \equiv \cos \theta = \vec{k} \cdot \hat{n} / k \) [30].

In a spatially-flat Robertson-Walker metric with linear density fluctuations described by gauge invariant potentials \( \Phi, \Psi \), and after angular integration of the collisional term of the Boltzmann equations [3], made easier by the axial symmetry, the evolution equations for the Fourier mode of wavevector \( \vec{k} \) of the gauge-invariant temperature fluctuation and polarization read [19,11,31]

\[
\dot{\Delta}_T + ik \mu (\Delta_T + \Psi) = -\dot{\Phi} - \dot{k} \{ \Delta_T - \Delta_{T0} - \mu V_b - \frac{1}{2} P_2(\mu) [\Delta_{T2} + \Delta_{P2} - \Delta_{T0}] \}
\]

\[
\dot{\Delta}_P + ik \mu \Delta_P = -\dot{k} \{ \Delta_P + \frac{1}{2} [1 - P_2(\mu)] [\Delta_{T2} + \Delta_{P2} - \Delta_{T0}] \}
\]

where a dot means derivative with respect to the conformal time \( \tau = \int dt a_o / a \), with \( a(t) \) the scale factor of the spatially-flat Robertson-Walker metric, and \( a_o = a(t_o) \) its value at present time. We shall write in this work the present value of the Hubble coefficient as \( H_o = h100 km/s/Mpc \). \( k = x_e n_e \sigma_T a / a_0 \) is the differential optical depth for Thomson scattering, with \( x_e \) the fraction of ionized electrons with number density \( n_e \), and \( \sigma_T \) the Thomson scattering cross section. \( V_b \) is the velocity of the baryons. \( \Phi, \Psi \) are the Fourier modes of the gravitational potentials [27,28].

The equation of motion for the baryons reads

\[
\dot{V}_b = -\frac{\dot{a}}{a} V_b - ik \Psi + \frac{\dot{k}}{R} (3\Delta_{T1} - V_b)
\]

where \( R \equiv 3 \rho_b / 4 \rho_\gamma \), the ratio between baryonic and radiation densities.

Equations (2.2) can be formally integrated

\[
(\Delta_T + \Psi) = \int_{\tau_0}^{\tau} d\tau e^{ik\mu(\tau-\tau_0)} e^{-k(\tau_0-\tau)} \{ \dot{k} (\Delta_{T0} + \Psi + \mu V_b + \frac{1}{2} P_2(\mu) [\Delta_{T2} + \Delta_{P2} - \Delta_{T0}] ) - \dot{\Phi} + \dot{\Psi} \}
\]

\[
(\Delta_P) = -\int_{\tau_0}^{\tau} d\tau e^{ik\mu(\tau-\tau_0)} \dot{k} e^{-k(\tau_0-\tau)} \frac{1}{2} [1 - P_2(\mu)] [\Delta_{T2} + \Delta_{P2} - \Delta_{T0}]
\]
where

$$\kappa(\tau_0, \tau) = \int_{\tau}^{\tau_0} x e_n z_T a(\tau) d\tau$$  \hspace{1cm} (2.5)$$

is the optical depth to photons emitted at conformal time \(\tau\). The combination \(\kappa e^{-\kappa}\) is called the conformal time visibility function. It is the probability that photons last scattered within \(d\tau\) of \(\tau\). For standard recombination this function has a sharp peak at the conformal time of decoupling \(\tau_D\) [32]. Thus, the integral for \(\Delta P\) in eq. (2.4) is dominated by the value of the integrand around decoupling. In other words, for standard recombination histories, with no reionization, the polarization of the CMB we observe today was produced just before decoupling.

In order to approximately solve the integral for \(\Delta P\) in eq. (2.4), we need to know the value of the combination \(S_P \equiv [\Delta P_0 - \Delta T_2 - \Delta P_2]\) around decoupling. We can evaluate the first multipoles of the CMB temperature fluctuations and polarization before decoupling in the tight coupling approximation [19], i.e. by a perturbative expansion in inverse powers of the differential optical depth \(\kappa\), which is high before decoupling. In order to perform this perturbative expansion we first rewrite the evolution equations (2.2) in the following form

$$\Delta T - \Delta T_0 - \mu V_b - \frac{1}{2} P_2(\mu)(\Delta T_2 + \Delta P_2 - \Delta P_0) = -\tau_C[\dot{\Delta} T + ik\mu(\Delta T + \Psi) + \Phi]$$

$$\Delta P + \frac{1}{2}(1 - P_2(\mu))|\Delta T_2 + \Delta P_2 - \Delta P_0| = -\tau_C[\dot{\Delta} P + ik\mu\Delta P]$$

$$3\Delta T_1 - V_b = \tau_C R[a^{-1}\frac{d}{d\tau}(aV_b) + ik\Psi]$$  \hspace{1cm} (2.6)$$

and expand all quantities of interest in powers of \(\tau_C \equiv \kappa^{-1}\), the conformal time between Compton (or Thomson) scatterings, assumed large for the tight coupling to be a sensible approximation.

In the strict tight coupling limit, that is to order zero in \(\tau_C = \kappa^{-1}\), the solutions to these equations read

$$\Delta T_1 = \frac{1}{3} V_b \; ; \; \Delta T_1 = 0 \; \text{if} \; l \geq 2$$

$$\Delta P = 0 \; .$$  \hspace{1cm} (2.7)$$

The interpretation of these formulae is very simple. In the lowest order approximation the photons and baryons are so strongly coupled that the photon distribution is isotropic in the baryon’s rest frame. The photon distribution being isotropic, Thomson scattering does not polarize the CMB.

Now we expand eqs. (2.6) to first order in \(\tau_C \equiv \kappa^{-1}\), and get
\[
\Delta P_2 = -\frac{1}{5}\Delta P_0 = \frac{1}{4}\Delta T_2 \quad ; \quad \Delta T_2 = -\frac{8}{15}ik\tau_C \Delta T_1
\]
\[
\Delta T_1 = \frac{i}{k}(\Delta T_0 + \dot{\Phi})
\]
\[
\Delta T_I = \Delta P_I = 0 \quad \text{if} \quad l \geq 3 \quad ; \quad \Delta P_I = 0
\]

These equations also have a simple interpretation. The polarization of the CMB is proportional to the quadrupole of the photon distribution function (a dipole does not induce polarization). The quadrupole in the temperature fluctuation, in its turn, is produced by the “free streaming” of the dipole between collisions. We see this from the relation \(\Delta T_2 \propto k\tau_C \Delta T_1\). The tight coupling approximation is actually valid when \(k\tau_C \ll 1\), i.e. for wavelengths much larger than the photon mean free path. The dipole in the temperature fluctuation can be derived from the gravitational potential and the monopole, through the relation \(ik\Delta T_1 = -(\Delta T_0 + \dot{\Phi})\). The monopole itself obeys, in the tight coupling limit, the equation of a forced and damped oscillator \([19]\)

\[
\ddot{\Delta} T_0 + \frac{\dot{R}}{1 + R} \dot{\Delta} T_0 + \frac{k^2}{3(1 + R)} \Delta T_0 = -\ddot{\Psi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2 \Psi}{3}
\]

The solution to this equation, for a given cosmological model, explains the properties of the CMB anisotropies \([19]\), and will also determine its polarization properties, as we show in the next subsection. We postpone to Section IV the solution of eq. (2.9) for a specific, realistic cosmological model, and the analysis of its most significant features, such as Doppler peaks, etc.

**B. Polarization**

Here we evaluate the integral for the polarization \(\Delta P\) in eq. (2.4), which can be rewritten as:

\[
\Delta P = \frac{3}{4}(1 - \mu^2) \int_0^{\tau_0} d\tau e^{ik(\tau - \tau_0)} \dot{\kappa}e^{-\kappa(\tau_0, \tau)} S_P
\]

where we have defined

\[
S_P \equiv \Delta P_0 - \Delta T_2 - \Delta P_2
\]

Because the visibility function \(\dot{\kappa}e^{-\kappa(\tau_0, \tau)}\) is strongly peaked around the time of decoupling, \(\tau_D\), to calculate the polarization today it is only necessary to know \(S_P\) near decoupling. Right before decoupling, and to first order in the tight coupling expansion, valid for scales such that \(k\tau_C \ll 1\), \(S_P\) can be approximated, using eqs. (2.8), as

\[
S_P \approx \frac{4}{3}ik\tau_C \Delta T_1
\]
But since \( \tau_C = \kappa^{-1} \) grows very fast during recombination, we need to know the time dependence of \( S_P \) around decoupling with better approximation than this. From (2.2) we find that \( S_P \) satisfies the following equation

\[
\dot{S}_P + \frac{3}{10} \kappa S_P = i k \left[ \frac{2}{3} \Delta T_1 + \frac{3}{5} (\Delta T_3 + \Delta P_3 - \Delta P_1) \right]
\]

(2.13)

During recombination we can approximate the right hand side of this equation by its tight coupling expansion to first order, and thus keep just \( \Delta T_1 \), as given by eq. (2.8). In this case the approximate solution for \( S_P \) is

\[
S_P(\tau) = \frac{2}{5} i k \int_0^\tau \Delta T_1 e^{-\frac{3}{10} \kappa(\tau, \tau')} d\tau' \]

(2.14)

We now replace this in the integrand of eq. (2.10) for \( \Delta P \). The integral is dominated by the contribution around decoupling, since the visibility function is strongly peaked around \( \tau_D \), the conformal time of decoupling. The conformal time visibility function is well approximated by a gaussian, of width \( \Delta \tau_D \) \[33,32\]. This means that photons were able to travel a distance of order \( \Delta \tau_D \) between their last two scatterings. This is the time the quadrupole had to grow, and thus the final polarization should be proportional to \( k \Delta \tau_D \Delta T_1 \). To see that this is indeed the case, we perform the integrals leading to \( S_P(\tau) \) around decoupling and to \( \Delta P \) under the following approximations, analogous to those in refs. \[15,13\]. We first approximate \( \kappa(\tau_0, \tau) \approx -\frac{e^{\kappa(\tau_0, \tau)}}{\Delta \tau_D} \), which is justified by the gaussian nature of the visibility function during recombination. Notice also that \( \kappa(\tau, \tau') = \kappa(\tau_0, \tau') - \kappa(\tau_0, \tau) \). We also neglect the time variation of \( \Delta T_1 \) during the decoupling transition. Then we can write, for \( \tau \) around decoupling:

\[
S_P(\tau) \approx \frac{2}{5} i k \Delta T_1(\tau_D) \Delta \tau_D e^{\frac{3}{10} \kappa(\tau_0, \tau)} \int_1^\infty \frac{dx}{x} e^{-\frac{3}{10} \kappa x},
\]

(2.15)

where we have changed the integration variable from \( \tau' \) to \( x = \kappa(\tau_0, \tau)/\kappa(\tau_0, \tau') \). Now, neglecting also the time variation of \( e^{ik\rho(\tau-\tau_0)} \) during recombination, we get

\[
\Delta P = \frac{3}{4} (1 - \mu^2) e^{ikp(\tau_D-\tau_0)} \frac{2}{5} i k \Delta T_1(\tau_D) \Delta \tau_D \int_0^\infty d\kappa e^{\frac{-3}{10} \kappa} \int_1^\infty \frac{dx}{x} e^{-\frac{3}{10} \kappa x}
\]

\[
= (1 - \mu^2) e^{ikp(\tau_D-\tau_0)} 0.51 i k \Delta T_1(\tau_D) \Delta \tau_D \equiv (1 - \mu^2) e^{ikp(\tau_D-\tau_0)} \beta(k)
\]

(2.16)

We here defined, for shortness and later reference, the quantity \( \beta(k) \). Expression (2.16) is one of our main analytic results. It gives, for standard recombination histories, the polarization induced upon the CMB by one Fourier mode of wavevector \( \vec{k} \) of the linear density fluctuations, in terms of the value of the dipole in the total temperature fluctuation at the time of decoupling. It is proportional to the width of the last scattering surface, \( \Delta \tau_D \), because it is actually the quadrupole in the temperature fluctuation during the last few scatterings what induces the polarization, and the quadrupole itself is proportional to the dipole times the width of the last scattering surface.
Expression (2.16) is strictly valid only for scales such that \( k \Delta \tau_D \ll 1 \), since we took the exponential out of the integral and simply evaluated it at \( \tau = \tau_D \). For scales such that \( k \Delta \tau_D \gg 1 \), the oscillations in the integrand produce a cancellation. In other words, the finite thickness of the last scattering surface damps the final polarization on these scales. The degree of polarization is thus largest for modes of wavelength comparable to \( \Delta \tau_D \).

In the long wavelength limit, the expression (2.16) reduces to our previous analytic estimate of the polarization on large angular scales [13], based on the method developed by Polnarev [15], once the dependence of \( \Delta T_1 \) with the gravitational potentials is replaced.

The polarization properties of the CMB are very sensitive to the details of the ionization history, and could very well serve to trace it back [5,34]. The proportionality in \( \Delta \tau_D \) in our solution is a hint of this. In a scenario with an appreciable reionization, the polarization would increase, because the quadrupole of the temperature anisotropy in the electron’s rest frame, which is the source of polarization, would be greatly enhanced.

A very important conclusion that can be drawn from eq. (2.16) is that one can determine the value of the dipole of the temperature fluctuations at recombination measuring the present polarization properties of the CMB radiation, at least if there was no reionization after recombination. This is a very interesting perspective, since it could also serve, for instance, to test alternative evolutions after recombination. The value of the dipole at recombination, \( \Delta T_{1}(\tau_D) \), can be derived from the monopole and the gravitational potentials using eq. (2.8), and the monopole itself solving eq. (2.9), in the tight coupling approximation. The oscillatory behaviour of \( \Delta T \) corresponds to the so-called Doppler peaks. In the case of adiabatic density fluctuations, the monopole turns out to be proportional to \( \cos \phi \) with \( \phi = k \int_0^\tau d\tau c_s \), where \( c_s = [3(1 + \mathcal{R})]^{-1/2} \) is the photon-baryon fluid sound speed, while the dipole is proportional to \( \sin \phi \) [19]. Thus, in the case of adiabatic perturbations the peaks in the polarization \( \Delta T \) are located at wavevectors such that \( \phi(\tau_D) = (m + \frac{1}{2})\pi \) with \( m \) an integer. For models with low \( \Omega_b \), where \( c_s \sim \text{const.} \sim \frac{1}{\sqrt{3}} \), the peaks are at \( \frac{k \tau_D}{\sqrt{3}} = (m + \frac{1}{2})\pi \). In the case of isocurvature perturbations the monopole is proportional to \( \sin \phi \), and the peaks in the polarization are instead at \( \phi(\tau_D) = m\pi \). A test that the relative locations of the peaks in \( \Delta T \) and \( \Delta P \) verify these relations may serve as a test if the recombination process was the standard one assumed here or not.

The relative heights of the different peaks is also a potential source of information about cosmological parameters. As can be seen from eq. (2.4), when \( \Delta T_0 \) and \( \Psi \) have opposite sign a suppression in the height of the corresponding peak in \( \Delta T \) may occur. In the case of adiabatic fluctuations, this can happen to even peaks. This pattern of suppression of even peaks relative to odd ones is sensitive to the value of \( \Omega_b h^2 \); the suppression grows with \( \Omega_b h^2 \). [19] This pattern of relative suppression in the heights of even peaks does not occur to the polarization. This is so because the gravitational infall represented by \( \Psi \), which affects the anisotropy, does not change the degree of polarization.

The CMB polarization is proportional to the dipole in the temperature fluctuation at recombination, which in turn, being proportional to a time derivative of the monopole, is proportional to \( c_s \), in the region where \( \Delta T_0 \propto \cos \phi \), with \( \phi = k \int_0^\tau d\tau c_s \). Since \( c_s \propto \frac{1}{\sqrt{3}} \),
$$(1 + R)^{-1/2}$$ and $R \propto h^2 \Omega_b$, then the height of the peaks in the polarization decreases for larger $\Omega_b$ (any other dependence on $\Omega_b$ in eq. (2.16) is not very significant).

More conclusions will be drawn in Sections IV and V, in the context of more specific cosmological models. Now we turn our attention to the behaviour of the temperature and polarization fluctuations at smaller scales, where the tight coupling approximation starts to break down.

**C. Diffusion damping**

The approximations that lead to eq. (2.9) break down for scales much smaller than $\tau_C$ ($k \tau_C \gg 1$), when the coupling between photons and electrons is not so tight. To find the qualitative, and approximately quantitative, behaviour at small scales it is necessary to expand the temperature, polarization and velocity fluctuations to second order in $\tau_C = \dot{k}^{-1}$. For these scales the role of gravity is unimportant, so we solve equation (2.2) neglecting the gravitational potentials [19,35]. Assuming solutions of the form

$$\Delta_T(\tau) = \Delta_T e^{i\omega \tau} , \quad \Delta_P(\tau) = \Delta_P e^{i\omega \tau} , \quad V_\delta(\tau) = V_\delta e^{i\omega \tau} ,$$

substituting this ansatz into the evolution equations (2.6), and expanding to second order in $\tau_C$, we obtain for $\omega = \omega_0 + i\gamma$

$$\omega_0 = \frac{k}{\sqrt{3(1+R)}} \equiv k c_s$$
$$\gamma = \frac{k^2 \tau_C}{6(1+R)} \left[ R^2 + \frac{2}{3} (1 + R)^2 \right]$$

where we have defined $c_s$, the photon-baryon sound speed.

The most important lesson to derive from this result is that at small scales, those such that $k \gg k_D$ defined below, the CMB temperature fluctuations and polarization are damped by an exponential factor $e^{-\bar{\gamma}}$, with

$$\bar{\gamma} = \frac{k^2}{k_D^2} \equiv k^2 \int_0^\tau d\tau' \frac{1}{\dot{k}} \frac{1}{6(1+R)^2} \left[ R^2 + \frac{2}{3} (1 + R)^2 \right] ,$$

where we have taken into account the evolution of $R$ and $\dot{k}$ with $\tau$. This is just Silk damping due to photon diffusion [36,35,19].

It is worth to stress at this point that the rigorous derivation of the damping factor $\bar{\gamma}$ requires to take into account the polarization dependence of Thomson scattering, as we did here. If the evolution equations for $\Delta_T$ are solved neglecting polarization (ignoring $\Delta_P$ in eqs. (2.2)), then a factor $4/5$ would appear instead of the factor $2/3$ in expression (2.19).
for $\gamma$ [35,19]. The polarization dependence is important because after each scattering the radiation is partially polarized. The polarization acts as a source of anisotropy and the anisotropy as a source of polarization. This coupling between anisotropy and polarization makes the fluctuations decay more slowly than if there were no polarization dependence. The difference is not insignificant. For a CDM dominated universe such that $R \ll 1$, before recombination ($x_e = 1$) we get

$$k_D^{-2} = 0.6 \times 10^7 (1 - Y_P/2)^{-1}(\Omega_b h)^{-1}(1 + z)^{-5/2}h^{-2}Mpc^2$$ (2.20)

which differs by about 20% from the result obtained neglecting that the radiation is polarized [19].

III. CORRELATION FUNCTION AND TOTAL POLARIZATION

In this section we perform the multipole expansion of the CMB polarization correlation function, and evaluate the total polarization produced by the scalar fluctuations. In the previous section we evaluated the polarization observed in a direction $\hat{n}$, $\Delta P(\hat{n}, \vec{k})$, induced by just one single Fourier mode of the gravitational fluctuations. We were able to choose Stokes parameters $Q \equiv \Delta P$ and $U = 0$ by the choice of the $\hat{z}$ axis along the direction of $\vec{k}$, and defining the Stokes parameters through projections of the CMB intensity along $\hat{\theta}$ and $\hat{\phi}$. But we can not do the same for all wavevectors. Now we must fix, for each direction of observation, a pair of orthogonal axis, the same for all wavevectors, to project the CMB intensity and evaluate the Stokes parameters. After rotation of the $\vec{k}$-dependent basis to the fixed one, for each direction of observation, there is a relatively simple relation between $Q$ and $U$ with $\Delta P$ as calculated in the previous section [11]. Let $\hat{n}$ be a direction of observation on the sky. The Stokes parameters induced by a given Fourier mode of wavevector $\vec{k}$ read, in terms of $\Delta P$ as calculated in the previous section, as

$$Q(\hat{n}) = \Delta P(\hat{n}, \vec{k}) \cos(2\phi_{\vec{k}}) \quad ; \quad U(\hat{n}) = \Delta P(\hat{n}, \vec{k}) \sin(2\phi_{\vec{k}})$$ , (3.1)

where $\phi_{\vec{k}}$ is, for a given direction of observation, the angle of rotation between the two basis.

Given two directions of observation $\hat{n}_1$ and $\hat{n}_2$ the polarization correlation function is defined as

$$C_P(\hat{n}_1, \hat{n}_2) = \langle Q(\hat{n}_1)Q(\hat{n}_2) + U(\hat{n}_1)U(\hat{n}_2) \rangle$$ (3.2)

where $\langle \ldots \rangle$ is an ensemble average. We assume that the density perturbations are gaussian. Thus

$$\langle \Delta_P^*(\vec{n}_1, \vec{k})\Delta_P(\vec{n}_2, \vec{k}') \rangle = |\beta(k)|^2(1 - \mu_1^2)(1 - \mu_2^2)\delta^3(\vec{k} - \vec{k}')$$ (3.3)
where \( \mu_1 \) and \( \mu_2 \) are the cosines of the angles formed by \( \vec{k} \) with \( \hat{n}_1 \) and \( \hat{n}_2 \) respectively, \( r \equiv \tau_0 - \tau_D \) is the distance to the last scattering surface, and we have written the result in terms of the quantity \( \beta(k) \) as defined in eq. (2.16).

The correlation function can be written as

\[
C_P(\hat{n}_1, \hat{n}_2) = \int d^3k |\beta(k)|^2 (1 - \mu_1^2)(1 - \mu_2^2)e^{ir\vec{k}\cdot(\hat{n}_1 - \hat{n}_2)} \cos[2(\phi_{1\vec{k}} - \phi_{2\vec{k}})]
\]  

(3.4)

The multipole coefficients of the polarization correlation function are given by

\[
\langle a_l^2 \rangle \equiv \sum_m \int d\Omega_1 d\Omega_2 Y_{lm}^*(\hat{n}_1)Y_{lm}(\hat{n}_2)C_P(\hat{n}_1, \hat{n}_2)
\]  

(3.5)

In the previous section we have seen that the most interesting structure in the polarization of the CMB occurs for wavelengths comparable or smaller than the horizon at decoupling. These wavelengths subdivide an angle in the sky of less than a few degrees. Small wavelengths have an effect on high multipoles of the correlation function expansion, \( l \gg 1 \). Thus, we will make simplifying approximations, valid for high multipoles only, the most interesting at any rate. Recall that \( \phi_{1\vec{k}} \) and \( \phi_{2\vec{k}} \) are the angles of rotation from the basis used to define the Stokes parameters when we took the \( \hat{z} \) axis in the direction of \( \vec{k} \), to the fixed basis we use now. They are different for each direction of observation. However, if \( \hat{n}_1 \) and \( \hat{n}_2 \) form a small angle \( \theta \), then \( \cos[2(\phi_{1\vec{k}} - \phi_{2\vec{k}})] \approx 1 - O(\theta^2) \). Thus, replacing \( \cos[2(\phi_{1\vec{k}} - \phi_{2\vec{k}})] \) by unity in equation (3.4) constitutes a very good approximation for high \( l \). Under this approximation the multipoles have a simple expression

\[
\langle a_l^2 \rangle \approx \frac{\pi}{4}(2l + 1) \int d^3k |\Delta_P(k)|^2 \left[c_{l+2}j_{l+2}(kr) + 2c_lj_l(kr) + c_{l-2}j_{l-2}(kr)\right]^2
\]  

(3.6)

where

\[
c_{l+2} = \frac{4(l + 1)(l + 2)}{(2l + 1)(2l + 3)} , \quad c_l = \frac{4(l^2 + l - 1)}{(2l - 1)(2l + 3)} , \quad c_{l-2} = \frac{4(l - 1)l}{(2l + 1)(2l - 1)} \quad .
\]  

(3.7)

Notice that for large \( l \), \( c_{l+2} \), \( c_{l-2} \) and \( c_l \) all tend to unity.

Another important quantity is the power spectrum of the total polarization, \( W(k) \) defined as

\[
C_P(\theta = 0) = \langle Q^2 + U^2 \rangle = \int_0^\infty \frac{dk}{k}W(k)
\]  

(3.8)

\( W(k) \) measures the contribution of each wavevector \( k \) to the total squared polarization. From equation (3.4) we see that it is given by

\[
W(k) = 2\pi k^3|\beta(k)|^2 \int_{-1}^1 d\mu (1 - \mu^2)^2
\]  

(3.9)

The total degree of linear polarization is \( P = (\int dk W(k)/k)^{1/2} \).
IV. POLARIZATION IN A COBE-NORMALIZED $\Omega_0 = 1$ CDM MODEL

In this section we specialize the evaluation of the polarization power spectrum and its correlation function multipoles to a specific cosmological model, namely a spatially-flat, $\Omega_0 = 1$ cold dark matter cosmology with scale invariant, adiabatic scalar fluctuations normalized to the large angle anisotropy measured by COBE, under the assumption that the baryonic energy density is much smaller than the radiation energy density ($R \ll 1$) at the time of decoupling. We will also compare our analytic results with previous numerical calculations.

The main ingredient in eq. (2.16) necessary to evaluate $\Delta_P(\hat{n}, \hat{k})$ is the value of the dipole in the temperature anisotropy around decoupling, which in turn can be derived from the monopole through the relation

$$ik\Delta T_1 = -(\dot{\Delta} T_0 + \dot{\Phi}) .$$

The monopole itself satisfies the equation

$$\dot{\Delta} T_0 + \frac{\dot{R}}{1+R} \dot{\Delta} T_0 + \frac{k^2}{3(1+R)} \Delta T_0 = -\ddot{\Phi} - \frac{\dot{R}}{1+R} \dot{\Phi} - \frac{k^2\Psi}{3} .$$

In models with low baryon content, such that $R \ll 1$ at the time of recombination, a WKB approximation to the exact solution of eq. (4.2) is a good approximation for most wavelengths of interest [19]. If we define

$$\Delta_0 \equiv \Delta T_0 + \Phi$$

we get for $\Delta_0$, in the WKB approximation,

$$\Delta_0 = a \cos(\omega_0 \tau) + b \sin(\omega_0 \tau) + 2\omega_0 \int_0^\tau d\tau' \sin[\omega_0(\tau - \tau')]\Phi(\tau') ,$$

with $\omega_0 = kc_s \approx k/\sqrt{3}$. We have neglected anisotropic stresses, so that $\Phi = -\Psi$. The constants $a$ and $b$ are fixed by the initial conditions. For adiabatic fluctuations $a = \frac{3}{2} \Phi(0)$ and $b = 0$. In order to calculate the integral above we need to know the time dependence of the gravitational potential. Rather than resorting to a full numerical approach, we will make some simplifying analytic approximations. During matter domination the potential $\Phi$ remains constant on all scales, and the integral in (4.4) is easily performed. For wavelengths which were outside the horizon at the time of matter-radiation equality the behaviour of the potential is also very simple: it remains constant during both radiation and matter domination, but the value during matter domination is $9/10$ times the value during during radiation domination. In this case:

$$\Delta_0(\tau) = -\frac{3}{10} \Phi(0) \cos(\omega_0 \tau) + \frac{18}{10} \Phi(0) \quad \text{if} \quad k \ll k_{eq} .$$
For wavelengths which entered the horizon before matter-radiation equality, the situation is different, since the potential $\Phi$ oscillates and decays. For times well before matter-radiation equilibrium there is an analytic solution for the potential: [28]

$$\Phi = 3\Phi(0) \frac{\sin(\omega_0 \tau) - \omega_0 \tau \cos(\omega_0 \tau)}{\omega_0^2 \tau^3}$$

(4.6)

If $k \gg k_{eq}$ the contribution of the integral is negligible, and then

$$\Delta_0(\tau) = -\frac{15}{10} \Phi(0) \cos(\omega_0 \tau) \quad \text{if} \quad k \gg k_{eq}$$

(4.7)

With the solutions above for $\Delta_0(\tau)$ in the two different regimes, we can evaluate the temperature dipole from $i k \Delta T_1 = \Delta_0$ and then use (2.16) to find the polarization produced in this specific model by one single Fourier mode:

$$\Delta_P(k) = \begin{cases} 
\frac{3}{10} \Phi(0)(0.5\omega_0 \Delta \tau_D) \sin(\omega_0 \tau_D) e^{ik\mu(\tau_0-\tau_D)} & \text{if} \quad k \ll k_{eq} \\
\frac{15}{10} \Phi(0)(0.5\omega_0 \Delta \tau_D) \sin(\omega_0 \tau_D) e^{ik\mu(\tau_0-\tau_D)} & \text{if} \quad k \gg k_{eq}
\end{cases}$$

(4.8)

For intermediate wavelengths the solution is not simple, so we find an approximate solution using eq. (4.6) during the radiation dominated epoch, and matching the potential to a constant during matter domination.

We also have to take into account the damping of the temperature fluctuations and polarization at small wavelengths due to photon diffusion (Silk damping) and due to the finite width of the last scattering surface. We have already calculated the effect of Silk damping at times around decoupling. The exponential damping factor, $\tilde{\gamma} = (k/k_D)^2$ is given by eq. (2.19). Since $k_D$ is time dependent, and varies very rapidly during recombination because of the fast change in the ionization fraction, its effect on the damping of the polarization should be averaged over the width of the last scattering surface. The average net damping factor $\langle \gamma \rangle$ should be approximately given by

$$e^{-\langle \gamma \rangle} = \int_0^{\tau_0} d\tau \frac{dk}{d\tau} e^{-\kappa e^{(-\tau_D/\tau_0)^2}}$$

(4.9)

The other source of damping comes from the finite width of the last scattering surface. The oscillations of the imaginary exponential in equation (2.10) produce a cancellation in the integral for wavelengths smaller than $\Delta \tau_D$. To take this effect into account, the phase of the perturbation should be averaged over the width of the last scattering surface, and so we should replace $e^{ik\mu \tau}$ by

$$\int_0^{\tau_0} d\tau \frac{dk}{d\tau} e^{-\kappa e^{ik\mu \tau}}$$

(4.10)

This factor depends on $\mu$, the cosine of the angle between the wavevector $\vec{k}$ and the direction of observation $\hat{n}$. Photons moving in a direction perpendicular to $\vec{k}$ do not suffer the damping due to the finite thickness of the last scattering surface. To simplify the calculations,
we average the damping factor over the angles, before performing the multipole expansion. Our results for small wavelengths will be qualitative, and only approximately correct quantitatively. We denote by \( f(k \Delta \tau_D) \) the averaged damping factor due to the finite width of the last scattering surface. It is given by

\[
f(k \Delta \tau_D) = \frac{1}{2} \int_{-1}^{1} d\mu \int_{0}^{\tau_0} d\tau \frac{dk}{d\tau} e^{-\kappa} e^{ik}\mu(\tau - \tau_D) \tag{4.11}
\]

To summarize, our final expression for the polarization produced by one single Fourier mode is given by the product of the factor \( e^{-\kappa} f(k \Delta \tau) \) times expression (4.8). The damping factors and the behaviour at intermediate wavelengths do not have simple analytic expressions, so we handle them numerically.

Using the result for \( \Delta \rho \), we evaluate the polarization power spectrum \( W(k) \) as given by eq. (3.9). Figure 1 shows the result, with the gravitational potential normalized to adjust the COBE-DMR measurement of the quadrupole in the temperature anisotropy, under the assumption of scale invariance for the scalar fluctuations. To normalize we have taken on large scales \( \frac{9}{10} \Phi(0) = \Phi(\tau_D) = A k^{-3} \) with \( (\frac{1}{3} A)^2 = 4! \langle a_{T_2}^2 \rangle / 5(4\pi)^2 \), and \( \sqrt{\langle a_{T_2}^2 \rangle} = 2 \times 10^{-5} \) the quadrupole in the temperature anisotropy measured by COBE. Figure 1 displays the main features of the CMB polarization. Very long wavelengths contribute little to the final polarization because of the relatively short width of the last scattering and the very tight coupling between photons and electrons prior to decoupling. For intermediate scales there are oscillations which follow the oscillations of the temperature anisotropy dipole, since those are the ones that by “free streaming” during the decoupling transition originate the quadrupole anisotropy that gives rise to polarization. Finally, the contribution to the polarization by smaller scales decays due to Silk damping and due to the cancellations produced by the finite width of the last scattering surface.

We have also calculated the polarization correlation function multipoles according to eq. (3.6). Since the largest contribution and the most interesting structure in the polarization power spectrum occurs at intermediate wavelengths, which correspond to \( l \gg 1 \), we can approximate \( c_{l+2}, c_l \) and \( c_{l-2} \) by unity, their large \( l \) limit. In that case the combination of Bessel functions appearing in eq. (3.6) can be approximated by \( (\frac{2l}{k})^2 j_l(kr) \). On the other hand due to the large value of \( r \), this Bessel functions wildly oscillate, and it is a good approximation to replace them by their approximate average

\[
\langle j_l^2(x) \rangle \approx \begin{cases} 
[2x(x^2 - l^2)^{1/2}]^{-1} & \text{if } x > l \\
0 & \text{if } x < l 
\end{cases} \tag{4.12}
\]

The result for the multipoles of the polarization correlation function is plotted in Figure 2. The main features of this plot are easily understood from the shape of the power spectrum in Figure 1.

The results in Figures 1 and 2 are similar to the numerical results obtained by Bond and Efstathiou in reference [11], more specifically to their figures (4b) and (7a). The agreement in the positions and heights of the first peaks is very good, after their normalization of the
quadrupole in the temperature anisotropy is changed to the one used here, the COBE-DMR measurement. The agreement is not so good quantitatively for small wavelengths, where damping starts to be very significant, given the rough approximations we made in this section to average the damping effects. The most interesting structure of peaks is, however, very well reproduced by our analytic results.

V. POLARIZATION OF THE CMB IN OPEN UNIVERSES

We have seen in Section II that the polarization of the CMB is produced during the process of decoupling of matter and radiation, and is proportional to the width of the last scattering surface $\Delta \tau_D$, and to the value of the dipole in the temperature anisotropy at the time of decoupling, which in turn is determined by the value of the gravitational potential $\Phi(\tau_D)$. The anisotropy in the CMB temperature also depends on these quantities, but differently. For instance, it is very insensitive on large angular scales to the values of $\Delta \tau_D$ and $\Phi(\tau_D)$. It is the aim of this section to show that because of this different dependence, the ratio between anisotropy and polarization at relatively low multipoles is very sensitive, in an open universe, to the value of the matter density $\Omega_0$. Besides, while the polarization of the CMB is produced during the decoupling transition, the present temperature anisotropy receives contributions not only from the Sachs-Wolfe effect, which is proportional to $\Phi(\tau_D)$, but also from the integrated Sachs Wolfe effect (ISW), due to the time dependence of the gravitational potentials. In an open universe, or in a flat universe with a cosmological constant $\Lambda$, the potentials depend on time, and so the ISW effect gives an additional contribution to the anisotropy. We will show that this also makes the ratio between anisotropy and polarization dependent, upon $\Omega_0$.

We shall evaluate the ratio between the multipoles of the temperature fluctuations and of the polarization correlation functions, and investigate its dependence upon $\Omega_0$ for $\Omega_0 \leq 1$ in two special cases: when there is no cosmological constant, and when $\Omega_0 + \Omega_\Lambda = 1$, with $\Omega_\Lambda = \Lambda/3H_0^2$. In both cases we shall assume a scale invariant spectrum of adiabatic density fluctuations.

It has been shown [37] that for multipoles such that $l_{\text{curv}} < l < l_D$, where $l_{\text{curv}} = \pi \sqrt{(1 - \Omega_0)/\Omega_0}$ and $l_D = r/\tau_D$, with $r$ the distance to the last scattering surface, the coefficients of the multipole expansion of the temperature fluctuation correlation function can be approximated by

$$C^T_l = A(1 + \frac{g(\Omega_0)}{l}) I^T_l$$

(5.1)

where $A$ is the normalization of the scale-invariant energy-density power spectrum of fluctuations, $\Phi(\tau_D) = Ak^{-3}$, and $I^T_l \equiv [9\pi(l + 1)]^{-1}$, both independent of $\Omega_0$. The dependence upon $\Omega_0$ through the function $g(\Omega_0)$ originates in the ISW, and is given by

$$g(\Omega_0) = 36\pi \int_{\tau_L}^{\tau_0} \left( \frac{dF}{d\tau} \right)^2 (\tau_0 - \tau) d\tau$$

(5.2)
where \( F(\tau) \) gives the time dependence of the gravitational potential, \( \Phi(\tau) = \Phi(\tau_0) \frac{F(\tau)}{F(\tau_0)} \). The evolution function \( F \) is given by \( F(\tau) = D(\tau)/a \) [19] with

\[
D = H \int \frac{da/a_0}{(Ha/a_0)^3} \tag{5.3}
\]

The Hubble constant \( H \) satisfies

\[
H^2 = \left( \frac{a_0}{a} \right)^4 \left( \frac{a + a_{eq}}{a_0 + a_{eq}} \Omega_0 H_0^2 - \left( \frac{a_0}{a} \right)^2 \Lambda + \frac{\Lambda}{3} \right) \tag{5.4}
\]

with \( K = -1 \) in an open universe and \( K = 0 \) in a spatially flat model.

Our result of previous sections for the coefficients of the multipole expansion of the polarization correlation function is

\[
C^P_l = \frac{\pi}{4} \int d^3k \left| \beta(k) \right|^2 \left[ c_{l+2j_{l+2}}(kr) + 2c_{lj}(kr) + c_{-2j_{l-2}}(kr) \right]^2 \tag{5.5}
\]

This result, which was derived in a spatially-flat universe, is also valid for multipoles \( l \geq l_{\text{curv}} \). Indeed, for the wavelengths that contribute to these multipoles, the radial functions appropriate for an open universe at the time of decoupling are well approximated by the spherical Bessel functions of the spatially flat case [37,19]. Besides, the evolution of the gravitational potentials until the time of decoupling is basically the same as in a flat universe provided that \( \Omega_0 z_D >> 1 \). We shall thus work under this assumption, and consider multipoles such that \( l \geq l_{\text{curv}} \) only. We shall also restrict our attention to multipoles \( l << l_D \), which correspond to those before the first peak in Figure 2, just to be able to work analytically. The wavevectors that significantly contribute to these multipoles are such that \( k \tau_D << 1 \), and for them we can approximate

\[
\left| \beta(k) \right|^2 \approx \left[ 0.17 \Phi(\tau_D) \omega_0 \Delta \tau_D \sin(\omega_0 \tau_D) \right]^2 \approx \left[ 0.17 \Phi(\tau_D) \omega_0^2 \Delta \tau_D \tau_D \right]^2 \tag{5.6}
\]

and the multipoles can be rewritten as

\[
C^P_l = A \left( \frac{\Delta \tau_D \tau_D}{(1 + R)r^2} \right)^2 I^P_l \tag{5.7}
\]

with

\[
I^P_l = \frac{16\pi^2}{3} \int \left[ c_{l+2j_{l+2}}(x) + 2c_{lj}(x) + c_{j_{l-2}j_{l-2}}(x) \right]^2 \tag{5.8}
\]

Thus, the ratio between the temperature fluctuation and the polarization correlation functions multipoles can be written as
\[
\frac{C_i^T}{C_i^P} = \left( \frac{r^2(1 + R)}{\Delta \tau_D \tau_D} \right)^2 (1 + \frac{g(\Omega)}{l}) B_i \tag{5.9}
\]

with \(B_i\) independent of \(\Omega_0\) and \(H_0\).

We want to find the explicit dependence of this ratio upon \(\Omega_0\). The distance to the last scattering surface, \(r\), is given by

\[
r = \frac{\int^{z_0}_{(1+z_D)} \frac{da}{da}}{(1+z_D)} \tag{5.10}
\]

When there is no cosmological constant this reduces to

\[
r = \frac{2\Omega_0 z_D + (2\Omega_0 - 4)\sqrt{\Omega_0 z_D + 1} - 1}{H_0 a_0 \Omega_0^2 (1 + z_D)} \tag{5.11}
\]

When \(\Omega_0 z_D >> 1\) it can be approximated by \(r \approx 2/\Omega_0 a_0 H_0\).

The dependence of \(\tau_D\) and \(\Delta \tau_D\) upon \(\Omega_0\) can be easily found exploiting the fact that \(z_D\) and \(\Delta z_D\) are approximately independent of \(\Omega_0\) and \(\Omega_i\) [33]. Taking this into account, for \(\Omega_0 z_D >> 1\) we get

\[
\Delta \tau_D = \frac{\Delta z_D}{a_0 \Omega_0 H_0 (1 + z_D)^{3/2}} \quad ; \quad \tau_D = \frac{2}{a_0 \Omega_0^{1/2} H_0 (1 + z_D)^{1/2}} \tag{5.12}
\]

Finally, the function \(g(\Omega)\), which can be shown to be function of the combination \((1 - \Omega_0) / \Omega_0\) only, is well approximated by

\[
g(\Omega_0) \approx 4.87 \frac{(1 - \Omega_0)}{\Omega_0} \tag{5.13}
\]

Using the above expressions we get for the ratio of multipoles (5.9)

\[
\frac{C_i^T}{C_i^P} = N_i G(\Omega_0, l) \equiv N_i [1 + \frac{4.87(1 - \Omega_0) / \Omega_0}{l}] \Omega_0^{-2} \tag{5.14}
\]

where \(N_i\) is a normalization factor, defined so that \(G(\Omega_0, l)\) measures the ratio between the temperature and polarization correlation function multipoles normalized to the value of the same ratio when \(\Omega_0 = 1\).

We plot \(G(\Omega_0, l)\) in Figure 3 for \(l = 30\). It is clear that the dependence upon \(\Omega_0\) is very significant. The ratio changes by a factor of order 40 for \(\Omega_0 \sim 0.2\) and 250 for \(\Omega_0 \sim 0.1\).
We now repeat the calculation for a model such that $\Omega_0 + \Omega_A = 1$. $g(\Omega_0)$ can be approximated in this case by $g(\Omega_0) = 0.33[(1 - \Omega_0)/\Omega_0]^{2.23}$. There is no simple analytic approximation for $r$ in this case, but it can be evaluated numerically and, when $\Omega_{0D} \gg 1$ the result is well fitted by $r \approx 2/H_0 a_0 \Omega_0^{0.39}$. Finally the ratio of anisotropy to polarization multipoles is given in this case by

$$\frac{C^T_l}{C^P_l} = N_l G(\Omega_0, l) \equiv N_l [1 + \frac{0.33[(1 - \Omega_0)/\Omega_0]^{2.23}}{l}]\Omega_0^{0.22} \tag{5.15}$$

which has a much weaker dependence upon $\Omega_0$ than in the open universe. The result is plotted in Figure 4 for a fixed value of $\Omega_b$ and $l = 30.$

VI. CONCLUSIONS

We have performed an approximate analytic evaluation of the polarization induced in the CMB on a wide range of angular scales by Thomson scattering prior to decoupling in the presence of density perturbations with adiabatic initial conditions, in a model with standard recombination. Eq. (2.16) gives the polarization induced by one single Fourier mode of the density perturbations, down to scales such that $k\Delta \tau_{D} \approx 1$. On smaller scales the finite width of the last scattering surface and photon diffusion damp the polarization of the CMB. The exponential damping factor due to photon diffusion (Silk damping), the same for the anisotropy and the polarization, is given by eq. (2.19). We stress the fact that the damping factor in the anisotropy that one would derive neglecting the polarization dependence of Thomson scattering would be slightly incorrect: instead of the factor 2/3 in eq. (2.19) one would get a factor 4/5 [35,19]. In conclusion, accurate calculations of the anisotropy on scales where the width of the last scattering surface is relevant should always include the polarization dependence of Thomson scattering.

Eq. (2.16) displays the fact that today’s polarization of the CMB was basically produced at the time when the tight coupling which kept the photon distribution isotropic in the electrons’ rest frame started to break down during recombination. The degree of polarization is proportional to the quadrupole in the temperature anisotropy around decoupling, which in turn is mainly due to the “free streaming” of the dipole during the last few scatterings. This is manifested in eq. (2.16) through the proportionality of the polarization upon the dipole in the temperature anisotropy at decoupling and upon the width of the last scattering surface. The faster decoupling occurs, smaller the degree of polarization induced in the CMB.

Figures 1 and 2 display our results for the CMB polarization power spectrum and correlation function multipoles respectively, for a $\Omega_0 = 1$ CDM model with adiabatic, scale-invariant scalar fluctuations normalized to the COBE-DMR measurement of the large angle CMB temperature anisotropy. We have assumed a standard recombination history and that the baryon density was much smaller than the radiation density around decoupling. The agreement in the height and position of the first peaks with respect to previous numerical computations [11] is very good.
If the ionization history was as assumed here, with a tight coupling between CMB photons and electrons until last scattering, and no later reionization, then the relative locations of the peaks in the temperature anisotropy and polarization correlation functions are in a simple and definite relation, as we discussed in section (II.B). Thus, measurement of the relative locations of the peaks in the anisotropy and polarization may give additional clues to the ionization history. Let us also stress again that the relative heights of the peaks in the polarization are less dependent on parameters such as $\Omega_b$ than those of the temperature fluctuations. Indeed, depending on the value of $\Omega_b$, it is possible that the heights of even-numbered peaks of the anisotropy are suppressed with respect to odd ones, due to a partial cancellation between the adiabatic oscillations and the Sachs-Wolfe effect. This does not happen to the degree of polarization, since it is not affected by the photons' redshift. The absolute height of the peaks in the polarization, on the other hand, is weakly dependent upon $\Omega_b$; they decrease with increasing baryonic density.

At last, but not at least, we have shown that the ratio between anisotropy and polarization multipoles is very sensitive to the value of $\Omega_0$ in an open Universe, as evidenced in Figure 3. This strong dependence upon $\Omega_0$ arises because anisotropy and polarization have a different scale-dependence. Roughly, the polarization induced by a Fourier mode scales as $k^2 \Delta \tau_D \Delta D$ with respect to the anisotropy induced by the same mode, on relatively large scales. In an open Universe with a scale invariant spectrum of density fluctuations, and just for geometric reasons, a change in the spatial curvature significantly changes the relation between the wavenumber $k$ of the density fluctuations and the value of the multipole $l$ to which that wavenumber contributes the most, significantly changing the value of the ratio between anisotropy and polarization multipoles. The ratio of anisotropy and polarization multipoles also depended upon $\Omega_0$ in an open Universe due to the time dependence of the gravitational potentials, which affects anisotropy through the integrated Sachs-Wolfe effect, while it does not affect the polarization.

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REFERENCES

[29] Notice that our definition of $\Delta_T$ and $\Delta_P$ differs by a factor 4 from that of ref. [11], while our definition for the polarization correlation function $C_P$ turns out to be identical.
[30] Notice that our multipole expansion differs from that in ref. [19] by a factor $(-i)^l/(2l+1)$.
FIGURE CAPTIONS

**Figure 1:** Polarization power spectrum, as defined by eq. (3.9), normalized to the COBE-DMR measurement of the quadrupole temperature anisotropy, for a cold dark matter model with $\Omega_0 = 1$, $h = 0.75$ and $\Omega_b = 0.03$.

**Figure 2:** Polarization correlation function multipoles normalized to COBE for a cold dark matter model with $\Omega_0 = 1$, $h = 0.75$ and $\Omega_b = 0.03$.

**Figure 3:** Ratio of the anisotropy to the polarization multipoles for $l = 30$ as a function of $\Omega_0$ in an open universe with no cosmological constant, normalized to its value when $\Omega_0 = 1$.

**Figure 4:** Ratio of the anisotropy to the polarization multipoles for $l = 30$ multipoles as a function of $\Omega_0$, normalized its value when $\Omega_0 = 1$, in a spatially-flat model with $\Omega_0 + \Omega_\Lambda = 1$. 