REPORTS OF THE WORKING GROUP
ON PRECISION CALCULATIONS
FOR THE Z RESONANCE

Editors: D. Bardin
W. Hollik
G. Passarino

GENEVA
1995
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          W. Hollik
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ABSTRACT

This Report summarizes the results of 12 months' activities of the Working Group on Precision Calculations for the Z Resonance, run at CERN in 1994.

The main goal of the Working Group was to present an update of studies on radiative corrections for Z-resonance processes, integrating all new results that had appeared since the previous Workshop on 'Z Physics at LEP 1', held in 1989.

The Report is, however, more than a mere collection of the proceedings of the three general meetings held on January 14, March 31 and June 13, 1994. Three subgroups have been working in the three related fields: electroweak physics, QCD at the Z resonance and Bhabha scattering in the luminosity region. An attempt has been made to present the final reports from these subgroups in a complete and homogeneous form. The subgroups' contributions in the three fields correspondingly comprise the three main parts of the Report.
Preface

After five years of $e^+e^-$ collisions around the $Z$ resonance impressive accuracy has been achieved in the data on $Z$ observables. The sensitivity in radiative corrections for electroweak observables, allowing precision tests of the standard model at the level of its quantum structure, requires the highest standards on the theoretical side as well. In 1989 the CERN Report 'Z Physics at LEP 1' provided as basic documentation the theoretical basis for the physics analysis of the LEP results. Although it remains quite comprehensive, an update of the discussion of radiative corrections has become necessary for at least two reasons:

- the experimental accuracy has reached a level much higher than originally expected
- a sizeable amount of theoretical work contributing to a steady improvement of the standard model predictions has appeared following the Yellow Report of 1989.

The idea of presenting an update on the calculation of the $Z$ resonance observables was triggered by experimentalists and is substantiated as far as possible in this Report. New theoretical input appearing after 1989 is included, both for the purely electroweak sector and for hadronic final states involving QCD corrections. In particular, a crucial amount of work has been performed in providing higher-order QCD corrections to the partial and total $Z$ widths and in pinning down the theoretical uncertainty of the standard model predictions: the $\alpha_s^3$ final-state corrections in the massless limit and, for the massive quark case, the $\alpha_s^3$ contributions to the vector and $\alpha_s^2$ contributions to the axial-vector parts of the $Z \to q\bar{q}$ decay width. New in the electroweak sector are the complete two-loop $O(\alpha_s)$ contributions to the vector boson self-energies, the two-loop electroweak leading contributions to the $\rho$-parameter and the $Zb\bar{b}$ vertex, the $O(\alpha_s G_F m_t^2)$ term in the $Zb\bar{b}$ vertex, and the three-loop $O(\alpha_s^2 G_F m_t^2)$ calculation for the $\rho$-parameter.

The structure of this Report centres on a description of the electroweak observable situation as obtained by various independent calculations, including the still-remaining theoretical uncertainties. This is followed by comprehensive descriptions of the QCD aspects of electroweak $Z$ physics and separate contributions on Bhabha scattering. In the electroweak part we have tried to give a broad and homogeneous summary covering the precision observables. The central electroweak documentation is preceded by an experimental overview of the current situation of electroweak precision measurements together with the basic requests experimentalists express to theoreticians. Comparisons among the various groups of authors for the central values and the range of uncertainty are documented and reflect the status of our theoretical knowledge. This may be understood as concluding a certain period of theoretical development and providing a consensus about the present situation in the calculation of the precision observables and estimates of their theoretical errors. Also in this section, an individual paper is included calculating the static $\rho$-parameter, $\rho(0)$, in neutrino scattering beyond the leading two-loop level, in order to demonstrate the potential importance of next-to-leading two-loop contributions.

In concluding, this part of the Report, we have collected and presented enough evidence to make the reasonable statement that the theoretical errors are sufficiently small for the current interpretation of LEP data, but we are not ready for a high luminosity LEP or for LEP with longitudinal beam polarization. After the electroweak part the QCD corrections are discussed in a section with several individual contributions.
A third major section is devoted to Bhabha scattering, where much work has been done since 1989 and crucial improvements have been achieved. Unlike the electroweak section, progress in Bhabha scattering is presented as a status report — it is less homogeneous and final, and appears as a collection of individual contributions, where numerical comparisons between the contributions have either not been performed or are not possible at present.

This Report is the result of a workshop started as a private initiative at the very end of 1993 and an appeal by experimentalists. Its progress over the subsequent twelve months to its final form was possible only with the support of CERN. We are obliged to CERN for acting as the host of the Workshop and for publication of the results — in particular to the Theory Division, and especially to Guido Altarelli, John Ellis and Torleif Ericson for their support and encouragement, to Janice Navarria and Stephen Kennedy for copy editing, and to Suzy Vascotto for organizational and technical assistance. One of us (DB) is very much indebted to the DESY-IfH directorate for creating the best conditions possible for completing the project.

We also gratefully acknowledge partial support from the European Union under CHRX-CT92-0004 and INTAS-93-744 and from INFN (Istituto Nazionale di Fisica Nucleare).

D. Bardin
W. Hollik
G. Passarino
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Experimental Aspects

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In the last few years, the $e^+e^-$ colliders, LEP and the SLC, have proven to be crucial tools for exploring the Standard Model, and in particular the electroweak theory, to a level which is sensitive to its quantum structure. Many very high precision measurements performed by the four LEP collaborations and the SLD experiment today allow the determination of important parameters of the theory with an unprecedented level of accuracy. For many of these measurements the present experimental accuracy is by far higher than foreseen a few years ago, and is expected to improve even more before the next generation of accelerators is operational.

From the experimental point of view, the success in achieving accuracies far beyond expectations can be assigned to both the accelerators and the experiments. The aspects of the LEP machine which have contributed to the high accuracy of the measurements are twofold:

- The impressive performance of the LEP machine lead to a steady increase of the luminosity but kept nevertheless a low background environment at the interaction zones. Several times intrinsic limitations of the machine were overcome by a thorough study of its behaviour and by the creativity of the machine physicists, who constantly tried to improve it by applying new ideas, such as the Pretzel scheme or bunch trains. This enabled the experiments to collect $1.5 \times 10^7$ visible $Z$ decays by the end of 1994, and leaves scope for a significant increase of the integrated luminosity before the start of LEP 200 [1].

- A big step forward in understanding the LEP beam energy was made possible by establishing a reproducible transverse polarization of the $e^+e^-$ beams. This allowed a new energy calibration technique to be applied which relates the beam energy to the frequency of a weak oscillating transverse magnetic field at which a resonant depolarization is observed. The intrinsic accuracy of this method amounts to a fraction of an MeV, much smaller than the time variation of the LEP energy caused by many now known effects. An important key to the exploitation of this intrinsic accuracy was the development of techniques to monitor orbit changes of the beam introduced by small shifts of the magnet positions. These techniques today provide us with a consistent interpretation of tiny deformations of the LEP ring which could be related to tidal forces of the sun and the moon, periods of heavy rainfall, and even the water level of Lake Léman. As a result the accuracy of $M_Z$ and $\Gamma_Z$ is now at the level of a few MeV.
At the SLC, substantial improvements have been achieved in increasing the electron longitudinal polarization and calibrating it. This has allowed a complementary determination of the effective electroweak mixing angle with an accuracy which matches the most precise measurements at LEP. Their measurement has remarkable prospects as it is based on 50,000 Z decays only, and the error is dominated by the statistics.

When discussing the detector aspects of the experiments it should be noted that their design and good performance together with the clean background conditions have allowed the understanding of the data to the level of the statistical precision or better almost from the very beginning. Typical systematic uncertainties in event selections are at the few per mil level. Nevertheless, in recent years, most of the experiments have upgraded their detectors to achieve even higher performances for some applications. Two detector improvements of this kind deserve to be mentioned:

- The luminosity monitors have been upgraded by installing SiW calorimeters or by improving the tracking capabilities. This has enabled the experiments to master the detector systematics in the luminosity determination below the per mil level, a limit inconceivable just a few years ago.

- The installation and steady improvement of microvertex detectors has drastically decreased the uncertainties in electroweak measurements with heavy flavours. These measurements now play a very important role when analysing the implications of electroweak precision measurements.

Table 1 gives an idea of the present experimental accuracy of the electroweak measurements which are most relevant for constraining the Standard Model parameters. These results are based on a preliminary analysis of data collected up to the end of 1993. As a rule of thumb, most of the results are expected to improve by a factor of two in the near future because in most cases the experimental accuracy is still limited by the statistics. Measurements like $M_Z$ and especially $\Gamma_Z$ will still benefit from more statistics in the wings of the Z peak, which will probably be collected in 1995. The ratio, $R_h$, of the hadronic and leptonic partial decay widths of the Z and all asymmetries will benefit from high event statistics at the Z peak. Observables related to the partial decay widths of the Z into $b\bar{b}$ or $c\bar{c}$ have systematic errors roughly as large as those in the statistics, mainly due to limitations in the modelling of fragmentation and decay. Substantial improvements are still expected for the error of $M_W$, both at the Tevatron and especially at LEP 200 where a relative accuracy in the few times $10^{-4}$ range seems feasible [2]. When predicting experimental systematics one should take into account that high statistics often allow cross-checks with the data, obviating or de-emphasizing model uncertainties.

In order to demonstrate the internal consistency of the results it is useful to project them onto a common scale. Within the Standard Model framework this can be a parameter of the Standard Model Lagrangian or any observation predicted from it. All asymmetry measurements, in particular, can be projected on to the effective electroweak mixing angle, $\sin^2\theta_{\text{eff}}$, without making any strong model-specific assumptions. Table 2 displays the values of the effective electroweak mixing angle derived from the asymmetry measurements in Table 1.

The mandate of an experimental data analysis consists in the presentation of results in a way which allows a confrontation with the theory without any specific detector
Table 1: Status of published and preliminary electroweak precision tests at the time of the 27th International Conference on High Energy Physics, Glasgow, Scotland, 20–27 July 1994 from Ref. [3], which should be consulted for the definition of the effective parameters listed. Section a) summarizes LEP averages, section b) electroweak precision tests from \( p\bar{p} \) colliders and \( \nu N \)-scattering, and section c) gives the result for \( \sin^2 \theta_{\text{eff}}^{\text{lept}} \) from the measurement of the left–right polarization asymmetry.

<table>
<thead>
<tr>
<th>a) LEP[3]</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Line-shape and lepton asymmetries: (weakly correlated)</td>
<td></td>
</tr>
<tr>
<td>( M_Z ) [GeV]</td>
<td>91.1888 ± 0.0044</td>
</tr>
<tr>
<td>( \Gamma_Z ) [GeV]</td>
<td>2.4974 ± 0.0038</td>
</tr>
<tr>
<td>( \sigma^0_h ) [nb]</td>
<td>41.49 ± 0.12</td>
</tr>
<tr>
<td>( R_t )</td>
<td>20.795 ± 0.040</td>
</tr>
<tr>
<td>( A_{FB}^{0,\ell} )</td>
<td>0.0170 ± 0.0016</td>
</tr>
<tr>
<td>( \tau ) polarization: (nearly uncorrelated)</td>
<td></td>
</tr>
<tr>
<td>( A_{\tau} )</td>
<td>0.143 ± 0.010</td>
</tr>
<tr>
<td>( A_{e} )</td>
<td>0.135 ± 0.011</td>
</tr>
<tr>
<td>b and c quark results: (strongly correlated)</td>
<td></td>
</tr>
<tr>
<td>( R_b )</td>
<td>0.2202 ± 0.0020</td>
</tr>
<tr>
<td>( R_c )</td>
<td>0.1583 ± 0.0098</td>
</tr>
<tr>
<td>( A_{FB}^{0,b} )</td>
<td>0.0967 ± 0.0038</td>
</tr>
<tr>
<td>( A_{FB}^{0,c} )</td>
<td>0.0760 ± 0.0091</td>
</tr>
<tr>
<td>( q\bar{q} ) charge asymmetry: ( \sin^2 \theta_{\text{eff}}^{\text{lept}} (\langle Q_{FB}\rangle) )</td>
<td>0.2320 ± 0.0016</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b) ( p\bar{p} ) and ( \nu N )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_W ) [GeV] (( p\bar{p}[4] ))</td>
<td>80.23 ± 0.18</td>
</tr>
<tr>
<td>( 1 - M_W^2 / M_Z^2 ) (( \nu N[5, 6, 7] ))</td>
<td>0.2253 ± 0.0047</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c) SLC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^2 \theta_{\text{eff}}^{\text{lept}} (A_{LR}[8]) )</td>
<td>0.2294 ± 0.0010</td>
</tr>
</tbody>
</table>
Table 2: Comparison of several determinations of $\sin^2 \theta_{\text{eff}}^{\text{lep}}$ from the asymmetries listed in Table 1. Averages are obtained as weighted averages assuming no correlations.

<table>
<thead>
<tr>
<th>Measurement</th>
<th>$\sin^2 \theta_{\text{eff}}^{\text{lep}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LEP:</strong></td>
<td></td>
</tr>
<tr>
<td>$A_{\text{FB}}^{0, \tau}$</td>
<td>0.2311 ± 0.0009</td>
</tr>
<tr>
<td>$A_{\tau}$</td>
<td>0.2320 ± 0.0013</td>
</tr>
<tr>
<td>$A_{e}$</td>
<td>0.2330 ± 0.0014</td>
</tr>
<tr>
<td>$A_{\text{FB}}^{0, b}$</td>
<td>0.2327 ± 0.0007</td>
</tr>
<tr>
<td>$A_{\text{FB}}^{0, c}$</td>
<td>0.2310 ± 0.0021</td>
</tr>
<tr>
<td>$\langle Q_{\text{FB}} \rangle$</td>
<td>0.2320 ± 0.0016</td>
</tr>
<tr>
<td><strong>Average LEP:</strong></td>
<td>0.2321 ± 0.0004</td>
</tr>
<tr>
<td><strong>SLC:</strong></td>
<td></td>
</tr>
<tr>
<td>$A_{\text{LR}}$ (SLD)</td>
<td>0.2294 ± 0.0010</td>
</tr>
<tr>
<td><strong>Average LEP+SLC:</strong></td>
<td>0.2317 ± 0.0004</td>
</tr>
</tbody>
</table>

Knowledge. The aim of the LEP and SLC experiments is to probe the quantum structure of the Standard Model. This requires a very careful presentation of the results, necessitating a close collaboration between experimentalists and theorists. An important aspect is the full understanding of the interplay of effects of radiative corrections and experimental cuts. It can be shown that most of the correlations between radiative corrections and experiment specific treatment of the data arise from photonic corrections, i.e. those which can be attributed to diagrams which have additional real or virtual photons added to the Born diagrams. For some electroweak observables gluon radiation may also require a treatment which depends on experimental cuts. The remaining radiative corrections can be absorbed to a good approximation into effective Born-like parameters.

This motivates a data presentation which is commonly handled in two different steps:

- First, the data are described by Born-like expressions, convoluted for the effects of photon radiation. Thus the measurements are collapsed into a small set of effective parameters, also referred to as pseudo-observables in this report, which should describe the measurements with very high precision. The results given in Tables 1 and 2 can be regarded as pseudo-observables.

- Second, these effective parameters are interpreted in the framework of the the Standard Model to check its validity for describing all the data, and to determine the basic parameters of its Lagrangian.

Using such a scheme has several advantages, amongst which the most relevant one is that it makes the comparison and combination of many different high precision measurements much simpler. Its main drawback might be linked with the limitation of theoretical
accuracy introduced by the use of these effective parameters as an intermediate step in
the predictions, but this is something to be quantified.

Although measurements of such an accuracy were not foreseen a few years ago, the
most relevant conceptual pieces of the theoretical language needed to describe them al-
ready existed and were provided in detail for instance in the CERN Report ‘Z Physics at
LEP 1’ [9] just before LEP 1 started operating. The classification and thorough analysis
of radiative corrections discussed in that report and in other contemporaneous publica-
tions, has been the conceptual basis of the effective parameter language. These concepts,
enriched with the exchange of experimental experience during the first years of LEP op-
eration, for instance in the framework of the LEP Electroweak Working Group, constitute
nowadays the standard way in which precision measurements are expressed. This clas-
sification also made it possible to understand which aspects of the calculations needed
more work to be able to cope with the experimental accuracy. For instance, the relevant
higher-order corrections for a heavy top were identified as well as the corrections in which
QCD aspects played an important role. Their calculation has been pursued since many
years and is still a very interesting activity in the field of radiative corrections.

Nevertheless, in the same way that progress in the understanding of the data is quan-
tified by quoting systematic uncertainties of experimental origin, it is necessary to give a
realistic estimation of our current understanding of the theoretical prediction by quoting
systematic theoretical uncertainties. So far, a detailed analysis on the actual limitations
of the theory when confronting the experimental measurements was lacking, and is the
final goal of this report. Together with the experimental analysis strategy the following
questions arise:

- With which accuracy can the effective electroweak parameters be predicted within
  the Standard Model with the present knowledge of radiative corrections?

- With which accuracy can these be determined from the data, i.e. what is the theo-
  retical uncertainty introduced when expressing the whole set of direct measurements
  in terms of a number of effective parameters?

If necessary, an effort could be made to confront the raw data directly with the
Standard Model prediction without using effective parameters as intermediate results.
In this context the question arises:

- What is the accuracy with which we are presently able to predict our direct obser-
  vations in terms of the Standard Model Lagrangian parameters?

It should be noted that the interpretation of several aspects of the electroweak precision
measurements requires a precise knowledge of $\alpha(M_Z^2)$. The present error of $\alpha(M_Z^2)$ arises
from the uncertainty of the hadronic contribution to the photon vacuum polarization and
translates into an error of $\Delta \sin^2 \theta_{\text{eff}}^{\text{lep}} = 0.0003$, which is not negligible with the present
experimental accuracy. This error should not be accepted as an intrinsic limitation of
electroweak precision tests as it could and should be reduced by precise measurements of
the hadronic cross section in the centre-of-mass energy range of 1–10 GeV.
References


Electroweak Working Group Report

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*On leave from INFN, Sezione di Pavia, Italy.*
1 Theoretical basics

1.1 Introduction

The radiative corrections (RC) for observables related to the $Z$ resonance have been described in great detail in the 1989 CERN Yellow Report 'Z-Physics at LEP 1' [1]. About $1.5 \times 10^7$ $Z$ decays have been recorded and analysed during the years of operation of the four LEP experiments — from autumn of 1989 to the end of 1994. In order to match the actual experimental precision, a completely revised analysis of radiative corrections at the $Z$ resonance is needed. The aim of this contribution to the present Report is threefold. We will:

(i) introduce a common language for presentation of the results emerging from different approaches;
(ii) update the predictions of $Z$ resonance observables within the Minimal Standard Model (MSM);
(iii) estimate the intrinsic theoretical uncertainties of these predictions, which are mainly caused by the neglect of higher order contributions.

The results, which are presented in this Report, are based on several different approaches and on a comparison of their numerical predictions. The findings of the Report are based on the following computer codes:

- BHM [2], LEPTOP [3], TOPAZO [4], WOH [5], and ZFITTER [6].

The design of some of these codes can be traced back to the 1989 Workshop on 'Z-Physics at LEP 1'; others have been developed later. All of them contain a so-called electroweak library, which allows the calculation of virtual higher-order electroweak and QCD corrections to selected quantities — for example, a weak mixing angle or a partial or total $Z$ width. Strictly speaking, these are 'pseudo-observables' and not directly measurable in an experiment. Some packages (codes) contain, in addition, virtual and real photonic corrections (bremsstrahlung) with simple kinematical and geometrical cuts similar to those used in experiments, thus allowing for the calculation of (idealized) measurable cross-sections and asymmetries, which we will call 'realistic observables'.

Apart from a comparison of the numerical results of different codes, we attempted a simulation of the theoretical uncertainties of each code. To do so we had to reach an agreement on the principles on which these simulations are based. We hope that as a result of this project the prospects of the quantitative tests of the MSM at LEP and SLC, and their sensitivity to potential New Physics beyond the MSM, will become clearer.

The Report is organized as follows. In the first section we define the exact framework for the analysis of the electroweak data in the MSM, beginning with 'Input parameters' and ending with 'Basic notions', which are needed for discussing the numerical results collected in the subsequent sections. These deal with pseudo-observables and realistic observables in the sense introduced above. The first three sections represent a homogenous presentation of the material related to or obtained with the above mentioned codes. In the fourth and fifth sections we have collected items particular to the different codes, such as
explicit analytical formulae and descriptions of the design and of the use of the packages. All the authors of the Report agree on the following conclusions from this study:

- The differences between results of different codes are small compared to existing experimental uncertainties. Thus improvement of experimental accuracy at LEP 1 and SLC is welcome even at the present level of theoretical accuracy.

- At present the most promising are measurements of $g_\nu/g_A$ in various $P$- and $C$-violating asymmetries and polarizations.

- The real bottleneck for improved theoretical accuracy in $g_\nu/g_A$ is presented by the uncertainty of the input parameter $\hat{\alpha} \equiv \alpha(M_Z)$. The improved accuracy of this important parameter calls for new accurate measurements of the cross-section $e^+e^- \rightarrow \text{hadrons}$ at low energies (Novosibirsk and Beijing accelerators, etc.)

- The estimates of theoretical uncertainties are highly subjective and their values partly reflect the internal philosophy of the actual implementation of radiative corrections in a given code.

- In many cases the one-loop approximation in the electroweak gauge coupling is adequate enough at the present level of experimental accuracy. At the same time, however, it should be stressed that a complete evaluation of the sub-leading corrections, $\mathcal{O}(G^2_\mu, M_Z^2 m_t^2)$, would greatly reduce the uncertainty that we observe, one way or another, for all observables.

- In case the next generation of experiments at LEP 1 and SLC improves accuracy considerably (a problem not only of statistics but mainly of systematics) the full program of two-loop electroweak calculations should be carried out.

1.2 Input parameters

Within the MSM, any measured quantity can be calculated in terms of a small set of input parameters. Once all possible relations between the parameters of the Lagrangian in the MSM are exploited, we may choose a set of them which, in the electroweak part, consists of one interaction constant — say, the fine structure constant $\alpha$ — of the masses of all particles and of the CKM fermionic mixing angles. The latter ones are of little importance for $Z$ resonance observables, which are dominated by flavour-diagonal neutral current interactions.

In practical calculations one prefers to make use of the most precisely measured parameters. Three of them characterize the gauge sector of the MSM:

\begin{align}
\alpha &\equiv \alpha(0) = 1/137.0359895(61), \\
G_\mu & = 1.16639(2) \times 10^{-5} \text{GeV}^{-2}, \\
M_Z & = 91.1887 \pm 0.0044 \text{GeV},
\end{align}

where $G_\mu$, the four-fermion $\mu$ decay coupling constant, is defined through the muon lifetime $\tau_\mu$ by
\[
\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \left( 1 - \frac{8m_e^2}{m_\mu^2} \right) \left[ 1 + \frac{\alpha}{2\pi} \left( 1 + \frac{2\alpha}{3\pi} \ln \frac{m_\mu}{m_e} \right) \left( \frac{25}{4} - \pi^2 \right) \right].
\]

Light fermion masses contribute via the electromagnetic coupling constant \( \alpha \), running from very small momenta up to \( M_Z \), the typical scale of the \( Z \) resonance, and yielding

\[
\alpha(M_Z) \equiv \tilde{\alpha} = \frac{\alpha}{1 - \Delta \alpha},
\]

with \( \Delta \alpha \) consisting of leptonic and hadronic contributions

\[
\Delta \alpha = \Delta \alpha_l + \Delta \alpha_h.
\]

The leptonic contribution is known explicitly:

\[
\Delta \alpha_l = \frac{\alpha}{3\pi} \sum_l \left[ -\frac{5}{3} - 4 \frac{m_l^2}{M_Z^2} + \beta_l \left( 1 + 2 \frac{m_l^2}{M_Z^2} \right) \ln \frac{\beta_l + 1}{\beta_l - 1} \right],
\]

where

\[
\beta_l = \sqrt{1 - 4 \frac{m_l^2}{M_Z^2}}.
\]

With lepton masses taken from the last edition of the Review of Particle properties [7], from (5) we get

\[
\Delta \alpha_l = 0.0314129.
\]

In the small mass approximation, (5) reduces to the well-known expression

\[
\Delta \alpha_l = \frac{\alpha}{3\pi} \sum_l \left( \ln \frac{M_Z^2}{m_l^2} - \frac{5}{3} \right) = 0.0314177.
\]

The hadronic contribution is being calculated by a dispersion relation,

\[
\Delta \alpha_h = \frac{\alpha M_Z^2}{3\pi} \Re \int_{4m_e^2}^{\infty} ds \frac{R(s)}{s(M_Z^2 - i\epsilon - s)},
\]

from experimental data on the cross-section ratio \( R(s) = \sigma_{e^+e^-}(s)/\sigma_0(s) \), where \( \sigma_{e^+e^-}(s) = \sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons}) \) and \( \sigma_0 = 4/3\pi \alpha^2/s \). In 1990, the value \( \Delta \alpha_h \) has been updated to

\[
\Delta \alpha_h = 0.0282(9),
\]

leading to \( \alpha^{-1}(M_Z) = 128.87 \pm 0.12 \) [8].

By using \( \tilde{\alpha} \) instead of \( \alpha \), the electromagnetic coupling constant at the correct scale for \( Z \) physics is chosen and one automatically avoids the problem of the light quark mass

\[1\text{Very recently, three new analyses were published. The first, [9], leads to } \Delta \alpha_h = 0.02666 \pm 0.00075 \text{ and } \alpha^{-1}(M_Z) = 129.08 \pm 0.16. \text{ The second, [10], leads to } \Delta \alpha_h = 0.02732 \pm 0.00042 \text{ and } \alpha^{-1}(M_Z) = 128.99 \pm 0.06. \text{ The third, [11], leads to } \Delta \alpha_h = 0.0280 \pm 0.0007 \text{ and } \alpha^{-1}(M_Z) = 128.899 \pm 0.090.\]
singularities. So the relevant input parameter is $\tilde{\alpha}$, and not $\alpha$ — in spite of the extremely high accuracy of the latter. It should be stressed that the quantity $\tilde{\alpha}$ used by us is not only numerically, but also in principle, different from the quantity $\hat{\alpha}(M_Z) = 1/127.9(1)$ (see, for example Ref. [7], page 1304). The latter is defined in the modified minimal subtraction scheme ($\overline{MS}$) in terms of a bare electric charge, while the former is a physical quantity expressed in terms of charged fermion masses.

The treatment of the light fermion masses (all but the top quark mass) in the electroweak corrections is not the same in different codes. First we should mention that the exact expression (5) for the photonic vacuum polarization is not the only place where fermionic power corrections, $m_f^2/M_Z^2$, may appear $^2$. There are fermionic contributions to the self-energy insertions of the heavy gauge bosons, which also contain power mass corrections. Moreover, additional power corrections appear, due to the presence of axial couplings. The latter may be explicitly calculated for leptons; for quarks, they are not taken into account by the dispersion integral (9). In principle, there are also finite mass corrections in three- and four-point functions. In some codes these finite mass terms are neglected completely, some other codes retain them in the two- and three-point functions. In the latter case, for the quarks some effective masses are used, which are selected in order to reproduce with a sufficiently high accuracy the hadronic vacuum polarization contribution $\Delta\alpha_b$ $^3$. We emphasize that these finite mass terms have no numerical relevance. They are being retained just in order to investigate the sensitivity of corrections to finite light fermion masses. They also help sometimes to compare results of independent calculations. Such a comparison has been done once for $\Delta r$, and an agreement of up to 12 digits (computer precision) was found [14].

The $\tau$ lepton mass has been retained in all codes in the phase-space corrections and in the Born-level matrix element. The masses of the heavy quarks $b$ and $c$ also give rise to phase-space corrections, but are treated together with QCD final state interaction corrections (see the QCD part of this Report). We take the following input values for $m_\tau$, $m_b$ and $m_c$ [7]:

$$m_\tau = 1.7771 \pm 0.0005 \text{ GeV},$$
$$m_b = 4.7 \pm 0.3 \text{ GeV},$$
$$m_c = 1.55 \pm 0.35 \text{ GeV},$$

(11)

where $m_b, m_c$ are the pole masses converted in the actual calculation of radiative corrections to $\overline{MS}$ masses using QCD perturbation theory. The meaning of these $\overline{MS}$ mass definitions and their relation to the pole masses are explained in the QCD part of this Report by Chetyrkin, Kühn and Kwiatkowski. The QCD part of the MSM is characterized by these running quark masses, and by the strong coupling constant $\alpha_s(M_Z)$, which will be assumed to be Refs. [15] and [16]:

$$\alpha_s(M_Z) = \hat{\alpha}_s = 0.125 \pm 0.007.$$  

(12)

$^2$One should emphasize that the dispersion integral (9) automatically includes all these power corrections for the hadronic part of the photonic vacuum polarization.

$^3$Details on effective quark masses are to be found in Refs. [12] or [13].
Note, that from pure QCD observables at the $Z$ peak one obtains [17]

$$\alpha_s(M_Z) \equiv \hat{\alpha}_s = 0.123 \pm 0.006 .$$  \hspace{1cm} (13)

Finally, there are two as yet unknown input parameters left: the top quark mass and the Higgs boson mass,

$$m_t, \quad M_H,$$  \hspace{1cm} (14)

although the situation has considerably improved with the recent CDF indication [18] of evidence for the $t$ quark with $m_t = 174 \pm 17$ GeV. In the following, by $m_t$ we will always imply the $t$-quark pole mass.

### 1.3 Codes to calculate electroweak observables

The radiative corrections for the measured physical observables must be included in the theoretical predictions for them, and the consistency of the theory is verified by comparison with the data. This particular step may assume different aspects related to the actual implementation of the comparisons, but in the end it usually takes the form of some constraint on $m_t$ and $\hat{\alpha}_s$, and to a lesser extent on $M_H$ and $\hat{\alpha}$. It is our main goal to present and discuss the most accurate theoretical predictions for the measured quantities and to introduce a reliable estimate of the associated theoretical uncertainties.

The results of this Report have been obtained by a critical comparison and examination of:

- the variations in the results following from the different, although not antithetic, formulations of the various groups;
- the *internal* estimates of the uncertainties induced by the still missing higher order corrections.

The groups participating in this project can be identified by the names of the codes that they have assembled, namely:

**BHM [2]**
Burgers, Hollik, Martinez, Teubert;

**LEPTOP [3]** – ITEP Moscow group
Novikov, Okun, Rozanov, Vysotsky;

**TOPAZ [4]** – Torino-Pavia group
Montagna, Nicrosini, Passarino, Piccinini, Pittau;

**WOH [5]**
Beenakker, Burgers, Hollik;

**ZFITTER [6]** – Dubna-Zeuthen group
All these codes may be used to calculate pseudo-observables. Three of them, BHM, TOPAZ0, and ZFITTER, may also calculate realistic observables. One code, TOPAZ0, is additionally able to calculate the full Bhabha cross-section, including complete s and t channel exchange contributions. Concerning the electroweak corrections, four codes are based on completely independent theoretical computational schemes, while BHM/MS use basically the same framework. The treatment of QED corrections in the three QED dressers is based on completely independent theoretical methods. As a consequence, it is extremely difficult to describe them in a common language. On the other hand, the treatment of QCD corrections is to a large extent common in all five codes. It is based on papers included in Part II of this volume and on references quoted therein.

In this Report, we will attempt to present the collected material homogeneously for as long as possible. Because of this, all complicated description of what is entering in the various, alternative theoretical formulations has been shifted to the last two sections of the Report.

1.4 The language of effective couplings

A natural and familiar but approximate language for the basic ingredients of the physical observables of the Z resonance is that of effective couplings. How it will translate into the particular realizations and schemes of the various codes can be studied in Refs. [2-6] and in the references quoted in these. The more theoretically oriented reader should, however, be aware that basic differences do indeed exist and that the language of effective couplings is not universally realized in all the approaches. The basic notions of effective couplings, which are common to all approaches, can be easily introduced. Nevertheless, this deserves to be seen in several logical steps.

1.4.1 The Born Approximation for $e^+e^- \rightarrow f\bar{f}$

The matrix element of the process

$$e^+e^- \rightarrow (\gamma, Z) \rightarrow f\bar{f}, \quad f \neq e,$$

depicted in Fig. 1 may be written as follows:

![Feynman graph for the reaction (15) in the Born approximation.](image)

Figure 1: Feynman graph for the reaction (15) in the Born approximation.
\[ M^{\text{Born}} \sim \frac{1}{s} \left[ Q_e Q_f \gamma_\alpha \otimes \gamma^\alpha + \chi \gamma_\alpha \left( g_\nu^e - g_\alpha^e \gamma_5 \right) \otimes \gamma^\alpha \left( g_\nu^f - g_\alpha^f \gamma_5 \right) \right] \]

\[ = \frac{1}{s} \left[ Q_e Q_f \gamma_\alpha \otimes \gamma^\alpha + \chi \left( g_\nu^e g_\nu^f \gamma_\alpha \otimes \gamma^\alpha - g_\alpha^e g_\alpha^f \gamma_\alpha \otimes \gamma^\alpha \gamma_5 ight. \\
\left. - g_\alpha^e g_\alpha^f \gamma_5 \otimes \gamma^\alpha + g_\alpha^e g_\alpha^f \gamma_5 \otimes \gamma^\alpha \gamma_5 \right) \right], \]

with \( \chi \) being the propagator ratio

\[ \chi = \frac{s}{s - M^2_z + i s \Gamma_z / M_z}. \]

And in the Minimal Standard Model (MSM)

\[ g_\alpha^f = I_f^{[3]}, \quad g_\nu^f = I_f^{[3]} - 2 Q_f^\dagger \sin^2 \theta, \]

where \( g_\alpha^f \), \( g_\nu^f \) are vector and axial vector couplings of the \( Z \) boson, \( Q_f \) are the electric charges of fermions in units of position charge, \( I_f^{[3]} \) the projection of weak isospin, and \( \theta \) the weak mixing angle in the Born approximation. Its value depends on the choice of computational scheme. In (16) and (17) a short notation for bilinear combinations of spinors \( u \) and \( v \) is used:

\[ A_\beta \otimes B^\beta = [\bar{u}_e A_\beta u_e] \times [\bar{u}_f B^\beta v_f]. \]

In the matrix element (16) and (17) the contributions from \( \gamma \) and \( Z \) exchange diagrams are unambiguously separated, and the \( Z \) exchange contribution is presented in a factorized form (16).

### 1.4.2 Electroweak non-photonic corrections

The higher order electroweak non-photonic corrections are indicated symbolically by the blobs in Fig. 2. They include all possible self-energy and \( Z \bar{f}f \) vertex insertions, together with non-photonic box insertions (the \( WW \) and \( ZZ \) boxes in the one-loop approximation), and lead to a slightly more complicated structure in the matrix element, which is valid as long as we neglect the external fermion masses:

\[ M^{\text{eff}} \sim \frac{1}{s} \left\{ \alpha(s) \gamma_\alpha \otimes \gamma^\alpha + \chi \left[ \mathcal{F}^{e_f}_{\nu\alpha}(s, t) \gamma_\alpha \otimes \gamma^\alpha - \mathcal{F}^{e_f}_{\alpha\nu}(s, t) \gamma_\alpha \otimes \gamma^\alpha \gamma_5 \\
- \mathcal{F}^{e_f}_{\nu\alpha}(s, t) \gamma_\alpha \gamma_5 \otimes \gamma^\alpha + \mathcal{F}^{e_f}_{\alpha\nu}(s, t) \gamma_\alpha \gamma_5 \otimes \gamma^\alpha \gamma_5 \right] \right\}. \]

The products of interaction constants in (17) are replaced by the running \( \alpha(s) \) — this time including the imaginary part — and by four running electroweak form factors, \( \mathcal{F}^{e_f}_{ij} \). In the MSM, the explicit expressions for the form factors result from an order-by-order
calculation and are certain explicit functions of the input parameters (1), (3), (11), (12) and (14).

Several comments should be given on the structure of the matrix element (21) after inclusion of higher order electroweak corrections:

- This structure is unique, but implies the introduction of complex valued form factors which depend on the two Mandelstam variables \( s, t \); the dependence on \( t \) is due to the weak box diagrams.

- The separation into insertions for the \( \gamma \) and \( Z \) exchanges is lost. In the proposed realization, the first term in (21) contains the running QED coupling where only fermionic insertions are retained. This ensures a gauge invariant separation.

- The weak boxes are present in (21) as non-resonating (\( \sim s - M_Z^2 \)) insertions to the electroweak form factors \( \mathcal{F}_{ij}^{\gamma} \). At the \( Z \) resonance, the one-loop weak \( WW \) and \( ZZ \) box terms are small, with relative contribution \( \leq 10^{-4} \). If we neglect them, the \( t \) dependence is turned off. The \( t \)-dependence would also spoil factorization of the form factors into products of effective vector and axial vector couplings.

- Full factorization is re-established by neglecting, in addition, the other non-resonating loop contributions, such as the bosonic insertions to the photon propagator, and photon-fermion vertex corrections. All the neglected terms are of the order \( \mathcal{O}(\alpha^{\Gamma_Z/M_Z}) \). The resulting effective vector and axial-vector couplings are complex valued and dependent on \( s \). The factorization is the result of a variety of approximations, valid at the \( Z \) resonance to the accuracy needed, and indispensable in order to relate the pseudo-observables to actual measured quantities.

In the complete codes — BHM, TOPAZO, ZFITTER — it is possible to control the numerical influence of all of these approximations.

### 1.4.3 \( Z \) pole approximation

After the above-mentioned series of approximations, we arrive at the so called \( Z \) boson pole approximation, which is actually equivalent to the setting \( s = M_Z^2 \) in the form factors. In the \( Z \) pole approximation, the one-loop diagrams contributing to the blobs in Fig. 2 can be visualized in the diagrams of Fig. 3.
In Fig. 3 all particles \( (Z, H, W, q, l, \nu) \) contribute through their weak neutral currents, while in the \( Z\gamma \) mixing (b) only charged ones contribute. This, of course, is strictly true only in the unitary gauge, while in the renormalizable gauge, Faddeev-Popov and Higgs-Kibble ghosts will also appear. The vertex diagrams (c) and (d) contain \( Z, W \), as vertical lines. The vertex diagrams (e) and (f) contain trilinear gauge boson interactions. The vertex diagrams contain, in principle, also Higgs boson exchanges — as vertical lines in diagrams (c) and (d), and as \( Z, H \) virtual states in diagrams (e) and (d). They give, however, a negligible contribution because of small \( H f \bar{f} \) Yukawa couplings.

The set of the diagrams of Fig. 3 has to be complemented by the contributions from the bare 'unphysical' parameters (masses and couplings) in order to get the physical amplitude of Fig. 2. This is depicted in Fig. 4 as an example, the cross representing the contribution from the bare mass \( M_0^Z \).

The \( W \) boson self-energy enters the analysis only through the \( W \) mass (see Fig. 5). The \( W \) self-energy contains all types of particles entering through their charged weak currents. Explicit expressions for the decay width of the \( W \) are not discussed in this Report, (see the existing literature [19]).

If the mass of the \( W \) boson were known with the same accuracy as that of the \( Z \), the conceptual picture of the electroweak corrections would be as simple as that of QED.

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4In some renormalization schemes one should also add the self-energy insertions for external fermion lines. In some other schemes they vanish when the field renormalization counterterm contributions are added.
Every physical observable would be expressed in terms of $\alpha$, $M_Z$, $M_W$, the masses of all fermions and the mass of the Higgs boson. Unfortunately, $M_W$ is not known with an accuracy comparable to that of $M_Z$. Therefore, instead of $M_W$ we have to choose $G_\mu$ as the most accurately measured dimensional observable.

1.4.4 Effective couplings for $Z$ decay.

Formulæ (17) and (21) are very similar in their structure, a property which makes the language of effective couplings so convenient. These formulæ refer to a general $f\bar{f} \to f'\bar{f}'$ annihilation process. After the simplifications described in 1.4.2 and 1.4.3, the factorization allows us to write for the form factors

$$F_{ij}^f(s = M_Z^2, t = 0) = G_i^f(M_Z^2)G_j^f(M_Z^2).$$

This implies that the $Z$ part of the amplitude (21) is obtained from that of (16) by substituting

$$g_{\epsilon,\alpha}^e \to G_{\epsilon,\alpha}^f(M_Z^2).$$

Simultaneously, we obtain for the matrix element of the decay process $Z \to f\bar{f}$, Fig. 6, as follows:

$$M_{Z\bar{f}f}^e = \bar{u}_f\gamma_\alpha \left[ G_i^f(M_Z^2) - G_i^f(M_Z^2)\gamma_\alpha \right] v_f e_\alpha,$$

where $\bar{u}_f, v_f, e_\alpha$ are wave-functions of the fermion $f$, antifermion $\bar{f}$ and $Z$ boson.
In the one-loop approximation, the blob in Fig. 6 can be visualized as the diagrams of Fig. 7. Note that Fig. 3a contributes both to initial and final state virtual corrections. The effective decay couplings $G^{f}_{V,A}$ are nothing else but the reduced electroweak form factors, which are now constant because of the kinematical constraint $s = M_{Z}^2$, which is itself equivalent to the $Z$ pole approximation. They will be the basic objects in the following presentation. We will use for them the simplified notations,

$$g^{f}_{V,A} = \Re G^{f}_{V,A}(M_{Z}^2),$$

(25)

where the capital letters $V, A$ are introduced in order to distinguish effective couplings $g_{V,A}$, dressed by higher order interactions, from their Born-like analogs $g_{v,a}$.

Figure 7: Feynman graphs of the $Z$ decay in the one-loop approximation.

In the next two subsections we will define the quantities to be computed and compared: the pseudo-observables, and the realistic observables.
1.5 Pseudo-observables

The pseudo-variables are related to measured cross-sections and asymmetries by some de-convolution or unfolding procedure. The concept of pseudo-observability itself is rather difficult to define. One way to introduce it is to say that the experiments measure some primordial (basically cross-sections and thereby also asymmetries) quantities which are then reduced to secondary quantities under some set of specific assumptions. Within these assumptions, the secondary quantities, the pseudo-observables, also deserve the label of observability. Just to give an example, we quote the de-convoluted forward-backward asymmetry, where, typically, only the $Z$ exchange is included and initial and final state QED corrections, plus eventual final state QCD corrections, are assumed to be subtracted from the experimental data. We have analyzed 25 such pseudo-observables in detail, namely:

- mass of the W
- hadronic peak cross-section
- partial leptonic and hadronic widths
- the total width
- the total hadronic width
- the total invisible width
- ratios
- asymmetries and polarization
- effective sine

The effective sine are defined by

$$ 4 |Q_f| \sin^2 \theta^f_{\text{eff}} = 1 \frac{g^f_V}{g^f_A}, $$

with $Q_f$ being the electric charge of the fermion $f$ in units of the positron charge. By definition, the total and partial widths of the $Z$ boson include final state QED and QCD radiation. The explicit formulae for partial widths will be presented in Subsection 1.9. Moreover, we have defined

$$
\begin{align*}
\Gamma_h &= \Gamma_u + \Gamma_d + \Gamma_e + \Gamma_s + \Gamma_b, \\
\Gamma_{\text{inv}} &= \Gamma_e - \Gamma_\mu - \Gamma_\tau - \Gamma_h, \\
R_l &= \frac{\Gamma_h}{\Gamma_e}, \\
R_{b,c} &= \frac{\Gamma_{b,c}}{\Gamma_h},
\end{align*}
$$

---

5 With lepton universality there exists only one leptonic effective sine, $\sin^2 \theta^\ell_{\text{eff}} = \sin^2 \theta^\ell_{\text{eff}}$. 

19
\[
\sigma_h = 12\pi \frac{\Gamma_e \Gamma_h}{M_Z^2 \Gamma_Z^2}.
\]  

(31)

The quantity \(\sigma_h\) is the de-convoluted hadronic peak cross-section, which by definition includes only the Z exchange. To this end we would like to emphasize that in our calculations we indeed assumed that

\[
\Gamma_{\text{inv}} = 3\Gamma_{\nu},
\]

(32)

then the total Z width becomes

\[
\Gamma_Z = 3\Gamma_{\nu} + \Gamma_e + \Gamma_\mu + \Gamma_\tau + \Gamma_h.
\]

(33)

Unlike the widths, asymmetries and polarizations do not contain, by definition, QED and QCD corrections; furthermore, they will only refer to pure Z exchange: they are nothing but simple combinations of the effective Z couplings, introduced above,

\[
A^{f}_{\text{FB}} = \frac{3}{4} A^e A^f,
\]

\[
A^{e}_{LR} = A^e,
\]

\[
P^f = -A^f,
\]

\[
P_{\text{FB}}(\tau) = -\frac{3}{4} A^e,
\]

(34)

where we define

\[
A^f = \frac{2 g^f_{\nu} g^f_{A}}{(g^f_{\nu})^2 + (g^f_{A})^2},
\]

(35)

and \(g^f_{\nu}, g^f_{A}\) defined by Eq. (25) are the effective neutral current vector and axial-vector couplings of the Z to a fermion pair \(f \bar{f}\). \(P_{\text{FB}}(\tau)\) is the \(\tau\) polarization forward–backward asymmetry. One should realize that as a consequence of the adopted definition of pseudo-observables, there exist even more relations among them than are indicated above; it is, for example, \(A^{e}_{LR} = -P^e\), if in addition lepton universality is assumed, which is granted automatically in the MSM. For this reason we will present numerical and graphical results only for a selected subset of them.

1.6 Realistic observables

Realistic observables are the cross-sections \(\sigma^f(s)\) and asymmetries \(A^{f}_{\text{FB}}(s)\) of the reactions

\[
e^+e^- \rightarrow (\gamma, Z) \rightarrow f \bar{f}(n\gamma),
\]

(36)

for a given interval of \(s = 4E^2\) around the Z resonance, including real and virtual photonic corrections (‘dressed by QED’). We will present results for \(f = \mu\) and \(f = \text{had}\), both in a fully extrapolated set-up or with simple kinematical cuts. For \(f = e\) both the \(s\) channel exchange cross-section and the complete Bhabha reaction will be treated. In total, we
considered 14 cross-sections and asymmetries for different combinations of flavour $f$ and cuts; they will be defined in Section 3. The kinematical cuts that will be imposed are of two types:

- $\theta_{\min} < \theta_{-} < \pi - \theta_{\min}$
- $s' > s'_{\min}$

and:

- $\theta_{\min} < \theta_{-} < \pi - \theta_{\min}$
- $\theta_{\text{acoll}} < \xi$
- $E_{\pm} > E_{\text{th}}$.

Where $s'$ is the invariant mass of two final-state fermions, $\theta_{-}$ refers to the outgoing scattering angle of the outgoing fermion, $\theta_{\text{acoll}}$ is the acollinearity angle between the outgoing fermions and $E_{\pm}$ the outgoing fermion (anti-fermion) energies. By $s$ channel Bhabha scattering we mean that the $s - t$ and $t - t$ exchange interferences have been subtracted from the observable under consideration.

We would like to mention in passing that the QED dressers have options which allow a model independent interpretation of realistic observables. Since this is beyond our present scope, we refer the reader for details to the existing literature [20, 21].

### 1.7 Calculational schemes

Before entering into a detailed study of the numerical results it is important to underline how an estimate of the theoretical uncertainty emerges from the many sets of numbers obtained with the five codes. First of all, one may distinguish between intrinsic and parametric uncertainties. The latter are normally associated with a variation of the input parameters according to the precision with which they are known. Typically, we have $|\Delta \alpha^{-1}(M_Z^2)| = 0.12$, $|\Delta m_b| = 0.3 \text{ GeV}$, $|\Delta m_c| = 0.35 \text{ GeV}$ etc. These uncertainties will eventually shrink when more accurate measurements become available. In this Report we have mainly devoted ourselves to a discussion of the intrinsic uncertainties associated with missing non-leading higher-order corrections, although some results on parametric uncertainties will be also given. An essential ingredient of all calculations for radiative corrections to pseudo-observables is the choice of the renormalization scheme. There are many renormalization schemes in the literature:

- the on-shell schemes in various realizations [22–28], [3]
- the $G_{\mu}$ scheme [29–31]
- the * scheme [32]
- the $\overline{\text{MS}}$ scheme [33, 34].
One cannot simulate the shift of a given quantity due to a change in the renormalization scheme with one code alone. Thus the corresponding theoretical band in that quantity will be obtained from the differences in the prediction of the codes, which use different renormalization schemes. On top of that we should also take into account the possibility of having different implementations of the full radiative corrections within one code — within one well specified renormalization scheme. Typically we are dealing here with the practical implementation of resummation procedures, of the exact definition of leading versus non-leading higher-order corrections and so on. Many of these implementations are equally plausible and differ by non-leading higher-order contributions, which, however, may become relevant in view of the achieved or projected experimental precision. This sort of intrinsic theoretical uncertainty can be very well estimated by staying within each single code. However, since there are no reasons to expect that these will be the same in different codes, only the full collection of different sources will, in the end, give reliable information on how accurate an observable may be considered from a theoretical point of view.

Of course, only a complete two-loop electroweak calculation, combined with some re-summations of the leading corrections (renormalization group improvements), would ultimately solve the problem of the theoretical accuracy.

1.8 Main features of different approaches

In this subsection we will discuss common features of and main differences between the electroweak libraries of the five codes.

Common features:

- All five codes use as input parameters the most accurately known electroweak parameters $G_\mu$, $M_Z$ and $\alpha$ (the codes, with the exception of LEPTOP, also use $\alpha$), in order to calculate the less precisely measured pseudo-observables.
- They use the same expressions for final state QED and QCD corrections (radiation factors).
- All codes include essentially the same internal gluon corrections of the order of $\alpha\alpha_s$ in the $W$ and $Z$ self-energy quark loops.
- All codes include leading two-loop corrections of the order of $G_\mu^2 m_t^4$; all gluonic corrections of the order of $\alpha_s G_\mu m_t^2$; leading gluonic corrections $\alpha\alpha_s^2$ in the vector boson self-energies.

Main differences:

- Each code uses a different renormalization framework.
- Some codes define an electroweak Born approximation, others give no physical emphasis to a Born approximation and employ only the notion of an Improved Born Approximation (IBA) which includes the leading electroweak loop corrections.
• Certain codes include higher-order electroweak corrections, which are documented in the literature but are numerically irrelevant, such as the irreducible two-loop photon vacuum polarization and the Higgs corrections proportional to $\alpha^2 m_t^2$. Some codes use, on top of these higher-order corrections, resummation of one-loop terms.

• They differ in the choice of the definition of the weak mixing angle.

1.8.1 BHM/WOH, ZFITTER

The three codes BHM [2], WOH [5] and ZFITTER [6] rely on different realizations of the on-mass-shell renormalization scheme. They perform the renormalization procedure with systematic use of the counterterm method for the basic parameters $\alpha$, $M_w$, $M_z$. The BHM and WOH codes lead back to one approach and are thus not completely independent of one another.

The description of the renormalization schemes may be found in Ref. [28] for BHM/WOH and in Ref. [24] for ZFITTER. The weak mixing angle formally appears as

\[
\begin{align*}
    c_w^2 &= \cos^2 \theta_w = \left( \frac{M_w}{M_z} \right)^2, \\
    s_w^2 &= \sin^2 \theta_w = 1 - c_w^2. 
\end{align*}
\]  

This corresponds to the definition proposed by Sirlin [35]. The masses in (37) are the physical $W$ and $Z$ boson masses. Thus, $s_w^2$ is not an independent quantity and is used mainly for internal book-keeping. For fixed values of $m_t$ and $M_H$, each observable, including the Fermi constant $G_\mu$, is expressed in terms of $\alpha$, $M_z$ and $M_w$ with the corresponding quantum corrections being taken into account. Because of lack of precision in the $W$ mass, $M_w$, it is not taken as an experimentally measured input quantity. It is instead replaced by the more accurate value of $G_\mu$.

The quantum corrections for $\mu$ decay (2) after removing the Fermi-model-like QED corrections are contained in the parameter $\rho_c$ related to $G_\mu$ by the equation \[^6\]

\[
G_\mu = \frac{\pi}{\sqrt{2}} \frac{\alpha}{s_w^2 c_w^2 M_z^2} \rho_c.  
\]  

Since this equation contains $\alpha$ instead of $\alpha(M_z)$ a large fraction of $\rho_c$ is of purely electromagnetic origin via $\Delta\alpha$. The parameter $\rho_c$ in (38) takes into account the $W$ propagator, vertex and box corrections due to one-loop diagrams and presently available higher order terms. The one-loop corrections were first calculated in Refs. [35] and [36]. The correction due to the photon self-energy is QED renormalization group improved: a geometrical progression brings $\Delta\alpha$ into the denominator of $\rho_c$. Hence, $\rho_c$ is usually written in the form

\[
\rho_c = \frac{1}{1 - \Delta r},  
\]  

\[^6\]The naïve Born approximation for muon decay of Eq. (38), $\rho_c = 1$, was used in the 70's for predictions of the masses of the $W$ and $Z$ bosons with a crude accuracy by using the value of $s_w^2$ from the ratio of neutral and charged currents in neutrino interactions.
where $\Delta r$ contains all the one-loop corrections to the muon-decay with the inclusion and the proper arrangement of the higher-order terms. (For more details see the explanations presented in Subsection 1.10.2.)

For given values of $m_t$ and $M_H$, Eq. (38) fixes $\theta_w$ and, hence, $M_w$ by the experimental value of $G_\mu$. In practice, Eq. (38) is solved with respect to $M_w$ iteratively, since $\Delta r$ is a complicated function of $M_w$. This equation for iteration of $M_w$ reads

$$M_w = \frac{M_Z}{\sqrt{2}} \left[ 1 + \frac{4\pi\alpha}{\sqrt{2}G_\mu M_Z^2[1 - \Delta r(M_w)]} \right]. \quad (40)$$

Therefore, $M_w$ appears as an $m_t$, $M_H$, $\hat{\alpha}_s$ dependent prediction, which can be compared with the experimental values from UA2 and CDF [37].

After fixing $M_w$ in this way all the other observables are expressed in terms of $G_\mu, \alpha, M_Z, m_t, M_H$ and $\hat{\alpha}_s$. As a consequence of this procedure, purely leptonic Feynman graphs (for example, a vertex correction in the $Z \rightarrow \mu\mu$ decay) also turn out to be implicitly $m_t$ dependent through the $M_w$.

In conclusion, Eq. (38) uses the amplitude of $\mu$ decay with the inclusion of weak corrections in order to establish the interdependence between $M_w$, $m_t$ and $M_H$ and the best measured parameters $\alpha, G_\mu, M_Z$. Thus, $M_w$ appears as a prediction as well as an intermediate parameter for the calculation of $Z$-boson observables.

The flowchart of the BHM/WOH, ZFITTER approach is shown in Fig. 8. One should have in mind that in spite of a common presentation of the basics of these codes there are certain differences in the realization of the on-mass-shell renormalization schemes between BHM/WOH on one side and ZFITTER on the other — for example, in the use of different gauges and in the different treatment of field renormalization.

---

7In leptonic processes the dependence on $\hat{\alpha}_s$ is of higher order; it comes from gluonic exchanges between virtual quarks in $W$, $Z$, $Z\gamma$ and $\gamma$ self-energy diagrams with quark loops.
FLOWCHART OF ZFITTER/BHM/WOH

Select minimal set of parameters in the MSM Lagrangian:
\( \alpha_0, M_{w0}, M_{z0}, M_{h0}, m_{f0} \) (including \( m_{t0} \)); note that \( \alpha_{w0}, \alpha_{z0} \) and VEV \( \eta \) are not among these.

Define renormalization Z-factors for each bare parameter and each field (Z-matrices for \( Z-\gamma \) and fermion mixing — for ZFITTER only).
Fix Z-factors on mass shell. Use dimensional regularization \((1/\epsilon, \mu)\).

Lagrangian now depends only on physical fields, couplings and masses, and on counterterms (Z-factors).

Expand Z-factors; \( Z_i = 1 + \alpha f_i \), where \( \alpha = \alpha(0) \) and \( f_i \)'s are functions of physical input \( M_w, M_z, M_h, m_f \) and \( 1/\epsilon \) and \( \mu \).

Calculate one–loop electroweak amplitudes with graphs, including loops and counterterms; \( 1/\epsilon \) and \( \mu \) drop out.

Improve one-loop results by RG-techniques and by proper resummation of the higher-order e.w. terms. Define improved Born approximation.

Select experimental inputs: \( \alpha(0), M_z, G_\mu (\tau_\mu) \).

Get \( M_w \) from \( G_\mu = \left( \frac{\pi}{\sqrt{2}} \right) \left( \frac{\alpha}{s_w^2 c_w^2 M_z^2} \right) \rho_c \), where \( \rho_c \) depends on \( m_t, M_h, \alpha(0), M_w, M_z \) and \( s_w^2 = 1 - M_w^2/M_z^2 \).

Calculate \( Z^0 \) decay observables, with \( m_t \) and \( M_h \) free,
in terms of \( G_\mu, \alpha(0), M_z \).

Introduce gluonic corrections into quark loops and QED + QCD final state interactions in terms of \( \bar{\alpha}, \bar{\alpha}_s(M_z), m_b (M_z), m_t \).

Compare the results with electroweak experimental data, exhibit \( M_z, m_t, M_h \), and \( \bar{\alpha}_s(M_z) \) dependence.

Figure 8: BHM/WOH ZFITTER flowchart.
1.8.2 LEPTOP

The authors of LEPTOP avoid the use of counterterms in the formulation of the theoretical framework. They do not use resummation of one-loop electroweak corrections, thus limiting themselves to a simple one-loop approximation, completed by selected leading electroweak two-loop corrections. In contrast to all the other codes, LEPTOP uses a Born approximation. According to LEPTOP, the weak mixing angle $\theta$ is defined by $(s = \sin \theta, c = \cos \theta)$:

$$G_\mu = \frac{\pi \tilde{\alpha}}{\sqrt{2} s^2 c^2 M_z^2},$$

$$\sin^2 2\theta = \frac{4\pi \tilde{\alpha}}{\sqrt{2} G_\mu M_z^2}.$$

This gives $s^2 = 0.23117(33)$, $c = 0.87683(19)$. Such a $\theta$, which by definition does not depend on $m_t$ and $M_H$, is used to determine the so-called $\tilde{\alpha}$-Born approximations for electroweak observables.

The bare gauge couplings $\alpha^0, \alpha^0_w$ and the bare mass $M^0_Z$ are expressed in terms of $\tilde{\alpha}$, $G_\mu$ and $M_z$, $1/\epsilon$ and $\mu$, where $\mu$ is the 't Hooft's scale parameter and $2\epsilon = 4 - D$ in the dimensional regularization scheme. The $W$ mass, $M_W$, is treated on an equal basis with the other observables.

The $\tilde{\alpha}$-Born approximation automatically includes purely electromagnetic corrections. In terms of $s$ and $c$, the expressions for the hadron-free pseudo-observables are very simple in that approximation:

$$(M_W/M_z)^B = c,$$

$$(g^f_A)^B = f^B_3,$$

$$(g^f_v/g^f_A)^B = 1 - 4|Q_f|s^2.$$ 

The $\tilde{\alpha}$-Born approximation with the due account of the final state QED and QCD radiation factors gives simple expressions for the observables in the hadronic decays of the $Z$ boson as well.

The electroweak corrections for all observables are calculated in the framework of LEPTOP in terms of $\alpha^0, \alpha^0_w, M^0_z, m^0_t, m^0_H$, and $(1/\epsilon, \mu)$ in one-loop approximation. In this approximation $m^0_t$ and $M^0_H$ can be replaced by the physical masses $m_t$ and $M_H$.

By expressing $\alpha^0, \alpha^0_w, M^0_z$ in terms of $\tilde{\alpha}, G_\mu, M_z$ and $(1/\epsilon, \mu)$ one derives formulae in which terms $(1/\epsilon, \mu)$ cancel out. Thus introduction of counterterms is avoided.

The explicit analytical expressions for the LEPTOP electroweak corrections in terms of $G_\mu, \tilde{\alpha}, c, s, t = (m_t/M_z)^2$ and $h = (M_H/M_z)^2$ are given in Subsection 4.1.

The flowchart for the LEPTOP approach is shown in Fig. 9.
Select the three most accurate observables:
\[ G_\mu, M_z, \alpha(M_z) \equiv \tilde{\alpha} \]

Define angle \( \theta \) (\( s \equiv \sin \theta, c \equiv \cos \theta \))
in terms of \( G_\mu, M_z \) and \( \tilde{\alpha} \):
\[ G_\mu = (\pi/\sqrt{2})\tilde{\alpha}/s^2c^2M_z^2 \]

Define the Born approximation for other electroweak
observables in terms of \( G_\mu, M_z \) and \( \theta \).

Introduce bare couplings \( (\alpha_0^0, \alpha_z^0, \alpha_w^0) \), masses \( (M_z^0, M_w^0, m_q^0) \) — including \( m_t^0 \) and VEV \( \eta \) in the framework of MSM.

Express \( \alpha^0, \alpha_z^0, M_z^0 \) in terms of \( G_\mu, M_z \) and \( \tilde{\alpha} \) in the one-loop
approximation, using dimensional regularization \( (1/\epsilon, \mu) \)

Express one-loop corrections to all other electroweak observables
in terms of \( \alpha_0^0, \alpha_z^0, M_z^0, m_t, M_H \) and hence in terms of \( G_\mu, M_z, \tilde{\alpha}, m_t, M_H \). Check cancellation of \( 1/\epsilon \) and \( \mu \).

Introduce gluonic corrections into quark loops and QED and QCD
final state interactions for hadronic decays
(in terms of \( \tilde{\alpha}, \tilde{\alpha}_s(M_z), m_b(M_z), m_t \)).

Compare the Born results and Born + one loop results with
experimental data on \( Z \)-decays and \( M_w \).

Make a global fit for three parameters \( m_t, M_H, \tilde{\alpha}_s(M_z) \).

Predict the central values of all electroweak observables
and their uncertainties.

**Figure 9**: LEPTOP flowchart.
1.8.3 TOPAZ

The \textsc{Topaz} code [4] uses the $\overline{\text{MS}}$ (modified minimal subtraction) scheme for all types of interactions, including the electroweak ones, as introduced in [34]. Its approach is quite different both from those of \textsc{bhm/woh}, \textsc{zfitter}, and \textsc{leptom}. Its main steps are presented in the following items:

- The MSM Lagrangian is assumed. Excluding fermion masses and mixing angles the MSM is a three-parameter theory: i.e. $g^0$, $M^0_w$ and $\sin \beta_0$. At the level of defining a renormalization scheme, two essentially different approaches may be distinguished: to prescribe precisely what a parameter of the Lagrangian is or to prescribe precisely what a counterterm is. It is a matter of convention, since only their combined effect appears in the confrontation with the data, and once the latter is chosen it would then seem natural to follow the consensus with respect to QCD. The bare parameters are fixed by considering three data points — $\alpha(0)$, $G_{\mu}$, and $M_Z$. These quantities are computed up to one-loop diagrams and \textit{fitting equations} are written:

$$\rho_i^0 = f_i(\alpha, G_{\mu}, M_Z, \Delta), \quad i = 1, 2, 3,$$

where $\rho_i^0$ are the bare parameters and $\Delta = -2/(n - 4) + \gamma_E - \ln \pi$. Everything is worked out explicitly in the \textsc{'t Hooft–Feynman} gauge.

- The \textit{fitting equations} are solved perturbatively (order-by-order renormalization), but gauge invariant (fermionic) higher-order leading terms are always re-summed.

Strictly speaking, one usually solves an implicit equation $f(a, \lambda) = 0$, where $a$ is some parameter and $\lambda$ is a coupling constant, by expanding $a$ around $a_0$, the solution of $f(a_0, 0) = 0$. However, one could just as well expand $a$ around $\tilde{a}$, the solution of some other equation $h(\tilde{a}, \lambda) = 0$, as long as $\tilde{a}$ is gauge invariant. Thus in our implementation $\alpha(0)$ will evolve into $\alpha(M_Z)$ and the lowest-order approximation for $s^2_\beta = \sin^2 \theta_0$ is

$$s^2 = \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{4 \pi \alpha(M_Z)}{\sqrt{2} G_{\mu} M_Z^2 \rho^R_Z}} \right],$$

$$\rho^R_Z = 1 + \frac{G_{\mu} M_Z^2}{2\sqrt{2}\pi^2} \times \{\text{UV finite combination of two-point functions}\},$$

where $(\rho^R_Z)^{-1} - 1$ is determined by an ultraviolet finite combination of two-point functions. The detailed definition of $\rho^R_Z$ is left for Subsection 4.2 — Eqs. (227), (232) and (233).

Eq.(47) is an algebraic solution of the following relation \footnote{For analogous relations for \textsc{bhm/woh}, \textsc{zfitter} see Eq. (38) and for \textsc{leptom} Eq. (41).}

$$G_{\mu} = \frac{\pi \tilde{\alpha}}{\sqrt{2} M_Z^2 s^2 \rho^R_Z}. \quad (49)$$
If no resummation at all is performed in TOPAZO then $\rho^u_Z = 1$ and, in this limit, one recovers the weak mixing angle as defined by LEPTOP. The $\rho^u_Z$ parameter is strictly connected to the $Z$ wave function renormalization and usually all bosonic contributions are expanded up to first order, in this case the $\rho$ parameter being instead denoted by $\rho_Z$, Eq. (227), but the possibility of a resummation for a certain gauge invariant subset in the $\overline{MS}$ framework is built in. It should be noticed that $\rho_Z$ or its variant $\rho^u_Z$ represent the natural extension of the Veltman's $\rho$-parameter, as defined at low energy [34], to the scale $M_Z$. Its asymptotic behaviour, for large $m_t$, is exactly the same as dictated by $\Delta \rho$ — see Eq. (80). That is why all the higher-order leading $(m_t)$ corrections will simply be added to $\rho^u_Z(\rho_Z)$.

- By a proper redefinition of the bare coupling $g^0$ [31], corresponding to a gauge field re-diagonalization in the quadratic part of the Lagrangian, in the $\xi = 1$ (t'Hooft–Feynman) gauge the following properties are fulfilled: the sum of all $Z f \bar{f}$ vertices, $\sum_v \{\text{vertex}\}$, is ultraviolet finite and the $Z - \gamma$ transition satisfies $\Sigma_{Z \gamma}(0) = 0$. In the renormalization procedure of TOPAZO it is observed that, unlike QED, no one-to-one correspondence exists between the bare parameters and experimental data points. Therefore no attempt is made to give an all-orders relation as for $s^2_W$, but rather it is observed that the mixing angle defines the distribution of the vector current between $Z$ and $\gamma$. It is an accident of the minimal Higgs system (or more generally of representations where the Higgs scalars only occur in doublets) that the vector boson masses are not both free and beyond lowest order different definitions of the weak mixing angle receive different radiative corrections. Also, the $W$ mass has no special role in TOPAZO, being computed like any other quantity, once the bare parameters are substituted through their explicit expressions. Typically,

$$\sin^2 \theta_0 = \hat{s}^2 + \{O(\alpha)\ \text{UV finite bosonic corrections}\}$$

$$+ \{O(\alpha) \ \Delta\text{-dependent terms}\}, \quad (50)$$

and in computing for instance $Z \rightarrow f \bar{f}$ the latter cancel their $\Delta$ dependence against the $Z - \gamma$ transition, leading to an ultraviolet finite result. According to the chosen strategy, all bosonic terms are expanded or, alternatively, a gauge invariant sub-set of the bosonic corrections is re-summed in $\hat{s}^2$ (in the $\overline{MS}$ environment). In the second case a (usually small) remainder is left, which compensates against vertex corrections. The corrected $Z$ propagator, leading to mass renormalization and to a $Z$ wave-function renormalization factor, is ultraviolet finite by inspection.

- The processes $Z \rightarrow f \bar{f}, e^+ e^- \rightarrow f \bar{f}$ are computed up to one-loop diagrams with lowest-order (partially re-summed) parameters plus tree diagrams with one-loop corrected parameters. Ultraviolet finiteness is checked both analytically and numerically (independence of $\Delta$). Roughly speaking, the central objects in TOPAZO are the $S$-matrix elements for a given process, so particular care is devoted to the proper treatment of the wave-function renormalization factors (see Ref. [38]).
1.9 Phenomenology of $Z$ boson decays

The decay width of $Z \to f \bar{f}$ is described by the following equation used by all five codes:

$$\Gamma_f \equiv \Gamma(Z \to f \bar{f}) = 4 N_c \Gamma_0 [(g^f_v)^2 R^f_v + (g^f_A)^2 R^f_A],$$  \hspace{1cm} (51)

where $g^f_v$ and $g^f_A$ are effective electroweak couplings defined in Subsection 1.4.4, $N_c = 1$ or 3 for leptons or quarks ($f = l, q$). The factors $R^f_v$ and $R^f_A$ describe the final state QED and QCD interactions and take into account the mass $m_f$. The radiation factors are universal, to a large extent, and will be described below in this subsection.

The standard width $\Gamma_0$ in Eq. (51) is known with very high accuracy:

$$\Gamma_0 = \frac{G_F M_Z^3}{24\sqrt{2}\pi} = 82.945(12) \text{ MeV}. \hspace{1cm} (52)$$

For the decay into leptons $l\bar{l}$ the radiation factors $R^f_{v,A}$ are especially simple. For charged leptons:

$$\Gamma_l = 4\Gamma_0 \left( (g^l_v)^2 \left(1 + \frac{3}{4\pi} \tilde{\alpha} \right) + (g^l_A)^2 \left(1 - 6 \frac{m^2_l}{M_Z^2} + \frac{3}{4\pi} \tilde{\alpha} \right) \right). \hspace{1cm} (53)$$

The mass term in Eq. (53) is negligible for $l = e, \mu$ and is barely visible only for $l = \tau$ ($m^2_{\tau}/M_Z^2 = 3.8 \times 10^{-4}$).

For the neutrino decay:

$$\Gamma_\nu = 8(g^\nu)^2 \Gamma_0, \hspace{1cm} g^\nu = g^\nu_v = g^\nu_A. \hspace{1cm} (54)$$

For the decays into quarks $q\bar{q}$ the radiation factors are given by [39–45]

$$R^q_v(s) = 1 + \delta^q_v + \delta_v a^2_s - 12.76706 \ a^3_s + 12 \frac{\tilde{m}^2_q(s)}{s} a_s \delta_{vm}, \hspace{1cm} (55)$$

$$R^q_A(s) = 1 + \delta^q_A + \left[ \delta_v - 2 I^{(3)}_q \bar{T}^{(2)} \left( \frac{s}{m^2_t} \right) \right] a^2_s + \left[ -12.76706 - 2 I^{(3)}_q \bar{T}^{(3)} \left( \frac{s}{m^2_t} \right) \right] a^3_s$$

$$- 6 \frac{\tilde{m}^2_q(s)}{s} \delta^1_{am} - 10 \frac{\tilde{m}^2_q(s)}{m^2_t} a_s^2 \delta^2_{am}, \hspace{1cm} (56)$$

where

$$\delta^q_v = \frac{3}{4} Q^2 q a + a_s - \frac{1}{4} Q^2 a a_s, \hspace{1cm} \delta_v = 1.40923 + \left( \frac{44}{675} - \frac{2}{135} \ln \frac{s}{m^2_t} \right) \frac{s}{m^2_t},$$

$$\delta_{am} = \frac{3}{4} \frac{Q^2}{Q^2 q} a_m + a_s - \frac{1}{4} Q^2 a a_s,$$

$$\delta^1_{am} = 1.40923 + \left( \frac{44}{675} - \frac{2}{135} \ln \frac{s}{m^2_t} \right) \frac{s}{m^2_t},$$

$$\delta^2_{am} = \frac{3}{4} \frac{Q^2}{Q^2 q} a_m + a_s - \frac{1}{4} Q^2 a a_s.$$
\[ \delta_{vm} = 1 + 8.7 \, a_s + 45.15 \, a_s^2, \]
\[ \delta_{am}^1 = 1 + \frac{11}{3} \, a_s + \left( 11.286 + \ln \frac{s}{m_i^2} \right) a_s^2, \]
\[ \delta_{am}^2 = \frac{8}{81} - \frac{1}{54} \ln \frac{s}{m_i^2}, \]  
(57)

and

\[ a_s = \frac{\alpha_s(s)}{\pi}, \quad a = \frac{\alpha(s)}{\pi}, \]
\[ T^{(2)}(x) = -\frac{37}{12} + l + \frac{7}{81} x + 0.0132 \, x^2, \]
\[ T^{(3)}(x) = -\frac{5651}{216} + \frac{8}{3} + \frac{23}{36} \pi^2 + \zeta(3) + \frac{67}{18} l + \frac{23}{12} l^2, \]  
(58)

where \( l = \ln x, \) \( m_i = m_{i\text{pole}} \) (and \( s = M_z^2 \) for \( Z \) decay).

Note that we have automatically absorbed the finite mass terms into the QCD correction factors and that in \( R_A \) the large logarithms \( \ln(m_i^2/s) \) have been absorbed through the use of running parameters

\[ \bar{m}(s) = \bar{m}(m^2) \exp \left\{ - \int_{a_s(m^2)}^{a_s(s)} dx \, \frac{\gamma_m(x)}{\beta(x)} \right\}, \]
\[ m = \bar{m}(m^2) \left[ 1 + \frac{4}{3} a_s(m) + K a_s^2(m) \right], \]  
(59)

where \( m = m_{\text{pole}} \) and \( K_b \approx 12.4, \) \( K_c \approx 13.3. \) The various codes may differ in the construction of \( g_V^f \) and \( g_A^f, \) but a general consensus has been reached for final state radiation.

In this way we obtain the hadronic and total \( Z \) width as

\[ \Gamma_h = \sum_{f=udcsb} \Gamma(Z \rightarrow f\bar{f}) + 4 \Gamma_0 N_c^f R_h^\nu, \]
\[ R_h^\nu = -0.41318 \left( \sum_q v_q \right)^2 a_s^3, \]
\[ \Gamma_z = \Gamma_h + \sum_{f=e\mu\tau} \Gamma_f + \Gamma_{\text{inv}}. \]  
(60)

Actually, the hadronic width and the partial \( q\bar{q} \) widths deserve some partial comment, while the complete theoretical framework is described in [45]. The singlet QCD contribution, which is simple and unambiguous for the hadronic width, becomes ambiguous, starting at \( \mathcal{O}(\alpha_s^2), \) for individual \( q\bar{q} \) channels. In fact, some sort of agreement has recently been reached on these matters but we want to summarize the roots of the problem. From a pragmatic point of view there is a hierarchical description where
\[ \Gamma_q = \Gamma [Z \to q\bar{q}(g) + q'\bar{q}'] \]  

(61)

for all \( q' \), such that \( m_{q'} < m_q \). However, we could have a democratic description where the final states \( q\bar{q} + q'\bar{q} \) are assigned half to \( \Gamma_q \) and half to \( \Gamma_{q'} \). The two descriptions agree — fortunately for the leading terms. In general, one could also decide that such final states should not be assigned to any specific channels in such a way that

\[ \sum_q \Gamma_q \neq \Gamma_h = \sum_q \Gamma(Z \to q\bar{q}) + \sum_{q,q'} \Gamma(Z \to q\bar{q}'+q'\bar{q}) + \ldots \]  

(62)

In particular, the \( \mathcal{O}(\alpha_s^3) \) contribution to \( \Gamma_V^S \) cannot be assigned to any specific flavour. In our approach the \( \mathcal{I}^{[2,3]} \) correction terms are included according to isospin — i.e., proportional to \( I_q^{[3]} \) — and therefore cancel in the hadronic width while summing over the first four flavours \( u, d, c, s \). However, it must be pointed out that no general consensus has been reached on this particular point, so that the prediction for \( \Gamma_u, \Gamma_d, \Gamma_c \) may differ from one code to the other, although the total hadronic width will remain the same.

1.10 Electroweak corrections: Basic notions

Above, we introduced a universal language describing the general features of radiative corrections, the language of effective couplings. At a secondary level, each computational scheme, being a particular representation of the same general concept, may use different building blocks, which, however, are so deeply related that one could even attempt the construction of some sort of ‘Rosetta stone’. Since one of our main motivations was to build up a theoretical framework that includes some estimate of its own uncertainty, we must necessarily spend some time in discussing some of the specific building blocks.

Indeed, even though there is a high degree of universality in the realization of the theoretical uncertainty, in practice the way in which this realization becomes effective is strictly related to the actual implementation of higher-order radiative corrections into the various codes. In order to discuss the numerous effects involved in the calculations we have to introduce additional notions.

1.10.1 Comparison of notations of different codes

As explained earlier, all electroweak loop effects in \( Z \) boson decays are concealed in the effective couplings \( g_V^f \) and \( g_A^f \). The treatment of these couplings, unlike that of the radiation factors \( R^f_{V,A} \), differs from one code to another (having several different realizations). In order to concentrate on these realizations let us for a moment forget about radiative factors \( R^f_{V,A} \). Then

\[ \Gamma_f = 4 N_c \Gamma_0 [(g_V^f)^2 + (g_A^f)^2] . \]  

(63)

The five codes deal with three main realizations.

In BHM/WH, ZFITTER the following notations are introduced:

\[ \rho_f = \frac{1}{1 - \delta \rho_f} = 4 (g_A^f)^2 , \]  

(64)

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\[ \frac{g_v^f}{g_A^f} = 1 - 4|Q_f|s_w^2 \kappa_f, \quad (65) \]

where
\[ \kappa_f = 1 + \delta \kappa_f, \quad (66) \]

so that Eq. (63) may be rewritten in the form
\[ \Gamma_f = \Gamma_0 N_c^f \rho_f [4(I_f^3 - 2Q_fs_w^2 \kappa_f)^2 + 1]. \quad (67) \]

Here the electroweak corrections affect \( \delta \rho_f \), \( \delta \kappa_f \) and \( s_w \). They will be discussed in the next subsection.

In TOPAZ0,
\[ \Gamma_f = 4 \Gamma_0 N_c^f \rho_z \left[ \left(I_f^3 - 2Q_fs^2 + \delta g_v^f\right)^2 + \left(I_f^3 + \delta g_A^f\right)^2 \right], \quad (68) \]

where \( s^2 \) is defined in (47), \( \rho_z \) in (227) and \( \delta g_v^f, \delta g_A^f \) — see (91) — contain (part of) the bosonic corrections as well as the vertex corrections.

In LEPTOP,
\[ g_A^f = I_f^3 \left[ 1 + \frac{3}{32\pi s^2 c^2} \tilde{V}_A^f \right], \quad (69) \]
\[ \frac{g_v^f}{g_A^f} = 1 - 4|Q_f|s^2 + \frac{3|Q_f|}{4\pi(c^2 - s^2)} \tilde{V}_R^f, \quad (70) \]

and \( V_{A,R} \) are simple functions of \((m_t/M_z)^2\) and \((M_H/M_z)^2\), as described in Subsection 4.1.

1.10.2 Basic notions of different codes

Here we will concentrate on a specific realization of the language of effective couplings in different codes. In particular it is important to clarify the notion of leading, non-leading and eventually of remainder terms in BHM/WH, TOPAZ0 and ZFITTER.

We start our presentation by considering two well known objects, \( \Delta r \) (which is used by BHM, TOPAZ0, ZFITTER) and the partial \( Z \) width. There are many notions which are common to both these objects.

The quantity \( \Delta r \) is nothing but the effective coupling of the \( \mu \)-decay, when it is being implemented with higher order corrections — see Eqs. (38) and (39). In this case there is only one effective coupling, as the \( \mu \) decay is a purely weak process mediated by a charged current. For the partial \( Z \) width, the situation is more involved, since we have a decay which is mediated by the neutral current. That is why it is described by two effective electroweak couplings. The introduction of \( \Delta r \) will be very useful in clarifying the notion of leading and of remainder contributions to radiative corrections. However, from a general point of view this is only one of the many specific realizations of the effective coupling in the \( \mu \) decay.
As an example of the leading remainder splitting we consider $\rho_c$ and $\Delta r$ (39, 40), $\rho_f$ and $\delta \rho_f$ (64), and $\kappa_f$ and $\delta \kappa_f$ (65, 66) — the electroweak corrections to $\Delta r$ and to $\Gamma_f$. Next we subdivide $\rho_c$ as introduced in Eq. (39) and $\rho_f$ and $\kappa_f$ of (64, 65) into a leading term, $\Delta_L$, and remainder, $\Delta_{\text{rem}}$, terms as follows:

$$\rho_c = \frac{1}{1 - \Delta r} = \frac{1}{1 - \Delta r_L - \Delta r_{\text{rem}}},$$  \hspace{1cm} (71)

$$\rho_f = \frac{1}{1 - \delta \rho_f} = \frac{1}{1 - \Delta \rho - \Delta \rho_{f,\text{rem}}},$$  \hspace{1cm} (72)

$$\kappa_f = 1 + \delta \kappa_f = (1 + \Delta \kappa_{f,\text{rem}}) \left( 1 + \frac{c_w^2}{s_w^2} \Delta \rho_x \right).$$  \hspace{1cm} (73)

From Eqs. (71–73) one may notice that each coupling contains some universal, flavour-independent piece ($\Delta \alpha$, $\Delta r_L$, $\Delta \rho$, $\Delta \rho_x$), and some flavour-dependent remainder. The former comprises the leading terms ($\Delta \alpha$, $\Delta \rho$) re-summed to all orders in accordance with the renormalization group equation for $\Delta \alpha$ and with the proper inclusion of leading irreducible terms for $\Delta \rho$ [46]. The latter are normally small, since all potentially large contributions have already been subtracted and shifted to the leading terms. For this reason, they are placed freely either into the denominators, (71), (72), or as a factor of the leading contribution (73); in doing so we follow Ref. [47]. In one case however, namely Eq. (71), there exist arguments in favour of keeping the remainder in the denominator. In Ref. [48] it was proved that the right-hand side of Eq. (71) actually means, on the two-loop level at least, all fermionic mass singularities are located exclusively in $\Delta \alpha$. This may be argued by a partial expansion of a simplified case ($\Delta \rho = 0$) up to the two-loop order terms,

$$\rho_c \approx \frac{1}{1 - \Delta \alpha - \Delta r_{\text{rem}}} \approx \frac{1}{(1 - \Delta \alpha)} \left( 1 + \frac{\Delta r_{\text{rem}}}{1 - \Delta \alpha} \right),$$  \hspace{1cm} (74)

from which one can see that the scale of $\Delta r_{\text{rem}}$ is actually $\alpha(M_Z)$.

One should emphasize that since there is no $\Delta \alpha$ in Eqs. (72) and (73) and since the non-leading terms of the order $\Delta_L \Delta_{\text{rem}}$ are unknown, no arguments in favour of putting $\Delta_{\text{rem}}$ as it is done in (72) and (73) can be presented. The size of these uncontrolled terms should be treated as an intrinsic theoretical error.

All the remainder terms have a similar structure:

$$\Delta r_{\text{rem}} = \Delta r^{1\text{loop},\alpha} + \Delta r^{2\text{loop},\alpha\alpha_s} + \frac{c_w^2}{s_w^2} \Delta \tilde{\rho}_x - \Delta \alpha,$$  \hspace{1cm} (75)

$$\Delta \rho_{f,\text{rem}} = \Delta \rho_f^{1\text{loop},\alpha} + \Delta \rho_f^{2\text{loop},\alpha\alpha_s} - \Delta \tilde{\rho},$$  \hspace{1cm} (76)

$$\Delta \kappa_{f,\text{rem}} = \Delta \kappa_f^{1\text{loop},\alpha} + \Delta \kappa_f^{2\text{loop},\alpha\alpha_s} - \frac{c_w^2}{s_w^2} \Delta \tilde{\rho}_x.$$  \hspace{1cm} (77)
They contain all the terms known at present: the complete one-loop \(O(\alpha)\) corrections (two-, three-, four-point functions) and complete two-loop \(O(\alpha\alpha_s)\) insertions to two-point functions, from which the leading \(O(\alpha)\) and \(O(\alpha\alpha_s)\) terms are subtracted:

\[
\Delta \tilde{\rho}_{(x)} = \Delta \tilde{\rho}^\alpha + \Delta \tilde{\rho}^{\alpha\alpha_s} + (\bar{X}) = \frac{3\alpha}{16\pi s_w^2 c_w^2 M_Z^2} \left[ 1 - \frac{2}{3} \left( 1 + \frac{\pi^2}{3} \right) \frac{\alpha_s(m_t^2)}{\pi} \right] + (\bar{X}), \quad (78)
\]

where braces in (78) and below mean that this expression describes simultaneously both quantities \(\Delta \tilde{\rho}\) and \(\Delta \tilde{\rho}_x\), which appeared in Eqs. (75)–(77).

The term \(\bar{X}\) in (78) is a next-to-leading order term, whose proper treatment is rather important:

\[
\bar{X} = \text{Re} \left[ \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_W(M_W^2)}{M_W^2} \right]^{\text{1loop}}_{\overline{\text{MS}}} - \Delta \rho^\alpha. \quad (79)
\]

In Ref. [49] it was argued that in the on-mass-shell renormalization scheme this term should be re-summed together with \(\Delta \tilde{\rho}^\alpha\), since in the \(\overline{\text{MS}}\) scheme it appears to be automatically incorporated. Similar arguments in favour of such a resummation were presented in Ref. [50]. In \(\bar{X}\) the UV divergences are removed according to the \(\overline{\text{MS}}\) renormalization scheme with \(\mu = M_Z\). The separation of \(\bar{X}\) is not unique; it makes the resummation dependent on the renormalization procedure.

The leading contribution \(\Delta \rho\) is built out of the same terms as (78). But they are normalized by \(G_\mu\) rather than by \(\alpha/s_w^2 M_W^2\), as is required by the resummation proposed in Ref. [46]:

\[
\Delta \rho_{(x)} = \Delta \rho^\alpha + \Delta \rho^{\alpha^2} + \Delta \rho^{\alpha\alpha_s} + \Delta \rho^\alpha\alpha_s^2 + (X) = N_c x_t \left[ 1 + x_t \Delta \rho^{[2]} \left( \frac{m_t^2}{M_H^2} \right) + c_1 \frac{\alpha_s(m_t^2)}{\pi} + c_2 \left( \frac{\alpha_s(m_t^2)}{\pi} \right)^2 \right] + (X), \quad (80)
\]

where

\[
x_t = \frac{G_\mu m_t^2}{\sqrt{2} 8\pi^2}, \quad (81)
\]

\[
X = 2 s_w^2 \frac{G_\mu M_Z^2}{\sqrt{2} \pi \alpha} \bar{X}, \quad (82)
\]

and \(x_t\) is the Veltman heavy top factor [51]. The coefficients \(c_1\) and \(c_2\) describe the first- and second-order QCD corrections for the leading \(x_t\) contribution to \(\Delta \rho\), calculated in Refs. [52] and [53]. Correspondingly:

\[
c_1 = -\frac{2}{3} \left( 1 + \frac{\pi^2}{3} \right), \quad (83)
\]

\[
c_2 = -\pi^2 (2.155165 - 0.180981 n_f). \quad (84)
\]

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The function $\Delta \rho^{(2)}(m_t^2/M_H^2)$ describes the leading second order $x_t$ (two-loop electroweak) correction to $\Delta \rho$, calculated first in the $M_H = 0$ approximation in Ref. [54] and later in Ref. [55] for an arbitrary relation between $M_H$ and $m_t$.

The partial decay width $Z \rightarrow b\bar{b}$ contains an additional $m_t$ dependence due to vertex diagrams (see Fig. 10).

\[ m^2_t = \frac{M^2}{2} W \]

Figure 10: Top quark exchange diagrams which contribute to $\Gamma_b$ (unitary gauge).

As a consequence, the effective couplings $\rho_b$ and $\kappa_b$ contain additional leading terms of the order $O(G^2 m_t^2)$.

The complete one-loop approximation for the $Z \rightarrow b\bar{b}$ partial width was calculated in Ref. [56]. We first redefine remainder terms by an additional subtraction of the leading one-loop term originating from these diagrams:

\[
\begin{align*}
\Delta \rho_{b,\text{rem}} &\rightarrow \Delta \rho_{b,\text{rem}} - 2\Delta \tilde{\rho}_b, \\
\Delta \kappa_{b,\text{rem}} &\rightarrow \Delta \kappa_{b,\text{rem}} + \Delta \tilde{\rho}_b,
\end{align*}
\]

where

\[
\Delta \tilde{\rho}_b = \frac{\alpha}{8\pi s_w^2} \frac{m_t^2}{M_w^2}.
\]

Following Refs. [55] and [57], the two-loop order QCD and electroweak leading terms in the $Zb\bar{b}$ vertex are implemented by an additional re-definition of effective couplings $\rho_b$ and $\kappa_b$:

\[
\begin{align*}
\rho_b &\rightarrow \rho_b (1 + \tau_b)^2, \\
\kappa_b &\rightarrow \frac{\kappa_b}{1 + \tau_b},
\end{align*}
\]

where $\tau_b$ is given by

\[
\tau_b = -2x_t \left[ 1 - \frac{\pi}{3} \alpha_s(m_t) + x_t \tau^{(2)} \left( \frac{m_t^2}{M_H^2} \right) \right].
\]

A compact analytic representation for the two-loop functions $\rho^{(2)}$ and $\tau^{(2)}$ was also given in Ref. [58].

We have just discussed what is known in the literature about the treatment of the corrections of the order $O(G^2 m_t^2)$. The $Zb\bar{b}$ vertex also contains a logarithmically enhanced
term, $O[\alpha \ln(m_t^2/M^2_{W})]$, whose contribution is comparable to the leading one. Recently, QCD corrections were also calculated for this term (see contribution by Kwiatkowski and Steinhauser in the QCD Part of this Report). This correction, however, can be nearly completely absorbed into the final-state QCD corrections. What remains is approximately one hundred times less than the QCD correction for the leading term as given in (89).

What has been presented so far in this subsection, is to an extent inherent in all five codes. However, in order to reach a better understanding of our set of theoretical predictions and of the strategy adopted to extract their related intrinsic errors we have to devote more time to discussing other realizations of the effective coupling language. In TOPAZ — see Eq. (68) — the partial $Z$-widths read:

$$\Gamma_f = 4 \Gamma_0 N_c f \rho_\pi \left[ \left( I_f^{(2)} - 2 Q_f \hat{s}^2 + \delta g_v^f \right)^2 + \left( I_f^{(3)} + \delta g_A^f \right)^2 \right], \quad (90)$$

where $\hat{s}^2$, defined by (47) and (229) (see also Ref. [31]), and is to be put in partial correspondence with the $s^2 \kappa_f$ of Eq. (65). Moreover, $\delta g_v^f, \delta g_A^f$ contain bosonic corrections as well as vertex corrections. Again, the basic point under examination is the leading remainder splitting. The main idea beyond this realization is to write a system of equations that connect the $\overline{MS}$ parameters of the theory in terms of $\alpha, G_\mu$ and $M_\pi$. These equations contain the effects of radiative corrections up to a certain order in perturbation theory and must be solved consistently and respecting gauge invariance. As a result of this procedure we end up in Eq. (90) with a $\rho_\pi$ defined according to Eq. (227). This parameter, which properly re-sums the gauge invariant fermionic corrections, contains all isospin breaking terms and includes, beyond $O(\alpha)$, all the presently known higher-order terms — the $O(\alpha^2)$ [55, 58], $O(\alpha \alpha_s)$ [52] and $O(\alpha^2)$ [53] corrections.

The bare weak mixing angle, according to the previous strategy, will always be expanded around $\hat{s}$, Eq. (229), where $\rho_\pi^a$ from Eqs. (232) and (233) is used. These two quantities ($\rho_\pi^a$ and $\rho_\pi$) may differ whenever a resummation of bosonic corrections is performed for the weak mixing angle. Once the divergent terms are treated in the $\overline{MS}$ framework, $s^2$ (the leading term) will receive a correction (the remainder) $\Delta s^2$. At this point, when the $Z$ wave function renormalization factor $\Pi_z$ is included, we usually expand all the remaining corrections up to first order — we decompose $\Pi_z$ into its leading, fermionic part and a remainder $\Delta \Pi_z$ — see Eq. (231). This $\Delta \Pi_z$ is always expanded. After inclusion of vertices and fermion wave-function renormalization factors we end up with

$$\delta g_v^f = \frac{\alpha}{4\pi} \left[ \frac{2F_v^f - \frac{1}{2} v_f \Delta \Pi_z}{c^2 s^2} - 2Q_f \Delta s^2 \right],$$

$$\delta g_A^f = \frac{\alpha}{4\pi} \left[ \frac{2F_A^f - \frac{1}{2} I_f^{(3)} \Delta \Pi_z}{c^2 s^2} \right], \quad (91)$$

where $s^2$ is again defined as (41) and $F_v^f, F_A^f$ are the flavour-dependent vertex corrections. Additional versions of this realization will be discussed in the framework of the theoretical options. Here we need only mention some of the problems connected to the resummation of bosonic contributions in this realization (as well as in others). It is strictly related to
the question of gauge invariance. As it is well known, the vertices and the bosonic parts of the vector boson self-energies are not separately gauge invariant. On the one hand there is no rigorous procedure for re-summing the vertices, and on the other hand any attempt to isolate the gauge variant parts and to throw them away is not unambiguously defined. Even when we identify the universal contributions from vertices and boxes (for instance by working in the $R_\xi$ gauge), to be combined with the vector boson self-energies in order to get the one-loop gauge invariant dressed propagators, we still have the freedom to shift from one part (to be re-summed) to the some other process-independent, $\xi$-independent, UV finite function of $q^2$ with the proper asymptotic behaviour. Strict enforcement of gauge invariance is one of the roots of Eq. (91), in that $\Delta \Pi_\rho$ is not absorbed into the $\rho_\rho$ parameter and that particular care must be devoted to the proper definition of the remainder, $\Delta s^2$, when resummation of irreducible terms is performed. This fact accounts for one of the many structural differences with other realizations.

In LEPTOP, due to the proper choice of the $\bar{a}$-Born approximation, the electroweak radiative corrections turned out to be small — for $m_t$ around 175 GeV. The smallness of corrections is a result of the mutual cancellation of different equally important terms. Therefore, functions $V_{A,R}^\phi$ representing loop corrections are not subdivided into leading and remainder terms and no resummation is performed.
2 Options, theoretical uncertainties

During the completion of this work it has become increasingly evident that there is a need to quantify the effect of our partial lack of knowledge of the missing higher-order terms in radiative corrections. Thus we have introduced the notion of option, which refers to a set of possible and plausible alternative implementations of the full machinery of radiative corrections within a given renormalization scheme. In order to explain the treatment of higher-order terms and the interplay between pure electroweak and QCD corrections we must once more remember that the effective coupling description of the $Z$ width has several different realizations.

Quite independent of specific details, all the realizations single out two main components in each observable:

$$O = O_B + \Delta O.$$  \hspace{1cm} (92)

The term $O_B$, giving the bulk of the answer, is often called the Born approximation (the $\bar{\alpha}$-Born approximation in LEPTOP, or improved Born approximation in BHM/WH, TOPAZ0 and ZFITTER), or the leading contribution to $O$. The term $\Delta O$ represents a small perturbative correction, often called remainder or non-leading contribution. Different realizations usually have different ways of performing this splitting so that, while they agree at the $O(\alpha)$, there are differences which start at $O(\alpha^2)$.

Independent of the particular realization of $g_V, g_A$, these effective couplings are complex valued functions, due to the imaginary parts of the diagrams. This, however, will have some relevance only for realistic distributions while for pseudo-observables they are taken to be strictly real. Given that the universal language has in practice many alternative realizations and that the actual implementation of the options is very much code-dependent, we have not created a common set-up for the electroweak options. From this simple fact follows the need to discuss at length the physical motivations behind our options. Although we have tried, as much as possible, to illustrate them from a general perspective, it is also evident that some space has been left to analyze certain specific issues. A description of the implementation of the options in the various codes, with the relative flags, will be given in Section 5.

2.1 Factorization of QCD Corrections

No matter which realization we are using, one problem naturally emerges when final-state QCD corrections are switched on. This problem is connected with the folding of the non-corrected widths using the QCD factors $R^{fA}_{V,A}$. One can take the point of view that non-universal and flavour-dependent couplings should also be folded or, to the contrary, that only the universal effective couplings should multiply the QCD radiation terms. This option merely reflects our ignorance of the mixed $O(\alpha \alpha_s)$ corrections, with the noticeable exception of the $b\bar{b}$ partial width where the leading $m_t$ part of these corrections is actually known [57]. If we consider the $b\bar{b}$ partial width we can write
\[ \Gamma_b^0 = \frac{G_F M_Z^2}{2\sqrt{2}\pi} \left[ (g_v^b)^2 + (g_a^b)^2 \right], \]
\[ \Gamma_{\text{rw}}^b = \Gamma_{\text{rw}}^b(Z \rightarrow \bar{b}b) \approx \Gamma_b^0 \left[ 1 - 4x_t \frac{(g_v^b + g_a^b) g_a^b}{(g_v^b)^2 + (g_a^b)^2} \left( 1 + 3 - \frac{\alpha_s}{\pi} \frac{\alpha_s}{\pi} \right) \right]. \tag{93} \]

At this point naïve factorization would imply
\[ \Gamma_b = \Gamma_{\text{rw}}^b \left( 1 + \frac{\alpha_s}{\pi} \right), \tag{94} \]
while the computed FTJR term [57] gives
\[ \Gamma_b = \Gamma_b^0 \left[ 1 + \frac{\alpha_s}{\pi} - 4x_t \frac{(g_v^b + g_a^b) g_a^b}{(g_v^b)^2 + (g_a^b)^2} \left( 1 + \frac{\pi^2}{3} \frac{\alpha_s}{\pi} \right) \right], \tag{95} \]
so that the correct QCD coefficient in front of the heavy top factor turns out to be \(-2.290 \alpha_s/\pi\) instead of \(\alpha_s/\pi\). Of course, an approximate but factorized expression is still possible and can be written as
\[ \Gamma_b = \Gamma_b^0 \left[ 1 - 4x_t \frac{(g_v^b + g_a^b) g_a^b}{(g_v^b)^2 + (g_a^b)^2} \left( 1 - \frac{\pi^2}{3} \frac{\alpha_s}{\pi} \right) \right] \left( 1 + \frac{\alpha_s}{\pi} \right), \tag{96} \]
where the uncertainty has now moved to order \(\alpha_s^2 G_\mu m_t^2\). Actually, we have not yet specified the scale of \(\alpha_s\) in the previous equations but these corrections have been implemented such that the universal QCD factor is computed as \(1 + \alpha_s(M_z)/\pi\), while the specific FTJR term is computed with \(\alpha_s(m_t)\). As a consequence the \(1 + \alpha_s(M_z)/\pi\) factor is not included for the \(b\)-quark asymmetries and for the effective \(b\)-quark mixing angle. In general, however, the \(\mathcal{O}(\alpha\alpha_s)\) corrections for the \(Zf\bar{f}\) vertex are presently unknown and we must accept an intrinsic uncertainty associated with the two procedures — i.e., factorization or non-factorization of electroweak and QCD corrections. Once again, if \(\Gamma_0^q\) is the (improved) Born \(q\bar{q}\) partial width and
\[ \Gamma_{\text{rw}}^q = \Gamma_0^q \left( 1 + \delta_{\text{univ}}^q \right) + \Delta_{\text{rw}}^q, \tag{97} \]
then
\[ \frac{\Delta_{\text{rw}}^q}{\Gamma_0^q} \frac{\alpha_s(M_z)}{\pi} \tag{98} \]
is roughly assumed as the corresponding uncertainty. This type of uncertainty can be illustrated very well by asking how correct it is to shrink an electroweak blob to a point before allowing for QCD radiation.
2.2 Genuinely Weak Uncertainties

In this section we briefly discuss the main ingredients which enter the pure weak corrections to the pseudo-observables — the resummation of the one-particle irreducible vector boson self-energies, the scale in vertex corrections and the linearization of the corresponding S-matrix elements. We have avoided any intensive usage of the lengthy formulae introduced in the first part of this Report and summarized in some detail in Section 5. Indeed, there is a simple set of problems in the implementation of radiative corrections which is inherent to perturbation theory and does not depend on any specific approach. Only formulae belonging to different realizations represent the technical transcription of these implementations. Real progress is usually achieved whenever a new term becomes available, otherwise we are left with heuristic arguments or with ingenious attempts to improve upon perturbation theory. Different paths along this road represent our present degree of inaccuracy and to understand these differences already gives a hint on how to proceed. The structural and logical essence of these differences can be explained without a massive use of equations.

2.2.1 Leading Remainder splitting

Before we move to discuss the next source of uncertainty we must recall again that, generally speaking, the effective couplings \( g^v_f, g^a_f \) contain a leading and usually re-summed part and a non-leading (remnant) one, quite independent of the specific realization. For instance, in one of the realizations the building blocks are \( \rho_f, \kappa_f \), while in another they are \( \rho_z, s^2 \) and \( \delta^v_f, \delta^a_f \), and in a third they are \( s^2, V_A \) and \( V_R \). The way in which the non-leading terms can be treated and the exact form of the leading remainder splitting give rise to several possible options in the actual implementation of radiative corrections that in turn become another source of theoretical uncertainty. For instance, for all the objects \( \Delta r, \Delta \rho_f, \Delta \kappa_f \), we can introduce the decomposition into leading and remainder. Since we know how to proceed with all objects in the leading approximation the only ambiguity is due to the treatment of the remainders. Clearly, after the splitting of \( \Delta r = \Delta r_L + \Delta r_{\text{rem}} \) there are in principle several possible ways of handling the remainder \(^9\):

\[
\frac{1}{1 - \Delta r} = \frac{1}{1 - \Delta r_L - \Delta r_{\text{rem}}} = \left\{ \begin{array}{l}
\frac{1}{1 - \Delta r_L - \Delta r_{\text{rem}}} \\
\frac{1}{1 - \Delta r_L} \left( 1 + \frac{\Delta r_{\text{rem}}}{\Delta r_L} \right) \\
\frac{1}{1 - \Delta r_L} + \Delta r_{\text{rem}}.
\end{array} \right. \tag{99}
\]

\(^9\)In the case of \( \Delta r \), the last two expansions are indeed not valid — see a discussion around Eq. (74). We nevertheless present them here for the sake of illustration.
Actually, these options differ among themselves but the difference can be related to the choice of the scale in the remainder term. A complete evaluation of the sub-leading $O(G^2 M_Z^2 m_t^2)$ would greatly reduce the associated uncertainty. At the moment these sub-leading terms are simulated just by varying the scale in the remainder. Typical choices are described in Section 5 and may vary from realization to realization. A different approach to the problem [3] will be illustrated in Subsection 2.2.5.

### 2.2.2 Scale in vertex corrections

Another possible option, which is realized by some of the codes (but not all), has to do with the scale of $\alpha$ in the non-leading corrections, in particular the vertex corrections. To make this point clear we use one of the realizations of the effective couplings and analyse the $Z \rightarrow b\bar{b}$ decay in some detail. The essential ingredient will be $\Delta \rho_b$, which we write as

\[
\Delta \rho_b = \Delta \rho_d + \delta \rho_b (m_t) \\
\delta \rho_b (m_t) = - \frac{G_\mu m_t^2}{2\sqrt{2}\pi^2} \left[ 1 + O(G_\mu m_t^2) + O(\alpha_s) \right] + O\left( \ln m_t^2 \right) + \delta \rho_b^{NL},
\]

where $\delta \rho_b$ is a correction specific to the $b\bar{b}$ channel. The question naturally arises as to what to use for the scale in $\delta^{NL}$ and in the sub-leading logarithmic term $-G_\mu \alpha (M_Z)$ or $\alpha(0)$. Obviously the same kind of option will be present in the vertex corrections for light fermions where the expansion parameter is formally $\alpha/(4\pi \sin^2 \theta \cos^2 \theta)$. Different ways to interpret $\alpha$ in these expressions give rise to different results, from $\alpha(0)/(4\pi \sin^2 \theta \cos^2 \theta)$ to $G_\mu M_Z^2/(2\sqrt{2}\pi^2)$. The difference between possible identifications of coupling constants in the $O(\alpha)$ corrections represents, of course, effects of $O(\alpha^2)$. The fact that the neutral current amplitude is automatically expressed in terms of $G_\mu M_Z^2$ is a possible heuristic argument to adopt the same strategy in the evaluation of the presently known $O(\alpha)$ corrections, but in order to be on the safe side, the differences should be considered as a theoretical uncertainty, at least according to many authors. For instance, we know from a specific calculation [41] that for inclusive quantities the final state QED radiation is controlled by $\alpha(M_Z)$ and not by $\alpha(0)$. The option of variation of the scale of non-leading corrections has been implemented by the majority of the codes.

### 2.2.3 Linearization

Another example to be discussed is the following. In almost any realization we have the possibility of using an expanded versus a non-expanded option — to linearize our expressions. To give a partial illustration of this possibility we use the realization of Ref. [4] where we have

\[
\sin^2 \theta_{\text{eff}} = \frac{1}{4} \left( 1 - \frac{g_{\nu}^l}{g_A^l} \right) = \begin{cases} \\
\hat{s}^2 + \frac{1}{2} \hat{g}_{\nu}^l + \frac{1}{2} (4 \hat{s}^2 - 1) \delta g_A^l & \text{expanded} \\
\hat{s}^2 + \frac{1}{2} (\hat{g}_{\nu} - \hat{g}_A) & \frac{1}{1 - 2\delta g_A^l} & \text{non-expanded}. \end{cases}
\]
On the same basis, the $\delta g^2$, $\delta g^4$ terms in each partial width will be expanded up to first order in the \textit{expanded} option. More generally, the difference between the two options in the evaluation of $O^2$, where $O$ is given by Eq. (92), is equal to $(\Delta O)^2$, a two-loop reducible but non-leading contribution, and from this point of view we clearly do not need a specific exemplification. By comparing the two options we obtain a rough estimate of the importance of the missing non-leading two-loop effects. It goes almost without saying that when this option is implemented on top of the leading remainder splitting and of the choice $G_\mu$ versus $\alpha(0)$ we can have rather different behaviours in the global theoretical error according to the size of the remainder. Thus whenever we need to quantify the effect then the remainder must be considered in detail and the effect becomes realization-dependent. This has some relevance for the de-convoluted leptonic forward–backward asymmetry, where the leading term is itself small due to accidental cancellations. To summarize we can say that the theoretical error on $A_{\mu}$ is very sensitive to the sum of various factors:

- \textit{expansion versus non-expansion}
- definition of the \textit{leading} part of $\sin^2 \theta_{\text{eff}}$
- scale of $\alpha$ for the non-leading terms.

A further exemplification of what we call option for radiative corrections, in part connected to a selection of the scale in the coupling but generalizable to all parameters appearing in the remainder, can be illustrated as follows. Suppose that a given quantity $A$, function of the parameter $a$, is given in perturbation theory by the following expansion:

\begin{align*}
A &= a + g \left[ a^2 + f_1(a) \right] + g^2 \left[ a^3 + f_2(a) \right] + \mathcal{O}(g^3) \\
&= \tilde{a} + g f_1(a) + \mathcal{O}(g^2) \\
\tilde{a} &= a / (1 - ga) , \quad (101)
\end{align*}

and that only the $f_1$ term is actually known. It could be decided that $\tilde{a}$ is the effective expansion parameter (or that in the full expression we change variable $a \rightarrow \tilde{a}$), which of course in the truncated expansion introduces the option,

\begin{align*}
A &= \tilde{a} + g f_1(a) \\
&= \tilde{a} + g f_1(\tilde{a}) , \quad (102)
\end{align*}

so that $\Delta A = g^2 a^2 f_1(a)$ would be our estimate of the associated theoretical uncertainty. Sometimes it is only a rough estimate, since there is no guarantee that the irreducible terms $g^2 f_2$ are essentially small in size.

### 2.2.4 Resummation

Another source of theoretical uncertainty is connected with the treatment of the physical Higgs contribution. As we know for large Higgs masses ($M_\mu \gg M_\nu$) there is a correction term in $\Delta \rho$ which is only logarithmic, in contrast to the heavy fermion case, the so-called Veltman's screening effect [59]. With respect to this correction $\Delta \rho_\mu$ we can:
• expand it to first order in $\alpha$ as is sometimes done for all bosonic corrections;

• re-sum the leading part of it, $\Delta\rho_H$, for relatively large values of $M_H$, for example, $M_H > M_W \exp 5/12$;

• re-sum in $\rho$ the whole physical Higgs contribution. This requires a further comment, since this term is not UV finite by itself and therefore the resummation procedure must be understood strictly in the $\overline{MS}$ scheme with a scale $\mu = M_Z$.

As an additional and rather general comment, which somehow collects many of the previous considerations, we stress once more that there are different ways of implementing the resummation of the vector boson self-energies. These choices, which in turn are deeply related to the proper definition of remainder, differ from code to code — at least in their default settings. We have already illustrated the splitting $\Delta r_L - \Delta r_{\text{rem}}$ and simply add a few additional considerations. Resummation is very often the main recipe for separating a small remainder from the bulk of the effect.

• One choice consists in a resummation which includes the square of the $Z - \gamma$ mixing-term [2, 5] with the option of strictly keeping in the resummation only the one-loop irreducible terms. The default thus corresponds to an additional mass counter-term which enters the field and coupling renormalization constants and modifies the quantity $\Delta r$. The resummation of the modified $\Delta r$ leads automatically to the factorization property which takes into account the proper summation of all the leading higher-order reducible terms.

• Even more generally we can distinguish among complete expansion of the one-loop self-energies, partial resummation of fermionic self-energies or partial inclusion of bosonic self-energies in the resummation. There are two considerations to be made at this point. Sometimes accidental cancellations occur among the fermionic and the bosonic sectors, which would suggest a similar treatment for both; however, the bosonic sector is not gauge invariant by itself. Thus any resummation of bosonic self-energy parts must properly identify some numerically relevant but gauge invariant sub-set.

For completeness we recall that one of these identifications gives in the $\xi = 1$ gauge [60]

\[
S_{\gamma\gamma}(p^2) = S_{\gamma\gamma}(p^2)|_{\xi=1} - 4e^2 p^2 I(p^2),
\]

\[
S_{Z\gamma}(p^2) = S_{Z\gamma}(p^2)|_{\xi=1} - 2e^2 \frac{\cos \theta}{\sin \theta} \left( 2p^2 - M^2 \right) I(p^2),
\]

\[
S_{ZZ}(p^2) = S_{ZZ}(p^2)|_{\xi=1} - 4e^2 \frac{\cos^2 \theta}{\sin^2 \theta} \left( p^2 - M^2 \right) I(p^2),
\]

where $I(p^2) = -B_0(p^2, M^2_Z, M^2_W)/(16\pi^2)$, $B_0$ is a scalar form factor [22] and $S$ denotes a possible identification of the gauge invariant part of $S$.

To utter an additional word on electroweak uncertainties and to understand the implications of some of the procedures used as a possible estimator of the theoretical error
we consider a fictitious quantity $X$ defined as $X = 1 - 4 \sin^2 \theta_{\text{eff}}$. Each code will define some effective Born approximation to $\sin^2 \theta_{\text{eff}}$ and here we distinguish between three basic possibilities: complete expansion of the corrections (E), a resummation which only includes the fermionic self-energies (FR), or a global resummation (R). We refer here to the complete set of formulae given in Section 4 and simply quote the adopted strategies:

- **BHM/WOH** adopted as the default of the resummation of the entire set of self-energies, including the $Z - \gamma$ mixing term;
- **LEPTOP**, which expands all those contributions not re-absorbed in the running of $\alpha$ into $\bar{\alpha}$;
- **TOPAZ**'s default, which re-sums in the $\overline{MS}$ framework the $\Sigma_R$ term of Eq. (233), while an option is left in which only its fermionic content is re-summed;
- **ZFITTER**'s default, which also re-sums in the $\overline{MS}$ framework the $X$ term of Eq. (82), allowing as options the resummations of the leading terms only.

Let us define $X_0$ as the effective Born approximation of $X$ and define a non-leading (remainder) part, $X = X_0 + [\alpha(M_Z)/\pi]X_1 + \mathcal{O}(\alpha^2)$. The three different procedures, E or FR or R, will find $X_0 = 0.07528, 0.08672, 0.07184$ (for $m_t = 175 \text{GeV}$, $M_H = 300 \text{GeV}$ and $\alpha_s = 0.125$). In fact, there is no unique way for global resummation (R), so that the last number can vary a little (say, from 0.07184 to 0.07416). Using the corresponding value for $\sin^2 \theta_{\text{eff}}$ we get $X_1 = -1.344, -5.976, -0.016$. Whenever we compute $X^2$ and use the square of the remainder to estimate part of the uncertainty (there are other options around) we are bound to see rather different behaviours, depending on the adopted leading remainder splitting. It should be clear that the remainder itself is subject to several possible options, from the scale of the coupling to the choice of those terms which are to be considered small and perturbative. These options themselves contribute, sometimes sizably, to the final uncertainty, quite independently from linearization (expansion) and in the end every adopted procedure is somehow equivalent to a proper choice of the scale. Thus a preferred set of options is equivalent, in some sense, to an ideal optimization of the perturbative expansion.

### 2.2.5 Estimate of the missing terms in higher orders

While some of the realizations of the effective coupling description make use of a (partial) resummation of higher-order terms, thus trying to improve upon ordinary perturbation theory, we have other realizations where the mixing angle is defined only in terms of $G_F$, $M_Z$ and of the running constant $\alpha(M_Z)$ while everything else is strictly expanded. Thus, a Born approximation is defined in terms of an $s^2$, the definition of which was given in Eq. (42) and the realization is constructed in terms of the one-loop approximation with respect to the genuinely electroweak interactions. As has became clear from the previous discussion, there is no common set of implemented procedures for describing the theoretical uncertainties but only some rather general guidelines. Certainly, when we move to a concrete implementation, it happens that every code has its own internal ways of estimating the missing higher-order effects. Codes which do not foresee, for theoretical
reasons, the possibility of expanding the remainders with respect to the leading terms or of playing with the choice of the scale in the remainders, introduce another set of options based on a different estimate of the not-yet-calculated diagrams or terms in a given diagram. The basic and alternative idea is that each of the Born relations (43–45) will receive a genuinely electroweak correction proportional to $\delta V_i$, as given by Eq. (180). The main point is related to the recent observation [61] that the sub-leading two-loop corrections to the vector boson self-energies, of order $O(G_\mu^2 m_t^4)$, could be numerically close to the leading ones, of $O(G_\mu^2 m_t^2)$. Thus the latter can be considered as an estimate of the uncertainties in the $V_i$. At the same level the $O(\alpha^2 m_t^2)$ corrections, to be discussed in the next section, can also be used as an estimate of the uncertainty. In order to have the correct asymptotic behaviour of the uncertainties for $m_t \gg M_z$, it is assumed that these universal corrections are multiplied by a factor $2/t = 2 M_z^2/m_t^2$. Indeed, $O(t^2)$ is completely under control, although the sub-leading $O(t)$ is not and thereby $1/t$ follows with the usual safety factor of two. Thus if the leading higher-loop corrections calculated in Refs. [55] and [53] are denoted respectively by

$$\delta V_i^{1^2}, \quad \delta V_i^{2^2},$$

then the corresponding estimates of the missing terms are assumed to be

$$\Delta V = (2/t) \delta V, \quad \text{with} \quad t = \frac{m_t^2}{M_z^2}. \quad (105)$$

There is also a specific correction to the $Z \rightarrow b\bar{b}$ vertex dependent on $M_H^2/m_t^2$, which is presently computed up to $O(G_\mu^2 m_t^4)$ term [55], while the sub-leading terms are still unknown. If we denote it by $\delta \phi^2$ — see the second term in Eq. (220) — then an upper bound on the ‘Higgs theoretical uncertainty’ can be estimated as

$$\Delta \Gamma_b = -\frac{\alpha(M_Z)}{\pi} \Gamma_0 \delta \phi^2. \quad (106)$$

For $m_t = 175$ GeV and $M_H = 300$ GeV this amounts to 0.02 MeV, which is much smaller than other uncertainties and could therefore be neglected.

### 2.3 QCD Corrections on Electroweak Loops

The effect of QCD corrections is not confined to the final-state radiation or the $O(\alpha\alpha_s)$ vertex corrections to $Z \rightarrow b\bar{b}$ but it will influence all vector boson self-energies through virtual gluon exchanges within quark-loop insertions. All codes include these two-loop diagrams [52] by first decomposing the $WW, ZZ, Z\gamma$, and $\gamma\gamma$ self-energies into two basic building blocks, $\Pi_v(m_i, m_j)$ and $\Pi_A(m_i, m_j)$, which are given by the expansion

$$\Pi_{v,A} = N_c \left( \Pi_v^{(1)} + \frac{\alpha_s}{\pi} C_F \Pi_v^{(2)} + \ldots \right), \quad (107)$$

with $C_F = (N_c^2 - 1)/ (2 N_c)$. For instance for each isodoublet,
\[
\gamma - \gamma \rightarrow e^2 \sum_{i=1}^{2} Q_i^2 \Pi_\nu \left( p^2, m_i, m_i \right),
\]

\[
Z - Z \rightarrow e^2 \sum_{i=1}^{2} \left[ \nu_i^2 \Pi_\nu \left( p^2, m_i, m_i \right) + a_i^2 \Pi_A \left( p^2, m_i, m_i \right) \right].
\]

(108)

In the limit where we neglect light quark masses, four different cases have to be considered:

\[
\Pi_\nu(m_t, m_t), \quad \Pi_A(m_t, m_t), \quad \Pi_{\nu,A}(m_t, 0), \quad \Pi_{\nu,A}(0, 0).
\]

(109)

An important issue is related to the renormalization scale to choose for \( \alpha_s \). The default that we have adopted is to select \( \mu = m_t \) for contributions from the \( t - b \) doublet while \( \mu = M_Z \) is assumed for light quark contributions. A practical difference emerges in the various implementations of \( \Pi^{(2)} \), where sometimes the full expression is used, while in other cases a Taylor expanded (in \( q^2/m_t^2 \)) version has been used. A step forward has been made with the evaluation of the \( O(\alpha_s^2) \) corrections to \( \Delta \rho \), the AFMT term [53]. Given the usual definition of \( \Delta \rho \) as

\[
\rho = \frac{1}{1 - \Delta \rho}, \quad \Delta \rho = N_c s_t \left( 1 + \delta^{\text{ew}} + \delta^{\text{QCD}} \right),
\]

(110)

we have the QCD contribution to \( \Delta \rho \), up to three loops and in the heavy top limit, as

\[
\Delta \rho^{\text{QCD}} = N_c s_t \delta^{\text{QCD}} = N_c s_t a_s \left( \delta_2^{\text{QCD}} + a_s \delta_3^{\text{QCD}} \right),
\]

(111)

with \( a_s = \frac{\alpha_s}{\pi} \). As stated, this correction has been computed in the heavy top limit and therefore only the leading part of \( \delta_3^{\text{QCD}} \) is available. Actually, this new calculation makes the QCD corrections to \( \Delta \rho \) much more stable with respect to the renormalization scale, since we now have

\[
\delta^{\text{QCD}}(\mu) \approx -0.910 \alpha_s(m_t) - 1.069 \alpha_s^2(m_t) + 2.609 \alpha_s^3(m_t) \ln \left( \frac{\mu^2}{m_t^2} \right) + O(\alpha_s^4),
\]

(112)

where \( \alpha_s(m_t) \) is evaluated with five flavours and the \( n_f \) that appears in the final AFMT result is interpreted as the total number of flavours contributing in the Feynman diagrams — \( n_f = 6 \). Recently the same result has been cast [62] into a slightly different form by use of the notion that the corrections to \( \Delta \rho \) in terms of \( m_t \) are almost entirely contained in \( \dot{m}^2(m_t)/m_t^2 \); \( \dot{m}(m_t) \) being the running mass evaluated at the pole mass. The corresponding differences in \( \delta^{\text{QCD}} \) amount to \( \approx 5\% \) of the total QCD correction. Concerning the
treatment of the AFMT term it should be noted that there is, at present, some disagreement on the value for $n_f$ — for instance, LEPTOP uses $n_f = 5$ on the basis of decoupling of heavy fermions in vector theories, leading to a 2% change in $\delta_{QCD}^\rho$.

A strictly related topic concerns the inclusion and the magnitude of the $t\bar{t}$ threshold effects on $\Delta \rho$ and therefore on all the electroweak parameters. These effects have recently been estimated by both dispersion relation and perturbative methods [64]. However, an uncertainty remains as to their magnitude (see Ref. [65] for a detailed discussion). Most of the QCD corrections to $\Delta \rho$ beyond the leading-order QCD terms can be discussed and evaluated by absorbing them into the $O(\alpha_s)$ term computed with an adjusted scale $\mu = \xi m_t$:

$$\Delta \rho_{QCD} = N_c x_t \left[ 1 - \frac{2}{3} \left( 1 + \frac{\pi^2}{3} \right) \frac{\alpha_s(\xi m_t)}{\pi} \right] + \Delta \rho_{NL}^{QCD}, \quad (113)$$

where $\Delta \rho_{NL}^{QCD}$ takes into account the non-leading top effect with the scale set to $\mu = m_t$, as well as the light quark effects with $\mu = M_Z$. For instance the result of Ref. [62] can be summarized by saying that it corresponds to a very high accuracy, to a scale, $\xi = 0.321^{+0.110}_{-0.073}$.

Incidentally, the original AFMT formulation corresponds in this language to a scale $\xi = 0.444$. We note that the absorption of the non-leading QCD terms into a rescaling of $\alpha_s$ has only been performed, so far, at the level of the leading $O(\alpha_s^2)$ term, even if we have at our disposal the full $O(\alpha_s^2)$ correction factor.

A comment is in order here. Our original idea was to incorporate the $t\bar{t}$ threshold effects through an opportune rescaling factor $\xi$. However, the intrinsic theoretical error on $\xi$ deriving from the threshold analysis is very difficult to assess, since non-relativistic approximations also play a certain role. In view of the present situation, the majority of the codes has agreed with a specific strategy:

- The default is represented by computing the pseudo-observables to include the complete three-loop AFMT term at a scale of $\xi = 1$, Eq. (112), which is equivalent to the use of Eq. (113) with $\xi = 0.444$.

- In order to estimate the size of the non-leading QCD effects, the $\delta_{QCD}^\rho$ correction factor has been implemented according to the formulation of Ref. [62], with a scale which gives the maximum variation with respect to the AFMT term — $\xi = 0.248$ — and the difference between this and the AFMT calculation is used as an estimate of the corresponding uncertainty. It is certainly not the ideal solution, but we cannot rely on other analyses not incorporating the $O(\alpha_s^2)$ term. In this context a further subtlety arises, since the effective scale for the AFMT term ($\xi = 0.444$) gives a correction outside the range required by the present Green function analysis [65].

---

[64] A recent and independent evaluation of the QCD corrections to the $\rho$-parameter has been presented by K.G. Chetyrkin, J.H. Kühn and M. Steinhauser [63], showing disagreement with the original AFMT result. Meanwhile, a revised version of the AFMT calculation has appeared in hep-ph (16.2.1995) showing agreement with the CKS result. For an understanding of the effect of this we have added a Note in proof at the end of this main part of the Report.
In fact, the result of Ref. [62] and the corresponding error estimate, Eq. (114), also leave the AFMT expansion at the edge or even a bit outside the error range.

2.4 Parametric Uncertainties

The parametric uncertainties are those related to the input parameters, $\alpha(M_z)$, $G_\mu$ and the masses, $M_Z, m_b$ etc. Among them the largest uncertainty comes from $\alpha(M_Z)$: $\alpha^{-1}(M_Z) = 128.87 \pm 0.12$ (see, however, the new results [9-11]), while the relative uncertainty in $M_Z$ is an order of magnitude smaller and that in $G_\mu$ is 50 times smaller. For the $b$ quark mass we used $4.7 \pm 0.3\text{GeV}$ as a sort of conservative average. We have tried to compare the effects of variations in the input parameters in a way which, hopefully, will also give useful information in the future, providing some sort of evolution of the uncertainties as a function of the errors in the input parameters. Assuming independence from the actual central values of $\alpha(M_Z)$ and of $m_b$ we computed the derivatives of the pseudo-observables with respect to $\alpha(M_Z), m_b$. Assuming also that the errors are small enough and that the dependence is therefore approximately linear, this result will allow us to give the uncertainties, even when the input parameters or the errors on them change with time. Actually, a linear approximation is good enough for the derivative with respect to $\tilde{\alpha}^{-1}$ but not with respect to $m_b$. In order not to have problems with the latter near a local extremum we have defined a maximum derivative given by

$$\mathcal{D}f = \frac{\text{sign}(f') \max(\delta_h, \delta_l)}{\Delta x},$$

$$\delta_h = |\max(f) - \bar{f}|,$$

$$\delta_l = |\bar{f} - \min(f)|,$$

(115)

where $\bar{f}$ is the corresponding central value. In order to avoid lengthy tables we have computed this set of derivatives for our pseudo-observables at the standard reference point, where $m_t = 175\text{GeV}$, $M_H = 300\text{GeV}$ and $\alpha_s(M_Z) = 0.125$. The results are shown in Tables 1 and 2 for BHM, LEPTOP, TOPAZ0 and ZFITTER.

2.5 Structure of the Comparisons

Having introduced the main ingredients of the calculations we are now ready to explain in more detail the structure of the comparison. As already pointed out, we focus on 25 pseudo-observables and vary $m_t, M_H$ and $\alpha_s(M_Z)$ in a given range of values. It is worth mentioning that we could introduce at this point the notion of an adapted set-up, in the sense that prior to the introduction of options we have made an effort to show that different codes running under as similar as possible conditions give very close answers.

In fact, each code will produce a set of results according to some well specified preferred set-up relative to the various options briefly discussed. As far as the external building blocks are concerned — QCD and QED correction factors — some effort has been made in order to reach a common default.
• The QED final state radiation for the partial widths is computed at the scale $M_z$ [41].

• The FTJR correction [57] has been split into a universal QCD factor computed at a scale $\mu = M_z$ and an internal correction factor computed at the scale $\mu = m_t$. The former is not included for the $b$-quark asymmetries (effective mixing angle $\sin^2 \theta_{\text{eff}}$).

• The AFMT three-loop effect [53] with $n_f = 6$ is included in the default with a scale $\mu = m_t$.

• The $\mathcal{O}(\alpha_\alpha_s)$ vector boson self-energies ($\Pi^{(2)}$) are included (almost always the full expression) with two scales: $\mu = m_t$ for the $t - b$ doublet and $\mu = M_z$ for the light quarks.

• The introduction of an effective scale $\xi m_t$ is not part of the preferred set-up but rather is included in the uncertainty.

This procedure for estimating the size of the non-leading QCD effects has been implemented in all codes but LEPTOP, which uses the $(2/t) \times \text{AFMT}$ term as an estimator of the uncertainty.

In this way the preferred set-up of a code refers to a specific choice of the electroweak options, everything of course embedded in a given choice of the renormalization scheme. The result of this procedure is given by five sets of predictions for the 25 pseudo-observables as functions of $m_t$, $M_H$ and $\alpha_s(M_z)$. On top of our predictions, each group has adapted the various codes to run under all its options (typically up to $2^5 \div 2^6$). It must be clearly realized that no common set of options has been created and that the options of one code have been designed independent of the options of all other codes. In the end, the $2^n$ electroweak options are folded with 2 QCD options, the inclusion of AFMT being the default versus a rescaling $m_t \rightarrow 0.248 m_t$ in the $\mathcal{O}(\alpha_\alpha_s)$ term representing the uncertainty. LEPTOP, however, uses a different procedure (see above). Note, that electroweak–QCD factorization simulates $\mathcal{O}(\alpha_\alpha_s)$ non-controlled terms, therefore it can also be considered a QCD option. For each pseudo-observable $O$ we have collected $O_{\text{adapt}}$ and $O_+, O_-$, i.e.

\[
O_+ = \max_i O_i \\
O_- = \min_i O_i
\]

where the index $i$ is running over the options. The differences $O_+ - O_{\text{adapt}}$ and $O_{\text{adapt}} - O_-$ calculated by a given code are our internal estimates of the theoretical uncertainty associated with $O$, while the different results for $O_{\text{adapt}}$ as obtained by the various codes may be considered as giving our estimation of the scheme dependence. Here internal must be understood as the estimate that one particular code can produce by varying internally its options on the implementation of radiative corrections. The corresponding error bars on the theoretical predictions are in some cases very asymmetric, merely reflecting the specific ideas or, even better, personal taste, beyond the choice of preferred set-up. Clearly the way in which the different realizations have been built into the codes is very indicative.
of the original strategy. To give an example, we may note that the reason why some code does not include the one-loop corrected axial-vector coupling of the Z to fermions in the definition of the $\rho$ parameter is because its preferred set-up is the expanded solution. In some cases, the theoretical uncertainty internally estimated by one code could turn out to be large with respect to those of other codes. Basically, we do not attribute any particular relevance to this fact, as the global indication should include some sort of average among the codes. In the general discussion of the results we have tried as much as possible to trace the roots of the phenomenon, whenever it appears, and any further consideration should be left to the potential users of our analysis. Each point in the error bars has the same content of probability and the width of the theoretical bands associated with each pseudo-observable should be taken as an indication of the relevance of that quantity for the analysis of the LEP data. Sometimes the theoretical error, which cannot be reduced unless we come up with a full two-loop calculation, comes to be of the same order as the present or projected experimental error. Even before entering into a full discussion of the results we may anticipate the predictions for $\sin^2 \theta_{\text{eff}}^l$, by choosing some reference point, such as $m_t = 175$ GeV, $M_H = 300$ GeV and $\alpha_s(M_Z) = 0.125$. The prediction is

$$\sin^2 \theta_{\text{eff}}^l = \begin{cases} 
0.23197^{+0.00004}_{-0.00007} & \text{BHM} \\
0.23206^{+0.00008}_{-0.00008} & \text{LEPTOP} \\
0.23200^{+0.00004}_{-0.00004} & \text{TOPAZO} \\
0.23194^{+0.00003}_{-0.00007} & \text{WOH} \\
0.23203^{+0.00004}_{-0.00014} & \text{ZFITTER} 
\end{cases}$$

The corresponding width for the theoretical band is therefore 0.00011, 0.00016, 0.00008, 0.00010 or 0.00018 for BHM, LEPTOP, TOPAZO, WOH and ZFITTER. This can be compared with what we obtain by combining the LEP results on all the asymmetries: $\sin^2 \theta_{\text{eff}}^l = 0.2321 \pm 0.0004$ [15, 16].

In order to present our results we have focused on the $W$ mass, on the primary set of pseudo-observables chosen by the LEP collaborations for fitting, $\Gamma_z$, $R_l$, $A_{\text{FP}}^l$, and on the $b$- and $c$-quark related charge asymmetries and ratios of partial widths.

### 2.6 Experimental data and theoretical predictions

To set the scene for our discussion we must first introduce, in Table 3, the relevant experimental data. Clearly a detailed discussion of our results should take into account the whole range of values for the input parameters, but many of the relevant conclusions can already be drawn by considering Tables 4–7, where we have reported $M_W$ [37] and a sample of 11 quantities as measured by the LEP collaborations. For comparison we have fixed a reference point, $m_t = 175$ GeV, $M_H = 300$ GeV and $\alpha_s = 0.125$, and reported the predictions from BHM/LEPTOP/TOPAZO/WOH/ZFITTER, including the estimated theoretical
errors and the average. The content of the tables should not be confused with a fit but should only be taken as a first introduction to the theoretical results. Our reference point is a mere consequence of the range indicated by CDF for the top quark mass and of the most recent prediction for $\alpha_s(M_Z)$ from LEP. As far as $M_H$ is concerned, we must admit a high degree of arbitrariness. As we have detailed, the differences among the (theoretical) central values for each quantity are basically (even though not totally) a measure of the effect induced by a variation in the renormalization scheme. As can be seen from Tables 4–7 the ratios of the maximal half-differences among the five codes over the experimental errors are:

\[
\begin{align*}
\Delta_c(M_w) &= 2.5 \times 10^{-2} \\
\Delta_c(\Gamma_e) &= 6.7 \times 10^{-2} \\
\Delta_c(\Gamma_x) &= 3.9 \times 10^{-2} \\
\Delta_c(R_l) &= 1.0 \times 10^{-1} \\
\Delta_c(R_b) &= 3.3 \times 10^{-2} \\
\Delta_c(R_c) &= 2.0 \times 10^{-3} \\
\Delta_c(\sin^2 \theta_{\text{eff}}) &= 1.4 \times 10^{-1} \\
\Delta_c(A_{\text{PB}}^l) &= 5.6 \times 10^{-2} \\
\Delta_c(A_{\text{PB}}^b) &= 7.8 \times 10^{-2} \\
\Delta_c(A_{\text{PB}}^c) &= 2.5 \times 10^{-2} .
\end{align*}
\]

Another piece of information is obtained by looking at the theoretical uncertainties as estimated internally by each code. A very conservative attitude would be to report the half-difference between the global maxima and minima among the five codes. By considering again the ratio with the experimental errors we obtain

\[
\begin{align*}
\Delta_g(M_w) &= 7.2 \times 10^{-2} \\
\Delta_g(\Gamma_e) &= 1.6 \times 10^{-1} \\
\Delta_g(\Gamma_x) &= 3.7 \times 10^{-1} \\
\Delta_g(R_l) &= 2.3 \times 10^{-1} \\
\Delta_g(R_b) &= 8.0 \times 10^{-2} \\
\Delta_g(R_c) &= 4.1 \times 10^{-2} \\
\Delta_g(\sin^2 \theta_{\text{eff}}) &= 2.8 \times 10^{-1} \\
\Delta_g(A_{\text{PB}}^l) &= 1.2 \times 10^{-1} \\
\Delta_g(A_{\text{PB}}^b) &= 1.6 \times 10^{-1} \\
\Delta_g(A_{\text{PB}}^c) &= 5.2 \times 10^{-2} .
\end{align*}
\]

By comparing the various $\Delta_c$ and $\Delta_g$ we can obtain a rough evaluation of the global theoretical error associated with the most relevant quantities in the analysis of the LEP data. As far as the differences among codes in their preferred (adapted) set-up is concerned,
we can safely conclude that the ratios of their predictions to the experimental errors are usually less than 0.15. However, the most important message to be derived from this simple exercise is that very often the theoretical uncertainty can be larger than what is to be expected on the basis of a simple comparison of the results from different calculations.

A wider sample of results is shown in Figs. 11–23, where, again, we have fixed some reference point — $M_{11} = 300$ GeV and $\alpha_s(M_{12}) = 0.125$ and where $100$ GeV $< m_t < 250$ GeV. In every figure the corresponding experimental points (data), as they are given in Ref. [15], are shown at $m_t = 178$ GeV. The full collection of our results refers instead to $M_{11} = 60, 300, 1000$ GeV and to $\alpha_s(M_{12}) = 0.118, 0.125, 0.132$.

Here we discuss the main features of the comparison and for each quantity we indulge in presenting an estimate of the global uncertainty by roughly considering the half-difference between the maximum and minimum among all predictions. Admittedly this is not a very rigorous procedure and therefore it should be treated with due caution. It should be stressed again that the $\Delta_c, \Delta_g$ factors presented above show the half-difference of the predictions over the experimental errors. In the following discussion we will mainly analyze two quantities: $d_c, d_g$ — the half-differences, either among central predictions or between the maximum and the minimum among all predictions. It should be noted that we will not address in this section the question of how the transformation from primordial distributions to the secondary quantities will affect their precision. In the following we have examined some of the pseudo-observables at the standard reference point and tried to present the state of the art for their theoretical predictions without including all sorts of parametric uncertainties; rather we have limited our discussion to the genuinely theoretical ones. Below we discuss 13 pseudo-observables.

- $M_w$
  There is a certain spread in the predictions for increasing values of $m_t$ substantially independent of $\alpha_s$ and increasing for large $M_{11}$, with the formation of two clusters, represented by BHM/WOH and ZFITTER on one side and LEPTOP/TOPAZO on the other. The maximum difference in central values is seen for $m_t = 250$ GeV where $d_c$ reaches a 10.5(12) MeV for $M_{11} = 300(1000)$ GeV, made even more significant by the additional fact that the bands essentially do not overlap. The situation improves for central values of $m_t$, where we see at most a half-difference of $\approx 4.5$ MeV even if the clustering is already evident.

- $\Gamma_c$
  No pattern of any particular relevance is observed. Almost independent of $\alpha_s$ we notice that for intermediate $m_t$ there is a substantial agreement for all $M_{11}$, while for high $m_t$ the agreement is better at large values of $M_{11}$, and for small $m_t$ it improves for low values of $M_{11}$. All the error bars tend to become wider for increasing $m_t$ and $M_{11}$, with the possible exception of TOPAZO/ZFITTER. The largest half-difference among central values is for low $M_{11}$ and high $m_t$ where $d_c$ can reach $\approx 0.03$ MeV. A safe estimate of the overall theoretical error at $m_t = 175$ GeV is of about 0.03 MeV ($0.030 < d_g < 0.031$ for the full range of $\alpha_s$ and $M_{11}$).

\[11\] A preliminary version of the comparison among our results has been presented in Ref. [66]
• $\Gamma_z$
Due to the final state QCD corrections $\Gamma_z$ is much more sensible to variations in $\alpha_s$. However, we have verified that there is no substantial variation among the codes as a function of $\alpha_s$, a sign that final state QCD corrections are well under control. For instance we find $1.4\,\text{MeV} < d_c < 1.5\,\text{MeV}$ for $0.118 < \alpha_s < 0.132$ at $m_t = 175\,\text{GeV}$ and $M_H = 300\,\text{GeV}$\textsuperscript{12}. For low $m_t$ the result of the comparison does not show any particular pattern, while for high $m_t$ the agreement improves with high $M_H$, showing again some sort of correlation between the two variables. For intermediate $m_t$, instead we find smaller differences around central values of $M_H$. In general for $m_t > 150 \div 175\,\text{GeV}$ the various error bands, while growing, have a tendency to overlap. The maximum deviation among codes is for high $m_t$ and low $M_H$, where $d_c$ can reach approximately 0.6 MeV. For $m_t = 175\,\text{GeV} \; \alpha_s = 0.125$ instead we get, as the global estimate of the uncertainty, 1.3, 1.4 and 1.4 MeV for $M_H = 60, 300$ and 1000 GeV (with tiny variations in respect of $\alpha_s$).

• $R_l$
This quantity has a role of its own since quite often it is used for extracting a precise determination of $\alpha_s(M_Z)$. Indeed, up to some degree of accuracy, the two variables are related by $\Delta \alpha_s \approx \pi \Delta R/R$ so that a difference of 0.01 in $R_l$ is equivalent to an error of 0.002 in $\alpha_s$. There is a common trend in all our results for $R_l$, namely the BHM predictions always stay a little higher, while the other codes tend to cluster, apart from the TOPAZO results, which, for very low $m_t$ and high $M_H$ tend to converge towards those of BHM. For $m_t > 175\,\text{GeV}$ the error bands tend to overlap so that each code has a central prediction within the other error bands, again apart from the BHM point, which sometimes is fully contained only within the WOH predictions and lies at the upper boundary of the LEPTOP/ZFITTER ones. Error bars are often very asymmetric, especially for WOH/ZFITTER. The maximal deviation is for high $m_t$, $M_H$, where $d_c$ may reach 0.006, whereas the global estimate of the uncertainty for $m_t = 175\,\text{GeV}$ gives 0.0085, 0.0090 and 0.0095 for $\alpha_s(M_Z) = 0.118, 0.125$ and 0.132, independent of $M_H$. These values correspond to an error of 0.002 $\div$ 0.003 in the determination of $\alpha_s(M_Z)$. We have also analyzed in more detail the $\alpha_s$ dependence of the ratio $R_l$ for $m_t = 175\,\text{GeV}$ and $M_H = 300\,\text{GeV}$, including the case when QCD is switched off. Indeed, a determination of $\alpha_s$ from $R_l$ is usually achieved by writing $R_l = R_l(\alpha_s = 0)(1 + \delta_{\text{QCD}})$ and by using the most updated formulation of QCD corrections (see, for instance, Ref. [67]). In this way, some relevance should also be attributed to a comparison of various predictions for the ratio $R_l$, unfolded of QCD correction terms. For $\alpha_s = 0$ the WOH prediction is lower than the others, which cluster around 19.946 (WOH is $-0.78 \times 10^{-2}$ below the average). When QCD corrections become active, BHM remains higher ($0.5 \div 0.6 \times 10^{-2}$ above average) while the other codes form a cluster.

\textsuperscript{12}This is not a trivial consequence of the fact that all five codes use the same radiation functions, since their implementation is usually different.
- $R_b, R_c$
  The ratios of the $b\bar{b}$ and $c\bar{c}$ partial widths to the total hadronic width share some common features. The experimental errors are 0.0020 and 0.0098 and the two quantities are $-38\%$ correlated. Our central predictions for $R_b$ are all within $2 \times 10^{-4}$ and even the inclusion of the theoretical uncertainty only gives $d_g \approx 6 \times 10^{-4}$. The overall theoretical error at $m_t = 175$ GeV is $1.7 \times 10^{-4}$. For $R_c$ the global uncertainty is well contained within $6.0 \times 10^{-5}$. $R_b$ shows some clustering which becomes more and more evident for large $m_t$, with BHM giving the higher prediction and LEPTOP/TOPAZ/WH/ZFITTER forming a lower cluster. The behaviour of $d_g$ as a function of $\alpha_s$ is practically flat.

- $\sin^2 \theta_{\text{eff}}^{\text{lep}}$
  The reported value of the leptonic effective weak mixing angle is the average of all forward–backward and polarization asymmetries from LEP. Therefore the analysis relies on the hypothesis that the peak forward–backward asymmetry can simply be connected with the remaining asymmetries through the use of the same $\sin^2 \theta_{\text{eff}}$. For all values of $M_H$ and $\alpha_s$ the agreement is less satisfactory for low $m_t$ where ZFITTER remains on the higher side, BHM/WH form a lower cluster and LEPTOP/TOPAZ are somewhere in between. The general trend is to have a convergence of all codes for high $m_t$. The maximal half-difference among central values is as for low $m_t$, where for all $M_H, \alpha_s$ we find $d_c \approx 9.0 \times 10^{-5}$. For $m_t = 175$ GeV the overall uncertainty is estimated to be $d_g \approx 1.1 \div 1.4 \times 10^{-4}$ over the whole range $M_H - \alpha_s$. For $M_H = 300$ GeV and $\alpha_s = 0.125$ we find $d_g = 1.3, 1.1, 2.1 \times 10^{-4}$ for $m_t = 100, 175, 250$ GeV. In view of the supposed relevance of this parameter and of its projected experimental error we have to admit that the status of the theoretical predictions is far from satisfactory but totally related to the unknown higher order effects. To give an example, we could say that the knowledge of the sub-leading $O(G_F^2 M_Z^2 m_t^2)$ corrections to $\Delta r$ would greatly improve the situation — for instance for ZFITTER, which dominates the error.

- $\sin^2 \theta_{\text{eff}}$
  From many points of view the situation is very similar to that described for $\sin^2 \theta_{\text{eff}}^{\text{lep}}$. Let us remember that $\sin^2 \theta_{\text{eff}}$ differs from $\sin^2 \theta_{\text{eff}}^{\text{lep}}$ because of flavour-dependent corrections, which are $m_t$-dependent and not negligible. There is some kind of crossed behaviour in our predictions, with an agreement substantially better for intermediate values of $m_t$ and deteriorating at the boundaries. For low $m_t$ the comparison is similar to that for $\sin^2 \theta_{\text{eff}}$: a higher prediction from ZFITTER, a central cluster in LEPTOP/TOPAZ and a lower one for BHM/WH. For high $m_t$ the highest prediction is from BHM, with a lower cluster in the other codes. The global estimate of the uncertainty for $m_t = 175$ GeV and $\alpha_s = 0.125$ is of $1.15(1.15, 1.45) \times 10^{-4}$ for $M_H = 60(300, 1000)$ GeV, with an uncertainty $\pm 0.1 \times 10^{-4}$ due to a variation in $\alpha_s$.

- $A_{\nu \beta}^{\ell}$
  In presenting results for the leptonic forward–backward (peak) asymmetry, as well as for any other leptonic asymmetries, we follow the indication of the experimentalists who keep $A_{\nu \beta}^{\ell}$ until the end in their standard model fits, without necessarily
identifying it directly with $\sin^2 \theta_{\text{eff}}^b$. We first start by analyzing the comparison among the central values. Here the result of the comparison is rather good and essentially we always start at low $m_t$ with two clusters — a higher one containing BHM/WOH and a lower one containing TOPAZ/ZFITTER and with LEPTOP somewhere in between. There is a fast convergence of all results for increasing $m_t$, especially for high $M_H$. Typically for $m_t = 175\,\text{GeV}$ and $M_H = 300\,\text{GeV}$ we find a half-difference of about $8.4 \times 10^{-5}$ for $\alpha_s = 0.118(0.125, 0.132)$. When we come to the inclusion of the theoretical uncertainty it is immediately evident that the previous comment still applies and the global error for the standard reference point becomes $1.9 \pm 2.0 \times 10^{-4}$ at $\alpha_s = 0.118, 2.0 \div 2.2 \times 10^{-4}$ for $\alpha_s = 0.125$ and $2.0 \div 2.3 \times 10^{-4}$ for $\alpha_s = 0.132$, where the variation with $M_H$ is illustrated. We will come back to $A_{\text{FB}}^b$ and to the related uncertainty while discussing additional theoretical options which have not been included in the working set.

- $A_{\text{LR}}$
  Also for the left–right (peak) asymmetry there is some general behaviour in our predictions. At low $m_t$ we start with maximal disagreement, two clusters with BHM/WOH in the higher one and LEPTOP/TOPAZ in the central one, to reach convergence of results for high $m_t$. For low $m_t$ there is also a considerable spreading of the error bands. Given the fact that the SLD measurement seems to require a much higher value of $m_t$, we have considered the overall uncertainty for $m_t = 250\,\text{GeV}$ with the result of $1.82(1.59, 1.74) \times 10^{-3}$ for $M_H = 60(300, 1000)\,\text{GeV}$ for $\alpha_s = 0.118$. These values become $1.84(1.61, 1.74) \times 10^{-3}$ and $1.86(1.63, 1.77) \times 10^{-3}$ for $\alpha_s = 0.125$ and 0.132.

- $A_{\text{FB}}^b$, $A_{\text{FB}}^c$
  Practically everything we state for the general behaviour of $A_{\text{LR}}$ can be repeated for the $b$- and $c$-quark charge asymmetries. As for $A_{\text{LR}}$, the agreement is worst at low $m_t$ and there is a general convergence at high $M_H$ when $m_t$ is growing. For low $M_H$ and large $m_t$ TOPAZ has the tendency to give a higher prediction. The global uncertainty for $A_{\text{FB}}^b$ at $m_t = 175\,\text{GeV}$ is $6.2 \div 7.4 \times 10^{-4}$. There is no general agreement among codes on the definition of $A_{\text{FB}}^b$. In particular LEPTOP and TOPAZ0 include mass effects into the definition of this pseudo-observable. For the $b$ quark, the effect of its non-vanishing mass leads to:

$$A^b = \frac{2g^b_v g^b_A}{[\frac{1}{2}(3-v^2)(g^b_v)^2 + v^2(g^b_A)^2]^v}, \quad (119)$$

where $v$ is the $b$ quark velocity:

$$v = \sqrt{1 - \frac{4 \tilde{m}_b(M_z^2)}{M_z^2}}, \quad (120)$$

and where $\tilde{m}_b(M_z^2)$ is the running $b$-mass in $\overline{\text{MS}}$-scheme defined by Eq. (59) below. However this is a very tiny effect. An estimate of it, based on TOPAZ0 results, gives
To summarize our results we have also presented in Table 8 the largest half-differences between central values or between maximal and minimal predictions among codes in the range $150 \text{ GeV} < m_t < 200 \text{ GeV}$, $60 \text{ GeV} < M_H < 1 \text{ TeV}$ and $0.118 < \alpha_s < 0.125$. In Tables 9 and 10 we have a fixed $m_t = 175 \text{ GeV}$ and have illustrated the largest half-differences among central values or those between the maximum of the maxima and the minimum of the minima among the five codes in two situations: $\alpha_s = 0.125$ fixed and $60 \text{ GeV} < M_H < 1 \text{ TeV}$, or $M_H = 300 \text{ GeV}$ and $0.118 < \alpha_s < 0.125$. In this way the separate contributions from $M_H$ or $\alpha_s$ to the theoretical errors are indicatively given. A final but partial indication of our results can be provided by computing some of the quantities which usually enter the SM fits (at $m_t = 175 \text{ GeV}$) and by collecting all available sources of error:

\begin{align}
\text{BHM} & \quad \Gamma_z = 2497.4^{+0.9}_{-1.0}(t.h.)^{+7.9}_{-8.8}(M_H, \alpha_s) \pm 6.7 \Delta \tilde{\alpha}^{-1} \pm 0.8 \Delta m_b \text{ MeV} \\
\text{LEPTOP} & \quad \Gamma_z = 2497.2 \pm 1.1(t.h.)^{+8.0}_{-8.5}(M_H, \alpha_s) \pm 6.8 \Delta \tilde{\alpha}^{-1} \pm 0.8 \Delta m_b \text{ MeV} \\
\text{TOPAZO} & \quad \Gamma_z = 2497.4^{+0.9}_{-0.9}(t.h.)^{+8.2}_{-9.1}(M_H, \alpha_s) \pm 6.8 \Delta \tilde{\alpha}^{-1} \pm 0.8 \Delta m_b \text{ MeV} \\
\text{WOH} & \quad \Gamma_z = 2497.4^{+1.5}_{-1.0}(t.h.)^{+8.0}_{-8.5}(M_H, \alpha_s) \text{ MeV} \\
\text{ZFITTER} & \quad \Gamma_z = 2497.5^{+0.6}_{-0.5}(t.h.)^{+7.9}_{-8.7}(M_H, \alpha_s) \pm 6.8 \Delta \tilde{\alpha}^{-1} \pm 0.8 \Delta m_b \text{ MeV} (121)
\end{align}

\begin{align}
\text{BHM} & \quad \sigma_h^b = 41.436^{+0.006}_{-0.003}(t.h.) \pm 0.042(M_H, \alpha_s) \pm 0.013 \Delta \tilde{\alpha}^{-1} \pm 0.007 \Delta m_b \text{ nb} \\
\text{LEPTOP} & \quad \sigma_h^b = 41.439^{+0.003}_{-0.004}(t.h.) \pm 0.040(M_H, \alpha_s) \pm 0.012 \Delta \tilde{\alpha}^{-1} \pm 0.008 \Delta m_b \text{ nb} \\
\text{TOPAZO} & \quad \sigma_h^b = 41.437^{+0.007}_{-0.002}(t.h.)^{+0.041}_{-0.040}(M_H, \alpha_s) \pm 0.012 \Delta \tilde{\alpha}^{-1} \pm 0.007 \Delta m_b \text{ nb} \\
\text{WOH} & \quad \sigma_h^b = 41.449^{+0.002}_{-0.003}(t.h.)^{+0.041}_{-0.040}(M_H, \alpha_s) \text{ nb} \\
\text{ZFITTER} & \quad \sigma_h^b = 41.441^{+0.000}_{-0.005}(t.h.)^{+0.041}_{-0.040}(M_H, \alpha_s) \pm 0.013 \Delta \tilde{\alpha}^{-1} \pm 0.007 \Delta m_b \text{ nb} (122)
\end{align}
where we have allowed, as usual, $60\text{ GeV} < M_H < 1\text{ TeV}$ and $0.118 < \alpha_s < 0.132$. Whenever available, the parametric uncertainties have been inserted. It is assumed that $\Delta m_b$ is given in GeV.

2.7 More on Theoretical Uncertainties

There are some options on electroweak radiative corrections which, although implemented in the codes, have not been used as working options for producing our comparisons with the experimental data. The main argument for this exclusion is related to their tendency to produce results deviating sensibly from the average. As a matter of fact, this is the visible consequence of their inclusion, but quite often we have theoretical reasons against them. This short section is devoted to a summary of those effects and of their eventual influence on the comparisons. At the end one should not forget that the design and the implementation of options is indeed something very peculiar to a given realization. The following short considerations give another illustration of a very important fact: theoretical uncertainties should be treated with due caution, realizing that they contain, in any case, a large degree of arbitrariness.

2.7.1 TOPAZ

With TOPAZ, the most striking effect is related to an expansion of the non-leading terms to $O(\alpha)$, once bosonic self-energies were not re-summed. Roughly speaking, we can assume that a certain quantity $X$ is given by (see also 2.2.4)

$$X = X_0 + \frac{\alpha}{\pi} X_1 + O(\alpha^2),$$

where $X_0$, by construction, will include all the re-summed contributions. As soon as we allow the two options,

$$X^2 = \begin{cases} 
(X_0 + \frac{\alpha}{\pi} X_1)^2, \\
X_0^2 + 2 \frac{\alpha}{\pi} X_0 X_1,
\end{cases}$$

and interpret the resulting difference as a theoretical uncertainty, then, under some circumstances, we end up with errors considerably larger than those presented above. The main problem is represented here by a clash between accidental cancellations and gauge
invariance (even the notion of cancellation between fermionic and bosonic sectors is gauge-dependent). As it is well known, the bosonic self-energies are gauge-dependent and, moreover, the fermionic ones tend to dominate away from the intermediate $m_t$ region. That is how TOPAZ0 allows, among other options, for a strict resummation of the fermionic self-energies alone. Of course, one could just avoid resummation altogether, but the option of expanding versus non-expanding the remainders, with respect to the leading terms, still applies and the theoretical uncertainties would become, in this case, sensibly $m_t$ dependent. Finally, we stress once more that the identification of a gauge invariant part of the bosonic self-energies is not at all a unique procedure and therefore some degree of arbitrariness is always hidden in a global resummation. We admit that this option allows for a nice construction of a small remainder, in a situation, however, where we are working with one-loop contributions and higher-order reducible ones. What is left out of our analysis is, in any case, related to the two-loop irreducible terms, about which nothing is presently known. To summarize we could say that in one case it is approximately the ‘square’ of the one-loop bosonic corrections that we use to estimate the theoretical error, while in the other we can decide to re-sum part of it into the leading term and make the remainder small, even though we are still missing information about not-yet-computed higher orders and their effects, which could make the smallness of the remainder inadequate.

The final reason why TOPAZ0 has excluded this option in presenting the pseudo-observable tables is therefore totally related to the abnormal (as compared to the other codes) size of the errors for some of the quantities, noticeably $A_{FB}$. Here we say again that the expansion option is bounded to produce larger errors whenever the remainders are not (one way or another) kept small and in some codes this effect is not seen, simply because nothing equivalent to the expansion has been implemented. To give a quick idea of the effect we present in Table 11 the shift in the central values and in the error bands for some of the pseudo-observables. Clearly the largest effect is seen for the leptonic forward-backward asymmetry where the theoretical error becomes comparable in size to the experimental counterpart. Needless to say, when this option is activated, a large fraction of the uncertainties become TOPAZ0 dominated. Additional consequences will be introduced and discussed in the next chapter.

### 2.7.2 ZFITTER

About ZFITTER, one can also mention several peculiar moments, related to the specific design of its options. We begin by making clear that nothing resembling what is described in the previous section was observed. To a large extent this is due to the fact that the coefficient $X_1$ (see discussion in 2.2.4) happened to be small in the framework of the ZFITTER renormalization scheme. Having accepted this statement, however, one should not conclude that it has particular advantages over the other schemes. The size of the coefficient is simply a numerical accident, without any deep physical meaning.

However, ZFITTER does contain additional options, which eventually were not left among its working options. An example is given by the array of expansions (99). All four expansions have been implemented, and it was noted that the third and fourth expansions enlarge the theoretical uncertainty for some observables ($M_w$ and $\sin^2 \theta_{\text{eff}}$), roughly by factors of two and four correspondingly. As it was pointed out in Subsection 1.10.2, the
third and the fourth expansions contradict the conclusion of Ref. [48] about fermionic mass singularities. It not surprising that this was the argument in favour of excluding these options from the working set.

Another interesting example is related to the leading remainder splitting problem. Three variants of the resummation of the leading terms in $\Delta \rho$, see Eq. (80), were implemented as different sub-options:

1) only the first term, i.e. $\Delta \rho^\alpha$, is re-summed;

2) all but $X$ terms of (80) — the content of square brackets is re-summed;

3) the whole expression (80) is re-summed (ZFITTER default).

There is a noticeable increase in the uncertainty when we include the second option with respect to a situation where only the first and the third are retained among the working options. One should emphasize, however, that this increase in the error is not dramatic: for example for $\sin^2 \theta^\text{eff}_L$ it amounts to a nearly $m_t$ independent uncertainty, of the order $5 \times 10^{-5}$. Examining the reasons for this enhancement, it was revealed that for the second option the remainder terms are about 5-10 times bigger compared to those for the first and the third options. Moreover, they are only several times smaller than the leading terms. On the basis of this observation, the second option was termed a pathological option and initially excluded from the working set, since its effect contradicted an accepted strategy.

In playing more with ZFITTER options a striking property was observed. Some of the ZFITTER options do not possess the additivity property. As an example, we mention that the ‘scale of remainder option’ alone produces nearly the same uncertainty as when it is applied ‘in conjunction’ with the following two other options:

- the one dealing with expansion (99), first two rows;
- the one dealing with the three above-mentioned variants of resummation for $\Delta \rho$: 1) $\div 3$).

As a consequence, the combined effect of all three options leads practically to the same uncertainty, and the latter is independent from the actual number of options included — two or three $\Delta \rho$ resummation sub-options are left in the working set. This eventually lead to a decision to retain all 1) $\div 3$) among working options.
3 Realistic distributions

To give predictions for pseudo-observables has not been our only task and we have also devoted a noticeable effort to presenting the most updated analysis for realistic observables. This process requires as fundamental ingredients a QED dresser and therefore the comparisons have been restricted to BHM, TOPAZ0 and ZFITTER, since they allow for the treatment of QED diagrams involving the emission of real photons with results which are dependent on energies and experimental cuts. Roughly speaking we can distinguish between s-channel processes and Bhabha scattering, \( e^+e^- \rightarrow e^+e^- \). For Bhabha scattering the commonly accepted procedure is the so-called t-channel subtraction, where the \( s-t \) and \( t-t \) contributions are subtracted from the data by using the code ALIBABA [68]. In the procedure the ‘most reasonable’ values of \( m_t \) and of \( M_H \) are used leading to additional sources of errors in the analysis. Given the present situation we have also performed a comparison between ALIBABA and TOPAZ0 in spite of the fact that the electroweak library of ALIBABA has not been constantly updated. Thus an intrinsic difference will emerge in the comparison due to the improved electroweak and QCD formulation of TOPAZ0. Our comparisons can be divided according to the following scheme:

- Fully extrapolated set-up for muonic and hadronic channel with the possible inclusion of a cut on the invariant mass of the final fermion pair, the so-called \( s' \) cut (BHM, TOPAZ0, ZFITTER);

- \( e^+e^- \rightarrow \mu^+\mu^- \), \( s \)-channel for \( 40^\circ < \theta_- < 140^\circ \), \( \theta^\text{max}_{\text{coll}} = 10^\circ, 25^\circ \), and \( E_{\text{th}}(\mu^\pm) = 20 \text{ GeV} \) (\( E_{\text{th}}(e^\pm) = 1 \text{ GeV} \)), (TOPAZ0, ZFITTER);

- \( e^+e^- \rightarrow e^+e^- \) for \( 40^\circ < \theta_- < 140^\circ \), \( \theta^\text{max}_{\text{coll}} = 10^\circ, 25^\circ \), and \( E_{\text{th}}(e^\pm) = 1 \text{ GeV} \), (ALIBABA, TOPAZ0).

Moreover, we have fixed the following set of values for the c.m. energy: 88.45, 89.45, 90.20, 91.1887, 91.30, 91.95, 93.00, and 93.70 GeV. All the results refer to a given reference point, \( m_t = 175 \text{ GeV}, M_H = 300 \text{ GeV} \) and \( \alpha_s(M_Z) = 0.125 \). A first comment concerns the \( s' \) cut, which should not be confused with a cut on the invariant mass of the event after initial state radiation (sometimes also used in the experiments). For realistic observables we have avoided all reference to any specific set of effective formulae and to their realizations, the interested reader having available the existing literature [68–69]. Weak radiative corrections, depending on the assumptions of the electroweak theory, have already been discussed from the point of view of the options that arise in their practical implementation. Here we only mention that we have fully propagated those theoretical uncertainties from the pseudo-observables to the realistic ones with the result that — for the fully convoluted cross-sections and forward–backward asymmetries — the final results include a theoretical error bar. In the following we present a short discussion of the main ingredients entering the calculation of realistic observables and critically compare some of the results obtained with BHM, TOPAZ0 and ZFITTER.
3.1 De-Convoluted Distributions

To illustrate in more detail the construction of realistic observables (RO) we start from the concept of de-convoluted quantities. For a given process we construct $\sigma_F(\sigma_B)$, the forward (backward) kernel cross-section, including electroweak corrections, and eventually the comprehensive of a cut on the angular acceptance. QCD corrections are included while all QED corrections are left out. After a first comparison at this level we proceeded by introducing QED final-state radiation (FSR), initial-state leptonic and hadronic pair production (PP), initial–final-state QED interference (INT) and, finally, the kernel distributions folded with initial-state QED radiation (ISR). One should emphasize, however, that the INT contribution was simply added — at $\mathcal{O}(\alpha)$ — and was not folded with the ISR. As far as the ISR is concerned we did not find it opportune to fully review the various treatments and implementations but have tried as much as possible to illustrate the origin of possible discrepancies whenever they arise. For instance, the differences that we find in the results of the various codes are dominated by pure weak (and QCD) corrections around the region of the peak and by QED radiation along the tails, where, however, the experimental error is considerably larger. Among the de-convoluted quantities, the most relevant are those computed at $s = M_Z^2$, which have an obvious counterpart in the pseudo-observables, $A_{pp}'$ and $\sigma^h$, which we have already computed, and which hereafter will be characterized by an index 0, as $A_{pp}^0$ and $\sigma_0^h$. There is, however, a noticeable difference between the two sets, represented by the interference of the $Z-\gamma$ $s$-channel diagrams and by the presence of imaginary parts in the form factors, the latter being particularly relevant for the leptonic forward–backward asymmetry.

As far as the propagation of electroweak uncertainties from pseudo-observables to realistic ones is concerned we notice that all three codes see an enhancement of the theoretical errors. For instance, for the standard reference point we denote with $\sigma_0^h$ the PO hadronic (peak) cross-section and with $\sigma^h(M_Z^2)$ the realistic one and get:

\[
\begin{align*}
\text{BHM} & \quad \sigma_0^h = 41.436^{+0.006}_{-0.006} \text{nb} \quad \sigma^h(M_Z^2) = 30.366^{+0.015}_{-0.005} \text{nb} \\
\text{TOPAZO} & \quad \sigma_0^h = 41.437^{+0.007}_{-0.006} \text{nb} \quad \sigma^h(M_Z^2) = 30.375^{+0.016}_{-0.005} \text{nb} \\
\text{ZFITTER} & \quad \sigma_0^h = 41.441^{+0.006}_{-0.006} \text{nb} \quad \sigma^h(M_Z^2) = 30.373^{+0.005}_{-0.005} \text{nb}
\end{align*}
\]

the enhancement factor being 1.9, 2.3 and 1.4 for the three codes respectively. The reason behind this increase in the induced theoretical error is that for $\sigma_0^h$, as defined in Eq. (31), we first compute $\Gamma_e, \Gamma_h$ and $\Gamma_Z$ and then construct the combination $\Gamma_e \Gamma_h / \Gamma_Z$ without any further expansion in $\alpha$, while for $\sigma^h(M_Z^2)$ we include the expansion in the option set (linearization of the cross-section), thus enlarging the error.

3.2 Final-State Radiation

A substantial difference exists between fully extrapolated RO and RO in the presence of cuts. This is illustrated by the fact that without cuts we can simply use the well known correction factor $1 + \frac{2}{3} Q_X^2 f_X^2$ for each partial channel and there is therefore no ambiguity in FSR. A possible source of discrepancy can instead be introduced when cuts are present, due to a different treatment of final-state higher-order QED effects. This can lead to
differences which in general depend on the experimental cuts required and may grow for particularly severe cuts. It has already been shown [4] that two possible prescriptions — completely factorized final-state QED correction versus factorized leading-terms and non-leading contributions summed up — lead to differences in the cross-section for Bhabha scattering of the order of 0.5% far from the peak, whereas the asymmetry is substantially left unchanged. To be more specific, the final-state QED corrections amount to a leading correction term:

\[ F_{\text{cut}}^l(s) = 2\alpha / \pi Q_f^2 \ln \left( 1 - \frac{s_0}{s} \right) \left[ \ln \left( \frac{s}{m_f^2} \right) - 1 \right], \]  

where \( s_0 \) represents a cut in the reduced invariant mass. Renormalization group arguments suggest that such a leading term should be exponentiated, which is of no practical importance at low thresholds but could give sizeable effects at high thresholds. By defining

\[ F_{\text{cut},\pm}(s) = F_{\text{cut},\pm}(s) - F_{\text{cut}}^l(s), \]  

the leading term resummation can be implemented as follows:

\[ F_{\text{cut}}^\pm(s) = \exp \left[ F_{\text{cut}}^l(s) \right] \left[ 1 + F_{\text{cut},\pm}(s) - F_{\text{cut},\pm}(s) F_{\text{cut}}^l(s) \right], \]

where spurious terms \( F_{\text{cut}}^l(s) \times F_{\text{cut},\pm}(s) \) are confined at least at \( \mathcal{O}(\alpha^3) \). Indeed, several prescriptions for treating final-state correction are possible, all equivalent at \( \mathcal{O}(\alpha) \). One reasonable recipe could be to define the leading term in a different way. For instance, in the presence of an acollinearity cut, the infrared logarithm could be defined as

\[ l = \ln(1 - x), \]

where \( x \) is given by

\[ x = \max(s_0 / s, y_T), \]

\( y_T \) being

\[ y_T = \frac{1 - \sin(\zeta/2)}{1 + \sin(\zeta/2)}, \]

and \( \zeta \) the maximum acollinearity allowed. Another possibility would be to exponentiate the full \( \mathcal{O}(\alpha) \) contribution \( F_{\text{cut},\pm}(s) \), even though there is no guarantee that the experimental cut-dependent terms do exponentiate; in this case spurious terms appear already at \( \mathcal{O}(\alpha^2) \). Alternatively, one could choose to factorize only a leading \( \mathcal{O}(\alpha) \) term and simply add the \( \mathcal{O}(\alpha) \) correction due to the acollinearity cut (this simulates the choice in Ref. [68]).

### 3.3 Initial-State Pair Production and QED Interference

Next we come to the inclusion of initial-state pair production in the realistic distributions. A fermionic pair of four-momentum \( q^2 \) radiated from the \( e^+ \) or \( e^- \) line gives a correction [70]:
\[
\sigma_{\text{pair}} = \sigma_{\text{pair}}^{S+V} + \sigma_{\text{pair}}^{H} \; , \\
\sigma^{S} = \int_{4m^2}^{\Delta^2} dq^2 \int_{(\sqrt{s}-\sqrt{s'})}^{(\sqrt{s}-\sqrt{s'})} ds' \frac{d^2\sigma}{dq^2 ds'} \; , \\
\frac{d\sigma^{H}}{dz} = s \int_{4m^2}^{(1-\sqrt{\varepsilon})^2} dq^2 \frac{d^2\sigma}{dq^2 ds'} .
\]

(132)

Also for this term there are different treatments — we can exponentiate the pair production according to the YFS formalism [71] or the same pairs can be included at \(O(\alpha^2)\). In the end, however, we found a reasonable agreement among the results of the three codes relative to the specific inclusion of pair production. The main features of this correction term are as follows.

- **TOPAZ/ZFITTER** employ the KKKS formulation [70] without exponentiating the soft+virtual part, which is added linearly to the cross-section. **BHM**, instead employs the YFS formalism [71]. The independence of the results from the soft–hard separator has also been successfully investigated.

- \(\tau\)-pairs are not included.

- The lower limit of integration \(z_{\text{min}}\), adopted for leptonic pair production, is 0.25 and the soft–hard separator \(\Delta\) has been fixed in the region where we see a plateau of stability.

Initial–final QED interference has been introduced in the calculations, including the effects of hard photons. For a fully extrapolated set-up this means that all photons, up to the maximum available energy, are taken into account, while \(s'\) cuts or energy thresholds and acollinearity cuts will restrict the available phase space. In order to proceed step-by-step we have introduced the procedure of comparing our results in the sequence \(\sigma_{\text{pair}}(NN, NY, YN, YY)\), where the first argument in parenthesis denotes inclusion or exclusion of PP, and the second refers to the interference. In this way the relative influence of QED corrections has been checked and kept under control. To illustrate the trend, we present in Tables 12 and 13 a comparison for the hadronic cross-section and the muonic forward–backward asymmetry. In particular, to continuously keep under control our comparisons, we have introduced and analyzed the ratios

\[
r(C-\text{TOPAZ0}, \text{conf}, O) = \frac{dO(C-\text{TOPAZ0}, \text{conf})}{dO(C-\text{TOPAZ0}, NN)} ,
\]

(133)

where \(O = \sigma^b, \sigma^i, A^\mu_{\text{pair}}, \text{conf} = YN/NY/YY\) and \(C = \text{BHM, ZFITTER}\). Moreover, \(dO\) denotes the relative variation for cross-sections and the absolute deviation for the asymmetry. Whenever pair production or QED interference have similar effects, the corresponding ratio assumes values of around 1. When the ratio goes to zero it is a signal that the apparent agreement does not reflect a similar agreement in the NN quantities and is therefore due to accidental compensations. Finally, if this ratio grows in modulus it gives
an indication that the agreement at the NN level is not respected when PP or INT are introduced.

To illustrate this fact, consider Table 12, which refers to $\sigma^h$. We find, for instance, that the YY cross-sections for ZFITTER-TOPAZO agree in five digits at $\sqrt{s} = 89.45$ GeV, 90.20 GeV and 91.30 GeV, giving 10.042 nb, 17.992 nb and 30.514 nb respectively. Moreover, a closer look reveals that

\[
\begin{align*}
\sqrt{s} &= 89.45 \text{ GeV} \quad 10.067 - 10.068 \text{ nb} \\
\sqrt{s} &= 90.20 \text{ GeV} \quad 18.039 - 18.040 \text{ nb} \\
\sqrt{s} &= 91.30 \text{ GeV} \quad 30.590 - 30.590 \text{ nb}
\end{align*}
\]

for ZFITTER/TOPAZO in the NN configuration.

### 3.4 Imaginary Parts of the Form factors

The presence of imaginary parts in the weak form factors introduces additional possibilities for the implementation of radiative corrections with respect to pseudo-observables. At the level of the kernel cross-sections and remembering the quite general subdivision into leading parts and remainders introduced in Subsection 2.2.1, we can basically select two possible options:

- Imaginary parts confined in the remainders
- Imaginary parts inserted into the leading terms.

To illustrate the effect of imaginary parts we select $f = \mu$ and consider the asymmetric part of the angular distribution for $e^+ e^- \rightarrow \mu^+ \mu^-$. If we denote with $\chi_\gamma$ and $\chi_z$ the corrected $\gamma - \gamma$ and $Z - Z$ propagators, this asymmetric part is proportional to

\[
\sigma_F - \sigma_B \propto \Re\left[\left(g_A^\ast\right)^2 \chi_\gamma \chi_z^\ast\right] + \left[\Re\left(g_V g_A^\ast\right)\right]^2 |\chi_z|^2 + \text{boxes}
\]

\[
= \Re g_A^2 \Im \chi_\gamma \Im \chi_z + 2 \Re g_A \Im g_A \Re \chi_\gamma \Im \chi_z
\]

\[
+ \left(\Re g_A\right)^2 \left(\Re g_A\right)^2 + O(\Re \chi_\gamma) + O\left(\Re g_A \times \Im \chi_z\right) + O\left(\Im^4\right),
\]

where $\Re \chi_z$ is suppressed around the $Z$ peak and $\Re g_A$ is usually small.

### 3.5 Initial-State QED Uncertainties

An estimate of the theoretical uncertainty due to QED radiation can also be derived, so that our results may contain two sources of theoretical error, $\pm \Delta(EW) \pm \Delta(QED)$. For example, such an estimate can be performed by using the following algorithm. Since the main source of QED theoretical uncertainty is the treatment of final-state radiation, namely the use of a completely factorized formula versus a leading-log factorized plus a non-log additive one, whenever the theoretical error is required, we can run over the two
possible options and return the corresponding uncertainty. Moreover, when the large-angle Bhabha scattering is considered, we must realize that the main approximation adopted by TOPAZ0 is to treat the $t$ and $s+t$ contributions to the cross-section at the leading logarithmic level. Thus the size of the convoluted $t$ and $s+t$ terms is computed as the difference between the full Bhabha prediction and the pure $s$-channel one and the theoretical error is estimated by assuming 1% of the difference. In particular, this means that the QED theoretical error depends on the detailed experimental set-up, growing when tightening the experimental cuts and, for the Bhabha case, when enlarging the angular acceptance to smaller angles, since this increases the contribution of the non-$s$ terms to the Bhabha cross-section. In the case of Bhabha scattering, the calorimetric measurement problem arises if a cross-section not inclusive of the energy of the outgoing fermion is considered. For thresholds of the order of $\approx 1$ GeV the effect is of order $0.01 \div 0.02\%$, but for higher energy thresholds ($E_{th}$) the contribution can grow considerably.

### 3.6 Comparisons

Let us now consider the question of de-convoluting the peak quantities in order to extract pseudo-observables like $\sigma^h_0$ and $A_{FB}^{\Gamma_0}$. By definition the de-convoluted asymmetry does not include final-state QED radiation, while the de-convoluted cross-sections include both QED and QCD final-state corrections. For $\sqrt{s} = M_Z$ GeV the three codes predict a hadronic cross-section of 30.366 nb, 30.375 nb and 30.373 nb respectively. The corresponding de-convoluted quantities — no QED corrections — are 41.400 nb, 41.409 nb and 41.402 nb, which in turn means that for BHM the effect of extracting QED corrections amounts to 11.034 nb, for ZFITTER 11.029 nb, and for TOPAZ0 also 11.034 nb with a 0.045% difference. If we introduce

$$\sigma^h = D[\sigma^h] (1 + \delta_{\text{conv}}) ,$$

where $D$ denotes de-convolution, we get

$$\delta_{\text{conv}}(B,T,Z) = -0.2665, \\ -0.2665, \\ -0.2664 .$$

From our previous tables we also find that for the same choice of input parameters

$$\sigma^h_0(B,T,Z) = 41.436, \\ 41.437, \\ 41.441 \text{ nb} .$$

The corresponding differences as compared to the de-convoluted observables, which are 0.036 nb, 0.028 nb and 0.039 nb, give an estimate of about 0.011 nb for the uncertainty in the effect of the imaginary parts from the weak form factors. Coming now to the asymmetry we find:

$$A_{FB}(B,T,Z) = -0.00082, \\ -0.00125, \\ -0.00109$$

at $\sqrt{s} = M_Z$, becoming after de-convolution,

$$D A_{FB}(B,T,Z) = 0.0169, \\ 0.0166, \\ 0.0166 .$$
Thus the effect of de-convolution is $0.0177$, $0.0179$ and $0.0177$ in BHM, TOPAZ and ZFITTER, with a BHM/ZFITTER-TOPAZ difference of $2.0 \times 10^{-4}$. It must be noted that the quantity usually reported in the literature is the de-convoluted peak asymmetry with a pure $Z$ exchange. In this case our predictions become

\[
DA_{\gamma\gamma}^p (B,T,Z) = 0.01544(0.01544), \ 0.01536(0.01536), \ 0.01528(0.01531),
\]

where in parentheses we have included the corresponding prediction for $A_{\gamma\gamma}^{\mu,0}$. Therefore this additional and conventional filtering of the asymmetry brings the BHM-TOPAZ difference to $8.5 \times 10^{-5}$ and the ZFITTER-TOPAZ difference to $-8.0 \times 10^{-5}$. The various combinations of de-convoluted asymmetries are presented in Table 14. The complete set of de-convoluted quantities is presented in Table 15, where, around the peak, we see the largest difference among codes of 0.04% and of 0.02% for $\sigma^\mu, \sigma^h$. For the muonic cross-section we find at $\sqrt{s} = M_z$ 1.4785nb, 1.4790nb and 1.4794nb, while the corresponding de-convoluted quantities are 2.0015nb, 2.0019nb and 2.0022nb respectively. Thus the effect of de-convolution in $\sigma^\mu$ is $-0.2613, -0.2611$ and $-0.2611$ for BHM, TOPAZ and ZFITTER. An interesting question, which arises in this contest, is related to the possibility of performing a real and significant test of the QED corrections by comparing the predictions of different codes. We could define

\[
\delta_{QED}^i (O^i) = \frac{O^i}{O^i_0} - 1
\]

for a given observable $O$, with $O^i_0$ being the de-convoluted observable as predicted by the $i$-th code. For a given code this $\delta_{QED}$ represents the effect of the convolution over an electroweak-corrected observable. It must however be noted that the comparison of $\delta_{QED}$ of different codes does not give an unambiguous information on QED corrections since we start already from slightly different kernels. Let us quantify this statement. Let $\Delta_{QED}^i$ be the absolute QED correction for some observable $O$ as computed by the $i$-th code:

\[
O^i = O^i_0 + \Delta_{QED}^i.
\]

Then

\[
\delta_{QED} (O^i) = \frac{\Delta_{QED}^i}{O^i_0},
\]

so that the difference between $\delta_{QED}$ of different codes can be written as

\[
\delta_{QED} (O^i) - \delta_{QED} (O^j) = \frac{\Delta_{QED}^i}{O^i_0} - \frac{\Delta_{QED}^j}{O^j_0}.
\]

By adding and subtracting $\Delta_{QED}^i/O^i_0$ one obtains

\[
\delta_{QED} (O^i) - \delta_{QED} (O^j) = \frac{\Delta_{QED}^i}{O^i_0} \frac{O^j_0 - O^i_0}{O^j_0} + \frac{1}{O^j_0} (\Delta_{QED}^i - \Delta_{QED}^j).
\]
Now it is clear that the difference between $\delta_{QED}$ of different codes depends also on the differences in the pure weak and QCD libraries of the codes. In particular, for the leptonic asymmetries in the $Z$ peak region, which are small, the first term in the r.h.s. of the last equation becomes very large, so that $\delta_{QED}(O') - \delta_{QED}(O'')$ is weak- and QCD-dominated, whereas it becomes QED-dominated far from the peak only.

Coming back to our original strategy we have made a constant effort to understand the systematics inherent in the extraction of pseudo-observables from the realistic distributions that the experiments should consider as a theoretical uncertainty. The main ingredient contributing to the systematics is of course the QED radiation, inclusive of initial-final interference and of initial-state pair production. But in the extraction, some relevance must also be attributed to the imaginary parts of the form factors and to the $Z - \gamma$ interference. The latter may have some influence, since not all the codes have the same splitting of the $\gamma f \bar{f}$ vertices among the form factors. We have already devoted some detailed discussion to the differences among the convolutions around the peak. In Table 16, we give all the remaining results, corresponding to the full set of the energy points. With due caution as to the correct interpretation of our comparison we observe a somehow larger difference in the convolution around the tails with respect to the peak region, notably 0.23% at $\sqrt{s} = 88.45$ GeV and 0.59% at $\sqrt{s} = 93.70$ GeV. Since these are differences in $\delta_{\text{conv}}$ and not in the total prediction, even 0.6% away from the peak is quite reasonable.

Our results are presented in Figs. 24–37 where we have reported $\sigma^\mu$ and $A^\mu_{pp}$ for four different set-ups: fully extrapolated, $s'>0.5$ $s$ and $40^\circ < \theta_\perp < 140^\circ, E_{th} > 20$ GeV, $\theta_{\text{coll}} < 10^\circ, 25^\circ$. Also reported are $\sigma^h$ for two set-ups, fully extrapolated and $s'>0.01$ $s$ and the $s$ channel $\sigma^e, A^e_{pp}$ where, however, $E_{th} > 1$ GeV.

In all these figures we have also shown the deviations (relative for cross-sections and absolute for the asymmetry) BHM–TOPAZ (when available) and ZFITTER–TOPAZ with the corresponding theoretical error bars. The choice of TOPAZ as a reference point is purely technical and avoids the necessity of introducing an average among the codes. It emerges from these comparisons that for a fully extrapolated set-up, the agreement for the muonic cross-section for energies below the peak and around it is quite reasonable, always taking the intrinsic error as a reference. Given the reasonable agreement at the level of de-convoluted cross-sections and once we have observed that the de-convolution is satisfactory for hadrons, we come to the conclusion that for muons the low-$q^2$ region, where mass effects may become relevant, gives the dominant difference in $\sigma^\mu$. Indeed, a comparison for $s'>0.5$ $s$ shows a much better agreement, giving $1.4391$ nb, $1.4396$ nb and $1.4397$ nb for the three codes (instead of $1.4785, 1.4790, 1.4794$). Since the corresponding de-convoluted quantities are $1.9599$ nb for BHM and $1.9605$ nb for TOPAZ we conclude that de-convolutions amount to $-0.2643$ versus $-0.2611$ with a 0.08% of relative difference. Around the peak, the energy-dependence of the observables as predicted by BHM looks different — a difference which should probably be related to the way in which BHM implements the effective coupling language. For the hadronic cross-section we observe a consistent agreement among the three codes. Finally, for the muonic asymmetry the agreement at the peak is again quite reasonable, but the energy dependence again looks different, with BHM/TOPAZ agreeing.
below the peak and TOPAZ0/ZFITTER agreeing above it. To summarize the status of the comparison for extrapolated set-up or for $s'$-cut we have from Figs. 24–29 the following.

- $\sigma^\mu$
  Around the peak the maximum deviation is $9 \times 10^{-4}$ nb corresponding to 0.06%, the BHM predictions are always lower, those of ZFITTER higher, and TOPAZ0 is in-between. On the high energy side the BHM behaviour with energy tends to differ. For an $s'$-cut of 0.5 s the maximum deviation around the peak corresponds to 0.04% and on the low-energy side of the resonance we observe a rather remarkable agreement among all codes.

- $\sigma^h$
  For the hadronic cross-section we have found an impressive agreement around the peak, with 0.03% of maximum relative deviation, which is only slightly worse around the tails — 0.08% and 0.06%. The comparison remains substantially unchanged for a low $s'$ cut.

- $A_{\mu}^\mu$
  Here the agreement is quite reasonable at the peak, with an absolute deviation of $4.3 \times 10^{-4}$ between BHM and TOPAZ0. Not completely satisfactory is the energy dependence, which registers a substantial disagreement with ZFITTER giving higher predictions below the peak and BHM after it. At least on the low-energy side the situation improves if an $s' = 0.5$ s cut is imposed.

The agreement between TOPAZ0 and ZFITTER remains rather remarkable even when the geometrical acceptance is constrained, and as well, final-state energies and the acollinearity angle are bounded with or without QED initial–final interference. For instance, for the $s$-channel leptonic cross-sections at $\sqrt{s} = M_z$ and $40^\circ < \theta_- < 140^\circ$, $E_{th}(\mu) > 20$ GeV, $E_{th}(e) > 1$ GeV, we find

$$
\begin{align*}
\sigma^\mu, & \quad \theta_{acoll} < 10^\circ & 0.9802 - 0.9801 \text{ nb} \\
\sigma^\mu, & \quad \theta_{acoll} < 25^\circ & 0.9905 - 0.9900 \text{ nb} \\
\sigma^e, & \quad \theta_{acoll} < 10^\circ & 0.9886 - 0.9884 \text{ nb} \\
\sigma^e, & \quad \theta_{acoll} < 25^\circ & 1.0012 - 1.0011 \text{ nb} ,
\end{align*}
$$

(147)

where the first entry is ZFITTER and the second TOPAZ0. More generally, and remembering that we use $E_{th} > 20$ GeV for muons and a lower cut of 1 GeV for electrons, we find from Figs. 30–37:

- $\sigma^{\mu, e}$
  for the muonic cross-section there is very good agreement for all energies and $\theta_{acoll} < 10^\circ$ — agreement which slightly deteriorates for $\theta_{acoll} < 25^\circ$. At the peak we have a relative difference of 0.02% and 0.05% respectively for the two above-mentioned $\theta_{acoll}$ (0.03% for a fully extrapolated set-up). For $s$-channel electrons the agreement is everywhere of the same quality: in particular at the peak we find a 0.02%, 0.01% of relative TOPAZ0/ZFITTER deviation — for $\theta_{acoll} < 10$, 25°. Thus for $\sigma^l$ our agreement is not altered by introducing cuts.
The two leptonic forward–backward asymmetries agree at the peak at the level of $0.3, 4.3 \times 10^{-4}$ for muons ($\theta_{\text{acoll}} < 10^\circ$) and $2.9, 0.1 \times 10^{-4}$ for electrons. For $\theta_{\text{acoll}} < 10^\circ$ the agreement tends to deteriorate at larger energies, reaching $9.6 \times 10^{-4}$ for muons and $8.7 \times 10^{-4}$ for electrons at $\sqrt{s} = 93.70$ GeV, while remaining always very good for $\theta_{\text{acoll}} < 25^\circ$. Globally for $A_{FB}^\mu$ the TOPAZO–ZFITTER comparison shows for the fully extrapolated set-up a larger difference on the low-energy side of the resonance ($1.0 \times 10^{-3}$ at $\sqrt{s} = 88.45$ GeV), a maximal agreement over the whole range of energies for $\theta_{\text{acoll}} < 25^\circ$, and a larger difference on the high-energy side for $\theta_{\text{acoll}} < 10^\circ$ ($9.6 \times 10^{-4}$ at $\sqrt{s} = 93.70$ GeV).

We illustrate the electroweak theoretical error by considering $\sigma^\mu$ at $\sqrt{s} = M_Z$. In the four different set-ups considered — fully extrapolated, $s'$-cut of 0.5 or $40^\circ < \theta_\perp < 140^\circ$, $E_{\text{th}} > 20$ GeV, and $\theta_{\text{acoll}} < 10^\circ$ or $< 25^\circ$ — we find 0.095%, 0.097%, 0.066% and 0.068%, respectively, in the relative deviation between the maximum and minimum predictions among the codes.

Finally, we illustrate the effect of different treatments of final-state QED radiation in the presence of severe kinematical cuts. By adopting different strategies TOPAZO predicts for $\sigma^\mu$ at $\sqrt{s} = M_Z$ and $40^\circ < \theta_\perp < 140^\circ$, $E_{\text{th}} > 20$ GeV, $\theta_{\text{acoll}} < 10^\circ$:

$$\sigma^\mu = 0.9801 - 0.9817\text{nb ,}$$

while for a loose cut of $E_{\text{th}} > 1$ GeV we obtain

$$\sigma^\mu = 0.9892 - 0.9893\text{nb .}$$

As already discussed, the 0.16% difference at large $E_{\text{th}}$ reduces to a mere 0.01% at low $E_{\text{th}}$.

Coming now to the full Bhabha cross-section and forward–backward asymmetry, we have considered in the following a comparison between the results of TOPAZO and of ALIBABA. As already explained, the comparison is not fully consistent as it stands now, because the electroweak and QCD libraries of ALIBABA are not up-to-date. Nevertheless, we show it because it gives an impression of the state-of-the-art. For this particular comparison the input parameters are slightly different from our default — namely $m_t = 174$ GeV and $\alpha_s = 0.124$ have been used. The ALIBABA-TOPAZO comparison for $\sigma^e$ is shown in Table 17 for $40^\circ < \theta_\perp < 140^\circ$, $E_{\text{th}} > 1$ GeV and $\theta_{\text{acoll}} < 10^\circ$. For TOPAZO we have shown three sets of numbers, all including QED interference, with the first two showing the effect of a different treatment of QED final-state radiation — I is the TOPAZO default, while II is the ZFITTER-like default — and III showing the effect of initial state pair production, not, however, included in ALIBABA. It should be mentioned at this point that the insertion of pair production is strictly valid in TOPAZO only for $s$-channel processes [4] and that it is approximate for full Bhabha. The cross-section is shown for ALIBABA with its numerical error, while for TOPAZO we give first the electroweak theoretical error and then the numerical one. Expressing due caution in comparing the two codes we observe a relative difference of 0.05% at the low-energy side of the resonance, which becomes $0.003 \div 0.004\%$ around it, with a visible deterioration at high energies where we reach 0.86%. As already
discussed [4], this is mainly due to a different treatment of higher-order QED final-state corrections.

A more detailed comparison between ALIBABA and TOPAZ0 is shown in Tables 18–25. We have shown the predictions for the full Bhabha cross-section and the forward-backward asymmetry for two different values of $\theta_{\text{coll}}$ in Tables 17–20. In Tables 21–22 we give the relative deviation between the central values of ALIBABA–TOPAZ for full Bhabha or for the s-channel alone. Next, in Table 23 we give the difference between full Bhabha and s-channel results both in ALIBABA and in TOPAZ0 with the relative contribution — i.e., $\delta = \sigma/\sigma(s) - 1$ and $\delta(A)/\delta(T)$. In Table 24 we show the $s + t$, $s$ and $t$ forward-backward asymmetries. Finally, in Table 25 we give a rough estimate of the theoretical uncertainty by considering the difference between maximal and minimal predictions from the two codes. For ALIBABA this takes into account the numerical error alone, while for TOPAZ0 we have added linearly the electroweak and the numerical uncertainty. With due caution in the interpretation we extract a $0.1 \div 0.2\%$ before and around the $Z$ resonance, which becomes as large as $0.9 \div 1.0\%$ at higher energies.
4 Basic Formulae for Electroweak Radiative Corrections

In this section we give a more detailed description of the realizations of the effective couplings $g_\nu$ and $g_A$.

4.1 BHM/WOH basics

4.1.1 Self-energies, propagators, and $\Delta r$

The radiative corrections to the photon-$Z$ propagator system (considering only the transverse parts $\sim g_{\mu\nu}$) can be obtained by inversion of the matrix

$$\left( D_{\mu\nu} \right)^{-1} = i g_{\mu\nu} \begin{pmatrix} k^2 + \hat{\Sigma}^{\gamma\gamma}(k^2) & \hat{\Sigma}^{\gamma z}(k^2) \\ \hat{\Sigma}^{z\gamma}(k^2) & k^2 - M_Z^2 + \hat{\Sigma}^{zz}(k^2) \end{pmatrix}, \quad (150)$$

with the renormalized self-energies specified below yielding

$$D_{\mu\nu} = -i g_{\mu\nu} \begin{pmatrix} D_\gamma & D_{\gamma z} \\ D_{\gamma z} & D_z \end{pmatrix}, \quad (151)$$

where $(s = k^2)$

$$D_\gamma(s) = \frac{1}{s + \hat{\Sigma}^{\gamma\gamma}(s) - \frac{\hat{\Sigma}^{\gamma z}(s)^2}{s - M_Z^2 + \hat{\Sigma}^{zz}(s)}},$$

$$D_z(s) = \frac{1}{s - M_Z^2 + \hat{\Sigma}^{zz}(s) - \frac{\hat{\Sigma}^{\gamma z}(s)^2}{s + \hat{\Sigma}^{\gamma\gamma}(s)}},$$

$$D_{\gamma z}(s) = -\frac{\hat{\Sigma}^{\gamma z}(s)}{[s + \hat{\Sigma}^{\gamma\gamma}(s)][s - M_Z^2 + \hat{\Sigma}^{zz}(s)] - [\hat{\Sigma}^{\gamma z}(s)]^2}. \quad (152)$$

The building blocks of Eq. (150) are the renormalized self-energies $\hat{\Sigma}$, which are decomposed into unrenormalized $\Sigma$ and counter terms, as follows:

$$\hat{\Sigma}^{\gamma\gamma}(k^2) = \Sigma^{\gamma\gamma}(k^2) + \delta Z_2^\gamma k^2,$$

$$\hat{\Sigma}^{zz}(k^2) = \Sigma^{zz}(k^2) - \delta M_Z^2 + \delta Z_Z^z (k^2 - M_Z^2),$$

$$\hat{\Sigma}^{WW}(k^2) = \Sigma^{WW}(k^2) - \delta M_W^2 + \delta Z_W^W (k^2 - M_W^2),$$

$$\hat{\Sigma}^{\gamma z}(k^2) = \Sigma^{\gamma z}(k^2) - \delta Z_2^{\gamma z} k^2 + (\delta Z_2^\gamma - \delta Z_1^\gamma) M_Z^2. \quad (153)$$

In the last line the abbreviations $(i = 1, 2)$

$$\delta Z_i^{\gamma z} = \frac{c_w s_w}{c_w^2 - s_w^2} (\delta Z_i^{\gamma z} - \delta Z_i^\gamma)$$

72
and

\[ s_w^2 = 1 - M_w^2 / M_z^2, \quad c_w^2 = 1 - s_w^2 \]

were used.

The self-energies \( \Sigma^{ij} \) in (153) are the sum of the electroweak one-loop diagrams [2], completed in the quark loops by the \( \mathcal{O}(a_s) \) two-loop QCD-electroweak contributions.

The mass counter terms for \( W \) and \( Z \) follow from the on-shell conditions for the \( W \) and \( Z \) propagators:

\[
\begin{align*}
\delta M_w^2 &= \mathcal{R} e \left[ \Sigma^{ww}(M_w^2) \right], \\
\delta M_Z^2 &= \mathcal{R} e \left\{ \Sigma^{zz}(M_Z^2) - \left[ \frac{\Sigma^{\gamma\gamma}(M_Z^2)}{M_Z^2} \right]^2 \right\}.
\end{align*}
\]

(154)

The other renormalization constants in (153) are given by the following set of equations:

\[
\begin{align*}
\delta Z_2^\gamma &= -\Pi^\gamma(0) = -\frac{\partial \Sigma^{\gamma\gamma}}{\partial k^2}(0), \\
\delta Z_1^\gamma &= -\Pi^\gamma(0) - \frac{s_w}{c_w} \Sigma^{\gamma\gamma}(0), \\
\delta Z_2^z &= -\Pi^z(0) - 2 \frac{c_w^2 - s_w^2}{c_w} \Sigma^{\gamma\gamma}(0) + \frac{c_w}{s_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right), \\
\delta Z_1^z &= -\Pi^z(0) - \frac{3c_w^2 - 2s_w^2}{s_w} \Sigma^{\gamma\gamma}(0) + \frac{c_w}{s_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right), \\
\delta Z_2^w &= -\Pi^w(0) - 2 \frac{c_w}{s_w} \Sigma^{\gamma\gamma}(0) + \frac{c_w}{s_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right), \\
\delta Z_1^w &= -\Pi^w(0) - \frac{3 - 2s_w^2}{s_w} \Sigma^{\gamma\gamma}(0) + \frac{c_w}{s_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right).
\end{align*}
\]

(155)

The photon vacuum polarization \( \Pi^\gamma(0) \) contains as its light fermion part the quantity \( \Delta \alpha \) discussed in Subsection 1.2. The last two constants \( \delta Z_i^w \) are not independent but are linear combinations of \( \delta Z_i^\gamma \) and \( \delta Z_i^z \). They are given here for completeness.

By the presence of the \( (\Sigma^{\gamma\gamma})^2 \) term on the r.h.s. the equations (155) are non-linear equations. It is, however, straightforward to solve them for the renormalization constants in terms of the unrenormalized quantities.

The higher-order irreducible contributions to the \( \rho \)-parameter, as far as available, are built in by means of substituting

\[
\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \rightarrow \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} + \Delta \rho^{(\text{HO})}
\]

(156)
for the r.h.s of (155), with

\[
\Delta \rho^{(\text{HO})} = 3 \bar{\varepsilon}_i \left[ 1 + x_i \Delta \rho^{(2)}(\xi) + \delta^{\text{QCD}}_{(3)} \right],
\]

(157)

\[
\bar{\varepsilon}_i = \frac{\alpha}{16\pi s_w^2 c_w^2 M_z^2} m_t^2, \quad x_i = \frac{G_\mu m_t^2}{8\pi^2 \sqrt{2},} \quad \xi = \frac{m_t^2}{M_H^2},
\]

(158)

comprising the two-loop electroweak and three-loop QCD contributions. For the functions \(\Delta \rho^{(2)}\) and \(\delta^{\text{QCD}}_{(3)}\) see Subsections 1.10.2 and 2.3.

Around the \(Z\) peak, the \(Z\) propagator has the form

\[
D_z(s) \simeq \frac{1}{1 + \hat{\Pi}^z(M_z^2)} \frac{1}{s - M_z^2 + i \frac{s}{M_z} \Gamma_z},
\]

(159)

with

\[
\hat{\Pi}^z(M_z^2) = \text{Re} \left( \int \frac{\hat{\Sigma}^z(s)}{ds} (M_z^2) \right),
\]

\[
\hat{\Sigma}^z(s) = \hat{\Sigma}^{\text{xx}}(s) - \frac{[\hat{\Sigma}^z(s)]^2}{s + \hat{\Sigma}^z(s)}.
\]

(160)

The \(Z\) width \(\Gamma_z\) is calculated from the effective coupling constants given below in Eq. (175) together with the general formulae of Subsection 1.10.1.

The vector boson masses \(M_w, M_z\) are correlated by the Fermi constant

\[
G_\mu = \frac{\pi \alpha}{\sqrt{2} M_w^2 s_w^2} \frac{1}{1 - \Delta r} = \frac{\pi \alpha}{\sqrt{2} M_z^2 c_w^2 s_w^2} \frac{1}{1 - \Delta r}.
\]

(161)

The quantity \(\Delta r(\alpha, M_z, M_w, m_t, M_H)\) has the following representation:

\[
\Delta r = \Pi'(0) - \frac{c_w^2}{s_w^2 M_z^2} \left( \frac{\delta M_z^2}{M_z^2} - \frac{\delta M_w^2}{M_w^2} \right) + \frac{\Sigma^{ww}(0) - \delta M_w^2}{M_w^2} \nonumber + 2 \frac{c_w}{s_w} \frac{\Sigma^{\gamma z}(0)}{M_z^2} + \frac{\alpha}{4\pi s_w^2} \left( 6 + \frac{7 - 4s_w^2}{2s_w^2} \log c_w \right).
\]

(162)

The last term is the sum of the box contributions and renormalized vertex corrections to the muon decay amplitude after removing the Fermi-model-like virtual photonic corrections. Due to the inclusion of the higher-order reducible and irreducible terms the way of writing \(\Delta r\) in the denominator of (161) automatically takes account of the proper resummation of the leading \(\Delta \alpha\) and \(\Delta \rho\) terms and some of the sub-leading terms as discussed in Subsection 1.10.2.
4.1.2 Vertex corrections

The vertex corrections can be summarized in terms of \( s \)-dependent vector and axial vector form factors if the masses \( m_f \) of the external fermions are small compared to \( M_W \), both for the electromagnetic and the weak NC vertex. In our terminology, ‘vertex corrections’ denote the renormalized \( \gamma(Z)f\bar{f} \) three-point functions in one-loop order, together with the finite wave function renormalizations for external fermions.

In contrast to the propagator corrections, the vertex corrections are not universal and depend on the fermion species. For this reason we have to list them separately for \( \nu, e, u \) and \( d \) type fermions. In addition, the \( b \) quark is exceptional due to the virtual top contributions in the vertex.

Our terminology is as follows. \( F_{V,A}^{Zf} \) and \( F_{V,A}^{\gamma f} \) denote the IR finite weak (without the virtual photon diagrams) form factors for the \( Zf\bar{f} \) and \( \gamma f\bar{f} \) vertex which, together with the lowest order terms, yield the dressed vertices:

\[
\begin{align*}
\hat{\Gamma}_\mu^{Zf\bar{f}} &= i \frac{e}{2s_W c_W} \gamma_\mu \left( v_f + F_{V}^{Zf}(s) - \gamma_5 \left[ a_f + F_{A}^{Zf}(s) \right] \right), \\
\hat{\Gamma}_\mu^{\gamma f\bar{f}} &= -i e Q_f \gamma_\mu - i e \gamma_\mu \left[ F_{V}^{\gamma f}(s) - F_{A}^{\gamma f}(s) \gamma_5 \right].
\end{align*}
\]

The lowest order coupling reads:

\[
v_f = I_f^{(3)} - 2Q_f s_w^2, \quad a_f = I_f^{(3)} ,
\]

and the weak form factors in (163) are explicitly given by the following set of formulae.

**Neutral current vertex:**

neutrinos:

\[
F_{V}^{Z\nu} = F_{A}^{Z\nu} = \frac{1}{4\pi} \left[ \frac{1}{8c_w^2 s_w^2} \Lambda_2(s, M_Z) + \frac{2s_w^2 - 1}{4s_w^2} \Lambda_2(s, M_w) + \frac{3c_w^2}{2s_w^2} \Lambda_3(s, M_w) \right],
\]

charged fermions:

\[
\begin{align*}
F_{V}^{Zf} &= \frac{\alpha}{4\pi} \left[ v_f (v_f^2 + 3a_f^2) \Lambda_2(s, M_Z) + F_L f \right], \\
F_{A}^{Zf} &= \frac{\alpha}{4\pi} \left[ a_f (3v_f^2 + a_f^2) \Lambda_2(s, M_Z) + F_L f \right].
\end{align*}
\]
with

\[
F_L^t = \frac{1}{4s_w^2} \Lambda_2(s, M_w) - \frac{3c_w^2}{2s_w^2} \Lambda_3(s, M_w), \\
F_L^u = -\frac{1 - \frac{2}{3}s_w^2}{4s_w^2} \Lambda_2(s, M_w) + \frac{3c_w^2}{2s_w^2} \Lambda_3(s, M_w), \\
F_L^d = -\frac{1 - \frac{4}{3}s_w^2}{4s_w^2} \Lambda_2(s, M_w) - \frac{3c_w^2}{2s_w^2} \Lambda_3(s, M_w). \tag{167}
\]

Electromagnetic vertex:

\[
F_{\gamma f} = \frac{\alpha}{4\pi} \left[ \frac{Q_f(v_f^2 + a_f^2)}{4s_w^2 c_w^2} \Lambda_2(s, M_z) + G_L^f \right], \\
F_A = \frac{\alpha}{4\pi} \left[ \frac{Q_f 2v_f a_f}{4s_w^2 c_w^2} \Lambda_2(s, M_z) + G_L^f \right], \tag{168}
\]

with

\[
G_L^t = -\frac{3}{4s_w^2} \Lambda_3(s, M_w), \\
G_L^u = -\frac{1}{12s_w^2} \Lambda_2(s, M_w) + \frac{3}{4s_w^2} \Lambda_3(s, M_w), \\
G_L^d = \frac{1}{6s_w^2} \Lambda_2(s, M_w) - \frac{3}{4s_w^2} \Lambda_3(s, M_w). \tag{169}
\]

The functions \( \Lambda_2, \Lambda_3 \) have the form

\[
\Lambda_2(s, M) = -\frac{7}{2} - 2w - (2w + 3)\log(-w) \\
+2(1 + w)^2 \left[ \text{Li}_2 \left( 1 + \frac{1}{w} \right) - \frac{\pi^2}{6} \right], \\
\Lambda_3(s, M) = \frac{5}{6} - \frac{2w}{3} + \frac{(2w + 1)}{3} \sqrt{1 - 4w} \log(x) + \frac{2}{3}w(w + 2)\log^2(x), \tag{170}
\]

with

\[
w = \frac{M^2}{s + i\varepsilon}, \quad x = \frac{\sqrt{1 - 4w} - 1}{\sqrt{1 - 4w} + 1}.
\]

The functions \( F_L^d \) and \( G_L^d \) cannot be used for \( b \) quarks. The full expressions for \( F_L^b, G_L^b \) can be found, for example, in Ref. [2] \(^{13}\).

\(^{13}\)Note that the normalization is different:

\[ F_L^b (\text{this report}) = 2s_w c_w F_L^b (\text{Ref. [2]}). \]
4.1.3 \( e^+e^- \rightarrow f\bar{f} \) amplitudes

Around the Z resonance, the amplitude for \( e^+e^- \rightarrow f\bar{f} \) can be cast into a form close to the lowest order amplitude:

\[
A(e^+e^- \rightarrow f\bar{f}) = A_\gamma + A_z + (box),
\]

where \( A_\gamma \) denotes the dressed photon, \( A_z \) the dressed Z exchange amplitude, and \( (box) \) summarizes the terms from the massive box diagrams, which, however, can be neglected around the Z.

The dressed photon exchange amplitude can be written in the following way:

\[
A_\gamma = \frac{e^2}{1 + \hat{\Pi}^\gamma(s)} \frac{Q_e Q_f}{s} \left[(1 + F^\gamma_{\gamma\gamma} \gamma_\mu - F^\gamma_A \gamma_\mu \gamma_5) \otimes (1 + F^\gamma_f \gamma_\mu - F^\gamma_{fA} \gamma_\mu \gamma_5)\right].
\]

\( \hat{\Pi}^\gamma \) is the subtracted vacuum polarization \( \hat{\Pi}^\gamma(s) = \Pi^\gamma(s) - \Pi^\gamma(0) \) with

\[
\Pi^\gamma(s) = \frac{\Sigma^{\gamma\gamma}(s)}{s}.
\]

The vertex form factors \( F^\gamma_{V,A} \) from Eq. (168) are evaluated for \( s = M_z^2 \).

The Z exchange amplitude without the box diagrams factorizes as follows:

\[
A_z = \sqrt{2} G_\mu M_z^2 (\rho_f \rho_f)^{1/2} \left[\gamma_\mu \left(I^{[3]}_f - 2 Q_f s_w^2 \kappa_f\right) - I^{[3]}_f \gamma_\mu \gamma_5\right] \times \left[\gamma_\mu \left(I^{[3]}_e - 2 Q_e s_w^2 \kappa_e\right) - I^{[3]}_e \gamma_\mu \gamma_5\right] \bigg/ \left(s - M_z^2 + i \frac{Q_f}{M_z^2} M_\pi \Gamma_z\right).
\]

The weak corrections appear in terms of fermion-dependent form factors \( \rho_f \) and \( \kappa_f \) in the coupling constants and in the width in the denominator.

4.1.4 Effective neutral current couplings

The factorized amplitude (173) allows us to define NC vertices at the Z resonance with effective coupling constants \( g_{V,A}^f \), synonymously with the use of \( \rho_f,\kappa_f \):

\[
J^{NC}_{\mu} = \left(\sqrt{2} G_\mu M_z^2 \rho_f\right)^{1/2} \left[I^{[3]}_f - 2 Q_f s_w^2 \kappa_f\right] \gamma_\mu - I^{[3]}_f \gamma_\mu \gamma_5\bigg/ \left(s - M_z^2 + i \frac{Q_f}{M_z^2} M_\pi \Gamma_z\right).
\]

The effective couplings read as follows:

\[
g_{V}^f = \left[v_f + 2 s_w c_w Q_f \hat{\Pi}^\gamma(M_z^2) + F^\gamma_{V}\right] \left[1 - \frac{1}{1 + \hat{\Pi}^\gamma(M_z^2)}\right]^{1/2},
\]

\[
g_{A}^f = \left[a_f + F^\gamma_{A}\right] \left[1 - \frac{1}{1 + \hat{\Pi}^\gamma(M_z^2)}\right]^{1/2}.
\]

The building blocks are:
• \( \Delta r \) from Eq. (162)
• \( \tilde{\Pi}^Z(M_z^2) \) from Eq. (160)
• \( \tilde{\Pi}^{\nu}(M_z^2) = \tilde{\Sigma}^{\nu}(M_z^2)/M_z^2 \) from Eq. (153)
• \( F_{V,A}^{Zf} \) from Eq. (166) for \( s = M_z^2 \).

For given \( m_t, M_H \) the values of \( s_w^2 \) and \( M_w \) respectively are chosen such that Eq. (161) is fulfilled.

For the \( b \) quark couplings the next-order leading corrections \( \sim G^2_m m_t^4, \sim \alpha_s G_m m_t^2 \) are taken into account by performing in the one-loop expression \( F^b_L \) the following substitution:

\[
F^b_L \to F^b_L - \frac{\alpha}{16\pi s^2 c^2 w} \frac{m_t^2}{M_z^2} + \Delta \tau_b,
\]

with

\[
\Delta \tau_b = x_t \left[ 1 + x_t \tau^{(2)}(\xi) - \alpha_s(m_t) \frac{\pi}{3} \right].
\]

The function \( \tau^{(2)} \) is taken from Refs. [55, 58], and the QCD correction term from Ref. [57].

The alternative form factors \( \rho \) and \( \kappa \) can then be obtained via

\[
\frac{g^f_V}{g^f_A} = 1 - 4 |Q_f| \kappa f s_w^2, \quad \left( \frac{g^f_A}{g^f_V} \right)^2 = \rho_f.
\]

Due to the imaginary parts of the self-energies and vertices, the form factors and the effective couplings, respectively, are complex quantities. The effective mixing angles are calculated from the real parts according to

\[
\sin^2 \theta^f_{\text{eff}} = \frac{1}{4 |Q_f|} \left( 1 - \frac{\Re g^f_V}{\Re g^f_A} \right).
\]

4.2 LEPTOP basics

4.2.1 Electroweak loops for hadron-free observables: functions \( V_i \)

For hadron-free observables we write the result of one-loop electroweak calculations in the form suggested in Ref. [72]:

\[
M_w/M_z = c + \frac{3c}{32\pi s^2(c^2 - s^2)} \tilde{\alpha} V_m(t, h),
\]

\[
g^f_A = -\frac{1}{2} - \frac{3}{64\pi s^2 c^2} \tilde{\alpha} V_A(t, h),
\]

\[
R = g^f_V/g^f_A = 1 - 4s^2 + \frac{3}{4\pi(c^2 - s^2)} \tilde{\alpha} V_R(t, h),
\]

\[
g^\nu = \frac{1}{2} + \frac{3}{64\pi s^2 c^2} \tilde{\alpha} V_\nu(t, h),
\]

\[
(180)
\]
where $t = m^2/M^2$, $h = M^2/H^2$ and functions $V_i(t, h)$ are normalized by the condition,

$$V_i(t, h) \simeq t,$$  

(181)

at $t \gg 1$.

Each function $V_i$ is a sum of five terms [72, 73]:

$$V_i(t, h) = t + T_i(t) + H_i(h) + C_i + \delta V_i(t, h).$$  

(182)

The functions $t + T_i(t)$ are due to the $(t, b)$ doublet contribution to self-energies of the vector bosons, $H_i(h)$ is due to $W^\pm, Z$ and $H$ loops, the constants $C_i$ include light fermion contribution to self-energies, vertex and box diagrams.

To give explicit expressions for $T_i(t)$ and $H_i(h)$ it is convenient to introduce three auxiliary functions $F_i(t), F_h(h)$ and $F'_h(h)$ (see Subsection 4.2.5 for their expressions).

The equations for $T_i(t)$ and $H_i(h)$ have the form [72]:

$$T_m(t) = \left( \frac{2}{3} - \frac{8}{9}s^2 \right) \ln t - \frac{4}{3} + \frac{32}{9}s^2 + \frac{2}{3}(c^2 - s^2) \left( \frac{t^3}{c^6} - \frac{3t}{c^2} + 2 \right) \ln \left| 1 - \frac{c^2}{t} \right|$$

$$+ \frac{2c^2 - s^2}{3t} + \frac{1}{3c^2} s^2 t + \left( \frac{2}{3} - \frac{16}{9}s^2 - \frac{2}{3} - \frac{32}{9}s^2 t \right) F(t),$$

$$H_m(h) = \frac{h}{h-1} \ln h + \frac{c^2 h}{h - c^2} \ln \frac{h}{c^2} - \frac{s^2}{18c^2 h} - \frac{8}{3}s^2 + \left( \frac{h^2}{9} - \frac{4h}{9} + \frac{4}{3} \right) F_h(h)$$

$$- (c^2 - s^2) \left( \frac{h^2}{9c^4} - \frac{4h}{9c^2} + \frac{4}{3} \right) F_h \left( \frac{h}{c^2} \right) + (1.1205 - 2.59\delta s^2),$$  

(183)

where $\delta s^2 = 0.23117 - s^2$.

$$T_A(t) = \frac{2}{3} - \frac{8}{9}s^2 + \frac{16}{27}s^4 - \frac{1 - 2tF_i(t)}{4t + 1}$$

$$+ \left( \frac{32}{9}s^4 - \frac{8}{3}s^2 - \frac{1}{2} \right) \left[ \frac{4}{3} F_i(t) - \frac{2}{3} (1 + 2t) \frac{1 - 2tF_i(t)}{4t + 1} \right],$$

$$H_A(h) = \frac{c^2 h}{1 - c^2/h} \ln \frac{h}{c^2} - \frac{8h}{9(h-1)} \ln h + \left( \frac{4}{3} - \frac{2}{3} h + \frac{4}{9} h^2 \right) F_h(h)$$

$$- \left( \frac{4}{3} - \frac{4}{9} h + \frac{1}{9} h^2 \right) F'_h(h) - \frac{1}{18} h + (0.7751 + 1.07\delta s^2).$$  

(184)

$$T_R(t) = \frac{2}{9} \ln t + \frac{4}{9} - \frac{2}{9} (1 + 11t) F(t),$$

$$H_R(h) = \frac{4}{3} - \frac{h}{18} + \frac{c^2 h}{1 - c^2/h} \ln \frac{h}{c^2} + \left( \frac{4}{3} - \frac{4}{9} h + \frac{1}{9} h^2 \right) F_h(h)$$

$$+ \frac{h}{1 - h} \ln h + (1.3590 + 0.51\delta s^2).$$  

(185)
\[ i = \nu \]

\[ T_{\nu}(t) = T_{\lambda}(t), \]
\[ H_{\nu}(h) = H_{\lambda}(h). \quad (186) \]

The constants \( C_i \) are rather complicated functions of \( \sin^2 \theta \) and we present their numerical values near \( s^2 = 0.23117 \):

\[ C_m = -1.3500 + 4.13 \, \delta s^2, \quad (187) \]
\[ C_A = -2.2619 - 2.63 \, \delta s^2, \quad (188) \]
\[ C_R = -3.5041 - 5.72 \, \delta s^2, \quad (189) \]
\[ C_\nu = -1.1638 - 4.88 \, \delta s^2. \quad (190) \]

### 4.2.2 Corrections \( \delta V_i \)

Functions \( \delta V_i(t, h) \) in Eq. (182) are small corrections to \( V_i \). They can be presented as a sum of five terms \( \delta_k V_i(k = 1 \div 5) \):

1. Corrections to the polarization of the electromagnetic vacuum due to W boson loop, \( \delta_{\alpha}W \), and t-quark loop, \( \delta_{\alpha}t \), are traditionally not included in the running of \( \alpha(q^2) \). We also prefer to consider them together with electroweak corrections. This is especially reasonable because W contribution \( \delta_{\alpha}W \) is gauge-dependent, while \( \delta_{\alpha}t \) is negligibly small. Here and for all other electroweak corrections we use the 't Hooft–Feynman gauge. The corrections \( \delta_{\alpha}W \) and \( \delta_{\alpha}t \) were neglected in Ref. [72] and were introduced in Ref. [74]:

\[ \delta_1 V_m(t, h) = - \frac{16}{3} \pi s^4 \frac{1}{\alpha} (\delta_{\alpha}W + \delta_{\alpha}t) = -0.055, \quad (191) \]
\[ \delta_1 V_R(t, h) = - \frac{16}{3} \pi s^2 c^2 \frac{1}{\alpha} (\delta_{\alpha}W + \delta_{\alpha}t) = -0.181, \quad (192) \]
\[ \delta_1 V_{\lambda}(t, h) = \delta_1 V_{\nu}(t, h) = 0, \quad (193) \]

where

\[ \frac{\delta_{\alpha}W}{\alpha} = \frac{1}{2\pi} \left[ (3 + 4c^2) \left( 1 - \sqrt{4c^2 - 1} \arcsin \frac{1}{2c} \right) - \frac{1}{3} \right] = 0.0686, \quad (194) \]
\[ \frac{\delta_{\alpha}t}{\alpha} = - \frac{4}{9\pi} \left[ (1 + 2t) F(t) - \frac{1}{3} \right] \simeq - \frac{4}{45\pi} \frac{1}{t} + ... \simeq -0.00768. \quad (195) \]

(Here and in Eqs. (196–209) we use \( m_t = 175 \) GeV for numerical estimates.)
2. Corrections of the order of $\bar{\alpha}\bar{\alpha}$, due to the gluon exchange in the quark electroweak loops [56] (see also Ref. [73]), $\delta_2 V_i = \delta_2^g V_i + \delta_2^t V_i$. For the two generations of light quarks ($q = u, d, s, c$) this gives:

\[
\begin{align*}
\delta_2^g V_m(t, h) &= 2 \left\{ \frac{4}{3} \left[ \frac{\alpha_s(M_Z)}{\pi} \right] (c^2 - s^2) \ln c^2 \right\} = \left[ \frac{\alpha_s(M_Z)}{\pi} \right] (-0.377) , \\
\delta_2^g V_A(t, h) &= 2 \left\{ \frac{4}{3} \left[ \frac{\alpha_s(M_Z)}{\pi} \right] (c^2 - s^2 + \frac{20}{9}s^4) \right\} = \left[ \frac{\alpha_s(M_Z)}{\pi} \right] (1.750) , \\
\delta_2^g V_R(t, h) &= 0 , \\
\delta_2^g V_\nu(t, h) &= \delta_2^g V_A(t, h) .
\end{align*}
\]

The results of calculations for the third generation are given by rather complicated functions of the top mass and $s^2$. Here we present an expansion for large values of $t$ and for the fixed value of $s^2 = 0.23117$:

\[
\begin{align*}
\delta_2^t V_m(t, h) &= \left[ \frac{\alpha_s(m_t)}{\pi} \right] \left[ -2.86 t + 0.46 \ln t - 1.540 - \frac{0.68}{t} - \frac{0.21}{t^2} \right] \\
&= \frac{\alpha_s(m_t)}{\pi} (-11.67) , \\
\delta_2^t V_A(t, h) &= \left[ \frac{\alpha_s(m_t)}{\pi} \right] \left[ -2.86 t + 0.493 - \frac{0.19}{t} - \frac{0.05}{t^2} \right] \\
&= \frac{\alpha_s(m_t)}{\pi} (-10.10) , \\
\delta_2^t V_R(t, h) &= \left[ \frac{\alpha_s(m_t)}{\pi} \right] \left[ -2.86 t + 0.22 \ln t - 1.513 - \frac{0.42}{t} - \frac{0.08}{t^2} \right] \\
&= \frac{\alpha_s(m_t)}{\pi} (-11.88) , \\
\delta_2^t V_\nu(t, h) &= \delta_2^t V_A(t, h) .
\end{align*}
\]

As these formulas are valid for $m_t > M_Z$, in order to go to the region $m_t < M_Z$ we either put $\delta_2^t V_i = 0$ or use a massless limit in which $\delta_2^t V_i = \frac{1}{2} \delta_2^g V_i$. In any case, this region gives a tiny contribution to the global fit.

3. Corrections of the order of $\bar{\alpha}\bar{\alpha}^2$ were calculated for the leading term $\bar{\alpha}\bar{\alpha}^2 t$ only [53]

\[
\delta_3 V_i(t, h) \simeq -(2.1552 - 0.18094 N_f) \bar{\alpha}^2 (m_t) t \simeq -1.250 \bar{\alpha}^2 (m_t) t = -0.06
\]

for $N_f = 5$ light flavours. [For the numerical estimate we use $\bar{\alpha}_s(M_Z) = 0.125$.]

4. The leading correction of the order $\bar{\alpha}^2 t^2$, which originates from the second-order Yukawa interaction, was calculated in Refs. [55–58]:

81
\[ \delta_4 V_i(t, h) = -\frac{\alpha}{16\pi s^2 c^2} A \left( \frac{h}{t} \right) t^2, \]  

(206)

where function \( A(M_H/m_t) \) can be found in Refs. [55, 58]. For \( m_t = 175 \text{ GeV} \) and \( M_H = 300 \text{ GeV} \) one has \( A = 8.9 \) and \( \delta_4 V_i(t, h) = -0.11 \).

5. In the second order in electroweak interactions quadratic dependence appears on the Higgs mass [75]

\[ \delta_5 V_m = \frac{\alpha}{24\pi} \left( \frac{h}{c^2} \right) 0.747 = 0.0011, \]  

(207)

\[ \delta_5 V_A = \frac{\alpha}{24\pi} \left( \frac{h}{s^2} \right) 1.199 = 0.0057, \]  

(208)

\[ \delta_5 V_R = -\frac{\alpha}{24\pi} \left( \frac{h}{c^2} \right) \frac{c^2 - s^2}{s^2} 0.973 = -0.0032. \]  

(209)

(Here for numerical estimates we use \( m_H = 300 \text{ GeV} \).)

4.2.3 Hadronic decays of Z boson

For the partial width of the Z decay into a pair of quarks \( q\bar{q} \) (\( q = u, d, s, c, b \)), we use the equation

\[ \Gamma_q = 12 \left[ (g_q^2)^2 R_q^v + (g_A^q)^2 R_q^A \right] \Gamma_0, \]

where the final-state QED and QCD radiative functions \( R_q^v \) and \( R_q^A \) are given by Eqs. (55) and (56) respectively, and \( \Gamma_0 \) is defined by Eq. (52). The electroweak radiative corrections are included in \( g_q^2 \) and \( g_A^q \):

\[ g_A^q = I_q^{(3)} \left[ 1 + \frac{3\alpha}{32\pi s^2 c^2} V_{Aq}(t, h) \right], \]  

(210)

\[ g_A^q / g_A^q = 1 - 4|Q_q|^2 + \frac{3|Q_q|^2}{4\pi(c^2 - s^2)} \delta V_{Aq}(t, h). \]  

(211)

The functions \( V_{Aq}(t, h) \) and \( V_{Rq}(t, h) \) in the one-loop electroweak approximation are related to the functions \( V_A(t, h) \) and \( V_R(t, h) \) from leptonic decays [76]:

\[ V_{Au}(t, h) = V_{Ac}(t, h) = V_A(t, h) + \frac{128\pi s^3 c^3}{3\alpha} (F_{Al} + F_{Au}), \]  

(212)

\[ V_{Ad}(t, h) = V_{As}(t, h) = V_A(t, h) + \frac{128\pi s^3 c^3}{3\alpha} (F_{Al} - F_{Ad}), \]  

(213)

\[ V_{Ru}(t, h) = V_{Rc}(t, h) = V_R(t, h) + \frac{16\pi s(c^2 - s^2)}{3\alpha} \times [F_{Vl} - (1 - 4s^2)F_{Al} + \frac{3}{2}(-(1 - \frac{8}{3}s^2)F_{Au} + F_{Vl})], \]  

(214)
\[ V_{Rd}(t, h) = V_{Rd}(t, h) = V_{Rd}(t, h) + \frac{16\pi s c(s^2 - s^2)}{3\delta} \times \left\{ F_{VI} - (1 - 4s^2)F_{AI} + 3 \left[ \left(1 - \frac{4}{3}s^2\right)F_{Ad} - F_{Vd}\right] \right\}, \]  

where \( F_{VI} = 0.00197 \), \( F_{AI} = 0.00186 \), \( F_{Vu} = -0.00169 \), \( F_{Au} = -0.00165 \), \( F_{Vd} = 0.00138 \), \( F_{Ad} = 0.00137 \) [76]. The difference \( V_{Aq} - V_{i} \) is due to different electroweak corrections to the vertices \( Zq\bar{q} \) and \( Z\ell\ell \) \((s^2 = 0.23117 \text{ is assumed) formulae used in the code have explicit } s^2 \text{ dependence).}

The oblique corrections of the order of \( \delta_s \), \( \delta_s^2 t \) and \( \delta_s t^2 \) to the \( V_{Aq}(V_{Rq}) \) are the same as in the case of \( V_{A} \) and \( V_{R} \). But for \( Z \) boson decay into pair \( q\bar{q} \) there are additional \( \delta_s \) corrections to the vertices that have not yet been calculated. This brings additional uncertainty into the theoretical accuracy.

### 4.2.4 Specific features of the decay \( Z \to b\bar{b} \)

For \( Z \to b\bar{b} \) decays we have to take into account corrections to the \( Z \to b\bar{b} \) vertex which depend on \( t \) [52, 57].

\[
V_{Ab}(t, h) = V_{Ad}(t, h) - \frac{8s^2 c^2}{3(3 - 2s^2)} [\phi(t) + \delta\phi(t)] , \\
V_{Rb}(t, h) = V_{Rd}(t, h) - \frac{4s^2(c^2 - s^2)}{3(3 - 2s^2)} [\phi(t) + \delta\phi(t)] .
\]

For \( \phi(t) \) we use the following expansion [52]:

\[
\phi(t) = \frac{3 - 2s^2}{2s^2 c^2} \left\{ t + c^3 \left[ 2.88ln\left(\frac{t}{c^2}\right) - 6.716 + \frac{1}{t} \left( 8.368 c^2 ln\left(\frac{t}{c^2}\right) - 3.408 c^2 \right) \right. \\
+ \frac{1}{t^2} \left( 9.126 c^4 ln\left(\frac{t}{c^2}\right) + 2.26 c^4 \right) + \frac{1}{t^3} \left( 4.043 c^8 ln\left(\frac{t}{c^2}\right) + 7.41 c^6 \right) \\
+ \ldots \right\} ,
\]

and for \( \delta\phi(t) \) we use the leading approximation calculated in Refs. [57, 55, 58] and:

\[
\delta\phi(t) = \frac{3 - 2s^2}{2s^2 c^2} \left\{ \frac{\pi^2}{3} \left[ \frac{\delta_s(m_t)}{\pi} \right] t + \frac{1}{16s^2 c^2} \left( \frac{\delta_s}{\pi} \right) t^2 \tau_b^{(2)} \left( \frac{h}{t} \right) \right\} ,
\]

where function \( \tau_b^{(2)} \) has been calculated in Ref. [58].

For \( m_t = 175 \text{ GeV} \) and \( M_H = 300 \text{ GeV} \)

\[
\tau_b^{(2)} = 1.245 .
\]

Asymmetries are calculated with the loop corrected values of \( g_V \) and \( g_A \).
4.2.5 Appendix: Auxiliary functions $F_t$ and $F_h$

\[
F_t(t) = \begin{cases} 
2(1 - \sqrt{4t - 1} \arcsin \frac{1}{\sqrt{4t}}) & , \quad 4t > 1 \\
2(1 - \sqrt{1 - 4t \ln \frac{1}{\sqrt{4t}}}) & , \quad 4t < 1 
\end{cases} 
\]

\[
F_h(h) = \begin{cases} 
1 + \left( \frac{h}{h-1} - \frac{h}{2} \right) \ln h + h \sqrt{1 - \frac{4}{h}} \ln \left( \frac{h}{4} - 1 + \sqrt{\frac{4}{h}} \right) & , \quad h > 4 , \\
1 + \left( \frac{h}{h-1} - \frac{h}{2} \right) \ln h - h \sqrt{\frac{4}{h} - 1} \arctan \sqrt{\frac{4}{h} - 1} & , \quad h < 4 , 
\end{cases} 
\]

\[
F'_h(h) = \begin{cases} 
-1 + \frac{h-1}{2} \ln h + (3-h) \sqrt{\frac{h}{h-4}} \ln \left( \frac{h}{4} - 1 + \sqrt{\frac{4}{h}} \right) & , \quad h > 4 , \\
-1 + \frac{h-1}{2} \ln h + (3-h) \sqrt{\frac{h}{4-h}} \arctan \sqrt{\frac{4}{h} - 1} & , \quad h < 4 . 
\end{cases} 
\]

4.3 TOPAZO basics

The realization given in Ref. [77] describes the coupling of the $Z$ as:

\[
\sqrt{2} (G_{\mu} p^2)^{1/2} M_Z \gamma'^{\mu} \left[ I^{[3]}_{f} - 2 Q_f \hat{s}^2 + \delta g_{v}^{f} + \left( I^{[3]}_{f} + \delta g_{A}^{f} \right) \gamma_{6} \right] .
\]  (244)

Before giving the specific expression of the various quantities entering the previous equation, we stress that our metric is such that a time-like momentum squared is negative. Next we decompose the unrenormalized vector boson self-energies as:

\[
S_{\gamma\gamma} (p^2) = \frac{g^2 s_\theta^2}{16 \pi^2} \Pi_{\gamma\gamma} (p^2) p^2 , \\
S_{zz} (p^2) = \frac{g^2}{16 \pi^2 c_\theta^2} \Sigma_{zz} (p^2) , \\
S_{z\gamma} (p^2) = \frac{g^2 s_\theta}{16 \pi^2 c_\theta} \Sigma_{z\gamma} (p^2) , \\
S_{ww} (p^2) = \frac{g^2}{16 \pi^2} \Sigma_{ww} (p^2) , \\
\Sigma_{zz} (p^2) = \Sigma_{a3} (p^2) - 2 s_\theta^2 \Sigma_{q3} (p^2) + s_\theta^4 \Pi_{\gamma\gamma} (p^2) p^2 , \\
\Sigma_{z\gamma} (p^2) = \Sigma_{aQ} (p^2) - s_\theta^2 \Pi_{\gamma\gamma} (p^2) p^2 ,
\]

where $\theta$ denotes the bare mixing angle. After a re-diagonalization in the neutral sector, which makes $S_{zz}(0) = 0$ in the $\xi = 1$ gauge and replaces $\Sigma_{ww}, \Sigma_{zz}$ and $\Sigma_{q3}$ with $\Sigma_{ww} + 4\Gamma, \Sigma_{zz} + 4\Gamma$ and $\Sigma_{q3} + 2\Gamma$ with $\Gamma = M^2 B_0(0, M^2, M^2) - M$ being the bare $W$ mass — we consider the ultraviolet and infrared finite corrections to the $\mu$-decay $\delta_\alpha$ and introduce

\[
\Sigma^a_{ww} = \Sigma_{ww}(0) + \frac{\sqrt{2} \pi \alpha}{G_{\mu}} \delta_\alpha ,
\]

(226)
which has the virtue of being gauge invariant, a property not satisfied by $\Sigma_{ww}(0)$ alone. The extra term induced by the diagonalization is gauge invariant by construction. From now on we will denote the fermionic (bosonic) contributions to a given quantity $A$ with the notation $A^{\text{ferm}}(A^{\text{bos}})$. With these quantities we define

$$
\rho_z = \frac{1}{1 + \frac{G_{\mu}M_z^2}{2\sqrt{2}\pi^2} \Sigma_{\text{ferm}} + \text{h.o.}},
$$

$$
M_z^2 \Sigma = \Sigma_{ww}^G - \Re e \Sigma_{3\lambda}^G(M_z^2) + \Re e \Sigma_{3Q}^G(M_z^2) + s^2 c^2 M_z^2 \Pi_{\mu}^{\text{bos}}(M_z^2),
$$

(227)

where

$$
s^2 c^2 = \frac{\pi \alpha(M_z)}{\sqrt{2} G_{\mu} M_z^2},
$$

(228)

and $\Pi_{\mu}(M_z^2) = \Re e \Pi_{\gamma\gamma}(M_z^2) - \Pi_{\gamma\gamma}(0)$. Note that $\Sigma$ is ultraviolet finite. Next we can introduce $s^2$ as:

$$
\frac{1}{2} \left[ 1 - \left( 1 - \frac{4 \pi \alpha(M_z)}{\sqrt{2} G_{\mu} M_z^2 \rho_z} \right)^{1/2} \right],
$$

(229)

and give the definition of $\delta g^f_{\nu}$, $\delta g^f_{A}$,

$$
\delta g^f_{\nu} = \frac{\alpha}{4\pi} \left[ \frac{2 F^f_{\nu} - \frac{1}{2} v_f \Delta \Pi_{\nu}}{c^2 s^2} - 2 Q_f \Delta s^2 \right],
$$

$$
\delta g^f_{A} = \frac{\alpha}{4\pi} \left[ \frac{2 F^f_{A} - \frac{1}{2} I_{\nu}^{(3)}}{c^2 s^2} \right],
$$

(230)

where $v_f = I_{\nu}^{(3)} - 2 Q_f s^2$, and

$$
M_z^2 \Delta \Pi_{\nu} = \Re e \left\{ \left( \Sigma_{ww}^G \right)^{\text{bos}} - \Sigma_{3\lambda}^{\text{bos}}(M_z^2) - \Sigma_{3Q}^{\text{ferm}}(M_z^2) - M_z^2 \Sigma_{3\lambda}^{\text{ferm}}(M_z^2) 
\right.
\left. + 2 s^2 \left[ \Sigma_{3Q}^{G}(M_z^2) + M_z^2 \Sigma_{3Q}^{G}(M_z^2) \right] + \delta^4 M_z^2 \Pi_{\gamma\gamma}^{G}(M_z^2) \right\}.
$$

(231)

Here $f'$ stands for $-df/dp^2$. The $F$ refers to flavour-dependent vertex corrections.

- If no resummation of bosonic self-energies is performed we have

$$
\rho_z^R = \rho_z, \quad \text{and} \quad \Delta s^2 = \frac{\Sigma_{\text{bos}}}{c^2 - s^2};
$$

(232)

- otherwise $\rho_z^R$ has the same structure of $\rho_z$ with $\Sigma_{\text{ferm}}$ replaced by $\Sigma_{\text{ferm}}$: 85
\[
\rho_z^R = \frac{1}{1 + \frac{G_v M_z^2}{2 \sqrt{2} \pi} \Sigma_R + \text{h.o.}},
\]

\[
\Sigma_R = \left[ \Sigma_{\text{tot}} - (c^2 - s^2) \frac{\Re \Sigma_{z\gamma}(M_z^2)}{M_z^2} \right] \frac{1}{M_S},
\]

and

\[
M_z^2 \Delta s^2 = \left[ \frac{\Re \Sigma_{z\gamma}(M_z^2)}{M_S} \right] \frac{1}{M_S}.
\]

\(\Delta \Pi_z\) is the residual Z wave function factor which obtains by writing the Z propagator \(\chi_z\) as

\[
\Re \chi_z^{-1} = \left(1 + \frac{G_v M_z^2}{2 \sqrt{2} \pi} \Pi_z\right) (p^2 + M_z^2),
\]

\[
M_z^2 \Pi_z = \Sigma_{ww}^\gamma - \frac{1}{p^2 + M_z^2} \Re \left[ \frac{p^2}{M_z^2} \Sigma_{zz}(M_z^2) + \Sigma_{zz}(p^2) \right],
\]

\[
\Pi_z = \Sigma_{\text{term}} + \Delta \Pi_z.
\]

The resummation operates on a gauge invariant quantity since it can be proved that

\[
\left[ M_z^2 \frac{\Sigma}{c^2 - s^2} - \Re \Sigma_{z\gamma}(M_z^2) \right] \frac{\text{bos}}{M_z^2} = \frac{1}{c^2 - s^2} \left[ \Sigma_{ww}^\gamma - s^2 M_z^2 \Pi_{z\gamma}(0) - \Re \Sigma_{zz}(M_z^2) \right] \frac{\text{bos}}{M_z^2},
\]

and the sum of the terms in the last parenthesis is automatically gauge invariant. The same quantity, however, is not ultraviolet-finite and therefore has to be strictly understood in the \(\overline{\text{MS}}\) sense at the scale \(\mu = M_z\). The higher-order terms in Eq. (227) are given by:

\[
\text{h.o.} = \frac{G_v M_z^2}{2 \pi^2} \Delta \Sigma_2 + \frac{1}{2} \frac{\alpha_s(m_t)}{\pi} \left(1 + \frac{\pi}{3} \right) m_t^2 \left[1 + \mathcal{O} \left( \frac{M_z^2}{m_t^2} \right) \right] + \frac{\alpha_s(M_z)}{\pi} \Delta \Sigma_{tq} + 10.55 \left( \frac{\alpha_s(m_t)}{\pi} \right)^2,
\]

where \(\Delta \Sigma_2\) is the two-loop factor proportional to \(m_t^4/M_z^4\), \(\Delta \Sigma_{tq}\) denotes the \(\mathcal{O}(\alpha \alpha_s)\) light quark contribution and the last factor is the three-loop correction computed for six active flavours. In the special case of the \(Z \to b\bar{b}\) width we include the well known correction factor [55], which modifies the vector and axial-vector couplings into \(1 - 4/3 s^2 + \tau\) and \(1 + \tau\).

To illustrate the internal structure of the renormalization procedure we use a well defined and generalizable example. The three bare parameters of the MSM Lagrangian
are related to experimental data by one-loop relations. One example of a solution concerns the bare mixing-angle

\[
\sin^2 \theta = s^2 + \frac{\alpha}{4\pi} s_1 + \left( \frac{\alpha}{4\pi} \right)^2 s_2,
\]

\[
s_1 = \frac{1}{c^2 - s^2} \Sigma - \frac{1}{M_z^2} \Re e \Sigma_{\gamma\gamma}(M_z^2),
\]

\[
s_2 = \frac{1}{(c^2 - s^2)^2} \Sigma^2 - \Pi_{\gamma\gamma}(0)s_1.
\]

(238)

For the \( Z \to f\bar{f} \) amplitude the self-energy corrections give rise to

\[
\gamma^\mu \left[ I_f^{(3)} - 2Q_fV_f + I_f^{(3)} \gamma^5 \right].
\]

(239)

Thus,

\[
V_f = \sin^2 \theta + \frac{\alpha}{4\pi} \frac{1}{M_z^2} \frac{\Re e \Sigma_{\gamma\gamma}(M_z^2)}{1 - \frac{\alpha}{4\pi} \Pi_{\gamma\gamma}} + \mathcal{O}(\alpha^2)
\]

\[
= s^2 + \frac{\alpha}{4\pi} \frac{\Sigma}{c^2 - s^2} + \left( \frac{\alpha}{4\pi} \right)^2 \left[ \frac{\Sigma^2}{(c^2 - s^2)^3} + \frac{\Sigma \Pi_{\gamma\gamma}}{c^2 - s^2} \right] + \mathcal{O}(\alpha^3)
\]

\[
= s^2 + \frac{\alpha(M_z)}{4\pi} \frac{\Sigma}{c^2 - s^2} + \left[ \frac{\alpha(M_z)}{4\pi} \right]^2 \frac{\Sigma^2}{(c^2 - s^2)^3} + \mathcal{O}(\alpha^3).
\]

(240)

After this result we may perform a partial resummation —

\[
V_f = s^2 + \frac{\alpha(M_z)}{4\pi} \frac{1}{c^2 - s^2} \Sigma_{\text{bos}} + \ldots
\]

(241)

As a final comment let us consider the term proportional to the square of the \( Z - \gamma \) transition in the propagators. It would contribute to \( \Pi_z \) with

\[
\Delta \Pi_z \rightarrow \Delta \Pi_z - \frac{G_\mu}{2\pi^2 s^2 c^2} \frac{p^2}{M_z^2(p^2 + M_z^2)} \left[ \Re e \Sigma_{\gamma\gamma}(M_z^2) + \frac{M_z^2}{p^2} \Re e \Sigma_{\gamma\gamma}(p^2) \right]^2
\]

\[
= -\frac{G_\mu}{2\pi^2 s^2 c^2} \frac{p^2}{M_z^2} \left[ p^2 + M_z^2 \right] \left[ -\Re e \Sigma'_{\gamma\gamma}(M_z^2) + \frac{\Re e \Sigma_{\gamma\gamma}(M_z^2)}{M_z^2} + \ldots \right]^2,
\]

(242)

where \( \ldots \) indicates terms of \( \mathcal{O}(p^2 + M_z^2) \). Thus the additional term, being \( \mathcal{O}(p^2 + M_z^2) \), does not contribute to the \( Z \) wave function renormalization factor. It should be noted that there is an easy dictionary of translation with other realizations, for instance:

\[
\rho_f = 4 \rho_z \left( I_f^{(3)} + \delta g_A^f \right)^2,
\]

\[
2Q_f\delta_w \kappa_f = I_f^{(3)} \frac{2Q_f \delta_s^2 + \delta g_A^f - \delta g_V^f}{I_f^{(3)} + \delta g_A^f}.
\]

(243)
Once electroweak corrections are included in the formulation, TOPAZ0 implements initial state QED corrections by a convolution on the weakly corrected kernel distributions with a radiator function, or with structure functions — depending on the experimental set-up. Resummation of soft photon effects and hard photon emission up to $O(\alpha^2)$ is taken into account, and final-state QED radiation with realistic cuts, QED initial–final interference, and initial-state leptonic and hadronic pair production are also included. In the next two appendices further details are given for pure electroweak corrections.

4.3.1 Appendix 1: The self-energies

Starting from the decomposition of Eq. (225) for the vector boson self-energies we give their general expression in terms of two-point scalar form factors [22]. In the following, the first argument, $p^2$, is always left understood:

$$\Pi_{\gamma\gamma} = \frac{2}{3} - 12 B_{21}(M_W, M_W) + 7 B_0(M_W, M_W)$$
$$+ 4 \sum_g \left[ B(m_l, m_l) + \frac{4}{3} B(m_u, m_u) + \frac{1}{3} B(m_d, m_d) \right],$$

$$\Sigma_{3q} = p^2 \left( \frac{2}{3} - 10 B_{21}(M_W, M_W) + \frac{13}{2} B_0(M_W, M_W) \right)$$
$$+ \sum_g \left[ B(m_l, m_l) + 2 B(m_u, m_u) + B(m_d, m_d) \right] - 2 M_W^2 B_0(M_W, M_W),$$

$$\Sigma_{33} = p^2 \Pi_{33} + \sigma_{33},$$

$$\Sigma_{ww} = p^2 \left[ \Pi_{ww}^0 + s^2 \Pi_{ww}^1 \right] + \left[ \sigma_{ww}^0 + s^2 \sigma_{ww}^1 \right],$$

$$\Pi_{33} = \frac{2}{3} - 9 B_{21}(M_W, M_W) + \frac{25}{4} B_0(M_W, M_W) - B_{21}(M_Z, M_H)$$
$$- B_1(M_Z, M_H) - \frac{1}{4} B_0(M_Z, M_H)$$
$$+ \frac{1}{2} \sum_g \left[ B(m_l, m_l) + B(m_u, m_u) + 3 B(m_u, m_u) + 3 B(m_d, m_d) \right],$$

$$\sigma_{33} = -2 M_W^2 B_0(M_W, M_W) + \frac{1}{2} M_Z^2 B_1(M_Z, M_H) + \frac{5}{4} M_Z^2 B_0(M_Z, M_H)$$
$$- \frac{1}{2} M_H^2 B_1(M_Z, M_H) - \frac{1}{4} M_Z^2 B_0(M_Z, M_H)$$
$$- \frac{1}{2} \sum_g \left[ m^2 B_0(m_u, m_u) + m^2 B_0(m_l, m_l) \right]$$
$$+ 3 m^2 B_0(m_u, m_u) + 3 m_d^2 B_0(m_d, m_d),$$

$$\sigma_{ww}^0 = \frac{9}{2} (M_Z^2 - M_W^2) B_1(M_Z, M_W) + \frac{1}{4} (13 M_Z^2 - 21 M_W^2) B_0(M_Z, M_W)$$
$$+ \frac{1}{2} (M_Z^2 - M_H^2) B_1(M_W, M_H) + \frac{1}{4} (5 M_W^2 - M_H^2) B_0(M_W, M_H)$$

88
\[ + \sum_{g} \left[ (m_{l}^{2} - m_{e}^{2})B_{1}(m_{\nu}, m_{l}) - m_{\nu}^{2}B_{0}(m_{\nu}, m_{l}) + 3 (m_{d}^{2} - m_{u}^{2})B_{1}(m_{u}, m_{d}) \right. \\
\left. - 3 m_{u}^{2}B_{0}(m_{u}, m_{d}) \right] , \]

\[
\sigma_{w \bar{w}}^{1} = 2 (M_{w}^{2} - M_{Z}^{2}) \left[ 2B_{1}(M_{Z}, M_{w}) + B_{0}(M_{Z}, M_{w}) \right] - 2 M_{w}^{2} \left[ 2B_{1}(0, M_{w}) + B_{0}(0, M_{w}) \right], \\
\Pi_{w \bar{w}}^{0} = \frac{2}{3} - 9B_{21}(M_{Z}, M_{w}) - 9B_{1}(M_{Z}, M_{w}) + \frac{7}{4} B_{0}(M_{Z}, M_{w}) \\
- B_{21}(M_{w}, M_{\nu}) - B_{1}(M_{w}, M_{\nu}) - \frac{1}{4} B_{0}(M_{w}, M_{\nu}) \\
+ 2 \sum_{g} \left[ B_{21}(m_{\nu}, m_{l}) + B_{1}(m_{\nu}, m_{l}) + 3 B_{21}(m_{u}, m_{d}) + 3 B_{1}(m_{u}, m_{d}) \right], \\
\Pi_{w \bar{w}}^{1} = 8 \left[ B_{21}(M_{Z}, M_{w}) + B_{1}(M_{Z}, M_{w}) - B_{21}(0, M_{w}) - B_{1}(0, M_{w}) \right] \\
+ 2 \left[ B_{0}(0, M_{w}) - B_{0}(M_{Z}, M_{w}) \right], \tag{244} \]

where \( B = 2B_{21} - B_{0} \) and the sum is over the fermionic generations. The factor \( \Delta \) is given by \( \Delta = -2/(n - 4) + \gamma_{\mu} - \ln 4\pi \). The functions \( \chi(x) \), \( G_{n}(y) \) are defined by [78]

\[
\chi(x) = -p^{2}x^{2} + (p^{2} + m_{Z}^{2} - m_{1}^{2})x + m_{1}^{2} , \\
G_{n}(y) = \int_{0}^{1} dx x^{n-1} \ln(x - y) . \tag{245} \]

In terms of \( \chi \) (where \( m^{2} \to m^{2} - i\epsilon \)) we have

\[
B_{0} = \Delta - \int_{0}^{1} dx \ln \chi , \\
B_{1} = -\frac{1}{2} \Delta + \int_{0}^{1} dx x \ln \chi , \\
B_{21} = \frac{1}{3} \Delta - \int_{0}^{1} dx x^{2} \ln \chi , \tag{246} \]

and the corresponding integrals can be written in terms of the \( G_{n} \) functions, for which we write recursion formulae to be worked upwards or downwards, according to the magnitude of \( y \).

The \( \mathcal{O}(\alpha_{s}) \) contributions to \( \Pi_{\gamma \gamma} \ldots \Sigma_{ww} \) are computed according to the formulation of Kniehl (see Ref. [52]). For instance,

\[
\Pi_{\gamma \gamma}^{\alpha} = \Pi_{\gamma \gamma} + \Pi_{\gamma \gamma}^{\alpha_{s}}, \\
\Pi_{\gamma \gamma}^{\alpha_{s}} = \frac{64}{9} \frac{\alpha_{s}(m_{l})}{\pi} \frac{m_{l}^{2}}{p^{2}} [rX + V_{1}(r)] , \tag{247} \]

where \( r = -1/4 \frac{p^{2}}{m_{l}^{2}} \) and the functions \( X \) and \( V_{1} \) are explicitly given in Ref. [52].
4.3.2 Appendix 2: The $Zf \bar{f}$ vertices

In order to introduce the vertex contributions $F_{v,A}^{f}$ we first present the fermion wave function renormalization factors ($f \neq b$):

$$W_{v}^{f} = - \frac{1}{32} \left[ (I_{f}^{[3]} - 2Q_{f}s^{2})^{2} + 1 \right] F_{z} - \frac{1}{8} e^{2} F_{w},$$

$$W_{A}^{f} = - \frac{1}{16} (1 - 8I_{f}^{[3]} Q_{f}s^{2}) F_{z} - \frac{1}{8} e^{2} F_{w},$$

(248)

where $F_{z} = \Delta - \ln \left( M_{X}^{2} \right) - \frac{1}{2}$. With them we can write ($f \neq b$):

$$F_{v}^{f} = W_{v}^{f} v_{f} + W_{A}^{f} I_{f}^{[3]} - 2v_{f} \left( v_{f}^{2} + \frac{3}{4} \right) F_{1}^{f} + 4I_{f}^{[3]} c^{2} F_{2}^{f} - c^{4} I_{f}^{[3]} F_{3}^{f},$$

$$F_{A}^{f} = W_{v}^{f} I_{f}^{[3]} + W_{A}^{f} v_{f} - 2I_{f}^{[3]} \left( 3v_{f}^{2} + \frac{1}{4} \right) F_{1}^{f} + 4I_{f}^{[3]} c^{2} F_{2}^{f} - c^{4} I_{f}^{[3]} F_{3}^{f},$$

(249)

where again $v_{f} = I_{f}^{[3]} - 2Q_{f}s^{2}$. The functions $F_{i}^{f}$ are given in terms of three-point scalar form factors [22]. For instance,

$$F_{1}^{e} = - \frac{1}{4} C_{24}(m_{e}, M_{e}, m_{e}^{2}) - \frac{1}{8} p^{2} \left[ C_{11}(m_{e}, M_{e}, m_{e}^{2}) + C_{23}(m_{e}, M_{e}, m_{e}^{2}) \right] + \frac{1}{8},$$

$$F_{2}^{e} = - \frac{1}{4} C_{24}(m_{e}, M_{e}, m_{e}^{2}) - \frac{1}{8} p^{2} \left[ C_{11}(m_{e}, M_{e}, m_{e}^{2}) + C_{23}(m_{e}, M_{e}, m_{e}^{2}) \right] + \frac{1}{8},$$

$$F_{3}^{e} = -6 C_{24}(M_{e}^{2}, m_{e}, M_{e}^{2}) - p^{2} \left[ C_{0}(M_{e}^{2}, m_{e}, M_{e}^{2}) + C_{11}(M_{e}^{2}, m_{e}, M_{e}^{2}) \right]
+ C_{23}(M_{e}^{2}, m_{e}, M_{e}^{2}) \right) + 1 + \Delta - \ln \left( M_{e}^{2} \right).$$

(250)

In TOPAZO the most general (arbitrary internal and external masses) two-, three- and four-point scalar functions are available [79]. For $bb$ final states the expression for vertices contains additional $m_{t}$-dependent terms which can be found in Ref. [77].

4.4 ZFITTER basics

Here, we introduce explicit expressions for $\Delta r$ and the weak form factors of $Z$ decay and of fermion pair production process $e^{+}e^{-} \rightarrow f \bar{f} \rho, \kappa$ to order (1loop,$\alpha$) and order (2loop,$\alpha\alpha_{s}$) as used in Eqs. (71–73) and (75–77) of the main text.

4.4.1 Muon life-time

The virtual, non-photonic one-loop corrections to the muon life-time are:

$$\Delta r^{1\text{loop},\alpha} = \frac{\alpha}{4\pi} \left\{ -\frac{2}{3} \left( 1 + 2 \sum_{f} Q_{f}^{2} \ln \frac{m_{f}^{2}}{M_{w}^{2}} \right) + \frac{R}{(1-R)^{2}} \left[ W(M_{w}^{2}) - Z(M_{w}^{2}) \right]
+ \frac{1}{1-R} \left[ W(0) - W(M_{w}^{2}) - \frac{5}{8} R(1+R) + \frac{11}{2} + \frac{9}{4} \frac{R}{1-R} \ln R \right] \right\}. \quad (251)$$
Here and in the following sections, we use the abbreviations,

\[ R = \frac{M^2_w}{M^2_z}, \quad r_w = \frac{M^2_H}{M^2_w}, \quad r_z = \frac{M^2_H}{M^2_z}. \tag{252} \]

The \( \Delta r \) was introduced in (E.8) and (F.3) of Ref. [80]. The \( t \) mass dependent terms may be found in the appendix of [81].

### 4.4.2 Partial widths of the Z boson

The two form factors for each fermionic partial width of the Z boson are in one loop order and in the approximation of vanishing external fermion masses [81]:

\[
\rho^\text{loop,\alpha}_f = 1 + \frac{\alpha}{4\pi(1-R)} \left\{ \frac{Z(M^2_z) + Z^\prime(M^2_z) - W(0)}{1-R} + \frac{11}{2} - \frac{9R}{4(1-R)} \ln R + u_f + \delta\rho^t_{\text{et,}f} \right\}, \tag{253} 
\]

\[
\kappa^\text{loop,\alpha}_f = 1 + \frac{\alpha}{4\pi(1-R)} \left\{ \frac{R}{1-R} \left[ Z(M^2_z) - W(M^2_w) \right] + M(M^2_z) + \frac{(1-R)^2}{R}Q^2_f \left[ V_{1z}(M^2_z) + \frac{3}{2} \right] - \frac{1}{2} \left[ u_f + \delta\rho^t_{\text{et,}f} \right] \right\}, \tag{254} 
\]

\[
u_f = \frac{1}{2R} \left[ 1 - 6|Q_f|(1-R) + 12Q^2_f(1-R)^2 \right] V_{1z}(M^2_z) + \frac{3}{2} 
+ \left[ 1 - 2R - 2|Q_f|(1-R) \right] V_{1w}(M^2_z) + \frac{3}{2} + 2R \left[ V_{2w}(M^2_z) + \frac{3}{2} \right]. \tag{255} 
\]

### 4.4.3 Auxiliary functions

All \( W \) and \( Z \) boson self-energy functions are sums of bosonic and fermionic parts, for example,

\[ W(0) = W_b(0) + W_f(0). \tag{256} \]

The bosonic parts are given in appendix A of [80].\(^{14}\)

\[
W_b(0) = \frac{5R(1+R)}{8} - \frac{17}{4} + \frac{5}{8R} - \frac{r_w}{8} + \left[ \frac{9}{4} + \frac{3}{4R} - \frac{3}{1-R} \right] \ln R + \frac{3r_w}{4(1-r_w)} \ln r_w, \tag{257} 
\]

\[
W_b(M^2_w) = -\frac{157}{9} + \frac{23}{12R} + \frac{1}{12R^2} - \frac{12r_w}{2} + \frac{r_w^2}{12} + \frac{1}{R} \left[ -\frac{7}{2} + \frac{7}{12R} + \frac{1}{24R^2} \right] \ln R 
+ r_w \left( -\frac{3}{4} + \frac{r_w}{4} - \frac{r_w^2}{24} \right) \ln r_w + \left( \frac{1}{2} - \frac{r_w}{6} + \frac{r_w^2}{24} \right) \frac{L_{\text{wH}}(M^2_w)}{M^2_w}. 
\]

\(^{14}\)One should eliminate from there the (approximate) fermionic parts proportional to \( \text{Tr}Q^2_f \) and \( N_f \).
The vertex functions are in the limit of vanishing fermion masses:

\[ V_{1\nu}(s) = -\frac{7}{2} - 2R_\nu - (3 + 2R_\nu) \ln(-\bar{R}_\nu) + 2(1 + R_\nu)^2 \left[ \text{Li}_2(1 + \bar{R}_\nu) - \text{Li}_2(1) \right], \]

\[ V_{2\nu}(s) = -\frac{1}{6} - 2R_w - \left( \frac{7}{6} + R_w \right) \frac{L_{ww}(s)}{s} + 2R_w (R_w + 2) \mathcal{F}_3(s, M_w^2), \]

with

\[ \bar{R}_\nu = R_\nu - i\gamma_\nu, \quad \gamma_\nu = \frac{M_\nu \Gamma_\nu}{s}, \quad R_\nu = \frac{M_\nu^2}{s}. \]
The additional $t$ mass corrections to the $Z b \bar{b}$ vertex and the counter term are at the $Z$ resonance\footnote{The net one-loop finite $t$ mass effect from the two vertices and the counter term is taken into account in ZFITTER by a variable $\mathrm{VTB}$.}:

\[
V_{1w}^t(M_z^2) = \frac{1}{R} \int_{0}^{1} dy \left\{ \frac{1}{2} - 3y(1-y) \right\} \ln |r_1| + 2r R \ln |r_2| - r R + (2 + R) \\
\times \left[ \bar{F}_1(r) - \bar{F}_1(0) - \left( \frac{3}{2} + R \right) \left[ \bar{F}_2(r) - \bar{F}_2(0) \right] + \frac{1}{2} r R (2 + R) \bar{F}_2(r) \right] \\
- \frac{2r R(1 - R)}{(1 - 4 R)} \left[ 1 + \ln |r_2| - 4 \bar{F}_1(r) + \frac{1 - r}{2} \bar{F}_2(r) \right] \right\}, \quad (268)
\]

\[
V_{2w}^t(M_z^2) = \frac{1}{R} \int_{0}^{1} dy \left\{ -(2 + R) \left[ F_2(r) - F_2(0) + r \left[ 2 R \ln |r_3| + \frac{1 - 3 R}{4} (\ln |r_4| - 1) \right.ight. \\
\left. \left. + \frac{1}{2} (r - 2r R - 4 R - 4) F_1(r) + \frac{1}{4} (1 - r + 2 R + 2r R) F_2(r) \right] \right\}, \quad (269)
\]

\[
\delta \rho_{e,b}^t = - \frac{r (1 + 2 R)}{6(1 - r)} \left[ \frac{1}{2} (5r - 11) + \frac{3 r (r - 2)}{1 - r} \ln r \right], \quad (270)
\]

with $r = r_i$ and

\[
\begin{align*}
  r_1 &= \frac{r R}{y(1-y)} - 1, \quad r_2 = r - (1-y)\frac{y^2}{R}, \\
  r_3 &= y + (1-y)r, \quad r_4 = 1 - (1-y)\frac{y^2}{R}, \\
  F_i(r) &= f_i(1+r), \quad \bar{F}_i(r) = f_i(r,1), \quad i = 1, 2,
\end{align*}
\]

\[
\begin{align*}
  f_1(a,b) &= \frac{1}{a-b-y/R} \ln \frac{a-y(1-y)/R}{ay+b(1-y)}, \\
  f_2(a,b) &= \frac{1-y}{a-b-y/R} \left( b + \frac{y^2}{R} \right) f_1(a,b).
\end{align*}
\]

A completely analytical expression, valid at arbitrary $s$ may be found in [84]. In the next section, we will need in addition the photonic self energy function $A(s)$, and $D_Z(s)$:

\[
A(s) = \frac{34}{3} + 8 R_w + \left( \frac{17}{6} + 2 R_w \right) \frac{L_{ww}(s)}{s}, \quad (272)
\]

\[
D_{Z^s}(s) = \frac{34 R}{3} - \frac{35}{18} \ln \frac{R}{12 R} - 2 R^2 \frac{L_{ww}(s)}{s} + \left( -2 R^2 - \frac{17 R}{6} + \frac{2}{3} + \frac{1}{24 R} \right) \\
\times L_{ww}(s) - \mathcal{R} e L_{ww}(M_z^2) \\
+ \frac{1}{2} \left( \frac{R_z}{12} (1 - r_z)^2 + \left[ 1 + (1 - r_z) \right. \right. \\
\left. \left. \left( 10 - 5 r_z + r_z^2 \right) R_z + (1 - r_z)^3 R_z^2 \right] \frac{\ln r_z}{24} + \left[ 11 - 4 r_z + r_z^2 \right. \right. \\
\left. \left. + (1 - r_z) R_z \right] L_{Z^H}(s) + \left( \frac{1}{2} - \frac{r_z}{6} + \frac{r_z^2}{24} \right) \frac{L_{Z^H}(s) - \mathcal{R} e L_{Z^H}(M_z^2)}{s} \right), \quad (273)
\]

\[
D_{Z^f}(s) = \frac{M_z^2}{M_z^2 - s} \left[ Z_f(s) - \mathcal{R} e Z_f(M_z^2) \right]. \quad (274)
\]
4.4.4 Form factors of the process $e^+e^- \to f\bar{f}$

The virtual, non-photonic corrections to fermion pair production, including Bhabha scattering, may be described by four form factors per production channel as indicated in Eq. (21). The contributions from the $ZZ, WW$ box diagrams are vanishingly small at the $Z$ peak. Leaving them out\textsuperscript{16}, it is [83]:

\[
\rho_{e,f}(s)^{\text{loop}}, \alpha = 1 + \frac{\alpha}{4\pi(1-R)} \left\{ Z(M_z^2) - W_f(0) + \frac{19}{12} \frac{5}{8R} + \frac{r_w}{8} + \frac{3}{4} \left[ - \frac{r_w \ln r_w}{1 - r_w} + \left( \frac{1}{1 - R} - \frac{1}{R} \right) \ln R \right] + \frac{5L_{ww}(s)}{6s} + D_z(s) + 2R V_{z}(s) + \left[ -2R + \frac{1}{2} \right] \right. \\
+ \frac{1}{4} (v_e + v_f) V_{i}(s) + \frac{1}{8R} \left[ 1 + 3 (v_e^2 + v_f^2) \right] V_{iz}(s) + \frac{1}{2} \delta \rho_{ct,f}^t \left\} , \quad (275)\]

\[
\kappa_f(s)^{\text{loop}}, \alpha = 1 + \frac{\alpha}{4\pi(1-R)} \left\{ \frac{R}{1-R} \left[ Z(M_z^2) - W(M_w^2) \right] - \frac{9}{2} \frac{L_{ww}(s)}{12s} - \frac{M(s) - RA(s) + (R_w - 2R) V_{2w}(s)}{12s} \right. \\
+ \frac{2R - |Q_f| - \frac{1}{4} (v_e + v_f) + (|Q_f| - 1) R_z}{12s} V_{i}(s) \\
+ \left[ - \frac{1}{8R} v_e (1 + v_e) - \frac{1}{2} |Q_f| v_f (1 - R_z) \right] V_{iz}(s) - \frac{1}{2} \delta \rho_{ct,f}^t \left\} , \quad (276)\]

\[
\kappa_e,f(s)^{\text{loop}}, \alpha = 1 + \frac{\alpha}{4\pi(1-R)} \left\{ \frac{2R}{1-R} \left[ Z(M_z^2) - W(M_w^2) \right] - \frac{8}{9} - \frac{L_{ww}(s)}{6s} \right. \\
- \frac{2M(s)}{3} \frac{R - 2RV_{2w}(s)}{R_w} - \frac{A(s) - \frac{2}{3}}{12s} \right. \\
+ \frac{2R - \frac{1}{2} - \frac{1}{4} (v_e + v_f)}{12s} V_{i}(s) + \left[ - \frac{1}{8R} \left[ 1 - \frac{3}{16R} (v_e^2 + v_f^2) \right] \\
+ \frac{1 - R}{R} \left( Q_e^2 + Q_f^2 \right) (1 - R_w) \right] V_{iz}(s) - \frac{1}{2} \delta \rho_{ct,f}^t \left\} . \quad (277)\]

4.4.5 The $\mathcal{O}(\alpha \alpha_s)$ corrections to electroweak observables

Here we follow Ref. [85], based on the second Ref. [52].

\[
\Delta\alpha^{2\text{loop}}, \alpha_s = \frac{\alpha_s}{3\pi^2} Q_t^2 \frac{m_t^2}{Q_t^2} R e \left\{ \Pi_t^{VP}(Q^2) + \frac{45}{4} \frac{Q^2}{m_t^2} \right\} , \quad (278)\]

\[
\Delta r^{2\text{loop}}, \alpha_s = \frac{\alpha_s}{12\pi^2} \frac{m_t^2}{M_w^2} \frac{Q_t^2}{R e} \left\{ M_t^2 \Pi_t^{VP}(M_w^2) + Q_t^2 \Pi_t^{VP}(M_w^2) \right\} .
\]

\textsuperscript{16}The interested reader may find expressions for $WW, ZZ$ boxes in Ref. [83] and in the subroutine ROKANC($..., t - s_i, - s_i, - t_i, ...$) of ZFITTER for the $s$ channel ($t > 0$). For the $t$ channel (used in Bhabha scattering) the call is ROKANC($..., s_i, t_i, u_i, ...$).
Here, the two-loop functions are:

\[
\Pi_t^{VP}(Q^2) = \frac{1}{\alpha} \left\{ \frac{55}{4} - 26\alpha + 3(1 + x_\alpha)(1 - 6\alpha) f_\alpha - 2 \left[ \alpha \left( 2x_\alpha^2 - 3x_\alpha + 2 \right) + 2x_\alpha \right] f_\alpha^2 + 2 \left( 4\alpha^2 - 1 \right) I_\alpha + 4\alpha \left( 2\alpha - 1 \right) (4\alpha + 1) J_\alpha \right\},
\]

\[
\Pi_t^{AP}(Q^2) = \frac{1}{\alpha} \left\{ \frac{55}{4} - \frac{19}{2} \alpha + 12\alpha^2 + 3(1 + x_\alpha) \left( 1 + 12\alpha + 4\alpha^2 \right) f_\alpha + 2 \left[ 2x_\alpha \left( 3\alpha^2 - 1 \right) + \alpha \left( 7x_\alpha^2 - 3x_\alpha + 7 \right) \right] f_\alpha^2 - 2(1 + 2\alpha) (1 + 4\alpha) I_\alpha - 4\alpha (1 + 4\alpha)^2 J_\alpha \right\},
\]

\[
\Pi_t^{WP}(Q^2) = \frac{1}{4\alpha} \left\{ 55 - \frac{71}{2} \alpha - 10\alpha^2 - 8\alpha G(x_b) + 2 \left( 6 + 9\alpha - 5\alpha^2 \right) f_b + 2 \left[ 4x_b \left( \alpha^2 - \alpha - 1 \right) + \alpha (5 - 4\alpha) \right] f_b^2 + 4 \left( \alpha - 2 \right) \left( \alpha + 1 \right)^2 I_b + 8\alpha \left( \alpha - 2 \right) (\alpha + 1) J_b \right\},
\]

\[
\frac{d \Pi_t^{VP}(Q^2)}{dQ^2} = -\frac{1}{m_t^2} \left[ \frac{43}{4} + 18\alpha + (10\alpha + 3) (1 + x_\alpha) f_\alpha + 2 (5\alpha - 2) x_\alpha f_\alpha^2 \right]
\]
In this subsection we describe the main options implemented in the version of BHM used for the present study to estimate the theoretical uncertainties. Out of the actual different user-accessible flags and their possible values, only the ones used in the present analysis are discussed. The rest of the flags and/or values either have a negligible effect on the predictions or exist only for testing purposes.

- IRES=0, 1, 2
  The resummation of the one-loop, one-particle irreducible contributions in $\Delta r$ and, in general, in the whole set of self-energies is implemented in BHM in three different ways. These can be considered representative of at least two rather different philosophies. The basic difference comes from the treatment of the $(\Sigma_f^\gamma)^2$ mixing term. If IRES=0, then the resummation comes from the resolution of the renormalization equations keeping that term (see for instance W. Hollik in Ref. [2]). In this case,
not only leading top and Higgs terms are re-summed but also non-leading ones. This is the default working option. If IRES=1, 2, then the renormalization equations are solved keeping strictly one-loop contributions and the inclusion of higher-order terms coming from one-loop, one-particle irreducible diagrams is explicitly done for \( \Delta r \) and for each self-energy. If IRES=1, the prescription used to treat the top, Higgs and remainder terms follows the suggestions of Halzen, Kniehl and Stong [86], whereas if IRES=2, then it follows the suggestions from S. Fanchiotti and A. Sirlin [49]. Both differ in the detailed treatment of the Higgs and remainder terms.

- **ITWO=1, 2**
  This flag allows the choice of scale in vertex corrections. The dominant effect happens in the b-vertex. If ITWO=1, then the \( \alpha(0) \) scale is used, whereas if ITWO=2, then the \( G_\mu \) scale is used. The later is the default working option.

- **IFAC=0, 1, 2**
  This flag allows the choice of different factorization schemes for the final-state corrections. If IFAC=0 then no factorization at all is applied for weak vertex, QED and QCD final-state corrections. That means that these three kinds of corrections are independently applied to the vacuum-polarization dressed amplitudes. This is the default working option. If IFAC=1, then QED corrections are applied on top of weak vertex ones, and if IFAC=2, then QCD corrections are also applied on top of weak vertex and QED ones.

- **IQCD=3, 4**
  This flag allows the choice of different treatments of the QCD corrections to electroweak loops. Of its possible values, only two have been used in the present study: if IQCD=3, then the exact AFMT correction is implemented, whereas if IQCD=4, then the Sirlin’s scale \( \xi = 0.248 \) is used. These two approaches have already been discussed elsewhere in the text.

To summarize, BHM runs in \( 3 \times 2 \times 3 \) electroweak \( \times 2 \) QCD options. We have added Table 26, showing the effect of the working options of BHM on theoretical errors.

### 5.2 LEPTOP options

Several options to the preferred formulae used in LEPTOP have been chosen and variations of these formulae have been made. Usually each option consists of the addition of an extra term corresponding to a rough guess of the value of the uncalculated higher-order terms. We then make all possible combinations of these options — \( 2^n \) in total, where \( n \) is the number of options. Among all these \( 2^n \) combinations we locate those yielding the minimum and the maximum values of the observables and take as the estimate of the theoretical errors their deviations from the central values.

Theoretical uncertainties come from two different sources: 1) not-yet-calculated Feynman diagrams, 2) not-yet-calculated terms in a given diagram. The first source is represented in LEPTOP by the gluonic corrections to the electroweak vertices of light quarks, \( F_{iq} \), where \( q = u, d, s, c; \ i = V, A \). In this case a crude estimate of these corrections is used as a basis for options:
\[ \delta F_{i \bar{q}} = \frac{\alpha_s}{\pi} F_{i \bar{q}}. \] (291)

The second source is represented by non-leading terms of: a) higgs; b) \( \alpha_s^2 \) corrections to the \( t \bar{t} \) loop in the Z boson self-energy and the \( t - b \) loop in the W boson self-energy, denoted by \( \delta V_i^q \) and \( \delta V_i^{\alpha_s^2} \) respectively; and c) by the \( t \bar{t} \) vertex contribution to \( Z \to b \bar{b} \) decay denoted by \( \delta \phi^2 \).

In cases a) and b) the missing non-leading terms are estimated multiplying the corresponding leading terms by a factor \( 2/t \), where \( t = (m_t/M_Z)^2 \). In case c) the leading correction itself is so small that it is taken as a measure of uncertainty. Thus the LEPTOP uncertainties are:

\[ \Delta V_i^q = (2/t) \delta V_i^q = (2/t) \left( -\frac{\alpha}{\pi} \frac{A(M_H/m_t)t^2}{16s^2c^2} \right) \] (292)
in accord with Eq. (206);

\[ \Delta V_i^{\alpha_s^2} = (2/t) \delta V_i^{\alpha_s^2} = (2/t)(-1.25\alpha_s^2(m_t)t) \] (293)
in accord with Eq. (205);

\[ \Delta \phi^2 = -1.37 \frac{3 - 2s^2}{2s^2c^2} \frac{\pi^2}{3\alpha_s^2(m_t)t} \] (294)
in accord with the first term in Eq. (220);

\[ \Delta \phi^2 = \delta \phi^2 = \frac{3 - 2s^2}{2s^2c^2} \frac{\alpha s^2}{16\pi s^2c^2} \tau_b^{[2]} \] (295)
in accord with the second term in Eq. (220), and as \( \Delta \phi^2 \) is much larger than \( \delta \phi^2 \), the latter will be neglected;

\[ \Delta F_{i \bar{q}}^{\alpha_s} = \delta F_{i \bar{q}}^{\alpha_s} = \frac{\alpha_s}{\pi} F_{i \bar{q}} \] (296)
in accord with Eq. (291).

Explicit calculations [1, 76] give:

\[ F_{V_u} = -0.00169, \quad F_{A_u} = -0.00165, \] (297)

\[ F_{V_d} = 0.00138, \quad F_{A_d} = 0.00137. \] (298)

Note that the uncertainties \( \Delta F_{i \bar{q}}^{\alpha_s} \) produce corresponding uncertainties in electroweak corrections \( \delta \Gamma_q \) to the partial widths \( Z \to q \bar{q} \):

\[ \delta \Gamma_q = 24s \Gamma_0 2I_q^{[3]} [(1 - 4|Q_q|s^2)F_{Vq} + F_{Aq}]. \] (299)

With the above numbers

\[ \Delta \Gamma_u = \Delta \Gamma_c = -1.9 \left( \frac{\alpha_s}{\pi} \right) \text{MeV} \simeq -0.08 \text{MeV}, \] (300)
\[
\Delta \Gamma_d = \Delta \Gamma_s = -2.0 \left( \frac{\alpha_s}{\pi} \right) \text{MeV} \approx -0.08 \text{MeV}.
\] 

(301)

So that the sum over four light quarks is

\[
\Sigma^d_i \Delta \Gamma_q = -0.3 \text{MeV}.
\]

(302)

Now the options of LEPTOP can be formulated:

- **Options 1,2**

  Uncertainty \( \Delta V_i^{_{\pm2}} \), given by Eq. (292), is added to (Option 1) or subtracted from (Option 2) the functions \( V_i, i = m, A, R \).

- **Options 3,4**

  Uncertainty \( \Delta V_i^{_{\pm2}} \), given by Eq. (293) is added to (Option 3), or subtracted from (Option 4) the function \( V_i, i = m, A, R \).

- **Options 5,6**

  Uncertainties \( \Delta \Gamma_q (q = u, d, s, c) \) given by Eqs. (300), (301) are added to (Option 5) or subtracted from (Option 6) the partial widths \( \Gamma_q \).

- **Option 7**

  Uncertainty \( \Delta \phi^{_{\pm2}} \) is added to the function \( \phi(t) \) given by Eq. (219).

Theoretical uncertainties for observables, assuming \( M_H = 300 \text{GeV}, m_t = 175 \text{GeV}, \) \( \hat{\alpha}_s (M_Z) = 0.125 \) are shown in Table 27.

### 5.3 TOPAZ0 options

In this subsection we describe the options implemented in TOPAZ0 version 2.0 for studying the theoretical uncertainties. A general comment is in order here. Some of the TOPAZ0 electroweak options were originally designed to produce a conservative estimate of the uncertainty. If nothing stands against a certain option then we accept it, even if it goes against our own philosophy.

- **OU0.EQ. 'N' ('Y')**

  This is the primordial option since it controls the partial resummation of bosonic self-energies. If **OU0.EQ. 'N'**, then in \( s^2 \) and \( \Delta s^2 \), we use:

\[
\Sigma \to \Sigma^{_{\text{ferm}}}, \quad \Delta s^2 = \frac{\Sigma^{_{\text{bos}}}}{c^2 - s^2},
\]

(303)
i.e., all bosonic self-energies are expanded. Otherwise for \texttt{OU0.EQ.}'Y'\ we use:

$$\Sigma \rightarrow \Sigma_R = \Sigma^{\text{tot}} - \left(c^2 - s^2\right) \frac{\text{Re} \Sigma_{\gamma}(M_z^2)}{M_z^2},$$

$$M_z^2 \Delta s^2 = \text{Re} \Sigma_{\gamma}(M_z^2). \quad \text{(304)}$$

For \texttt{OU0.EQ.}'Y'\ TOPAZ0 will automatically select \texttt{OU2} = 'N'\ and \texttt{OU3} = 'Y'.

- \texttt{OU1.EQ.}'N'(Y')

The default choice requires that in

$$\delta g_f^v = \frac{\alpha}{4\pi} \left[ \frac{2F_f^v - \frac{1}{2} v_f \Delta \Pi_x}{c^2 s^2} - 2Q_f \Delta s^2 \right],$$

$$\delta g_f^A = \frac{\alpha}{4\pi} \left[ \frac{2F_f^A - \frac{1}{2} f_f^{(3)} \Delta \Pi_x}{c^2 s^2} \right], \quad \text{(305)}$$

everything must be expanded in terms of $\alpha \equiv \alpha(0) = 1/137.036\ldots$, while for \texttt{OU1.EQ.}'Y'\ the parameter expansion is selected as $\alpha \equiv \alpha(M_z)$.

- \texttt{OU2.EQ.}'N'(Y')

This option deals with the so-called problem of expansion. The default for TOPAZ0 is the expanded solution where, for instance, a $Z$ partial width is computed according to

$$\Gamma_f = \frac{G_F M_z^3}{6\sqrt{2}\pi} N_c \rho_z \left[ \hat{v}_f^2 + \frac{1}{4} + 2\hat{v}_f \delta g_f^v + 2f_f^{(3)} \delta g_f^A \right],$$

$$\hat{v}_f = f_f^{(3)} - 2Q_f \delta s^2, \quad \text{(306)}$$

where for the moment we assume that there is no final-state QED + QCD radiation, factor, whose treatment is explained by one of the following options. If, instead, \texttt{OU2.EQ.}'Y', then the electroweak corrected vector and axial-vector couplings are squared numerically.

- \texttt{OU3.EQ.}'Y'(N')

Two different procedures are introduced for dealing with the physical Higgs contribution to the self-energies. If \texttt{OU3.EQ.}'Y', then by working in a $\overline{MS}$ environment with a scale $\mu$ set to $M_z$ we extract from $\Sigma^{\text{bos}}$ the physical Higgs contribution, $\Sigma^H$, and redefine

$$\Sigma^{\text{ferm}} \rightarrow \Sigma^{\text{ferm}} + [\Sigma^H]_{\overline{MS}}, \quad \text{(307)}$$

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where \( \Sigma^H \) is subject to no additional expansion, such as leading or sub-leading behaviour with \( M_H \). Of course, OU3.EQ.'N' leaves the Higgs contribution expanded as for any other bosonic contribution. This option reflects and partially illustrates one of the defining rules of TOPAZ0 — a certain reluctance to accept the isolation and a different treatment for something which can be considered as the leading part of some quantity only for \emph{extraordinary} values of some of the parameters.

- **OU4.EQ.'N' ('\( \mathcal{Y} \)'**)

When we consider the mass corrections to the \( \bar{b}b \) partial decay rate there will be something like

\[
-6 \frac{m_b}{M_Z^2} \left[ 2I^{(3)}_j \delta g^f_A + \left( \delta g^f_A \right)^2 \right].
\]

In these mixed corrections there is an additional uncertainty connected with the choice of \( m_b \) — i.e., pole mass or running mass.

- **OU5.EQ.'N' ('\( \mathcal{Y} \)'**)

According to the strategy that all non-leading and gauge-variant quantities should be expanded, in Eq.(305) we use as the zero order approximation,

\[
s^2 = \frac{1}{2} \left[ 1 - \frac{4 \pi \alpha(M_Z)}{\sqrt{2} G_\mu M_Z^2} \right].
\]

However, since perturbation theory rearranges itself in such a way that the expansion of \( \sin^2 \theta^\text{eff} \) starts with \( \hat{s}^2 \), i.e.,

\[
\sin^2 \theta^\text{eff} = \hat{s}^2 + \frac{1}{2} \delta g^I_A + \frac{1}{2} \left( 4 \hat{s}^2 - 1 \right) \delta g^I_A,
\]

we have envisaged the possibility of reorganizing the structure of the pseudo-observables such that to all orders the bare weak mixing angle has an expansion starting with \( \hat{s}^2 \). Combined with OU1.EQ.'N' ('\( \mathcal{Y} \)') this option tells us that the expansion parameter, formally \( \alpha/(4 \pi s^2 c^2) \), can be set to

\[
\frac{G_\mu M_Z^2}{2 \sqrt{2} \pi^2} \times \left\{ 1; \rho_x^R, 1 - \Delta \alpha; \rho_x^R (1 - \Delta \alpha) \right\}.
\]

- **OU6.EQ.'N' ('\( \mathcal{Y} \)'**)

This option deals with factorization of electroweak corrected kernels and QCD radiation. For instance, if OU2.EQ.'N' — the expanded solution — we still distinguish between a non-factorized solution,

\[
\Gamma_j = \frac{G_\mu M_Z^3}{6 \sqrt{2} \pi} N_c \rho_x \left[ \hat{v}_j \hat{R}_j^f + \frac{1}{4} R_f^f + 2 \hat{v}_j \delta g^f_A + 2I^{(2)}_j \delta g^f_A \left( 1 - 6 \frac{m_f}{M_Z^2} \right) \right],
\]
and a fully factorized solution. A special treatment is of course devoted to \( b\bar{b} \) final state in order to reproduce the FTJ most correction term.

- **OU7.EQ.'N'(Y)**
  
  Higher-order QCD corrections to \( \rho_x \) are implemented with the exact AFMT term or by subtracting from \( \Sigma^{\text{term}} \) the leading \( m_t \) term and replacing it with the corresponding one evaluated at a scale of \( \xi = 0.248 m_t \).

To summarize: TOPAZ runs in 2\(^{a}\) or 2\(^{b}\) electroweak \( \times 2 \) QCD options. While we have devoted a special section to the effect of OU0.EQ.'N' and OU2.EQ.'Y', here we present a short Table 28 to indicate the increasing spread in the theoretical errors while the various flags are switched on.

### 5.4 ZFITTER options

In this subsection we describe some electroweak and QCD options implemented in ZFITTER (version 4.9). Simultaneously, we give a description of the flags, implemented in version 4.9 for studying the theoretical uncertainties.

- **OZ1: IAMT4=3,2,1**
  
  The realization of leading and remainder terms, given by formulae (71–84), is the default, **IAMT4=3**. If **IAMT4=2**, then \( X \) is not included in the leading terms and it stays a part of remainders [87]. If **IAMT4=1**, then both \( X \) and 2-loop-\( \alpha_\alpha \) terms are in the remainders. In the last case the 3-loop-\( \alpha_\alpha \) term is also placed in remainders. In the result of numerical investigations it was revealed that for **IAMT4=2** the remainder terms in \( \rho_f \) and \( \kappa_f \) are not small. Since this contradicts our philosophy to keep remainder terms small, one could exclude it from the set of working options. However, it was found that the difference between **IAMT4=3** and 1 is rather small, therefore this option doesn’t sizably influence the theoretical errors.

- **OZ2: IHIGS=0,1**
  
  This option governs the resummation of the leading Higgs contribution in \( \Delta r \)

\[
\Delta r_{\text{Higgs}} \sim \frac{\sqrt{2} G_\mu M_H^2}{4\pi^2} \frac{11}{12} \left( \ln \frac{M_H^2}{M_W^2} - \frac{5}{6} \right), \quad M_H \gg M_W . \tag{313}
\]

If **IHIGS=0** (the default), then the Higgs contribution is not re-summed. If **IHIGS=1** and if

\[
\ln \frac{M_H^2}{M_W^2} - \frac{5}{6} \geq 0 , \tag{314}
\]

then it is re-summed — i.e., it is extracted from remainders with the scale \( \alpha/s_W \) and put to the leading terms with the scale \( G_\mu \), as in (313). We observed, that 10/12 of \( \Delta r_{\text{Higgs}} \) is contained in \( X \). Therefore, if **IAMT4=3**, then only 1/12 of it is additionally re-summed. The influence of this option on theoretical errors was found to be tiny. For this reason, the Higgs resummation has not been implemented in \( \rho_f \) and \( \kappa_f \).
023: ISCRE=0, 1, 2
This option defines the scale of the remainder terms. If ISCRE=0 (the default), then the scale is \( \alpha/s_W^2 \), if ISCRE=2 it is \( G_\mu \). [More precisely, \( G_\mu \) in \( \rho_f \) and \( \kappa_f \) and \( G_\mu(1-\Delta \alpha) \) in \( \Delta r \).] For ISCRE=1, the scale of the remainder in \( \Delta r \) is set equal to \( G_\mu(1-\Delta r_L) \), which was not included in the set of working options, since its influence is much smaller than the previous one’s. The effect of the variation of the scale of the remainder term on the theoretical bands was found to be dominating. Especially influential is the scale variation in \( \Delta r \), which introduces terms of the order \( c_w^2/s_W^2 \Delta \rho \cdot \Delta r_{\text{rem}} \). This illustrates the importance in the calculation of the next-to-leading term of the order \( O(G_\mu^2 m_t^2 M_Z^2) \).

024: IFACR=0, 1, 2, 3
It realizes four subsequent expansions of \( \Delta r \) as they are given in (99). The first fully non-expanded option is the default. Only the first expansion (the second row) was retained in the set of working options. The last two were excluded, since they contradict Sirlin’s theorem on mass singularities. This leads to a visible spread of theoretical bands, although one much smaller than that of the previous option.

025: IFACT=0, 1, 2, 3, 4, 5
It realizes six subsequent expansions for \( \rho_f \) and \( \kappa_f \). The default, IFACT=0, corresponds to the non-expanded realizations (72–73). The first four options for \( \rho_f \), IFACT=0, 1, 2, 3, are exactly the same as are given in (99), while \( \kappa_f \) is expanded for IFACT=1, 2, 3:

\[
\kappa_f = \kappa_L + \Delta \kappa_{f,\text{rem}} = 1 + \frac{c_w^2}{s_W^2} \Delta \rho_x + \Delta \kappa_{f,\text{rem}}. \tag{315}
\]

For IFACT=4, 5, we linearize the full expression (63). Introducing

\[
\rho_L = \frac{1}{1-\Delta \rho}, \tag{316}
\]

\[
\tilde{g}_{vl}^f = 1 - 4|Q_f| s_w^2 \kappa_L, \tag{317}
\]

see (72) and (80), we have for IFACT=4,

\[
\Gamma_f = \frac{G_\mu M_Z^3}{24\sqrt{2}\pi} N_c \left\{ (\rho_L + \Delta \rho_{f,\text{rem}}) \left[ (\tilde{g}_{vl}^f)^2 R_v^f + R_A^f \right] \\
+ \rho_L \left[ -8s_w^2 \tilde{g}_{vl}^f \Delta \kappa_{f,\text{rem}} R_v^f \right] \right\}. \tag{318}
\]

Realizing, that

\[
R_{v,A}^f = 1 + \Delta R_{v,A}^f, \tag{319}
\]

for IFACT=5 finally ending up with a fully expanded equation for the partial widths:

\[
\Gamma_f = \frac{G_\mu M_Z^3}{24\sqrt{2}\pi} N_c \left\{ \rho_L \left[ (\tilde{g}_{vl}^f)^2 R_v^f + R_A^f \right] + \Delta \rho_{f,\text{rem}} \left[ (\tilde{g}_{vl}^f)^2 + 1 \right] \\
+ \rho_L \left[ -8s_w^2 \tilde{g}_{vl}^f \Delta \kappa_{f,\text{rem}} \right] \right\}. \tag{320}
\]

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It was found that the spread of error bands gradually grows while coming from IFACT=0 to IFACT=5. For this reason, only these two limiting cases were left among the working options. This option was found to be rather influential.

- **OZ6**: ISCAL=0,1,2,3,4
  
  is the only QCD option. At the default, ISCAL=0, we implemented the exact AFMT correction. For ISCAL=2,1,3 we implemented the $\xi$-factor as given in Ref. [65]. Finally, for ISCAL=4, Sirlin’s scale $\xi = 0.248$ (see Ref. [62]) was implemented. Only ISCAL=0,4 were left among working options.

To summarize, **ZFITTER** runs in $2^5$ electroweak $\times$ 2 QCD options 17. Table 29 has been added to show the effect of the working options of **ZFITTER** on theoretical errors.

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17The option OZ5, IFACT=5 also may be ranked among QCD options, since it simulates missing terms of the order $\mathcal{O}(\alpha_\beta^2)$. 

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Figure 11: The BHM, LEPTOP, TOPAZ, ZFITTER, WOH predictions for $M_w$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\hat{\alpha}_s = 0.125$. 

\[ \alpha_s(M_Z) = 0.125 \]

\[ M_w = 300 \text{ GeV} \]
Figure 12: The BHM, LEPTOP, TOPAZ0, ZFITTER, WOH predictions for $\Gamma_e$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 13: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $\Gamma_{\tau}$, including an estimate of the theoretical error as a function of $m_\tau$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 14: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $R_l$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 15: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $R_b$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 16: The BHM, LEPTOP, TOPAZ0, ZFITTER, WOH predictions for $R_c$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 

$R_c = 0.1583 \pm 0.0098$ 

$\alpha_s(M_Z) = 0.125$ 

$M_H = 300$ GeV
Figure 17: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $\sin^2 \theta_W$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 18: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $\sin^2 \theta_{\text{eff}}$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 19: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $A^\mu_{FB}$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 20: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $A_{\text{lep}}^R$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 21: The BHM, LEPTOP, TOPAZ0, ZFITTER, WOH predictions for $A_{FB}^b$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 

$\alpha_s(M_Z) = 0.125$ 
$M_H = 300$ GeV
Figure 22: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $A_{FB}^e$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Figure 23: The BHM, LEPTOP, TOPAZO, ZFITTER, WOH predictions for $\sigma^0$, including an estimate of the theoretical error as a function of $m_t$, for $M_H = 300$ GeV and $\alpha_s = 0.125$. 
Realistic-observables

Figure 24: The BHM (square), TOPAZ0 (diamond) and ZFITTER (cross) predictions, including an estimate of the theoretical error, for $\sigma^t$ in a fully extrapolated set-up. Here $m_t = 175$ GeV, $M_H = 300$ GeV and $\alpha_s = 0.125$. In the lower part a comparison is also made with the relative deviation of BHM, ZFITTER versus TOPAZ0.
Figure 25: The same as in Fig. 24 with an $s'$ cut, $s' = 0.5s$. 
Figure 26: The same as in Fig. 24 for the hadronic cross-section, $\sigma^h$. 
Figure 27: The same as in Fig. 26 with an $s'$ cut, $s' = 0.01s$. 
Figure 28: The same as in Fig. 21 for the leptonic forward–backward asymmetry. In the lower part a comparison is also shown with the absolute deviation of BHM, ZFITTER versus TOPAZ0.
Figure 29: The same as in Fig. 28 with an $s'$ cut, $s' = 0.5s$. 
Figure 30: The TOPAZ0 (diamond) and ZFITTER (cross) predictions, including an estimate of the theoretical error, for $\sigma^\prime$ in the following set-up: $40^\circ < \theta_\perp < 140^\circ$, $\theta_{\text{coll}} < 10^\circ$ and $E_{\text{th}} = 20\text{ GeV}$. Here $m_t = 175\text{ GeV}$, $M_{\tilde{t}} = 300\text{ GeV}$ and $\alpha_s = 0.125$. In the lower part a comparison is also shown with the relative deviation of ZFITTER versus TOPAZ0.
Figure 31: The same as in Fig. 30 for $\theta_{\text{coll}} < 25^\circ$. 
Figure 32: The TOPAZ0 (diamond) and ZFITTER (cross) predictions, including an estimate of the theoretical error, for $\sigma^e$ with $s$-channel electrons, in the following set-up: $40^\circ < \theta_\perp < 140^\circ$, $\theta_{\text{acoll}} < 10^\circ$ and $E_{\text{th}} = 1$ GeV. Here $m_t = 175$ GeV, $M_H = 300$ GeV and $\alpha_s = 0.125$. In the lower part a comparison is also shown with the relative deviation of ZFITTER versus TOPAZ0.
Figure 33: The same as in Fig. 32 for $\theta_{\text{acoll}} < 25^\circ$. 
Figure 34: The TOPAZ0 (diamond) and ZFITTER (cross) predictions, including an estimate of the theoretical error, for $A_{FB}$ in the following set-up: $40^\circ < \theta_\perp < 140^\circ$, $\theta_{\text{coll}} < 10^\circ$ and $E_{\text{th}} = 20\text{ GeV}$. Here $m_t = 175\text{ GeV}$, $M_{H} = 300\text{ GeV}$ and $\alpha_s = 0.125$. In the lower part a comparison is also shown with the absolute deviation of ZFITTER versus TOPAZ0.
Figure 35: The same as in Fig. 34 for $\theta_{\text{coll}} < 25^\circ$. 
Figure 36: The TOPAZ0 (diamond) and ZFITTER (cross) predictions, including an estimate of the theoretical error, for $A_{FB}^e$ with $s$-channel electrons in the following set-up: $40^\circ < \theta_- < 140^\circ$, $\theta_{\text{coll}} < 10^\circ$ and $E_{\text{th}} = 1$ GeV. Here $m_t = 175$ GeV, $M_H = 300$ GeV and $\hat{\alpha}_s = 0.125$. In the lower part a comparison is also shown with the absolute deviation of ZFITTER versus TOPAZ0.
Figure 37: The same as in Fig. 36 for $\theta_{\text{coll}} < 25^\circ$. 
Tables

Pseudo-observables

Table 1

Maximum Derivatives with respect to $\bar{\alpha}^{-1}$.

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<tr>
<th>Observables</th>
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<th>TOPAZO</th>
<th>ZFITTER</th>
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<td>1.2282</td>
<td>1.2330</td>
<td>1.2235</td>
</tr>
<tr>
<td>$\sin^2 \beta_{\text{eff}}$</td>
<td>$-0.25824 \times 10^{-2}$</td>
<td>$-0.25781 \times 10^{-2}$</td>
<td>$-0.25781 \times 10^{-2}$</td>
<td>$-0.25798 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\sin^2 \theta_{\text{eff}}$</td>
<td>$-0.25968 \times 10^{-2}$</td>
<td>$-0.25925 \times 10^{-2}$</td>
<td>$-0.25930 \times 10^{-2}$</td>
<td>$-0.25947 \times 10^{-2}$</td>
</tr>
<tr>
<td>$A^p_{\text{V}}$</td>
<td>0.43992 $\times 10^{-2}$</td>
<td>0.43863 $\times 10^{-2}$</td>
<td>0.43863 $\times 10^{-2}$</td>
<td>0.43781 $\times 10^{-2}$</td>
</tr>
<tr>
<td>$A^p_{\text{N}}$</td>
<td>0.20344 $\times 10^{-1}$</td>
<td>0.20312 $\times 10^{-1}$</td>
<td>0.20312 $\times 10^{-1}$</td>
<td>0.20327 $\times 10^{-1}$</td>
</tr>
<tr>
<td>$\Gamma_{\bar{\chi}}$ (MeV)</td>
<td>6.7115</td>
<td>6.7968</td>
<td>6.8041</td>
<td>6.7685</td>
</tr>
<tr>
<td>$R_{\gamma}$</td>
<td>0.46721 $\times 10^{-1}$</td>
<td>0.46367 $\times 10^{-1}$</td>
<td>0.46446 $\times 10^{-1}$</td>
<td>0.46719 $\times 10^{-1}$</td>
</tr>
<tr>
<td>$\sigma^p_{\gamma}$ (nb)</td>
<td>$-0.12890 \times 10^{-1}$</td>
<td>$-0.12305 \times 10^{-1}$</td>
<td>$-0.12363 \times 10^{-1}$</td>
<td>$-0.13001 \times 10^{-1}$</td>
</tr>
<tr>
<td>$R_{\bar{\alpha}}$</td>
<td>$-0.92933 \times 10^{-4}$</td>
<td>$-0.91531 \times 10^{-4}$</td>
<td>$-0.89791 \times 10^{-4}$</td>
<td>$-0.92040 \times 10^{-4}$</td>
</tr>
<tr>
<td>$A_{\gamma}^p_{\text{V}}$</td>
<td>0.14434 $\times 10^{-1}$</td>
<td>0.14411 $\times 10^{-1}$</td>
<td>0.14411 $\times 10^{-1}$</td>
<td>0.14421 $\times 10^{-1}$</td>
</tr>
<tr>
<td>$\Gamma_{\bar{\chi}}$ (MeV)</td>
<td>6.3768</td>
<td>6.4335</td>
<td>6.4408</td>
<td>6.4143</td>
</tr>
<tr>
<td>$P^b$</td>
<td>0.16748 $\times 10^{-2}$</td>
<td>0.16718 $\times 10^{-2}$</td>
<td>0.16722 $\times 10^{-2}$</td>
<td>0.16738 $\times 10^{-2}$</td>
</tr>
<tr>
<td>$\Gamma_{\text{inv}}$ (MeV)</td>
<td>$-0.19877 \times 10^{-1}$</td>
<td>$-0.37484 \times 10^{-2}$</td>
<td>$-0.38416 \times 10^{-2}$</td>
<td>$-0.56691 \times 10^{-2}$</td>
</tr>
<tr>
<td>$A_c^p_{\text{V}}$</td>
<td>0.11116 $\times 10^{-1}$</td>
<td>0.11095 $\times 10^{-1}$</td>
<td>0.11098 $\times 10^{-1}$</td>
<td>0.11098 $\times 10^{-1}$</td>
</tr>
<tr>
<td>$R_c$</td>
<td>0.14772 $\times 10^{-3}$</td>
<td>0.14744 $\times 10^{-3}$</td>
<td>0.14730 $\times 10^{-3}$</td>
<td>0.14698 $\times 10^{-3}$</td>
</tr>
</tbody>
</table>
Table 2

Maximum Derivatives with respect to $m_b$.

<table>
<thead>
<tr>
<th>Observables</th>
<th>BHM</th>
<th>LEPTOP</th>
<th>TOPAZO</th>
<th>ZFITTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_b$ (MeV)</td>
<td>-0.77934</td>
<td>-0.81333</td>
<td>-0.80247</td>
<td>-0.79613</td>
</tr>
<tr>
<td>$\Gamma_z$ (MeV)</td>
<td>-0.81949</td>
<td>-0.81344</td>
<td>-0.84278</td>
<td>-0.79462</td>
</tr>
<tr>
<td>$R_l$</td>
<td>$-0.92763 \times 10^{-2}$</td>
<td>$-0.96913 \times 10^{-2}$</td>
<td>$-0.95275 \times 10^{-2}$</td>
<td>$-0.94665 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\sigma_0^2$ (nb)</td>
<td>$0.73002 \times 10^{-2}$</td>
<td>$0.76691 \times 10^{-2}$</td>
<td>$0.74958 \times 10^{-2}$</td>
<td>$0.74927 \times 10^{-2}$</td>
</tr>
<tr>
<td>$R_b$</td>
<td>$-0.34685 \times 10^{-3}$</td>
<td>$-0.36581 \times 10^{-3}$</td>
<td>$-0.35739 \times 10^{-3}$</td>
<td>$-0.35819 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\Gamma_h$ (MeV)</td>
<td>$-0.80799$</td>
<td>-0.81344</td>
<td>-0.83069</td>
<td>-0.79462</td>
</tr>
<tr>
<td>$R_c$</td>
<td>$0.76013 \times 10^{-4}$</td>
<td>$0.80403 \times 10^{-4}$</td>
<td>$0.78998 \times 10^{-4}$</td>
<td>$0.79395 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Table 3

The experimental data.

<table>
<thead>
<tr>
<th>Observables</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$(GeV)</td>
<td>80.22 ± 0.18</td>
</tr>
<tr>
<td>$\Gamma_l$(MeV)</td>
<td>83.96 ± 0.18</td>
</tr>
<tr>
<td>$\Gamma_z$(MeV)</td>
<td>2497.4 ± 3.8</td>
</tr>
<tr>
<td>$\sigma^h$(nb)</td>
<td>41.49 ± 0.12</td>
</tr>
<tr>
<td>$R_l$</td>
<td>20.795 ± 0.040</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.2202 ± 0.0020</td>
</tr>
<tr>
<td>$R_c$</td>
<td>0.1583 ± 0.0098</td>
</tr>
<tr>
<td>$\sin^2 \theta^l_{\text{eff}}$</td>
<td>0.2321 ± 0.0004</td>
</tr>
<tr>
<td>$A_{\text{FB}}^l$</td>
<td>0.0170 ± 0.0016</td>
</tr>
<tr>
<td>$A_{\text{FB}}^b$</td>
<td>0.0967 ± 0.0038</td>
</tr>
<tr>
<td>$A_{\text{FB}}^c$</td>
<td>0.0760 ± 0.0091</td>
</tr>
<tr>
<td>$A_{\text{LR}}$(SLD)</td>
<td>0.1668 ± 0.0077</td>
</tr>
</tbody>
</table>
Table 4

The experimental data for $M_w, \Gamma_l = \Gamma_e, \Gamma_Z$ and the theoretical predictions corresponding to $m_t = 175$ GeV, $M_H = 300$ GeV and $\alpha_s(M_Z) = 0.125$. The first entry is BHM then LEPTOP, TOPAZ0, WOH and ZFITTER. The uncertainties quoted are obtained from a variation of program options as described in Section 5.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Exp.</th>
<th>Theor. Predictions</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$ (GeV)</td>
<td>80.22 ± 0.18</td>
<td>80.319$^{+0.003}<em>{-0.007}$, 80.312$^{+0.013}</em>{-0.013}$, 80.310$^{+0.000}<em>{-0.007}$, 80.319$^{+0.004}</em>{-0.000}$, 80.317$^{+0.007}_{-0.007}$</td>
<td>80.315</td>
</tr>
<tr>
<td>$\Gamma_l$ (MeV)</td>
<td>83.96 ± 0.18</td>
<td>83.919$^{+0.020}<em>{-0.013}$, 83.930$^{+0.022}</em>{-0.023}$, 83.931$^{+0.015}<em>{-0.012}$, 83.943$^{+0.022}</em>{-0.022}$, 83.941$^{+0.013}_{-0.021}$</td>
<td>83.933</td>
</tr>
<tr>
<td>$\Gamma_Z$ (MeV)</td>
<td>2497.4 ± 3.8</td>
<td>2497.4$^{+0.9}<em>{-1.0}$, 2497.2$^{+1.1}</em>{-1.1}$, 2497.4$^{+0.2}<em>{-0.5}$, 2497.4$^{+1.5}</em>{-0.6}$, 2497.4$^{+0.6}_{-0.5}$</td>
<td>2497.4</td>
</tr>
</tbody>
</table>
Table 5

The same as in Table 4 for $R_l, R_b, R_c$.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Exp.</th>
<th>Theor. Predictions</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_l$</td>
<td>20.795 ± 0.040</td>
<td>20.788$^{+0.004}<em>{-0.008}$ 20.780$^{+0.006}</em>{-0.005}$ 20.782$^{+0.002}<em>{-0.005}$ 20.780$^{+0.013}</em>{-0.000}$ 20.781$^{+0.006}_{-0.001}$</td>
<td>20.782</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.2202 ± 0.0020</td>
<td>0.21577$^{+0.00010}<em>{-0.00011}$ 0.21564$^{+0.00009}</em>{-0.00004}$ 0.21567$^{+0.00003}<em>{-0.00012}$ 0.21567$^{+0.00018}</em>{-0.00006}$ 0.21571$^{+0.00001}_{-0.00002}$</td>
<td>0.21569</td>
</tr>
<tr>
<td>$R_c$</td>
<td>0.1583 ± 0.0098</td>
<td>0.17236$^{+0.00002}<em>{-0.00002}$ 0.17240$^{+0.00003}</em>{-0.00003}$ 0.17237$^{+0.00004}<em>{-0.00000}$ 0.17240$^{+0.00001}</em>{-0.00003}$ 0.17236$^{+0.00002}_{-0.00000}$</td>
<td>0.17238</td>
</tr>
</tbody>
</table>
Table 6

The same as in Table 4 for $\sin^2 \theta^l_{\text{eff}}, \sin^2 \theta^b_{\text{eff}}, A^l_{\text{FB}}$.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Exp.</th>
<th>Theor. Predictions</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin^2 \theta^l_{\text{eff}}$</td>
<td>$0.2321 \pm 0.0004$</td>
<td>$0.23197^{+0.00004}<em>{-0.00007}$ $0.23200^{+0.00008}</em>{-0.00008}$ $0.23200^{+0.00004}<em>{-0.00004}$ $0.23194^{+0.00003}</em>{-0.00007}$ $0.23205^{+0.00004}_{-0.00014}$</td>
<td>$0.23199$</td>
</tr>
<tr>
<td>$\sin^2 \theta^b_{\text{eff}}$</td>
<td></td>
<td>$0.23331^{+0.00004}<em>{-0.00012}$ $0.23329^{+0.00008}</em>{-0.00010}$ $0.23330^{+0.00009}<em>{-0.00001}$ $0.23325^{+0.00004}</em>{-0.00007}$ $0.23335^{+0.00004}_{-0.00014}$</td>
<td>$0.23330$</td>
</tr>
<tr>
<td>$A^l_{\text{FB}}$</td>
<td>$0.0170 \pm 0.0016$</td>
<td>$0.01544^{+0.00011}<em>{-0.00007}$ $0.01539^{+0.00013}</em>{-0.00013}$ $0.01536^{+0.00008}<em>{-0.00007}$ $0.01549^{+0.00012}</em>{-0.00006}$ $0.01531^{+0.00024}_{-0.00007}$</td>
<td>$0.01540$</td>
</tr>
</tbody>
</table>
Table 7

The same as in Table 4 for $A_{LR}, A_{FB}^b, A_{FB}^c$.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Exp.</th>
<th>Theor. Predictions</th>
<th>Average</th>
</tr>
</thead>
</table>
| $A_{LR}$   | 0.1668 ± 0.0077 | $0.14346^{+0.00052}_{-0.00031}$  
$0.14326^{+0.00060}_{-0.00060}$  
$0.14327^{+0.00028}_{-0.00031}$  
$0.14372^{+0.00057}_{-0.00024}$  
$0.14289^{+0.00110}_{-0.00032}$ | 0.14332 |
| $A_{FB}^b$ | 0.0967 ± 0.0038 | $0.10053^{+0.00038}_{-0.00022}$  
$0.10040^{+0.00043}_{-0.00042}$  
$0.10033^{+0.00023}_{-0.00023}$  
$0.10072^{+0.00041}_{-0.00006}$  
$0.10013^{+0.00079}_{-0.00022}$ | 0.10042 |
| $A_{FB}^c$ | 0.0760 ± 0.0038 | $0.07169^{+0.00029}_{-0.00017}$  
$0.07158^{+0.00033}_{-0.00033}$  
$0.07159^{+0.00016}_{-0.00017}$  
$0.07183^{+0.00032}_{-0.00013}$  
$0.07138^{+0.00061}_{-0.00017}$ | 0.07161 |
Table 8

Largest half-differences among central values \( (d_c) \) and among maximal and minimal predictions \( (d_g) \) for 150 GeV < \( m_t < 200 \) GeV, 60 GeV < \( M_H < 1 \) TeV and 0.118 < \( \alpha_s(M_Z) < 0.125 \).

<table>
<thead>
<tr>
<th>Observable ( O )</th>
<th>( d_cO )</th>
<th>( d_gO )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_W ) (GeV)</td>
<td>( 6.5 \times 10^{-3} )</td>
<td>( 1.9 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \Gamma_e ) (MeV)</td>
<td>( 1.7 \times 10^{-2} )</td>
<td>( 3.7 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \Gamma_Z ) (MeV)</td>
<td>( 0.3 )</td>
<td>( 1.6 )</td>
</tr>
<tr>
<td>( \sin^2 \theta^l_{\text{eff}} )</td>
<td>( 6.5 \times 10^{-5} )</td>
<td>( 1.5 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \sin^2 \theta^r_{\text{eff}} )</td>
<td>( 6.0 \times 10^{-5} )</td>
<td>( 1.6 \times 10^{-4} )</td>
</tr>
<tr>
<td>( R_t )</td>
<td>( 4.0 \times 10^{-3} )</td>
<td>( 1.0 \times 10^{-2} )</td>
</tr>
<tr>
<td>( R_b )</td>
<td>( 7.0 \times 10^{-5} )</td>
<td>( 2.0 \times 10^{-4} )</td>
</tr>
<tr>
<td>( R_c )</td>
<td>( 3.0 \times 10^{-5} )</td>
<td>( 5.0 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \sigma^h_0 ) (nb)</td>
<td>( 7.5 \times 10^{-3} )</td>
<td>( 8.5 \times 10^{-3} )</td>
</tr>
<tr>
<td>( A^l_{\text{FB}} )</td>
<td>( 1.2 \times 10^{-4} )</td>
<td>( 2.5 \times 10^{-4} )</td>
</tr>
<tr>
<td>( A^f_{\text{FB}} )</td>
<td>( 3.5 \times 10^{-4} )</td>
<td>( 8.2 \times 10^{-4} )</td>
</tr>
<tr>
<td>( A^c_{\text{FB}} )</td>
<td>( 2.7 \times 10^{-4} )</td>
<td>( 6.3 \times 10^{-4} )</td>
</tr>
<tr>
<td>( A_{\text{LR}} )</td>
<td>( 5.0 \times 10^{-4} )</td>
<td>( 1.1 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

Table 9

Largest half-differences among central values \( (d_c) \) and among maximal and minimal predictions \( (d_g) \) for \( m_t = 175 \) GeV, 60 GeV < \( M_H < 1 \) TeV and \( \alpha_s(M_Z) = 0.125 \).

<table>
<thead>
<tr>
<th>Observable ( O )</th>
<th>( d_cO )</th>
<th>( d_gO )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_W ) (GeV)</td>
<td>( 4.5 \times 10^{-3} )</td>
<td>( 1.6 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \Gamma_e ) (MeV)</td>
<td>( 1.3 \times 10^{-2} )</td>
<td>( 3.1 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \Gamma_Z ) (MeV)</td>
<td>( 0.2 )</td>
<td>( 1.4 )</td>
</tr>
<tr>
<td>( \sin^2 \theta^l_{\text{eff}} )</td>
<td>( 5.5 \times 10^{-5} )</td>
<td>( 1.4 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \sin^2 \theta^r_{\text{eff}} )</td>
<td>( 5.0 \times 10^{-5} )</td>
<td>( 1.5 \times 10^{-4} )</td>
</tr>
<tr>
<td>( R_t )</td>
<td>( 4.0 \times 10^{-3} )</td>
<td>( 9.0 \times 10^{-3} )</td>
</tr>
<tr>
<td>( R_b )</td>
<td>( 6.5 \times 10^{-5} )</td>
<td>( 1.7 \times 10^{-4} )</td>
</tr>
<tr>
<td>( R_c )</td>
<td>( 2.0 \times 10^{-5} )</td>
<td>( 4.5 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \sigma^h_0 ) (nb)</td>
<td>( 7.0 \times 10^{-3} )</td>
<td>( 8.5 \times 10^{-3} )</td>
</tr>
<tr>
<td>( A^l )</td>
<td>( 9.3 \times 10^{-5} )</td>
<td>( 2.2 \times 10^{-4} )</td>
</tr>
<tr>
<td>( A^f_{\text{FB}} )</td>
<td>( 3.0 \times 10^{-4} )</td>
<td>( 7.4 \times 10^{-4} )</td>
</tr>
<tr>
<td>( A^c_{\text{FB}} )</td>
<td>( 2.3 \times 10^{-4} )</td>
<td>( 5.7 \times 10^{-4} )</td>
</tr>
<tr>
<td>( A_{\text{LR}} )</td>
<td>( 4.2 \times 10^{-4} )</td>
<td>( 8.7 \times 10^{-4} )</td>
</tr>
</tbody>
</table>
Table 10

Largest half-differences among central values ($d_c$) and among maximal and minimal predictions ($d_g$) for $m_t = 175$ GeV, $M_H = 300$ GeV and $0.118 < \alpha_s(M_Z) < 0.125$.

<table>
<thead>
<tr>
<th>Observable $O$</th>
<th>$d_c O$</th>
<th>$d_g O$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$ (GeV)</td>
<td>$4.5 \times 10^{-3}$</td>
<td>$1.3 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\Gamma_c$ (MeV)</td>
<td>$1.2 \times 10^{-2}$</td>
<td>$3.0 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\Gamma_z$ (MeV)</td>
<td>0.15</td>
<td>1.4</td>
</tr>
<tr>
<td>$\sin^2 \theta_{\text{eff}}$</td>
<td>$5.5 \times 10^{-5}$</td>
<td>$1.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\sin^2 \theta_{\text{eff}}$</td>
<td>$5.0 \times 10^{-5}$</td>
<td>$1.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_t$</td>
<td>$4.0 \times 10^{-3}$</td>
<td>$9.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>$R_b$</td>
<td>$6.5 \times 10^{-5}$</td>
<td>$1.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_c$</td>
<td>$2.5 \times 10^{-5}$</td>
<td>$4.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\sigma_0^h$ (nb)</td>
<td>$6.5 \times 10^{-3}$</td>
<td>$8.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>$A^l_{Pb}$</td>
<td>$8.9 \times 10^{-5}$</td>
<td>$1.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>$A^p_{Pb}$</td>
<td>$3.0 \times 10^{-4}$</td>
<td>$6.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$A^c_{Pb}$</td>
<td>$2.3 \times 10^{-4}$</td>
<td>$4.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>$A_{LR}$</td>
<td>$4.2 \times 10^{-4}$</td>
<td>$8.6 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Table 11

Effects of additional options in TOPAZ2 for $m_t = 175$ GeV, $M_H = 300$ GeV and $\alpha_s(M_Z) = 0.125$.

<table>
<thead>
<tr>
<th>Observables</th>
<th>Predictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$(GeV)</td>
<td>$80.310^{+0.000}<em>{-0.007} \rightarrow 80.310^{+0.005}</em>{-0.007}$</td>
</tr>
<tr>
<td>$\Gamma_I$(MeV)</td>
<td>$83.931^{+0.015}<em>{-0.012} \rightarrow 83.931^{+0.043}</em>{-0.031}$</td>
</tr>
<tr>
<td>$\Gamma_z$(MeV)</td>
<td>$2497.4^{+0.2}<em>{-0.5} \rightarrow 2497.4^{+1.3}</em>{-0.6}$</td>
</tr>
<tr>
<td>$R_I$</td>
<td>$20.782^{+0.002}<em>{-0.005} \rightarrow 20.782^{+0.010}</em>{-0.004}$</td>
</tr>
<tr>
<td>$R_b$</td>
<td>$0.21567^{+0.00003}<em>{-0.00012} \rightarrow 0.21569^{+0.00000}</em>{-0.00012}$</td>
</tr>
<tr>
<td>$R_c$</td>
<td>$0.17237^{+0.00004}<em>{-0.00000} \rightarrow 0.17237^{+0.00007}</em>{-0.00001}$</td>
</tr>
<tr>
<td>$\sin^2 \theta_{\text{eff}}$</td>
<td>$0.23200^{+0.00004}<em>{-0.00004} \rightarrow 0.23201^{+0.00004}</em>{-0.00000}$</td>
</tr>
<tr>
<td>$A^l_{FB}$</td>
<td>$0.01536^{+0.00008}<em>{-0.00007} \rightarrow 0.01538^{+0.00007}</em>{-0.00007}$</td>
</tr>
<tr>
<td>$A^b_{FB}$</td>
<td>$0.10033^{+0.00023}<em>{-0.00023} \rightarrow 0.10037^{+0.00037}</em>{-0.00037}$</td>
</tr>
<tr>
<td>$A^c_{FB}$</td>
<td>$0.07159^{+0.00016}<em>{-0.00017} \rightarrow 0.07157^{+0.00015}</em>{-0.00015}$</td>
</tr>
<tr>
<td>$A_{LR}$</td>
<td>$0.14327^{+0.00028}<em>{-0.00031} \rightarrow 0.14320^{+0.00019}</em>{-0.00019}$</td>
</tr>
</tbody>
</table>
Realistic-observables

Table 12

The hadronic cross-section (nb) in two different configurations, NN=NY/YN=YY for inclusion of initial state pair production.

<table>
<thead>
<tr>
<th>√s (GeV)</th>
<th>BHM</th>
<th>TOPAZ</th>
<th>ZFITTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>5.181</td>
<td>5.184</td>
<td>5.185</td>
</tr>
<tr>
<td></td>
<td>5.168</td>
<td>5.171</td>
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</tr>
<tr>
<td>89.45</td>
<td>10.062</td>
<td>10.068</td>
<td>10.067</td>
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<tr>
<td></td>
<td>10.036</td>
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<td>10.042</td>
</tr>
<tr>
<td>90.20</td>
<td>18.033</td>
<td>18.040</td>
<td>18.039</td>
</tr>
<tr>
<td></td>
<td>17.983</td>
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<td>17.992</td>
</tr>
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<td>91.1887</td>
<td>30.446</td>
<td>30.452</td>
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<tr>
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<td>30.366</td>
<td>30.375</td>
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<tr>
<td>91.30</td>
<td>30.585</td>
<td>30.590</td>
<td>30.590</td>
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<td>30.514</td>
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<td>93.70</td>
<td>10.070</td>
<td>10.065</td>
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<tr>
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<td>10.068</td>
<td>10.073</td>
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</tr>
</tbody>
</table>
The $\mu$ forward-backward asymmetry in four different configurations, NN/YN/NY/YY for inclusion of initial state pair production and initial-final QED interference.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>BHM</th>
<th>TOPAZ</th>
<th>ZFITTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>-0.2552</td>
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</tr>
<tr>
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<td>-0.2552</td>
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<td>-0.2544</td>
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<td>-0.2546</td>
<td>-0.2543</td>
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<td></td>
<td>-0.2546</td>
<td>-0.2549</td>
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<tr>
<td>89.45</td>
<td>-0.1631</td>
<td>-0.1632</td>
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</tr>
<tr>
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<td>-0.1631</td>
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<td>90.20</td>
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<td>-0.0912</td>
<td>-0.0917</td>
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<td>-0.0907</td>
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<td>91.1887</td>
<td>-0.0012</td>
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<td>-0.0011</td>
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<td>0.0080</td>
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<td>0.0083</td>
<td>0.0078</td>
<td>0.0083</td>
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<tr>
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<td>0.0549</td>
<td>0.0554</td>
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<tr>
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<td>0.0550</td>
<td>0.0553</td>
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<tr>
<td>93.00</td>
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</tr>
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<tr>
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<td>0.1380</td>
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<tr>
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<td>0.1380</td>
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<td>0.1368</td>
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</tbody>
</table>
Table 14

De-convoluted $\mathcal{A}_{\mu}^{\mu}$ at $s = M_{Z}^{2}$, first entry is the $ZZ$ part, then $ZZ + \gamma\gamma$, $ZZ + Z\gamma$ and total.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BHM</th>
<th>TOPAZO</th>
<th>ZFITTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ZZ$</td>
<td>0.015443</td>
<td>0.015358</td>
<td>0.015279</td>
</tr>
<tr>
<td>$ZZ + \gamma\gamma$</td>
<td>0.015351</td>
<td>0.015267</td>
<td>0.015189</td>
</tr>
<tr>
<td>$ZZ + Z\gamma$</td>
<td>0.017026</td>
<td>0.016735</td>
<td>0.016725</td>
</tr>
<tr>
<td>Total</td>
<td>0.016925</td>
<td>0.016636</td>
<td>0.016627</td>
</tr>
</tbody>
</table>
Table 15

The de-convoluted $\sigma^\mu$, $\sigma^h$ and $A^\mu_{\text{FB}}$.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>BHM</th>
<th>TOPAZ0</th>
<th>ZFITTER</th>
</tr>
</thead>
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<tr>
<td>88.45</td>
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<td>0.3492</td>
</tr>
<tr>
<td></td>
<td>7.020</td>
<td>7.026</td>
<td>7.022</td>
</tr>
<tr>
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<td>-0.2386</td>
<td>-0.2388</td>
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<tr>
<td>89.45</td>
<td>0.6884</td>
<td>0.6887</td>
<td>0.6887</td>
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<tr>
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<td>14.073</td>
<td>14.083</td>
<td>14.077</td>
</tr>
<tr>
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<td>-0.1454</td>
<td>-0.1456</td>
<td>-0.1453</td>
</tr>
<tr>
<td>90.20</td>
<td>1.2453</td>
<td>1.2457</td>
<td>1.2458</td>
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<td>25.650</td>
<td>25.663</td>
<td>25.655</td>
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<td>-0.0749</td>
<td>-0.0751</td>
<td>-0.0750</td>
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<td>91.1887</td>
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<td>2.0019</td>
<td>2.0022</td>
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<td>41.409</td>
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<td>0.0169</td>
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<td>0.0271</td>
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<td>0.0268</td>
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<td>91.95</td>
<td>1.4486</td>
<td>1.4488</td>
<td>1.4492</td>
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<td>29.937</td>
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<td>0.0855</td>
<td>0.0853</td>
<td>0.0851</td>
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<td>93.00</td>
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<td>0.6536</td>
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<td>13.408</td>
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<td>0.1751</td>
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<td>0.4104</td>
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<tr>
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<td>8.362</td>
<td>8.358</td>
<td>8.361</td>
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<tr>
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<td>0.2323</td>
<td>0.2320</td>
<td>0.2316</td>
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</tbody>
</table>
Table 16

$\delta_{\text{conv}}$, as defined by Eq.(136) for the hadronic cross-section.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>BHM</th>
<th>TOPAZO</th>
<th>ZFITTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.450</td>
<td>-0.2638</td>
<td>-0.2640</td>
<td>-0.2634</td>
</tr>
<tr>
<td>89.450</td>
<td>-0.2869</td>
<td>-0.2869</td>
<td>-0.2866</td>
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<tr>
<td>90.200</td>
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<td>-0.2989</td>
<td>-0.2987</td>
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<tr>
<td>91.189</td>
<td>-0.2665</td>
<td>-0.2665</td>
<td>-0.2664</td>
</tr>
<tr>
<td>91.300</td>
<td>-0.2556</td>
<td>-0.2556</td>
<td>-0.2555</td>
</tr>
<tr>
<td>91.950</td>
<td>-0.1607</td>
<td>-0.1607</td>
<td>-0.1606</td>
</tr>
<tr>
<td>93.000</td>
<td>+0.0526</td>
<td>+0.0530</td>
<td>+0.0532</td>
</tr>
<tr>
<td>93.700</td>
<td>+0.2040</td>
<td>+0.2052</td>
<td>+0.2049</td>
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</table>
Bhabha scattering

**Table 17**

The ALIBABA(A)–TOPAZ0(T) comparison for the full Bhabha cross-section (in pb) for the following set-up: $40^\circ < \theta_- < 140^\circ$, $\theta_{\text{coll}} < 10^\circ$ and $E_{\text{th}} = 1$ GeV. I is the TOPAZ0 default for QED final state radiation, II is the ZFITTER-like default for QED final state radiation and III includes initial state pair production.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ GeV</th>
<th>A</th>
<th>T(I) $^{\pm 0.17}_{-0.08} \pm 0.25$</th>
<th>T(II)</th>
<th>T (III)</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>457.52 ± 0.27</td>
<td>457.30 ± 0.17 $^{+0.17}_{-0.08} \pm 0.25$</td>
<td>457.20</td>
<td>456.14</td>
</tr>
<tr>
<td>89.45</td>
<td>643.95 ± 0.31</td>
<td>644.37 $^{+0.26}_{-0.09} \pm 0.24$</td>
<td>644.23</td>
<td>642.60</td>
</tr>
<tr>
<td>90.20</td>
<td>908.99 ± 0.39</td>
<td>910.46 $^{+0.38}_{-0.11} \pm 0.24$</td>
<td>910.26</td>
<td>907.86</td>
</tr>
<tr>
<td>91.19</td>
<td>1183.99 ± 0.39</td>
<td>1184.03 $^{+0.48}_{-0.13} \pm 0.24$</td>
<td>1183.78</td>
<td>1180.79</td>
</tr>
<tr>
<td>91.30</td>
<td>1163.56 ± 0.45</td>
<td>1163.51 $^{+0.47}_{-0.13} \pm 0.24$</td>
<td>1163.26</td>
<td>1160.38</td>
</tr>
<tr>
<td>91.95</td>
<td>876.90 ± 0.28</td>
<td>874.02 $^{+0.35}_{-0.12} \pm 0.24$</td>
<td>873.83</td>
<td>872.13</td>
</tr>
<tr>
<td>93.00</td>
<td>481.35 ± 0.14</td>
<td>477.51 $^{+0.17}_{-0.08} \pm 0.24$</td>
<td>477.41</td>
<td>477.18</td>
</tr>
<tr>
<td>93.70</td>
<td>355.57 ± 0.13</td>
<td>352.52 $^{+0.14}_{-0.06} \pm 0.25$</td>
<td>352.45</td>
<td>352.72</td>
</tr>
</tbody>
</table>
Table 18

The same as in Table 17 for the forward-backward asymmetry.

<table>
<thead>
<tr>
<th>√s GeV</th>
<th>A</th>
<th>T(I)</th>
<th>T(III)</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>0.44611 ± 1.06 × 10⁻³</td>
<td>0.44534⁺⁰.⁰²⁻₀.₀⁸ ± 0.79 × 10⁻³</td>
<td>0.44637</td>
</tr>
<tr>
<td>89.45</td>
<td>0.34250 ± 0.90 × 10⁻³</td>
<td>0.34166⁺⁰.⁰¹⁻₀.₀⁷ ± 0.51 × 10⁻³</td>
<td>0.34252</td>
</tr>
<tr>
<td>90.20</td>
<td>0.24956 ± 0.81 × 10⁻³</td>
<td>0.24977⁺⁰.⁰²⁻₀.₀⁵ ± 0.33 × 10⁻³</td>
<td>0.25043</td>
</tr>
<tr>
<td>91.19</td>
<td>0.13925 ± 0.66 × 10⁻³</td>
<td>0.13916⁺⁰.⁰³⁻₀.₀⁸ ± 0.23 × 10⁻³</td>
<td>0.13951</td>
</tr>
<tr>
<td>91.30</td>
<td>0.13050 ± 0.70 × 10⁻³</td>
<td>0.13035⁺⁰.⁰³⁻₀.₀⁸ ± 0.23 × 10⁻³</td>
<td>0.13067</td>
</tr>
<tr>
<td>91.95</td>
<td>0.10169 ± 0.63 × 10⁻³</td>
<td>0.10139⁺⁰.⁰³⁻₀.₀⁹ ± 0.30 × 10⁻³</td>
<td>0.10158</td>
</tr>
<tr>
<td>93.00</td>
<td>0.13110 ± 0.61 × 10⁻³</td>
<td>0.13055⁺⁰.⁰³⁻₀.₁² ± 0.57 × 10⁻³</td>
<td>0.13061</td>
</tr>
<tr>
<td>93.70</td>
<td>0.18157 ± 0.68 × 10⁻³</td>
<td>0.17957⁺⁰.⁰²⁻₀.₁⁰ ± 0.83 × 10⁻³</td>
<td>0.17944</td>
</tr>
</tbody>
</table>
**Table 19**

The same as in Table 17 for $\theta_{\text{coll}} < 25^\circ$.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ GeV</th>
<th>A</th>
<th>$T(\text{I})$</th>
<th>$T(\text{II})$</th>
<th>$T(\text{III})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>483.27 ± 0.25</td>
<td>484.97$^{+0.07}_{-0.06}$ ± 0.22</td>
<td>484.86</td>
<td>483.74</td>
</tr>
<tr>
<td>89.45</td>
<td>672.67 ± 0.29</td>
<td>673.99$^{+0.13}_{-0.09}$ ± 0.22</td>
<td>673.84</td>
<td>672.13</td>
</tr>
<tr>
<td>90.20</td>
<td>942.52 ± 0.34</td>
<td>942.96$^{+0.25}_{-0.12}$ ± 0.22</td>
<td>942.76</td>
<td>940.27</td>
</tr>
<tr>
<td>91.19</td>
<td>1218.66 ± 0.40</td>
<td>1219.24$^{+0.36}_{-0.13}$ ± 0.21</td>
<td>1218.98</td>
<td>1215.90</td>
</tr>
<tr>
<td>91.30</td>
<td>1198.23 ± 0.36</td>
<td>1198.42$^{+0.35}_{-0.13}$ ± 0.21</td>
<td>1198.17</td>
<td>1195.20</td>
</tr>
<tr>
<td>91.95</td>
<td>908.87 ± 0.30</td>
<td>905.35$^{+0.24}_{-0.12}$ ± 0.20</td>
<td>905.16</td>
<td>903.39</td>
</tr>
<tr>
<td>93.00</td>
<td>505.38 ± 0.15</td>
<td>504.24$^{+0.10}_{-0.08}$ ± 0.20</td>
<td>504.13</td>
<td>503.88</td>
</tr>
<tr>
<td>93.70</td>
<td>378.20 ± 0.13</td>
<td>377.94$^{+0.06}_{-0.06}$ ± 0.20</td>
<td>377.86</td>
<td>378.14</td>
</tr>
</tbody>
</table>
Table 20

The same as in Table 18 for $\theta_{\text{coll}} < 25^\circ$.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ GeV</th>
<th>A</th>
<th>$T(\bar{I})$</th>
<th>$T(\text{III})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>0.45843 ± 0.94 × 10^{-3}</td>
<td>0.46061_{-0.19}^{+0.02} ± 0.65 × 10^{-3}</td>
<td>0.46168</td>
</tr>
<tr>
<td>89.45</td>
<td>0.35479 ± 0.77 × 10^{-3}</td>
<td>0.35560_{-0.17}^{+0.01} ± 0.43 × 10^{-3}</td>
<td>0.67213</td>
</tr>
<tr>
<td>90.20</td>
<td>0.26121 ± 0.67 × 10^{-3}</td>
<td>0.26165_{-0.14}^{+0.02} ± 0.29 × 10^{-3}</td>
<td>0.26235</td>
</tr>
<tr>
<td>91.19</td>
<td>0.15114 ± 0.70 × 10^{-3}</td>
<td>0.15045_{-0.12}^{+0.03} ± 0.20 × 10^{-3}</td>
<td>0.15083</td>
</tr>
<tr>
<td>91.30</td>
<td>0.14067 ± 0.62 × 10^{-3}</td>
<td>0.14203_{-0.12}^{+0.03} ± 0.20 × 10^{-3}</td>
<td>0.14238</td>
</tr>
<tr>
<td>91.95</td>
<td>0.11466 ± 0.62 × 10^{-3}</td>
<td>0.11773_{-0.17}^{+0.03} ± 0.25 × 10^{-3}</td>
<td>0.11795</td>
</tr>
<tr>
<td>93.00</td>
<td>0.15628 ± 0.59 × 10^{-3}</td>
<td>0.15838_{-0.29}^{+0.03} ± 0.45 × 10^{-3}</td>
<td>0.15845</td>
</tr>
<tr>
<td>93.70</td>
<td>0.21239 ± 0.66 × 10^{-3}</td>
<td>0.21360_{-0.36}^{+0.02} ± 0.65 × 10^{-3}</td>
<td>0.21344</td>
</tr>
</tbody>
</table>


Table 21

The same as in Table 17 with the percentage relative deviation.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ GeV</th>
<th>A</th>
<th>T(II)</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>457.52</td>
<td>457.20</td>
<td>+0.07</td>
</tr>
<tr>
<td>89.45</td>
<td>643.95</td>
<td>644.23</td>
<td>−0.04</td>
</tr>
<tr>
<td>90.20</td>
<td>908.99</td>
<td>910.26</td>
<td>−0.14</td>
</tr>
<tr>
<td>91.19</td>
<td>1183.99</td>
<td>1183.78</td>
<td>+0.02</td>
</tr>
<tr>
<td>91.30</td>
<td>1163.56</td>
<td>1163.26</td>
<td>+0.03</td>
</tr>
<tr>
<td>91.95</td>
<td>876.90</td>
<td>873.83</td>
<td>+0.35</td>
</tr>
<tr>
<td>93.00</td>
<td>481.35</td>
<td>477.41</td>
<td>+0.82</td>
</tr>
<tr>
<td>93.70</td>
<td>355.57</td>
<td>352.45</td>
<td>+0.88</td>
</tr>
</tbody>
</table>
Table 22

The same as in Table 21 for $s$-channel alone.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>A</th>
<th>T(II)</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>173.01</td>
<td>172.71</td>
<td>+0.17</td>
</tr>
<tr>
<td>89.45</td>
<td>330.62</td>
<td>330.70</td>
<td>−0.02</td>
</tr>
<tr>
<td>90.20</td>
<td>588.47</td>
<td>588.81</td>
<td>−0.06</td>
</tr>
<tr>
<td>91.19</td>
<td>989.81</td>
<td>990.90</td>
<td>−0.11</td>
</tr>
<tr>
<td>91.30</td>
<td>994.07</td>
<td>995.38</td>
<td>−0.13</td>
</tr>
<tr>
<td>91.95</td>
<td>819.56</td>
<td>819.97</td>
<td>−0.05</td>
</tr>
<tr>
<td>93.00</td>
<td>463.72</td>
<td>461.68</td>
<td>+0.44</td>
</tr>
<tr>
<td>93.70</td>
<td>331.78</td>
<td>329.83</td>
<td>+0.59</td>
</tr>
</tbody>
</table>

Table 23

ALIBABA-TOPAZ0 comparison for $s-t$ and $t-t$ contributions with the following set-up: $40^\circ < \theta_- < 140^\circ, \theta_{\text{coll}} < 10^\circ$ and $E_{\text{th}} = 1$ GeV.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>$\sigma - \sigma(s)$ A</th>
<th>$\sigma - \sigma(s)$ T</th>
<th>$\delta(A)$</th>
<th>$\delta(T)$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>284.51</td>
<td>284.49</td>
<td>1.644</td>
<td>1.647</td>
<td>−0.18</td>
</tr>
<tr>
<td>89.45</td>
<td>313.33</td>
<td>313.53</td>
<td>0.948</td>
<td>0.948</td>
<td>+0.00</td>
</tr>
<tr>
<td>90.20</td>
<td>320.52</td>
<td>321.45</td>
<td>0.545</td>
<td>0.546</td>
<td>−0.18</td>
</tr>
<tr>
<td>91.19</td>
<td>194.18</td>
<td>192.88</td>
<td>0.196</td>
<td>0.195</td>
<td>+0.51</td>
</tr>
<tr>
<td>91.30</td>
<td>169.49</td>
<td>167.88</td>
<td>0.171</td>
<td>0.169</td>
<td>+1.18</td>
</tr>
<tr>
<td>91.95</td>
<td>57.34</td>
<td>53.86</td>
<td>0.070</td>
<td>0.066</td>
<td>+5.88</td>
</tr>
<tr>
<td>93.00</td>
<td>17.63</td>
<td>15.75</td>
<td>0.038</td>
<td>0.034</td>
<td>+11.11</td>
</tr>
<tr>
<td>93.70</td>
<td>23.79</td>
<td>22.62</td>
<td>0.072</td>
<td>0.069</td>
<td>+4.26</td>
</tr>
</tbody>
</table>

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Table 24

The same as in Table 23 for the forward-backward asymmetry.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ GeV</th>
<th>$A_{s+t}$</th>
<th>$T_{s+t}$</th>
<th>$A_s$</th>
<th>$T_s$</th>
<th>$A_t$</th>
<th>$T_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>0.44611</td>
<td>0.44534</td>
<td>-0.22019</td>
<td>-0.22060</td>
<td>0.66630</td>
<td>0.66594</td>
</tr>
<tr>
<td>89.45</td>
<td>0.34250</td>
<td>0.34166</td>
<td>-0.13754</td>
<td>-0.13847</td>
<td>0.48004</td>
<td>0.48013</td>
</tr>
<tr>
<td>90.20</td>
<td>0.24956</td>
<td>0.24977</td>
<td>-0.07774</td>
<td>-0.07649</td>
<td>0.32730</td>
<td>0.32626</td>
</tr>
<tr>
<td>91.19</td>
<td>0.13925</td>
<td>0.13916</td>
<td>-0.00102</td>
<td>0.00001</td>
<td>0.14027</td>
<td>0.13915</td>
</tr>
<tr>
<td>91.30</td>
<td>0.13050</td>
<td>0.13035</td>
<td>0.00745</td>
<td>0.00779</td>
<td>0.12305</td>
<td>0.12256</td>
</tr>
<tr>
<td>91.95</td>
<td>0.10169</td>
<td>0.10139</td>
<td>0.05059</td>
<td>0.04851</td>
<td>0.05110</td>
<td>0.05288</td>
</tr>
<tr>
<td>93.00</td>
<td>0.13110</td>
<td>0.13055</td>
<td>0.09863</td>
<td>0.09789</td>
<td>0.03247</td>
<td>0.03266</td>
</tr>
<tr>
<td>93.70</td>
<td>0.18157</td>
<td>0.17957</td>
<td>0.12237</td>
<td>0.12248</td>
<td>0.05920</td>
<td>0.05709</td>
</tr>
</tbody>
</table>
Table 25

Comparison for the full Bhabha cross-section (in pb) for the following set-up: $40^\circ < \theta_+ < 140^\circ$, $\theta_{\text{coll}} < 10^\circ$ and $E_{\text{th}} = 1$ GeV. First entry is ALIBABA, second entry is the TOPAZ0 default for QED final state radiation and no pair production, $\delta_{\text{FSR}}$ is the uncertainty on final-state QED radiation estimated by TOPAZ0, while $\Delta_\%$ is the relative difference between the maximal and minimal predictions of ALIBABA-TOPAZ0.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ GeV</th>
<th>A</th>
<th>$T(I)$</th>
<th>$\delta_{\text{FSR}}(T)%$</th>
<th>$\Delta_%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.45</td>
<td>457.52 ± 0.27</td>
<td>457.30$^{+0.17}_{-0.06}$ ± 0.25</td>
<td>0.02</td>
<td>0.15</td>
</tr>
<tr>
<td>89.45</td>
<td>643.95 ± 0.31</td>
<td>644.37$^{+0.26}_{-0.09}$ ± 0.24</td>
<td>0.02</td>
<td>0.19</td>
</tr>
<tr>
<td>90.20</td>
<td>908.99 ± 0.39</td>
<td>910.46$^{+0.38}_{-0.11}$ ± 0.24</td>
<td>0.02</td>
<td>0.27</td>
</tr>
<tr>
<td>91.19</td>
<td>1183.99 ± 0.39</td>
<td>1184.03$^{+0.48}_{-0.13}$ ± 0.24</td>
<td>0.02</td>
<td>0.10</td>
</tr>
<tr>
<td>91.30</td>
<td>1163.56 ± 0.45</td>
<td>1163.51$^{+0.47}_{-0.13}$ ± 0.24</td>
<td>0.02</td>
<td>0.10</td>
</tr>
<tr>
<td>91.95</td>
<td>876.90 ± 0.28</td>
<td>874.02$^{+0.35}_{-0.12}$ ± 0.24</td>
<td>0.02</td>
<td>0.37</td>
</tr>
<tr>
<td>93.00</td>
<td>481.35 ± 0.14</td>
<td>477.51$^{+0.17}_{-0.08}$ ± 0.24</td>
<td>0.02</td>
<td>0.86</td>
</tr>
<tr>
<td>93.70</td>
<td>355.57 ± 0.13</td>
<td>352.52$^{+0.14}_{-0.06}$ ± 0.25</td>
<td>0.02</td>
<td>0.95</td>
</tr>
</tbody>
</table>
Effect of working options for different codes

Table 26

The effect of the working options of BHM on theoretical errors.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Default</th>
<th>IRES err</th>
<th>+IQCD err</th>
<th>+IFAC err</th>
<th>+ITWO err</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_W$ (GeV)</td>
<td>80.319</td>
<td>0.004</td>
<td>0.010</td>
<td>0.010</td>
<td>0.010</td>
</tr>
<tr>
<td>$\Gamma_e$ (MeV)</td>
<td>83.919</td>
<td>0.020</td>
<td>0.032</td>
<td>0.033</td>
<td>0.033</td>
</tr>
<tr>
<td>$\Gamma_z$ (MeV)</td>
<td>2497.4</td>
<td>0.68</td>
<td>1.07</td>
<td>1.71</td>
<td>1.92</td>
</tr>
<tr>
<td>$R_l$</td>
<td>20.788</td>
<td>0.001</td>
<td>0.002</td>
<td>0.010</td>
<td>0.012</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.21577</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00011</td>
<td>0.00020</td>
</tr>
<tr>
<td>$A_{\ell}$</td>
<td>0.015435</td>
<td>0.00011</td>
<td>0.00018</td>
<td>0.00018</td>
<td>0.00018</td>
</tr>
<tr>
<td>$A_{\text{FB}}$</td>
<td>0.10053</td>
<td>0.00037</td>
<td>0.00059</td>
<td>0.00059</td>
<td>0.00059</td>
</tr>
<tr>
<td>$\sin^2 \theta_{\text{eff}}$</td>
<td>0.23197</td>
<td>0.00007</td>
<td>0.00011</td>
<td>0.00011</td>
<td>0.00011</td>
</tr>
<tr>
<td>$\sin^2 \phi_{\text{eff}}$</td>
<td>0.23331</td>
<td>0.00007</td>
<td>0.00011</td>
<td>0.00011</td>
<td>0.00016</td>
</tr>
</tbody>
</table>

Table 27

The effect of the working options of LEPT0P on theoretical errors. The first two lines indicate parametric uncertainties caused by $\delta s^2 = 0.0003$ (which is equivalent to $\delta \tilde{\alpha}^{-1} = 0.12$) and by $\delta m_b = 0.3\text{GeV}$. The next six lines refer to the intrinsic theoretical uncertainties. This table was calculated with $m_t=175 \text{GeV}$, $M_H=300 \text{GeV}$ and $\tilde{\alpha}_s=0.125$.

<table>
<thead>
<tr>
<th></th>
<th>$M_W$ (MeV)</th>
<th>$\Gamma_l$ (MeV)</th>
<th>$\sin^2 \theta_{\text{eff}}$</th>
<th>$\sigma^b_0 (nb)$</th>
<th>$\Gamma_z$ (MeV)</th>
<th>$\Gamma_h$ (MeV)</th>
<th>$R_l$ (MeV)</th>
<th>$R_b \times 10^5$</th>
<th>$R_c \times 10^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta s^2$</td>
<td>16</td>
<td>0.015</td>
<td>0.00031</td>
<td>0.0014</td>
<td>0.8</td>
<td>0.8</td>
<td>0.0055</td>
<td>0.15</td>
<td>1.1</td>
</tr>
<tr>
<td>$\delta m_b$</td>
<td>9</td>
<td>0.008</td>
<td>0.00005</td>
<td>0.0002</td>
<td>0.5</td>
<td>0.4</td>
<td>0.0009</td>
<td>0.08</td>
<td>0.2</td>
</tr>
<tr>
<td>$\Delta V_i^e$</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>0.0029</td>
<td>0.3</td>
<td>0.3</td>
<td>0.0005</td>
<td>0.04</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Delta V_i^{\alpha^2}$</td>
<td>13</td>
<td>-</td>
<td>-</td>
<td>0.00010</td>
<td>0.1</td>
<td>0.1</td>
<td>0.0012</td>
<td>0.10</td>
<td>4.6</td>
</tr>
<tr>
<td>$\Delta \Gamma_q$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+0.00032</td>
<td>-1.1</td>
<td>-0.9</td>
<td>-0.0051</td>
<td>-0.13</td>
<td>-4.1</td>
</tr>
<tr>
<td>total</td>
<td>+13</td>
<td>+0.023</td>
<td>+0.00008</td>
<td>+0.0032</td>
<td>+1.2</td>
<td>+1.0</td>
<td>+0.0063</td>
<td>+0.23</td>
<td>+8.7</td>
</tr>
<tr>
<td></td>
<td>-13</td>
<td>-0.023</td>
<td>-0.00008</td>
<td>-0.0042</td>
<td>-1.1</td>
<td>-0.9</td>
<td>-0.0051</td>
<td>-0.13</td>
<td>-4.1</td>
</tr>
</tbody>
</table>
Table 28

The effect of the working options of TOPAZ0 on theoretical errors.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Default</th>
<th>+OU1 err</th>
<th>+OU4 err</th>
<th>+OU5 err</th>
<th>+OU6 err</th>
<th>+OU7 err</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$ (GeV)</td>
<td>80.310</td>
<td>0.014</td>
<td>0.10</td>
<td>0.001</td>
<td>0.001</td>
<td>0.007</td>
</tr>
<tr>
<td>$\Gamma_e$ (MeV)</td>
<td>83.931</td>
<td>0.014</td>
<td>0.10</td>
<td>0.015</td>
<td>0.015</td>
<td>0.027</td>
</tr>
<tr>
<td>$\Gamma_z$ (MeV)</td>
<td>2497.4</td>
<td>0.10</td>
<td>0.10</td>
<td>0.30</td>
<td>0.70</td>
<td></td>
</tr>
<tr>
<td>$R_l$</td>
<td>20.782</td>
<td>0.004</td>
<td>0.004</td>
<td>0.006</td>
<td>0.007</td>
<td></td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.21567</td>
<td>0.00012</td>
<td>0.00012</td>
<td>0.00013</td>
<td>0.00014</td>
<td>0.00015</td>
</tr>
<tr>
<td>$A^l_{pb}$</td>
<td>0.015362</td>
<td>0.00006</td>
<td>0.00006</td>
<td>0.00008</td>
<td>0.00008</td>
<td>0.00015</td>
</tr>
<tr>
<td>$A^b_{pb}$</td>
<td>0.10033</td>
<td>0.00018</td>
<td>0.00019</td>
<td>0.00024</td>
<td>0.00024</td>
<td>0.00046</td>
</tr>
<tr>
<td>$\sin^2 \theta^l_{eff}$</td>
<td>0.23200</td>
<td>0.00004</td>
<td>0.00004</td>
<td>0.00004</td>
<td>0.00004</td>
<td>0.00008</td>
</tr>
<tr>
<td>$\sin^2 \theta^b_{eff}$</td>
<td>0.23330</td>
<td>0.00005</td>
<td>0.00006</td>
<td>0.00006</td>
<td>0.00006</td>
<td>0.00010</td>
</tr>
</tbody>
</table>

Table 29

The effect of the working options of ZFITTER on theoretical errors.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Default</th>
<th>+OZ1 err</th>
<th>+OZ2 err</th>
<th>+OZ3 err</th>
<th>+OZ4 err</th>
<th>+OZ5 err</th>
<th>+OZ6 err</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$ (GeV)</td>
<td>80.317</td>
<td>0.001</td>
<td>0.002</td>
<td>0.005</td>
<td>0.007</td>
<td>0.007</td>
<td>0.14</td>
</tr>
<tr>
<td>$\Gamma_e$ (MeV)</td>
<td>83.941</td>
<td>0.004</td>
<td>0.006</td>
<td>0.014</td>
<td>0.014</td>
<td>0.021</td>
<td>0.034</td>
</tr>
<tr>
<td>$\Gamma_z$ (MeV)</td>
<td>2497.5</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.6</td>
<td>1.1</td>
</tr>
<tr>
<td>$R_l$</td>
<td>20.781</td>
<td>-</td>
<td>0.001</td>
<td>0.002</td>
<td>0.003</td>
<td>0.005</td>
<td>0.007</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.21571</td>
<td>-</td>
<td>-</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00002</td>
<td>0.00003</td>
</tr>
<tr>
<td>$A^l_{pb}$</td>
<td>0.01531</td>
<td>0.0003</td>
<td>0.00009</td>
<td>0.00015</td>
<td>0.00024</td>
<td>0.00024</td>
<td>0.00030</td>
</tr>
<tr>
<td>$A^b_{pb}$</td>
<td>0.1001</td>
<td>0.0011</td>
<td>0.00029</td>
<td>0.00050</td>
<td>0.00078</td>
<td>0.00079</td>
<td>0.00101</td>
</tr>
<tr>
<td>$\sin^2 \theta^l_{eff}$</td>
<td>0.23205</td>
<td>0.00005</td>
<td>0.00009</td>
<td>0.00014</td>
<td>0.00014</td>
<td>0.00014</td>
<td>0.00018</td>
</tr>
<tr>
<td>$\sin^2 \theta^b_{eff}$</td>
<td>0.23335</td>
<td>0.00002</td>
<td>0.00009</td>
<td>0.00014</td>
<td>0.00014</td>
<td>0.00014</td>
<td>0.00018</td>
</tr>
</tbody>
</table>
Note added in proof

While proof-reading this contribution, we have been informed of some recent developments concerning the AFMT term in $\Delta \rho$. A recent calculation by K.G. Chetyrkin, J.H. Kühn and M. Steinhauser [63], as well as a revised version of the AFMT calculation [53], have shown that the correct coefficient of $\zeta(4)$ in a term proportional to $C_F^2 \delta_{[3]}^{\text{QCD}}$ is 4, and not 188/5. This will mean some shift in our predictions for the central values of the pseudo-observables — as a matter of fact, a shift common to all codes, since the AFMT term is a common external block. We have shown this shift in the following Table, where we compare our error bands with the shifted central values at the standard reference point. The result can simply be summarized by saying that the updated central values remain within our theoretical error bands. As for the theoretical bands themselves, note, first, that they are not dominated by the QCD-uncertainty in the calculation of the $\rho$-parameter, and, second — as seen from the Table — that the widths of the uncertainty bands are only marginally affected by the shift of the AFMT correction.
Change of some of the observables due to the introduction of the revised AFMT formulae. Here $m_t = 175 \text{ GeV}$, $M_{H^0} = 300 \text{ GeV}$ and $\alpha_s = 0.125$. In order to estimate the size of the non-leading QCD effects, the $\alpha_s^{(5)}$ correction factor has been implemented according to the formulation of Ref. [62], with a scale which gives the maximum variation with respect to the AFMT(revised) term — $\xi = 0.248(0.204)$ — and the difference between this and the AFMT (revised) calculation is used as an estimate of the corresponding uncertainty. The numbers are calculated by TOPAZ0, first row, and ZFITTER, second row.

<table>
<thead>
<tr>
<th>Observable</th>
<th>New AFMT</th>
<th>AFMT</th>
<th>Central difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_w$ (GeV)</td>
<td>$80.307^{+0.007}_{-0.007}$</td>
<td>$80.310^{+0.007}_{-0.007}$</td>
<td>$-3 \text{ MeV}$</td>
</tr>
<tr>
<td>$\Gamma_L$ (MeV)</td>
<td>$83.926^{+0.015}_{-0.012}$</td>
<td>$83.931^{+0.015}_{-0.013}$</td>
<td>$-0.005 \text{ MeV}$</td>
</tr>
<tr>
<td>$\Gamma_2$ (MeV)</td>
<td>$2497.2^{+0.2}_{-0.4}$</td>
<td>$2497.4^{+0.2}_{-0.5}$</td>
<td>$-0.2 \text{ MeV}$</td>
</tr>
<tr>
<td>$R_L$</td>
<td>$20.782^{+0.002}_{-0.005}$</td>
<td>$20.782^{+0.002}_{-0.005}$</td>
<td>$0.000$</td>
</tr>
<tr>
<td>$R_b$</td>
<td>$0.21568^{+0.00002}_{-0.00001}$</td>
<td>$0.21567^{+0.00003}_{-0.00001}$</td>
<td>$+1 \times 10^{-5}$</td>
</tr>
<tr>
<td>$A_P$</td>
<td>$0.14314^{+0.0003}_{-0.0002}$</td>
<td>$0.14327^{+0.0003}_{-0.0002}$</td>
<td>$-3 \times 10^{-5}$</td>
</tr>
<tr>
<td>$A_L$</td>
<td>$0.10024^{+0.00022}_{-0.00022}$</td>
<td>$0.10033^{+0.00023}_{-0.00023}$</td>
<td>$-9 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Two-loop Electroweak Top Corrections: Are they Under Control? *

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Abstract

The assumption that two-loop top corrections are well approximated by the $O(G_\mu^2m_t^4)$ contribution is investigated. It is shown that in the case of the ratio neutral-to-charged current amplitudes at zero momentum transfer the $O(G_\mu^2m_t^2M_Z^2)$ terms are numerically comparable with the $m_t^4$ contribution for realistic values of the top mass. An estimate of the theoretical error due to unknown two-loop top effect is presented for a few observables of LEP interest.

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1 Introduction

The constant improvement of the experimental precision on line shape and asymmetry parameters at LEP has stimulated the evaluation of two-loop corrections of a purely electroweak nature in order to assess the reliability of the theoretical predictions. Although the latter seem to be affected mainly by the uncertainty of the hadronic contribution on $\Delta\alpha$, it is not yet clear which error may be attributed to the ignorance of higher orders in the electroweak perturbative expansion. The first attempt made in this direction was the computation of the Higgs contribution to the $\rho$ parameter in the limit of large $M_H$ [1]. Subsequently, top effects were also investigated [2]. Concerning the top, at the moment we only have two-loop results obtained from the SM in the limit of vanishing gauge coupling constants [3–5]. Such contributions are of $O(G_\mu^2 m_t^4)$ and are of leading order in the limit of large top mass. They should be considered as the present best estimate of the top influence on higher-order corrections. This note deals with the next-to-leading corrections of $O(G_\mu^2 m_t^2 M_Z^2)$. Such terms are suppressed by a power $M_Z^2/m_t^2$ with respect to the leading ones, but the present range of values for $m_t$ [6, 7] does not exclude that these corrections may be numerically important. Our computation can be regarded as an attempt to check the validity of such an expansion, until the full two-loop results are available. At the same time we should be able to provide a more realistic estimate of the error associated with the two-loop electroweak effects.

To keep the computation as simple as possible we have focused on neutrino scattering on a leptonic target, of which we will compute the electroweak corrections of $O(G_\mu^2 m_t^2 M_Z^2)$ to the $\rho$ parameter, defined as the ratio of neutral-to-charged current amplitudes, at zero momentum transfer. To be more precise, we identify $\rho$ with the cofactor, expressed in units of $G_\mu$, the $\mu$-decay constant of the $J_\rho \ J_\rho$ interaction in neutral current amplitudes. It is well known that radiative effects also lead to a modification of the mixing angle, described by a parameter usually called $\kappa$. These effects will not be discussed in the present paper.

For the processes under examination, we found large subleading corrections of the same sign and of about the same magnitude as the leading one. Therefore, at least for the case we have investigated, the use of the first term of an expansion in inverse power of $m_t$ to approximate the full two-loop result appears to be doubtful. Our result, being obtained at $q^2 = 0$, cannot be directly applied to LEP physics, but can give us a flavour of the size of subleading effects that are due to one-particle irreducible contributions. In the concluding Section, we will elaborate this point, analysing the consequences of a naive extrapolation of our result to some LEP observables.

2 $O(G_\mu^2 m_t^2 M_Z^2)$ corrections to the $\rho$ parameter

In this Section we outline the computation of the electroweak corrections of $O(G_\mu^2 m_t^2 M_Z^2)$ to the $\rho$ parameter. We begin by writing the relation between the $\mu$-decay constant and the charged current amplitude expressed in terms of bare quantities. At the two-loop
level, neglecting contributions that will not give $O(G^2 \mu^2 M_Z^2)$ terms, we have

\[ G_{\mu} = \frac{g_0^2}{8 M_{w_0}^2} \left\{ 1 - \frac{A_{ww}}{M_{w_0}^2} + V_w + M_{w_0}^2 B_w + \frac{A_{ww}^2}{M_w^4} - \frac{A_{ww} V_w}{M_w^2} \right\} , \]

where $g_0$ and $M_{w_0}$ are the bare $SU(2)_L$ coupling and $W$ mass, respectively, $A_{ww}$ is the transverse part of the $W$ self-energy at zero momentum transfer, and the quantities $V_w$ and $B_w$ represent the relevant vertex and box corrections. At the bare level, using the fact that $M_{z_0}^2 c_0^2 = M_{w_0}^2$, where $c_0 \equiv \cos \theta_{w_0}$ with $\theta_{w_0}$ the weak mixing angle and $M_{z_0}$ the bare $Z$ mass, the $\rho$ parameter can be written as:

\[ \rho = \frac{1 - \frac{A_{zz}}{M_{z_0}^2} + V_z + M_{z_0}^2 c_0^2 B_z + \frac{A_{zz}^2}{M_z^2} - \frac{A_{zz} V_z}{M_z^2}}{1 - \frac{A_{ww}}{M_{w_0}^2} + V_w + M_{w_0}^2 B_w + \frac{A_{ww}^2}{M_w^4} - \frac{A_{ww} V_w}{M_w^2}} , \]

where $A_{zz}$, $V_z$ and $B_z$ are the corresponding self-energy, vertex, and box contribution in the neutral current amplitude. To the order we are interested in, Eq. (2) reduces to:

\[ \rho = 1 + \left( \frac{A_{ww}}{M_{w_0}^2} - \frac{A_{zz}}{M_{z_0}^2} \right) + (V_z - V_w) + (M_{w_0}^2 + A_{ww})(B_z - B_w) \]

\[ + \left( \frac{A_{ww}}{M_w^2} - \frac{A_{zz}}{M_z^2} \right) \left( - \frac{A_{zz}}{M_z^2} + (V_z - V_w) - M_w^2 B_w \right) . \]

We proceed by separating the self-energies into one-loop and two-loop contributions:

\[ A_{zz} = A^{(1)}_{zz} + A^{(2)}_{zz} ; \quad A_{ww} = A^{(1)}_{ww} + A^{(2)}_{ww} , \]

on the understanding that the one-loop term is still expressed in terms of bare parameters. The one-loop part can be decomposed further into pure bosonic ($b$) and fermionic ($f$) terms:

\[ A^{(1)}_{zz} = A^{(b)}_{zz} + A^{(1)}_{zz} ; \quad A^{(1)}_{ww} = A^{(b)}_{ww} + A^{(1)}_{ww} , \]

and the one-loop fermionic contribution to the $\rho$ parameter, assuming a vanishing bottom mass, can be expressed as follows:

\[ \chi_d^0 = \left( \frac{A_{ww}}{M_{w_0}^2} - \frac{A_{zz}}{M_{z_0}^2} \right)^{(1)} = \frac{g_0^2}{8 M_{w_0}^2} f(m_t^2, \epsilon) \]

\[ f(m_t^2, \epsilon) = \frac{3}{2 \pi^2} \frac{1}{(4 - 2 \epsilon)} m_t^2 \epsilon \Gamma(\epsilon) \left( \frac{4 \pi \mu^2}{m_t^2} \right)^{\epsilon} . \]

where $\epsilon$ is related to the dimension $d$ of the space-time by $\epsilon = (4 - d)/2$ and $\mu$ is the 't-Hooft mass scale.

We wish to express our final result in terms of the physical $Z$ mass, therefore we perform the shift $M_{z_0}^2 = M_Z^2 - \text{Re} \, \Pi_{zz}(M_Z^2)$, where $\Pi_{zz}(M_Z^2)$ is the transverse part of
the $Z$ self-energy at $q^2 = M_z^2$. Using the decompositions given in Eqs. (4) and (5), and keeping only terms up to $O(G^2 m_t^2 M_z^2)$, we obtain after simple algebra:

$$
\rho = 1 + X_d^0 + X_d \left( -\frac{A_{ww}}{M_w^2} + V_w + M_w^2 B_w \right)
$$

\[
+ \left( \frac{A_{ww}/c_0^2 - A_{zz}}{M_z^2} \right) \left[ \frac{A_{ww}/M_w^2 - A_{zz}/M_z^2} \right] + \left( V_z - V_w \right) + M_z^2 c_0^2 (B_z - B_w) - X_d (V_w + 2 M_w^2 B_w)
\]

\[
+ X_d \left[ \left( \frac{A_{ww}/M_w^2 - A_{zz}/M_z^2} \right) + (V_z - V_w) + M_w^2 (B_z - B_w) \right],
\]

(7)

where $X_d$ is the same quantity introduced in Eq. (6), but expressed in terms of renormalized parameters.

We observe that Eq. (7) simplifies further if we express the one-loop fermionic contribution in terms of the Fermi constant $G_F$. Indeed, as can be seen from Eq. (1), the first line of Eq. (7) reproduces the effective coupling in the charged sector:

$$
X_d^0 \left( 1 - \frac{A_{ww}}{M_w^2} + V_w + M_w^2 B_w \right) = \frac{g_0}{8 M_w^2 \left( 1 - \frac{A_{ww}}{M_w^2} + V_w + M_w^2 B_w \right)} f(m_t^2, \epsilon)
$$

\[
\simeq \frac{G_F}{\sqrt{2}} f(m_t^2, \epsilon).
\]

(8)

Until now, apart from the use of the physical $Z$ mass, we have not specified any particular renormalization condition. In order to simplify the structure of the counterterms, we have found it convenient to perform the calculation using the $\overline{MS}$ parameter $\sin^2 \theta_W(M_z)$ (henceforth abbreviated as $\hat{s}^2$). Indeed, while in the on-shell (OS) scheme the counterterm associated with the quantity $s^2 = 1 - M_w^2/M_z^2$ contains terms proportional to $m_t^2$ and gives rise to $O(G_F^2 m_t^2 M_z^2)$ contributions to $\rho$, the counterterm related to $\hat{s}^2$ does not exhibit any $m_t^2$ dependence and this greatly simplifies our task. Therefore, to the order we are interested in, we can directly replace $c_0^2$ with $\hat{c}^2$ in Eq. (7) ($\hat{c}^2 \equiv 1 - \hat{s}^2$). It will always be possible to recover the result in the pure OS scheme, by appropriately shifting $\hat{s}^2$ in the one-loop expression for $\rho$.

We now notice that the one-loop contribution is still written in terms of bare quantities. To put $\rho$ in its final form, we split it into the usual $O(\alpha)$ result, $\delta \rho^{(1)}$, plus the counterterm part, $\delta \rho_C$, namely

$$
\frac{G_F}{\sqrt{2}} f(m_t^2, \epsilon) + \left( \frac{A_{ww}/\hat{c}^2 - A_{zz}}{M_z^2} \right) + (V_z - V_w) + M_z^2 \hat{c}^2 (B_z - B_w) = \delta \rho^{(1)} + \delta \rho_C
\]

(9)

with

\[
\delta \rho^{(1)} = \delta \rho^{(1)}(1) + \delta \rho^{(1)}(2)
\]

(10a)

\[
\delta \rho^{(1)}(1) = N_c x_t = N_c \frac{G_F m_t^2}{8 \pi^2 \sqrt{2}}
\]

(10b)

\[
\delta \rho^{(1)}(2) = \frac{\hat{c}^2}{4 \pi \hat{s}^2} \left[ \frac{3}{4 \hat{s}^2} \ln \hat{c}^2 - \frac{7}{4} + 2 \hat{c}^2 + \hat{s}^2 G(\xi, \hat{c}^2) \right]
\]

(10c)

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where $N_c$ is the colour factor, and $\hat{\alpha} = \alpha/(1 + 2\delta/e)_{\overline{MS}}$ is the $\overline{MS}$ coupling as defined in [8]. In Eqs. (10)

$$c_\zeta = \frac{\zeta^2}{4}(5 - 3I_3) - 3 \left( \frac{I_3}{8} - \frac{\zeta^2}{2} Q + \frac{\zeta^4}{2} I_3 Q^2 \right),$$

(11a)

where $I_3$ and $Q$ are the isospin and electric charge of the target ($I_3 = -1$ for electrons) and

$$G(\xi, \zeta^2) = \frac{3 \xi}{4\zeta^2} \left[ \ln \frac{\zeta^2}{\xi} - \frac{1}{\zeta^2 - \xi} \right],$$

(11b)

with $\xi = M_{\pi}^2/M_\zeta^2$. Using Eqs. (7), (8), and (9) we can express $\rho$ as follows:

$$\rho = 1 + \delta\rho^{(1)} + N_c x_t \delta\rho^{(1)} + \delta\rho^{(2)},$$

(12)

where the previous relation defines the two-loop contribution, $\delta\rho^{(2)}$, as:

$$\delta\rho^{(2)} = \delta\rho_c + \left( \frac{A_{ww}}{M_w^2} - \frac{A_{zz}}{M_z^2} \right)[2] (V_x - V_w)[2] + M_z^2 \zeta^2 (B_x - B_w)[2]$$

$$- X_d (V_w + 2 M_w B_w)$$

(13)

Eq. (12) suggests that a possible way to take into account higher-order effects is to write $\rho$ as

$$\rho = \frac{1}{(1 - \delta\rho^{(1)})} (1 + \delta\rho^{(1)} + \delta\rho^{(2)}),$$

(14)

where the resummation of $\delta\rho^{(1)}$ can be justified theoretically on the basis of $1/N_c$ expansion arguments [9]. Explicitly we find, in units $N_c [\hat{\alpha}/(16\pi \bar{s} \bar{c}^2)] m_e^2 / M_\zeta^2 \approx N_c x_t^2$:

$$\delta\rho^{(2)} = 25 - 4 \zeta t + \left( \frac{1}{2} - \frac{1}{\zeta t} \right) \pi^2 + \frac{(-4 + \zeta t) \sqrt{\zeta t} g(\zeta t)}{2} + \left( -6 - 6 \zeta t + \frac{\zeta t^2}{2} \right) \ln \zeta t$$

$$+ \left( -15 + \frac{6}{\zeta t} + 12 \zeta t - 3 \zeta t^2 \right) \ln \zeta t + \left( -15 + 9 \zeta t - \frac{3 \zeta t^2}{2} \right) \phi \left( \frac{\zeta t}{4} \right)$$

$$+ \left( 2 - \frac{4}{\zeta t} - 8 \zeta t + \frac{288 \bar{s}^2}{\zeta t} \right) \ln \zeta t + \pi^2 \left( \frac{-7}{3} - \frac{2}{3 \zeta t^2} + \frac{1}{\zeta t} - \frac{56 \bar{s}^2}{27} + \frac{2 \bar{s}^2}{3 \zeta t^2 - \bar{s}^2} \right)$$

$$+ \left( -2 - \frac{8}{\zeta t} + 5 \bar{s}^2 + \frac{24 \bar{s}^2}{\zeta t} - \frac{10 \bar{s}^2}{\zeta t} + \zeta t \bar{s}^2 \right) \phi \left( \frac{\zeta t}{4} \right),$$

(15a)
for $M_H \gg M_z$, whilst in the region $M_H < M_z$,

\[
\delta \rho^{(3)} = 19 - 2\pi^2 - 4\pi \sqrt{ht} + ht \left( \frac{-27}{2} + 2\pi^2 - 6 \ln ht - 5 \ln \hat{c}^2 + 3 \ln zt \right) \\
+ zt \left[ -\frac{11}{2} + \frac{3}{\hat{s}^2} + \frac{319 \hat{s}^2}{9} + 6 I_\delta \hat{c}^2 + \pi^2 \left( -\frac{7}{3} - \frac{56 \hat{s}^2}{27} \right) \right] \\
+ \left( 7 + \frac{3}{\hat{s}^4} - \frac{6}{\hat{s}^2} - 4 \hat{s}^2 \right) \ln \hat{c}^2 + \left( 21 - 16 \hat{s}^2 \right) \ln zt \right].
\]

(15b)

In Eqs. (15) $ht \equiv (M_H/m_t)^2$, $zt \equiv (M_z/m_t)^2$,

\[
g(x) = \begin{cases} \\
\sqrt{4-x} \left( \pi - 2 \arcsin \sqrt{x/4} \right) & 0 < x \leq 4 \\
2\sqrt{x/4} - 1 \ln \left( \frac{1 - \sqrt{1-4/x}}{1 + \sqrt{1-4/x}} \right) & x > 4 
\end{cases}
\]

(16a)

\[
\Lambda(-1 + \frac{4}{x}) = \begin{cases} \\
-\frac{1}{2\sqrt{x}} g(x) + \frac{z}{2} \sqrt{4/x - 1} & 0 < x \leq 4 \\
-\frac{1}{2\sqrt{x}} g(x) & x > 4 
\end{cases}
\]

(16b)

\[
Li_2(x) = -\int_0^x dt \frac{\ln(1-t)}{t},
\]

(16c)

and

\[
\phi(z) = \begin{cases} \\
4\sqrt{\frac{z}{1-z}} Cl_2(2 \arcsin \sqrt{z}) & 0 < z \leq 1 \\
\frac{1}{\lambda} \left[ -4 Li_2(\frac{1-\lambda}{2}) + 2 \ln^2(\frac{1-\lambda}{2}) - 2 \ln^2(4z) + \pi^2/3 \right] & z > 1 
\end{cases}
\]

(16d)

where $Cl_2(x) = \text{Im} Li_2(e^{ix})$ is the Clausen function with

\[
\lambda = \sqrt{1 - \frac{1}{z}}.
\]

(16e)

The first two lines of Eq. (15b) represent the leading $O(G_F^2 m_t^4)$ result [3], which is completely independent of the gauge sector of the theory. Indeed this part can be computed in the framework of a pure Yukawa theory, obtained from the SM in the limit of vanishing gauge coupling constants. The rest of Eq. (15b) is proportional to $zt = M_Z^2/m_t^2$ and represents the first correction to the Yukawa limit. Equations (15) show a process-dependent contribution, i.e. $6 zt I_\delta \hat{c}^2$ that comes from $B_\delta^{[2]}$. This reflects the fact that, already at one-loop, the box diagrams in neutral current depend on the process under consideration [10] [cf. Eq. (11a)].
3 Numerical results

In the previous Section we derived the expression for the $\rho$ parameter up to $O(G_{\mu}^{2}m_{t}^{2}M_{Z}^{2})$ in the $\overline{MS}$ scheme. We expressed our result in terms of the $\overline{MS}$ quantities $\tilde{\alpha}$, $\tilde{s}^{2}$, and the physical mass of the $Z$ boson. To obtain the corresponding expressions in terms of $G_{\mu}$ and the on-shell (OS) parameter $c^{2} \equiv M_{w}^{2}/M_{Z}^{2}$, we use the relations [8]

$$\frac{\tilde{\alpha}}{4\pi \tilde{s}^{2}} = \frac{G_{\mu}M_{w}^{2}}{2\sqrt{2}\pi^{2}} \frac{1 - \Delta_{w}}{1 + (\frac{2\delta e}{c})_{\overline{MS}}} \approx \frac{G_{\mu}M_{w}^{2}c^{2}}{2\sqrt{2}\pi^{2}}$$ \hspace{1cm} (17a)

$$c^{2} = c^{2}(1 - Y_{\overline{MS}}) \approx c^{2}(1 - N_{c}x_{t})$$ \hspace{1cm} (17b)

Equation (17b) will create additional contributions to $\delta\rho^{(2)}$. The one-loop result is then given by Eqs. (10) with the substitutions $\tilde{\alpha}/(4\pi \tilde{s}^{2}) \rightarrow (G_{\mu}M_{w}^{2}c^{2})/(2\sqrt{2}\pi^{2})$, $\tilde{s}^{2}, c^{2} \rightarrow s^{2}, c^{2}$, while for the two-loop contribution we have

![Graph](image-url)
\[ \delta \rho_{d}^{(2)} = \delta \rho^{(2)}(s^2, c^2 \to s^2, e^2) + N_{c} x_{t} zt \left[ -\frac{3c}{s} \ln c^2 - \frac{3c^2}{s^2} - 3I_{3} + 12Q - 24s^2(1 + c^2)I_{3}Q^2 + 4c^2G'(\xi, c^2) \right] \] (18a)

where

\[ G'(\xi, c^2) = \frac{3}{4} \xi \left[ c^2 \frac{\ln(c^2/\xi)}{(c^2 - \xi)^2} - \frac{1}{c^2 - \xi} + \frac{1}{c^2} \ln \xi \right]. \] (18b)

In Eq. (18a) \( \delta \rho^{(2)}(s^2, c^2 \to s^2, e^2) \) represents a term obtained from Eqs. (15) applying the same substitutions as in the one-loop case.

From Eq. (18a) we notice that the process-dependence is more pronounced in the OS framework. This is easily understood by noticing that the expansion of the bare couplings in the one-loop box diagrams gives rise, unlike the \( \overline{MS} \) case, to \( m_t^2 \) contributions.

In Fig. 1 we plot \( \delta \rho^{(2)} \) [Eqs. (15)] as a function of \( m_t \) for few values of \( M_H \). As a comparison we also show the values obtained including only the \( O(G_\mu^2m_t^4) \) contribution. The process under consideration is \( \nu_e \mu \). From Fig. 1 it is evident that the inclusion of corrections suppressed by a factor \( M_Z^2/m_t^2 \) with respect to the leading term is quite substantial.

To have a better understanding of the size of these corrections in Table 1 we present the values of \( \delta \rho^{(2)} \) and \( \delta \rho_{d}^{(2)} \) for \( zt = 0, 0.2, \) and \( 0.3 \) as a function of \( r = M_H/m_t \). When preparing the Table we matched the values from (15a) and (15b) when the latter were very close \( (r \approx 0.5) \). We see that in the region of light Higgs the \( O(G_\mu^2m_t^2M_Z^2) \) corrections are much larger than the \( m_t^4 \) term that is actually suppressed by accidental cancellations, while for large Higgs mass in the TeV region, their contribution is still \( 50\% \) of the leading part. It is worth noticing that the numbers shown in Table 1 are very close to the corresponding ones obtained in Ref. [11] in the case of a model with \( SU(2) \) symmetry. That is not surprising \( s \) being a relatively small number \( (s^2 \approx 0.23) \).

### 4 Conclusions

We have seen that calculating the difference of self-energies is not sufficient to compute the \( O(G_\mu^2m_t^2M_Z^2) \) corrections to the \( \rho \) parameter [cf. Eq. (13)] but one has to resort to physical processes and this introduces process-dependent quantities. Our result, being obtained at \( q^2 = 0 \), cannot be directly applied to LEP physics. However one can ask general questions about the two-loop electroweak corrections involving the top and use the answers coming from the calculation of \( \delta \rho^{(2)} \) as a `warning bell' for the estimation of the theoretical error in the present knowledge of these corrections.

It is natural to ask whether we can expect that the \( O(G_\mu^2m_t^4) \) term will approximate well the completely unknown result for values of \( m_t \) not larger than 250 GeV. Table 1 shows that in the case of \( \delta \rho^{(2)} \) the answer is negative. We have looked for the asymptotic
regime of the top, namely for which value of $m_t$ $\delta \rho^{(2)}$ begins to be close to the $O(G^2 mu^4)$ contribution. We found that, typically, $\delta \rho^{(2)}$ starts to be within 10% of the leading $m_t^4$ value for $m_t \simeq 800$ GeV.

Table 1

$\delta \rho^{(2)}$ ($\overline{MS}$) and $\delta \rho^{(2)}_{OS}$ ($OS$) relevant to $\nu_x e$ scattering for $zt \equiv M_Z^2/m_t^2 = 0.2, 0.3$, in units $N_c x_t^2$ as a function of $r = M_H/m_t$. The column $zt = 0$ is the result of the Yukawa theory.

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To consider the top as an asymptotically heavy particle can be an unrealistic assumption also for electroweak quantities of LEP interest, like $\Delta r$ [12] and $\Delta \hat{r}$ [13, 8]. It is then important to have a feeling of how large the theoretical error one is making can be when these quantities are computed including only the $O(G^2 mu^4)$ correction. A possible way to obtain this is to assume that the ratio between the $O(G^2 mu^2 M_Z^2)$ and the $O(G^2 mu^4)$ contributions in $\delta \rho^{(2)}$ can be representative of the unknown two-loop top effects in $\Delta r$ and $\Delta \hat{r}$. We can then use this ratio to estimate the additional contributions to $\Delta r$ and $\Delta \hat{r}$ simply multiplying it by the known $O(G^2 mu^4)$ terms of these quantities. The shifts in the W mass and the effective sinus, $\sin^2 \theta_{eff}$, due to these additional contributions can be
Table 2

Calculated ratio ($R$), for few values of $m_t$ and $M_H$, between the $O(G^2_\mu m_t^2 M_Z^2)$ and the $O(G^2_\mu m_t^4)$ contributions in $\delta \rho^{(2)}$. The corresponding estimate of the shifts in the W mass and $\sin^2 \theta^{\text{lep}}_{\text{eff}}$ are also presented (see text).

<table>
<thead>
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<th>$m_t$ (GeV)</th>
<th>$M_H$ (GeV)</th>
<th>$R$</th>
<th>$\Delta M_W$ (MeV)</th>
<th>$\Delta \sin^2 \theta^{\text{lep}}_{\text{eff}}$ ($10^{-4}$)</th>
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estimated from the relations

$$\frac{\Delta M_W}{M_W} = -\frac{s^2}{2(c^2 - s^2)} \delta(\Delta r)$$

$$\Delta \sin^2 \theta^{\text{lep}}_{\text{eff}} = \frac{s^2 c^2}{c^2 - s^2} \delta(\Delta r) + \hat{s}^2 \delta \hat{k}_l(M_Z^2),$$

where the correction $\hat{k}_l$ is defined in Ref. [14].

In Table 2 we show, for few values of $m_t$ and $M_H$, the effect of our estimate of the unknown top contributions on the W mass and $\sin^2 \theta^{\text{lep}}_{\text{eff}}$. In our estimate we have put $\delta \hat{k}_l = 0$. The ratio between subleading and leading terms in $\delta \rho^{(2)}$ has been computed using expressions slightly different from Eqs. (15). In fact, we decided to maximize the one-loop result of our $\overline{MS}$ calculation by writing it in terms of the physical masses of both W and Z. Such a procedure is frequently used in one-loop calculations [8], and in our case has the further advantage of eliminating the process-dependent terms. From the third column, it can immediately be seen that, for a fixed value of the top mass the effect is more pronounced for light Higgs. This is not surprising, bearing in mind the fact that the $O(G^2_\mu m_t^4)$ term is a monotonically increasing (in modulus) function of $M_H$.

We want to stress that the numbers presented in Table 2, more than a definite estimate of the shifts in $M_W$ and $\sin^2 \theta^{\text{lep}}_{\text{eff}}$, should be taken as an indication that subleading two-loop $m_t$ effects could be larger than what is “naively” expected. Their size is probably comparable to, or may be larger than, the theoretical uncertainty due to the hadronic contribution to the photonic self-energy. The latter amounts to $\pm 16$ MeV and $\pm 3 \times 10^{-4}$ in $M_W$ and $\sin^2 \theta^{\text{lep}}_{\text{eff}}$, respectively.

To conclude, we think that our calculation of $\delta \rho^{(2)}$ shows that it is questionable to believe that two-loop electroweak top contributions are well approximated by the $O(G^2_\mu m_t^4)$
term and therefore sufficiently under control. However, the possibility of establishing top effects of a purely electroweak nature at the two-loop level seems quite remote. The experimental accuracy envisaged for the $W$ mass is $(\delta M_w)_{\text{exp}} = \pm 50 \text{ MeV}$, whilst $\sin^2 \theta_{\text{eff}}^{\text{exp}}$ is presently known with a precision $(\delta \sin^2 \theta_{\text{eff}}^{\text{exp}})_{\exp} \equiv \pm 4 \times 10^{-4}$. At this level of precision it is likely that only QCD corrections to a one-loop top contribution can be relevant. However, if the experimental precision improves in the future to reach $(\delta \sin^2 \theta_{\text{eff}}^{\text{exp}})_{\exp} = \pm 2 \times 10^{-4}$, or $\pm 1 \times 10^{-4}$, then a meaningful theoretical interpretation will require a complete study of the two-loop top effect of an electroweak nature.

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References


QCD Corrections to the $e^+e^-$ Cross-Section and the $Z$ Boson Decay Rate

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Abstract

QCD corrections to the electron–positron annihilation cross-section into hadrons and to the hadronic $Z$ boson decay rate are reviewed. Formal developments are introduced in a form particularly suited for practical applications. These include the operator product expansion, the heavy mass expansion, the decoupling of heavy quarks and matching conditions. Exact results for the quark mass dependence are presented whenever available, and formulae valid in the limit of small bottom mass ($m_b^2 \ll s$) or of large top mass ($m_t^2 \gg s$) are presented. The differences between vector and axial vector induced rates as well the classification of singlet and non-singlet rates are discussed. Handy formulae for all contributions are collected and their numerical relevance is investigated. Prescriptions for the separation of the total rate into partial rates are formulated. The applicability of the results in the low-energy region, relevant for measurements to around 10 GeV and below, is investigated and numerical predictions are collected for this energy region.

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kwiat@theor2.lbl.gov
1 Introduction

Since experiments at the $e^+e^-$ storage ring LEP started data-taking a few years ago, and by the end of the 1993 run by the four experiments, more than seven million hadronic events had been collected at the $Z$ resonance. The accuracy of the measurements is impressive. Numerous parameters of the standard model can be determined with high precision, allowing stringent tests of the standard model to be performed. Among them: the mass $M_Z = (91.188 \pm 0.0044)$ GeV and the width $\Gamma_Z = (2.4974 \pm 0.0038)$ GeV of the $Z$ boson or the weak mixing angle $\sin^2 \theta_{\text{eff}}^{\text{exp}} = 0.2322 \pm 0.0006$ [1]. All experimental results were in remarkable agreement with theoretical predictions and a triumphant confirmation of the standard model.

As well as the electroweak sector of the standard model, LEP provides an ideal laboratory for the investigation of strong interactions. Due to their purely leptonic initial state, events are very clean from both the theoretical and experimental point of view and represent the ideal place for testing QCD. From cross-section measurements $\sigma_{\text{had}} = (41.49 \pm 0.12) \text{ nbarn}$ [1] as well as from the analysis of event topologies the strong coupling constant can be extracted. Other observables measurable with very high precision are the (partial) $Z$ decay rates into hadrons $\Gamma_{\text{had}}/\Gamma_z = 20.795 \pm 0.040$ and bottom quarks $\Gamma_{b\bar{b}}/\Gamma_{\text{had}} = 0.2192 \pm 0.0018$. From the line shape analysis of LEP a value $\alpha_s = 0.126 \pm 0.005 \pm 0.002$ is derived. The program of experimentation at LEP is still not complete. The prospect of an additional increase in the number of events by a factor of two to four will further improve the level of accuracy. This means, for example, that the relative uncertainty of the partial decay rate into $b$ quarks $\Delta \Gamma_{b}/\Gamma_b$ falls below one percent and that an experimental error for $\alpha_s$ of 0.002 may be achieved.

Also at lower energies significant improvements can be expected in the accuracy of cross-section measurements. The energy region of around 10 GeV just below the $BB$ threshold will be covered with high statistics at future $B$ meson factories. The cross section between the charm and bottom thresholds can be measured at the BEPC storage ring in Beijing. These measurements could provide a precise value for $\alpha_s$ and — even more important — a beautiful proof of the running of the strong coupling constant.

In view of this experimental situation theoretical predictions for the various observables with comparable or even better accuracy become mandatory and higher-order radiative corrections are required. It seems appropriate to collect all presently available calculations and reliably estimate their theoretical uncertainties. The aim of this report is to provide such a review for the QCD sector of the standard model, as far as cross-section measurements are concerned, at the $Z$ peak as well as in the ‘low energy’ region from 5 to 20 GeV. (Related topics have been also discussed in recent reviews [2].) Higher-order QCD corrections to the $e^+e^-$ annihilation cross-section into hadrons will be discussed as well as the hadronic width of the $Z$ boson. Further interest lies in the partial rates for the decay of the $Z$ boson into specific quark channels. Of particular importance is the partial width $\Gamma(Z \rightarrow b\bar{b})$, as this quantity can be measured with high accuracy and provides important information about the top quark mass from the $Zb\bar{B}$ vertex. However, the decomposition of $\Gamma_{\text{had}}$ into partial decay rates of different quark species is possible in a simple, straightforward way only up to corrections of the order of $O(\alpha_s)$. Apart from diagrams where ‘secondary quarks’ are radiated off the ‘primary quarks’ one encounters
flavour singlet diagrams that first arise in order $O(\alpha_s^2)$ and lead to a confusion of different species. They therefore have to be carefully scrutinized.

For many considerations and experimental conditions quark masses can be neglected, compared to the characteristic energy of the problem. Accordingly, higher-order QCD corrections to the total cross-section were first calculated for massless quarks. At LEP energies this is certainly a good approximation for u, d, s and c quarks. In view of the accuracy reached at LEP much effort has been spent in estimating the size of mass effects of the bottom and the top quark. Whereas $b$ quarks are present as particles in the final state, top quarks can appear only through virtual corrections. A large part of this report is devoted to these effects. The application of these formulae and, if necessary, their numerical evaluation will also be covered.

In Part 2 topics of a general nature are addressed. In Section 2.1 the notation is introduced and the relation between cross-sections and decay rates on the one hand and the corresponding current correlators on the other is discussed. Furthermore, the classification of singlet versus non-singlet terms is introduced. The behaviour of coupling constant, masses, operators and correlators under renormalization group transformations is reviewed in Section 2.2 and the relevant anomalous dimensions are listed. The decoupling of heavy quarks and the resulting matching conditions for coupling constant masses and effective currents are treated in Section 2.3. Numerical values of quark masses are discussed in Section 2.4. Part 3 is concerned with calculational techniques relevant to the problems at hand. Emphasis is put on the behaviour of the current correlators at large momenta, the structure of mass corrections in the small mass limit and the resummation of large logarithms of $m^2/s$. And the other extreme, with $s/m^2 \ll 1$ also dealt with in this Part, which concludes with a discussion of $\gamma_5$ in $D \neq 4$ dimensions.

The analytical first-order QCD corrections to the cross-section are recalled in Part 4. Approximations in the limits of low and high energies are given.

Non-singlet and singlet contributions to the QCD corrections are presented in Parts 5 and 6, respectively, and the relevant formulae for various applications are given. First, the calculations are reviewed for massless quarks. This assumption is evidently not justified for the heavy top top mass, which appears as a virtual particle. Top mass corrections are described in Section 5.2. The dependence on the mass of the final-state quarks is given in Section 5.3. At low energies not only do the leading quadratic mass terms have to be taken into account, but quartic mass terms also become relevant. They are presented in Section 5.4. The influence of secondary quark production on determinations of the partial rate is treated in Section 5.5.

Flavour singlet contributions are discussed in Part 6. They arise for the first time in second order for the axial-induced rate and in third order for the vector current-induced rate. $O(\alpha_s^2)$ singlet corrections would be absent for six massless flavours, but do not vanish due to the large mass splitting in the (b, t) doublet. Massless contributions and bottom-mass corrections from singlet diagrams are covered in Sections 6.1 and 6.2 respectively. The assignment of the singlet contributions to a partial rate into a specific quark flavour is explained in Section 6.3 and the resulting ambiguity is discussed. In Part 7 the numerical relevance of the different contributions are studied. Different sources of theoretical uncertainties are investigated and their size estimated.
A collection of formulae is presented in the Appendix. It provides an overview and may serve as a quick and convenient reference for later use.

2 General Considerations

2.1 Notations

2.1.1 Cross-Sections and Decay Rates

We introduce our notations by casting the total cross-section for longitudinally polarized $e^+e^-$ into hadrons in leading order of the electroweak coupling as:

$$\sigma_{l,l} = \frac{4\pi\alpha^2}{3s} \left\{ \left(\frac{v_e + a_e}{y}\right)^2 \frac{s}{s - M_Z^2 + i M_Z \Gamma_Z}\right\}^2 \frac{R^V + R^A}{y^2} + 2Q_e \frac{v_e + a_e}{y} \text{Re} \left\{ \frac{s}{s - M_Z^2 + i M_Z \Gamma_Z} \frac{R^{\text{int}}}{y} + Q_e^2 R^{\text{em}} \right\},$$

(1)

with the weak couplings defined through

$$v_f = 2I_3^f - 4Q_f \sin^2 \theta_w, \quad a_f = 2I_3^f, \quad y = 4 \sin \theta_w \cos \theta_w.$$

(2)

$R$ and $L$ denote the electron beam polarization (positrons are assumed to be unpolarized). The functions $R^k$ with $k = V, A, \text{em}, \text{int}$ are the natural generalization of the Drell ratio $R \equiv \sigma_{\text{had}}/\sigma_{\text{point}} = R^{\text{em}}$, which is familiar from purely electromagnetic interactions at lower energies. They are induced by the vector and axial couplings of the $Z$ boson, the pure QED part and an interference term. In the massless parton model they are given by

$$R^V = 3 \sum_f v_f^2, \quad R^A = 3 \sum_f a_f^2, \quad R^{\text{em}} = 3 \sum_f Q_f^2, \quad R^{\text{int}} = 3 \sum_f Q_f v_f.$$

(3)

Here the sum extends over all flavours $f$.

The hadronic decay rate of the $Z$ can be expressed in a way similar to the incoherent sum of its vector- and axial-vector-induced parts:

$$\Gamma_Z^{\text{had}} = \Gamma^V + \Gamma^A$$

$$= \frac{\alpha}{3} \frac{M_Z}{y^2} (R^V + R^A).$$

(4)

Alternatively one may express $\alpha/y^2$ through the Fermi constant

$$\frac{\alpha}{y^2} = \frac{G_F M_Z^2}{8\pi\sqrt{2}}$$

(5)

and absorb the large logarithms from the running of QED. These formulae are equivalent for the present purpose, where higher-order electroweak corrections are ignored.
All relevant information needed for the correction factors $R^k$ is contained in the current correlation functions

$$
\Pi_{\mu}^{ij}(q) = \int dx e^{iqx} \langle 0 | T j^i_\mu(x) j^j_\mu(0) | 0 \rangle
= g_{\mu} \Pi_{1}^{ij}(-q^2) + q_{\mu} q_{\nu} \Pi_{2}^{ij}(-q^2),
$$

with $(i, j) = (V, V), (A, A), (e\ell, e\ell), (e\ell, V)$ for $k = V, A, e\ell, \text{int}$ respectively. The currents under consideration are defined through

$$
\begin{align*}
    j^V_\mu &= \sum_f v_f \bar{\psi}_f \gamma_\mu \psi_f, \\
    j^A_\mu &= \sum_f a_f \bar{\psi}_f \gamma_\mu \gamma_5 \psi_f, \\
    j^{e\ell}_\mu &= \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f,
\end{align*}
$$

where the sum extends over all six flavours.

The relation between the cross-section $\sigma_{\text{had}}$ and the corresponding current correlator is closely connected to the analytic properties of $\Pi_{\mu\nu}$. After the Lorentz decomposition into the functions $\Pi_1$ and $\Pi_2$, only $\Pi_1$ enters the cross-section, since the contraction of $q_{\mu} q_{\nu} \Pi_2$ with the lepton tensor is suppressed by the electron mass. The threshold energies for the production of fermion pairs are branch points of the vacuum polarization, and $\Pi_1(-s)$ is analytic in the complex plane cut along the real positive axis. For energies above the lowest-lying threshold ($s = 4m^2$) the function $\Pi_1(s)$ is discontinuous when $s$ approaches the real axis from above and below. The optical theorem relates the inclusive cross-section and thus the function $R(s)$ to the discontinuity of $\Pi_1$ in the complex plane

$$
R(s) = -\frac{12\pi}{s} \text{Im} \Pi_1(-s - i\epsilon) = \frac{6\pi i}{s} [\Pi_1(-s - i\epsilon) - \Pi_1(-s + i\epsilon)],
$$

where Schwarz's reflection principle has been employed for the second step. Conversely, the vacuum polarization is obtained through a dispersion relation from its absorptive part. Applying Cauchy’s theorem along the integration contour of Fig. 1 leads to:

$$
\begin{align*}
    \Pi_1(-s) &= \frac{1}{2\pi i} \int ds' \frac{\Pi_1(-s')}{s' - s} = \frac{1}{\pi} \int_0^\infty ds' \frac{\text{Im} \Pi_1(-s' - i\epsilon)}{s' - s} \mod \text{sub} \\
    &= -\frac{1}{12\pi^2} \int_0^\infty ds' \frac{s'}{s' - s} R(s') \mod \text{sub},
\end{align*}
$$

No subtraction is needed if $\Pi_1(-s)$ vanishes at infinity, since the large circle does not contribute to the integral in this case. If the spectral function is only bounded by $s^n$ at large distances, one may apply the dispersion relation to the function $\Pi_1/s^{n+1}$. This is achieved by $n + 1$ subtractions. For example, a twice-subtracted dispersion relation has to be applied for $\Pi_1(-s)$, is given by

$$
\tilde{\Pi}_1(-s) = \Pi_1(-s) - \Pi_1(0) - (-s) \Pi'(0).
$$

The absorptive part is not affected by these subtractions. For the vector current $\Pi_1(0)$ vanishes as a consequence of current conservation and the second subtraction corresponds to charge renormalization.
Let us add an additional remark concerning the applicability of perturbative QCD for the calculation of radiative corrections to the cross-section $\sigma_{\text{had}}$. Experimental $e^+e^-$ data are taken in the physical regime of timelike momentum transfer $q^2 > 0$. This region is influenced by threshold and bound state effects which make the use of perturbative QCD questionable. However, perturbative QCD is strictly applicable for large spacelike momenta ($q^2 = -Q^2 < 0$), since this region is far away from non-perturbative effects due to hadron thresholds, bound state and resonance effects [3]. Therefore, reliable theoretical predictions can be made for $\Pi_1(Q^2)$ with $Q^2 > 0$. To compare theoretical predictions and experimental results for time-like momenta, one has to perform suitable averaging procedures [4]. For large positive $s$ one may appeal to the experimentally observed smoothness of $R$ as a function of $s$ and to the absence of any conceivable non-perturbative contribution.

For later use it is convenient to introduce the Adler function

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2} \left[ \frac{\Pi_1(Q^2)}{Q^2} \right].$$

(10)

It is related to $R$ through a dispersion relation which allows a comparison between the perturbatively calculated Adler function ($Q^2 > 0$) and the experiment if the cross-section $R$ is known over the full energy scale $s' > 0$:

$$D(Q^2) = Q^2 \int_0^\infty ds' \frac{R(s')}{(s' + Q^2)^2} + 12\pi^2 \frac{\Pi_1(0)}{Q^2}.$$  

(11)

The relation inverse to Eq. (10) finally reads

$$R(s) = \frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} dQ^2 \frac{D(Q^2)}{Q^2}.$$  

(12)

Diagrammatically, current correlators are depicted as vacuum polarization graphs. Their absorptive parts are obtained from the sum of all possible cuts applied to the diagram (see Fig. 2). This means — according to Cutkosky's rule — that the absorptive
Figure 2: The absorptive part of a current correlator is obtained by cutting the diagram in all possible ways.

part of a Feynman integral is obtained, if the substitution

\[
\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow -2\pi i\delta(p^2 - m^2)\theta(p_0)
\]

is applied to those propagators associated with to the cut lines of the corresponding Feynman diagram. Calculating the two-point correlator and taking its absorptive part is equivalent to the evaluation of the matrix element squared with a subsequent integration over the phase space of the final-state particles. The former method has some advantages. Although the vacuum polarization graph contains one loop more than the amplitudes in the direct calculation of the rate, the problem is reduced to a propagator type integral, for which quite elaborate techniques have been developed and implemented in corresponding computer packages. Furthermore, the occurrence of infrared divergences is naturally circumvented, since virtual and bremsstrahlung corrections correspond only to different cuts of the same diagram and hence are combined in the same amplitude. The cancellation of infrared divergences is therefore inherent in each diagram. Depending on the cut, final states with a different number of particles are represented by the same diagram, as is shown in Fig. 2.

2.1.2 Classification of Diagrams

Higher-order QCD corrections to $e^+e^-$ annihilation into hadrons were first calculated for the electromagnetic case in the approximation of massless quarks. Considering the annihilation process through the $Z$ boson, numerous new features and subtleties become relevant at the present level of precision.

The different charge and chiral structure of electromagnetic and weak currents respectively has already been addressed in the previous section: The functions $R^k$ as defined above were classified according to the space–time structure of the currents (vector versus axial vector) and their electroweak couplings. Another important distinction, namely
‘singlet’ versus ‘non-singlet’ diagrams, originates from two classes of diagrams with intrinsically different topology and resulting charge structure. The first class of diagrams consists of non-singlet contributions with one fermion loop coupled to the external current. All these amplitudes are proportional to the charge structures given in Eq. (3), consisting of a sum of terms proportional to the square of the coupling constant or the trivial generalization in the interference term $R_{\text{int}}$. QCD corrections corresponding to these diagrams contribute a correction factor independent of the current under consideration as long as masses of final state quarks are neglected. Singlet contributions arise from a second class of diagrams where two currents are coupled to two different fermion loops and hence can be cut into two parts by cutting gluon lines only (see Fig. 3). They cannot be assigned to the contribution from one individual quark species. In the axial vector and the vector case the first contribution of this type arises in order $O(\alpha_s^2)$ and $O(\alpha_s^3)$ respectively. Each of them has a charge structure different from the one in Eq. (3). The lowest order term is therefore ultraviolet finite. Furthermore, singlet contributions are separately invariant under renormalization group transformations. These diagrams are obviously absent in charged-current-induced processes like the $W$ decay.

The functions $R^k$ are therefore conveniently decomposed as follows:

$$R^V = 3 \left[ \sum_f v_f^2 r^V_{\text{NS}}(f) + \sum_{f,f'} v_f v_{f'} r^V_S(f,f') \right]. \quad (14)$$

It will be shown below that $r^V_S(f,f')$ is independent of $f$ and $f'$ (meaning the respective quark masses) up to terms of order $\alpha_s^2 m_q^2/s$, where $q$ stands for one of the five light quarks. Hence

$$R^V \approx 3 \left[ \sum_f v_f^2 r^V_{\text{NS}}(f) + (\sum_f v_f)^2 r^V_S \right]. \quad (15)$$

The functions $r^V_{\text{NS}}$ and $r^V_S$ are independent of the quark charges and arise identically in the decompositions of $R_{\text{int}}$ and $R_{\text{em}}$:

$$R_{\text{int}} = 3 \left[ \sum_f v_f Q_f r^V_{\text{NS}}(f) + \sum_{f,f'} v_f Q_{f'} r^V_S(f,f') \right]$$

$$\approx 3 \left[ \sum_f v_f Q_f r^V_{\text{NS}}(f) + (\sum_f v_f)(\sum_{f'} Q_{f'}) r^V_S \right] \quad (16)$$

$$R_{\text{em}} = 3 \left[ \sum_f Q_f^2 r^V_{\text{NS}}(f) + \sum_{f,f'} Q_f Q_{f'} r^V_S(f,f') \right]$$

$$\approx 3 \left[ \sum_f Q_f^2 r^V_{\text{NS}}(f) + (\sum_f Q_f)^2 r^V_S \right].$$

A similar decomposition can be derived for $R^A$:

$$R^A = 3 \left[ \sum_f a_f^2 r^A_{\text{NS}}(f) + \sum_{f,f'} a_f a_{f'} r^A_S(f,f') \right]. \quad (17)$$
In the limit of massless $u,d,s$ and $c$ quarks the second term receives contributions from $f,f'=b$ or $t$ only, or — more precisely — the light quarks compensate mutually. The advantage of this decomposition becomes even more manifest in the limit $m_q^2/s \to 0$. Then the non-singlet functions $r_{NS}^V$ and $r_{NS}^A$ are identical and the corrections for non-vanishing, but small, masses are easily calculated.

### 2.2 $\beta$ Function and Anomalous Dimensions

In this section several aspects of the renormalization procedure in QCD are recalled, which will be of importance for the subsequent calculations. The renormalization of various currents and the corresponding current correlators will be considered. Green functions with the insertion of two external currents require subtractive renormalization. The corresponding renormalization constants lead to anomalous dimensions for the correlators. The presentation will be rather short, and for more detail the reader should consult, for example, Refs. [5–8].

#### 2.2.1 Renormalization in QCD

The QCD Lagrangian is given by

\[
\mathcal{L}\{\Phi;g,\mu\} = \mathcal{L}\{A^a_\mu,\Psi,\overline{\Psi},C^a,C^\alpha;g,\xi,\mu\} = \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \overline{\Psi} (i \not\!D - m) \Psi + \mathcal{L}_{GF} + \mathcal{L}_{FP},
\]

\[
G^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu,
\]

\[
D_\mu = \partial_\mu - ig \epsilon^{abc} A^a_\mu \lambda^b, \\
\nabla^{ab}_\mu = \delta^{ab} \partial_\mu - g \epsilon^{abc} A^c_\mu.
\]

---

Figure 3: Singlet contribution of order $\mathcal{O}(\alpha_s^3)$ and $\mathcal{O}(\alpha_s^2)$. 

---
Here \( f^{abc} \) are the structure constants of the colour \( SU_c(3) \) group, \( \xi \) is the gauge parameter, \( G_{\mu\nu}^a \) is the gluon field strength tensor and \( C^a \) is the ghost fields, with

\[
\mathcal{L}^{GF} = -\frac{1}{2\xi}(\partial_\mu A_\nu)^2 \quad \text{and} \quad \mathcal{L}^{FP} = (\partial_\mu C)^\mu C \, .
\] (20)

The quark masses are denoted as \( m = \{m_q\} \); \( \Psi = \{\Psi_q\} \) \( q = \{u, d, s, c, b, t\} \) represents the quark fields; while \( \Phi \) stands for the collection of all fields, and \( g = \{g, m, \xi\} \) for the ‘coupling constants’. Anticipating the use of dimensional regularization, the unit mass \( \mu \) has been introduced in (18) to keep the coupling constant \( g \) dimensionless even if the Lagrangian is considered in \( D = 4 - 2\epsilon \) dimensions.

A convenient representation of all (connected) Green functions is provided by the generating functional

\[
Z_c(I; S) = \left\{ \int [d\Phi] \exp(iI + \Phi \cdot S) \right\}_c \, ,
\] (21)

with the normalization condition \( Z_c(I_c, 0) = 1 \). Here the action

\[
I(\Phi, g, \mu) = \int \mathcal{L}(\Phi, g) dx
\] (22)

and the functional integration is to be understood in the standard manner within the perturbation theory framework.

Finite Green functions can be constructed from the Lagrangian Eq. (18) in three equivalent ways: The first method is based on the renormalized Lagrangian, obtained from the original one by a rescaling of fields and parameters, expressing them in terms of renormalized quantities:

\[
\mathcal{L}_R(\Phi; g, \mu) = \mathcal{L}\{Z_3^2 A_\mu^a, Z_2^\frac{3}{2} \Psi, Z_2^\frac{1}{2} \overline{\Psi}, \bar{Z}_3^\frac{3}{2} C^a, \bar{Z}_3^\frac{1}{2} \overline{C}^a, Z_\xi \xi, Z_g g, Z_m m\} \, .
\] (23)

The explicit form of the renormalization constants depends on the renormalization scheme adopted. The most powerful method, which is particularly suitable for applications in QCD, the procedure of dimensional regularization [9, 10] and minimal subtraction [11] is nowadays widely used. After continuation of the Feynman integrals to \( D = 4 - 2\epsilon \) space–time dimensions divergences reappear as poles in \( \epsilon \). The renormalization constants may then be expanded in the coupling constant

\[
Z = 1 + \sum_{i<j}^{0<i<j} Z_{ij} \left( \frac{\alpha_s}{\pi} \right)^i \frac{1}{\epsilon^j} \, .
\] (24)

In the minimal subtraction scheme (we will use the \( \overline{\text{MS}} \)-scheme [12] throughout this work) the coefficients \( Z_{ij} \) are just dimensionless constants. There exists a choice of the renormalization constants such that every Green function of elementary fields computed with the help of \( \mathcal{L}_R \) is finite in the limit \( \epsilon \to 0 \) in every order of perturbation theory. Hence, too, the generating functional

\[
Z_c(I_R; S) \quad \text{with} \quad I_R = \int \mathcal{L}_R dx
\] (25)
The functional change of variables is noted in every order of perturbation theory.

An example of a renormalized finite Green function obtained from $\mathcal{L}_R$ is the two-point function of two quark fields, namely the renormalized quark propagator

$$S(p,g,m,\mu) = i \int \text{d}e^{ipx} \langle 0| T[q(x)\bar{q}(0)] |0\rangle,$$

(26)

whose contribution will serve to define the quark field renormalization constant $Z_2$.

A second calculational method is based on the bare Lagrangian

$$\mathcal{L}_B\{\Phi_B; g_B\} = \mathcal{L}\{\Phi_B; g_B, 1\}$$

(27)

and the resulting generating functional of bare Green functions

$$Z_c(I_B; S_B) = \left\{ \int [\text{d}\Phi_B] \exp (iI_B + \Phi_B \cdot S_B) \right\}_c,$$

(28)

with the bare action

$$I_B(\Phi_B, g_B) = \int \mathcal{L}_B\{\Phi_B, g_B, 1\} \text{d}x.$$

(29)

The functional change of variables

$$A_{B,\mu}^a = (Z_3)^{1/2} A_{\mu}^a, \quad \Psi_B = (Z_2)^{1/2} \psi, \quad \overline{\Psi}_B = (Z_2)^{-1/2} \overline{\psi}, \quad C_B^a = (\bar{Z}_3)^{1/2} C^a$$

(30)

leads to the immediate conclusion that

$$Z_c(I_B; S_B) = Z_c(I_R; S),$$

(31)

provided bare and renormalized sources and parameters are related through

$$(S_{\alpha}^a)_{B} = (Z_3)^{-1/2} S_{\alpha}^a, \quad (S_{\psi}^a)_{B} = (\bar{Z}_3)^{-1/2} S_{\psi}^a, \quad (S_{\overline{\psi}}^a)_{B} = (Z_3)^{-1/2} S_{\overline{\psi}}^a,$$

(32)

and

$$g_B = \mu^e Z_g Z_3^{-1/2} Z_2^{-1} g, \quad m_B = Z_m Z_2^{-1} m, \quad \xi_B = (Z_3)^{-1} Z_2 \xi.$$

(33)

The bare Green function corresponding to Eq. (26) is given by

$$S_B(p, g_B, m_B) = i \int \text{d}xe^{ipx} \langle 0| T[q_B(x)\bar{q}_B(0)] |0\rangle.$$

(34)

Equations (26) and (34) show that after having introduced a renormalized coupling constant, masses and gauge-fixing parameters, all remaining divergences of the Green function can be eliminated by wave function renormalization:

$$S(p, g, m, \mu) = Z_2^{-1} S_B(p, g_B, m_B).$$

(35)

A third way of obtaining finite Green functions is based on the so-called $R$-operation — a recursive subtraction scheme — to remove ultraviolet (UV) divergences from a given (arbitrary) Feynman integral in a way compatible with adding local counterterms to the Lagrangian.\(^1\) Using the $R$-operation the renormalized generating functional (25) can be conveniently presented in the form:

$$Z_c(I_R; S) = Z_c^R(I; S) \equiv R_{\text{MS}} \left\{ \int [\text{d}\Phi] \exp (iI + \Phi \cdot S) \right\}_c.$$

(36)

\(^1\)A good pedagogical introduction to the apparatus of the $R$-operation in the MS-scheme and its applications may be found in Refs. [6, 13].
Running Coupling Constant and Masses

In comparison with the classical Lagrangian (18) (now considered in the physical \(D = 4\) number of space-time dimensions) the renormalized one (23) depends on an additional parameter — the 't Hooft unit mass \(\mu\). This naturally leads to the well-known renormalization group (RG) constraint: any physical prediction (that is measurable at least in principle) obtained with the help of (23) must not depend on the value of \(\mu\) provided bare fields and parameters are kept fixed. If \(P(\alpha_s, m, \mu)\) denotes a physical quantity computed with the Lagrangian (23) then it must meet the RG equation

\[
\mu^2 \frac{d}{d\mu^2} P(\alpha_s, m, \mu) = 0,
\]

where

\[
\mu^2 \frac{d}{d\mu^2} \equiv \mu^2 \frac{d}{d\mu^2} |_{g_n, m_n}
\]

or, equivalently,

\[
\mu^2 \frac{d}{d\mu^2} = \mu^2 \frac{\partial}{\partial \mu^2} + \pi \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + 2\bar{m}^2 \gamma_m(\alpha_s) \frac{\partial}{\partial \bar{m}^2}.
\]

Note that we follow the common convention by denoting

\[
\alpha_s \equiv \frac{g^2}{4\pi}.
\]

In addition, in complicated formulae we shall for brevity use the *couplant* \(a\) defined as

\[
a \equiv \frac{\alpha_s}{\pi} = \frac{g^2}{4\pi^2}.
\]

The \(\beta\)-function and the quark mass anomalous dimension \(\gamma_m\) are

\[
\mu^2 \frac{d}{d\mu^2} \left[ \frac{\alpha_s(\mu)}{\pi} \right] |_{g_n, m_n} = \beta(\alpha_s) \equiv -\sum_{i \geq 0} \beta_i \left( \frac{\alpha_s}{\pi} \right)^{i+2},
\]

\[
\mu^2 \frac{d}{d\mu^2} \bar{m}(\mu) |_{g_B, m_B} = \bar{m}(\mu) \gamma_m(\alpha_s) \equiv -\bar{m} \sum_{i \geq 0} \gamma_m^i \left( \frac{\alpha_s}{\pi} \right)^{i+1}.
\]

Their expansion coefficients to three loops are well known [14, 15] and read (\(n_f\) is the number of quark flavours; note that the results (42) and (43) have been recently confirmed in Refs. [16, 17] respectively)

\[
\beta_0 = \left( 11 - \frac{2}{3} n_f \right)/4, \quad \beta_1 = \left( 102 - \frac{38}{3} n_f \right)/16,
\]

\[
\beta_2 = \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right)/64,
\]

\[
\gamma_m^0 = 1, \quad \gamma_m^1 = \left( \frac{202}{3} - \frac{20}{9} n_f \right)/16,
\]

\[
\gamma_m^2 = \left\{ 1249 - \left[ \frac{2216}{27} + \frac{160}{3} \zeta(3) \right] n_f - \frac{140}{81} n_f^2 \right\}/64.
\]
According to (40) and (41) both the minimally renormalized coupling constant \( g \) and quark mass \( m_q \) run (that is depend on) with \( \mu \). This demonstrates clearly that \( g \) and \( m \) are just parameters entering the QCD Lagrangian and that their connection to measurable physical quantities is not direct. In this sense the \( \overline{\text{MS}} \) renormalization scheme is not unique or distinguished by physical considerations. However, it allows one to employ the RG equation (37) in order to very efficiently and conveniently ‘improve’ the perturbation expansion by neatly summing up potentially dangerous logarithms of momenta and masses appearing in higher-orders.

Indeed, in any necessarily finite order of perturbation theory, the master RG Eq. (37) is met only partially: that is, its r.h.s. does not vanish but rather is a polynomial in the coupling constant that includes only terms of order higher than the one taken into account in the calculation of \( P \). If additionally the characteristic scale \( Q \) on which the physical quantity \( P \) depends is taken large, then, as is well known, the general structure of \( P \) may be visualized as follows — it is understood that the \( P \) starts from an \( \alpha \), independent constant, with all power (suppressed) mass effects are neglected for the moment —

\[
P = \sum_{n_1 < n_2} c_{n_1 n_2} \left( \ln \frac{\mu^2}{Q^2} \right)^{n_1} \left( \frac{\alpha}{\pi} \right)^{n_2}.
\]

Even if a renormalization prescription has already been specified, there remain two problems: the residual \( \mu \) dependence and the invalidation of the perturbation expansion by large logarithms of \( Q \) irrespective of the smallness of the initial value of the coupling constant. The well-known solution of both problems is to fix the value of \( \mu \) to be of order \( Q \) or, in many cases, just equal to \( Q \). Such a prescription clearly eliminates the dangerous momentum logs and, as a side effect, helps to specify the value of \( \mu \).

Of course, the above consideration cannot fix the \( \mu \) value exactly. In other words, in the limit of asymptotically large \( Q \) the choices of \( \mu = Q \) or \( \mu = \sqrt{2}Q \) are mathematically equivalent but still generally lead to slightly different predictions. A considerable amount of literature discussing this ambiguity exists and various recipes for overcoming it have been suggested (we cite only a few of them [18–23]). Below, a commonly accepted and pragmatic approach will be adopted: the \( \overline{\text{MS}} \) scheme will be chosen with \( \mu \) set to a characteristic momentum of the problem at hand and, finally, \( \mu \) will be varied somewhere around the scale to test the sensitivity of the result with respect to not-yet-computed corrections of higher order.

In what follows we will always identify the \( \overline{\text{MS}} \) quark mass with the running one and occasionally denote the latter with a bar. If not stated otherwise a running mass without an argument will be understood as taken at scale \( \mu \); so that

\[ m(\mu) \equiv \overline{m}(\mu) = \overline{m} . \]

We finish this subsection by writing out the explicit solution for the running \( \alpha_s \), and quark mass. The solution of Eq. (40) reads (\( L \equiv \ln \mu^2/\Lambda_{\overline{\text{MS}}}^2 \))

\[
\frac{\alpha_s(\mu)}{\pi} = \frac{1}{\beta_0 L} \left\{ 1 - \frac{1}{\beta_0 L} \frac{\beta_1 \ln L}{\beta_0} + \frac{1}{\beta_0^2 L^2} \left[ \frac{\beta_2^2}{\beta_0^2} (\ln^2 L - \ln L - 1) + \frac{\beta_2}{\beta_0} \right] \right\}
\]

(45)
while Eq. (41) is solved by

$$
\bar{m}(\mu) = \bar{m}(\mu_0) \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\gamma_{\mu} / \beta_0} \{ 1 + \left( \frac{\gamma_{\mu}^1}{\beta_0} - \frac{\beta_1 \gamma_{\mu}^0}{\beta_0^2} \right) \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(\mu_0)}{\pi} \right] \\
+ \frac{1}{2} \left( \frac{\beta_1 \gamma_{\mu}^0}{\beta_0^2} \right)^2 \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(\mu_0)}{\pi} \right]^2 \}
$$

\[ (46) \]

2.2.3 **MS Mass Versus Pole Mass**

There are situations where it is convenient to deal with quark mass definitions different from that given by the \( \overline{\text{MS}} \) scheme. For instance, for very heavy quarks a non-relativistic description is believed to be relevant. In this case the pole mass seemingly should be used. The pole mass \( M_q \) presents a gauge-invariant, infrared-finite, scheme-independent object which is defined as the position of pole of a renormalized quark propagator. It should be heavily emphasized that by definition the renormalized quark propagator is to be understood in a strictly perturbative framework.

There is the firm belief that non-perturbative effects should change significantly the pole structure of the propagator of even a quite heavy quark. This belief has been supported by the recent observation [24] that there are some non-perturbative effects in the heavy quark propagator which defy their description in terms of familiar vacuum condensate contributions. However, with the qualification above the pole, mass remains a valuable characteristic of heavy quark masses.

The explicit relation between both masses was obtained in Refs. [25–29]. The most advanced calculation presented in Ref. [29] leads to the following result:
\[ m_q(\mu) = M_q \left\{ 1 - \frac{\alpha_s(\mu)}{\pi} \left[ \frac{4}{3} + \ln \frac{\mu^2}{M_q^2} \right] \right. \\
\left. - \left[ \frac{\alpha_s(\mu)}{\pi} \right]^2 \left[ K_q(M) - \frac{16}{9} - n_f \frac{13}{36} \ln \frac{\mu^2}{M_q^2} + \frac{7}{8} - \frac{n_f}{12} \right] \ln \frac{\mu^2}{M_q^2} \right\} + O(\alpha_s^3), \]

(47)

with \( M = \{ M_f \} \) and

\[ K_q(M) = \frac{3817}{288} + \frac{2}{3} (2 + \ln 2) \zeta(2) - \frac{1}{6} \zeta(3) - \frac{n_f}{3} \left[ \zeta(2) + \frac{71}{48} \right] + \frac{4}{3} \sum_f \Delta \left( \frac{M_f}{M_q} \right), \]

(48)

and \( \Delta(r) \) being a complicated function of \( r \). For our aims it is enough to know that it has the limiting behaviours [30]:

\[ \Delta(r) \xrightarrow{r \to \infty} \frac{1}{4} \ln^2 r + \frac{13}{24} \ln r + \frac{1}{4} \zeta(2) + \frac{151}{288} + O(r^{-2} \ln r), \]

(49)

\[ \Delta(r) \xrightarrow{r \to 0} \frac{3}{4} r \zeta(2) + O(r^2), \]

(50)

with \( \Delta(1) = \frac{3}{4} \zeta(2) - \frac{3}{8} \). Numerically Eq. (48) reads

\[ K_q = 16.00650 - 1.04137 n_f + \frac{4}{3} \sum_f \Delta \left( \frac{M_f}{M_q} \right) \]

(51)

or, equivalently,

\[ K_q = 17.1514 - 1.04137 n_f + \frac{4}{3} \sum_{f \neq q} \Delta \left( \frac{M_f}{M_q} \right). \]

(52)

If \( 0 \leq r \leq 1 \) then the function \( \Delta(r) \) may be conveniently approximated as follows

\[ \Delta(r) = \frac{\pi^2}{8} r - 0.597 r^2 + 0.230 r^3 \]

which is accurate to 1%.

2.2.4 External Currents

So far we have addressed the properties of Green functions of elementary fields. The discussion may be extended to Green functions with insertions of composite operators, for which we want to consider at first various external currents coupled to quark fields. Let

\[ j(x) = \overline{\gamma(x)} \Gamma q(x) \]

(54)

denote a general current where \( \Gamma \) stands for an arbitrary combination of \( \gamma \)-matrices. Green functions with the insertion of one current \( j \) remain in general divergent even after wave
function renormalization of all elementary fields, coupling constants and masses. The remaining divergence is removed by multiplicative renormalization:

\[ j_R = Z_j \bar{q} \gamma_R q, \]  

where

\[ Z_j = 1 + \sum_{i,k}^{0<k<i} (Z_j)_{ik} \left( \frac{\alpha_s}{\pi} \right)^i \frac{1}{e^k}. \]  

The renormalized Green function \( G^{(n)}_{j} \) with insertion of the external current \( \bar{q}j(0)q \) is given by

\[ G^{(n)}_{j} = \int \left( \prod_{i=1}^{n} dy_i e^{ip_i y_i} \right) \left\langle 0 | T j_R(0) \prod_{i=1}^{n} \Phi_i(y_i) | 0 \right\rangle e^{i \int dx \mathcal{L}_R(\Phi(x))} j_R(0) \prod_{i=1}^{n} \Phi_i(y_i) \right\rangle. \]  

The bare Green function is defined through the insertion of the bare current

\[ j_B(x) = Z_j^{-1} Z_2 j_R, \]  

and in an analogous manner

\[ G^{(n)}_{jB} = \int \left( \prod_{i=1}^{n} dy_i e^{ip_i y_i} \right) \left\langle 0 | T j_B(0) \prod_{i=1}^{n} \Phi_B(y_i) | 0 \right\rangle e^{i \int dx \mathcal{L}_B(\Phi_B(x))} j_B(0) \prod_{i=1}^{n} \Phi_B(y_i). \]  

Comparing eqs. (57) and (59) one may relate bare and renormalized Green functions

\[ G^{(n)}_{j} = (Z_j/Z_2) \prod_{i=1}^{n} (Z_i)^{-1/2} G^{(n)}_{jB}(p_f p, m, g, \mu, \epsilon). \]  

The connection between the renormalized and the bare Green functions allows the derivation of the current renormalization constant \( Z_j \) and its anomalous dimension

\[ \gamma_j = \mu^2 \frac{d}{d\mu^2} \ln \left( \frac{Z_j}{Z_2} \right). \]  

The relation between bare and renormalized quark propagator Eq. (35) can be combined with Eq. (60) employed for the special case of the vertex function

\[ G^{(2)}_{j}(p_1, p_2) = i^2 \int dy_1 dy_2 e^{ip_1 y_1 - ip_2 y_2} \left\langle 0 | T q(y_1) j(0) \bar{q}(y_2) | 0 \right\rangle, \]  

to discuss the renormalization of scalar, pseudoscalar, vector and axial vector currents respectively:

\[ j_s = \bar{q} q, \quad j_P = \bar{q} i \gamma_5 q, \quad j^\mu = \bar{q} \gamma_\mu q, \quad j^A_\mu = \bar{q} \gamma_\mu \gamma_5 q. \]
For the diagonal vector current $j^V_{\mu} = \overline{q} \gamma_{\mu} q$ the Ward–Takahashi identity

$$(p_1 - p_2)^\alpha G^{(2)}_{V\alpha}(p_1, p_2, g, m, \mu) = S(p_1, g, m, \mu) - S(p_2, g, m, \mu)$$ (64)

can be employed. A corresponding identity relates the bare vertex function with the bare quark propagator in the same manner, which implies in view of Eqs. (35) and (60) for the renormalization constant and the vector current anomalous dimension

$$Z_V = Z_2, \quad \gamma_V = 0.$$ (65)

These identities are valid also for a nondiagonal vector current $j^V_{\mu} = \overline{q}' \gamma_{\mu} q$ composed of two different quark fields, because $Z_V$ does not depend on the quark mass.

In the case of the axial vector current one may, in a first step, ignore the axial anomaly and assume a Hermitian, anticommuting $\gamma_5$ ($\gamma_5^2 = 1, \{\gamma_5, \gamma_{\mu}\} = 0$). As long as diagrams are considered which involve either nondiagonal axial currents or diagrams without traces of an odd number of $\gamma_5$ matrices, these assumptions are justified and lead to

$$Z_A = Z_2, \quad \gamma_A = 0.$$ (66)

The more involved discussion of the correct treatment of $\gamma_5$ in $D$ dimensions is given below in Section 3.3.

The quark propagator and the two-point Green function with a scalar current insertion are related by

$$G^{(2)}_S(p_1, p_1, g, m, \mu) = - \frac{\partial}{\partial m} S(p_1, g, m, \mu).$$ (67)

From the comparison of this identity with the analogous identity for the bare Green functions one then obtains

$$Z^S = Z_m Z_2.$$ (68)

The scalar current therefore has a non-vanishing anomalous dimension. This result holds true also for nondiagonal currents.

With the same qualifications as discussed above for the axial vector current and with similar arguments one obtains for the pseudoscalar current

$$Z^P = Z_m Z_2.$$ (69)

Whereas neither scalar nor pseudoscalar currents are RG invariant, this holds true for the combinations $m_B j_B^{S/P} = m_R j_R^{S/P}$.

### 2.2.5 Current Correlators

The renormalization properties and anomalous dimensions of the two-point current correlators, defined as the vacuum expectation value of the time-ordered product of the respective currents, will be discussed in this section. In coordinate space the renormalized correlator of a nondiagonal current $j = \overline{q} \Gamma q$ is given by

$$\Pi_j(x, g, m, m', \mu) = \langle 0 | T j(x) j^\dagger(0) | 0 \rangle.$$
The transversality of the polarization operator for the masses of the quark fields \( q \) and \( q' \) are denoted by \( m \) and \( m' \) and all other quarks are assumed to be massless. The correlator contains two renormalized composite operators and hence is finite in the limit \( \epsilon \to 0 \) as long as \( x \neq 0 \). However, in the vicinity of the point \( x = 0 \) this function does exhibit singularities which are not removed by renormalization of the coupling constant, the quark masses, the quark fields and the current as discussed above. In momentum space the renormalized polarization function \( \Pi_j(q, g, m, m', \mu) \) is thus obtained from its bare counterpart by adding new renormalization constants. For the vector current correlator two independent constants appear:

\[
\Pi_{\mu\nu}^V(q, g, m, m', \mu) = i \int dx e^{i q x} \langle 0 | T j_{\mu}(x) j_{\nu}^\dagger(0) | 0 \rangle = \mu^2 \Pi_{B,\mu\nu}^V(q, g_B, m_B, m'_B) + (g_q q - g_{\mu\nu} q^2) Z_q^V \frac{1}{16\pi^2} + g_{\mu\nu}(m - m')^2 Z_m^V \frac{1}{16\pi^2} .
\]  

The transversality of the polarization operator for \( m = m' \) is explicitly taken into account. The factor \( \mu^2 \) is introduced in order to make the dimension of the function \( \Pi_{\mu\nu}^V(q, g, m, m', \epsilon) \) independent of \( \epsilon \) and the factors \( 1/(16\pi^2) \) have been introduced for convenience.

The subtractive renormalization constants \( Z_q^V, Z_m^V \) can be expanded in the coupling constant and have the following form in the minimal subtraction scheme:

\[
Z_q^V = \sum_{0 \leq j-1 \leq i} (Z_q^V)_{ij} \left( \frac{\alpha_s}{\pi} \right)^i \frac{1}{\epsilon^j} ,
\]

\[
Z_m^V = \sum_{0 \leq j-1 \leq i} (Z_m^V)_{ij} \left( \frac{\alpha_s}{\pi} \right)^i \frac{1}{\epsilon^j} .
\]  

The dimensionless expansion coefficients \( (Z_q^V)_{ij} \) and \( (Z_m^V)_{ij} \) are pure numbers. The quadratic dependence of the subtractions in Eq. (70) on the momentum \( q \) and the quark masses is a trivial consequence of the mass dimension of the function \( \Pi_{\mu\nu}^V \) and the fact that renormalization constants are polynomial in masses and momenta [31].

Applying \( \mu^2 \frac{d}{d\mu^2} \) to both sides of Eq. (70), one obtains the RG equation

\[
\mu^2 \frac{d}{d\mu^2} \Pi_{\mu\nu}^V(q, g, m, m', \mu) = (g_q q - g_{\mu\nu} q^2) \gamma_q^V \frac{1}{16\pi^2} + g_{\mu\nu}(m - m')^2 \gamma_m^V \frac{1}{16\pi^2} ,
\]  

with the anomalous dimensions

\[
\gamma_q^V = \mu^2 \frac{d}{d\mu^2} (Z_q^V) - \epsilon Z_q^V ,
\]

\[
\gamma_m^V = \mu^2 \frac{d}{d\mu^2} (Z_m^V) - \epsilon Z_m^V + 2 \gamma_m Z_m^V .
\]  

After insertion of Eq. (71) one observes that \( \gamma_q^V \) and \( \gamma_m^V \) are already completely determined by the coefficients of the simple poles \( 1/\epsilon \) in the expansion of the renormalization constants:

\[
\gamma_q^V = - \sum_{i \geq 0} (i + 1)(Z_q^V)_{i1} \left( \frac{\alpha_s}{\pi} \right)^i ,
\]

\[
\gamma_m^V = - \sum_{i \geq 0} (i + 1)(Z_m^V)_{i1} \left( \frac{\alpha_s}{\pi} \right)^i .
\]
For the axial current correlator

\[ \Pi^A_{\mu \nu} = i \int dx e^{iqx} \langle 0 | T j^A_\mu(x) j^{A\dagger}_\nu(0) | 0 \rangle , \]  

(75)

the renormalization properties correspond to those of the vector case, since

\[ \Pi^A_{\mu \nu}(q, g, m, m', \mu) = \Pi^V_{\mu \nu}(q, g, \pm m, \mp m', \mu) , \]

(76)

which leads to the equivalent RG equation

\[ \mu^2 \frac{d}{d\mu^2} \Pi^A_{\mu \nu}(q, g, m, m', \mu) = (q_\mu q_\nu - g_{\mu \nu} q^2) \gamma^{AA}_q \frac{1}{16\pi^2} + g_{\mu \nu} (m + m')^2 \gamma^{AA}_m \frac{1}{16\pi^2} , \]

(77)

with

\[ \gamma^{AA}_q = \gamma^V_q , \quad \gamma^{AA}_m = \gamma^V_m . \]

(78)

Similar considerations apply to the two-point correlation function of pseudoscalar currents

\[ \Pi^P(-q^2, g, m, m', \mu) = i \int dx e^{iqx} \langle 0 | T j^P(x) j^{P\dagger}_P(0) | 0 \rangle . \]

(79)

The subtractive renormalization of the bare correlator (we limit ourselves below to the massless case)

\[ \Pi^P(Q^2, g, \mu) = (Z_m)^2 \mu^2 \Pi^P_B(Q^2, g_B) + Q^2 Z^{PP}_{PP} \frac{1}{16\pi^2} \]

(80)

leads to the RG equation

\[ \mu^2 \frac{d}{d\mu^2} \Pi^P(Q^2, g, \mu) = Q^2 \gamma^{PP}_q \frac{1}{16\pi^2} + 2\gamma_m \Pi^P(Q^2, g, \mu) , \]

(81)

with the anomalous dimension

\[ \gamma^{PP}_q = \mu^2 \frac{d}{d\mu^2} (Z^{PP}_q) - \epsilon Z^{PP}_q = - \sum_{i \geq 0} (i + 1) (Z^{PP}_q)_i \left( \frac{\alpha_s}{\pi} \right)^i . \]

(82)

The scalar current correlator

\[ \Pi^S = i \int dx e^{iqx} \langle 0 | T j^S(x) j^{S\dagger}(0) | 0 \rangle \]

(83)

and the pseudoscalar current correlator are related in a simple manner:

\[ \Pi^S(Q^2, m, m', \mu) = \Pi^P(Q^2, \mp m, \pm m', \mu) . \]

(84)

For vanishing quark masses scalar and pseudoscalar current correlators are therefore identical: \( \Pi^S = \Pi^P \).

The axial vector and pseudoscalar correlators are connected through the axial Ward identity

\[ q_\mu q_\nu \Pi^A_{\mu \nu} = (m + m') \langle \overline{\psi}_q \psi_q \psi_q \psi_q \rangle , \]

(85)

where the vacuum expectation values on the r.h.s. are understood within the perturbation theory framework. Equation (85) leads to the following relation between the corresponding anomalous dimensions [32]:

\[ \gamma^{AA}_m \equiv -\gamma^{PP}_q . \]

(86)

This relation was used in Ref. [32] in order to find the anomalous dimension \( \gamma^{AA}_m \) at the \( \alpha_s^2 \) order starting from the results of Ref. [33].
2.3 Decoupling of Heavy Quarks

This section deals with the issue of heavy quarks decoupling when MS-like renormalization schemes are employed. The matching conditions relating the parameters of minimally renormalized theories describing the physics well below and above a heavy quark threshold are formulated and a short discussion of power-suppressed effects is given.

2.3.1 Decoupling Theorem in MS-like Schemes

Masses of known quark species differ vastly in their magnitude. As a result, in many QCD applications the mass of a heavy quark \(h\) is much larger than the characteristic momentum scale \(\sqrt{s}\) intrinsic to the physical process. In such a situation there appear two interrelated problems when using an MS-like scheme.

- First, one has \emph{two} large but in general quite \emph{different} mass scales, \(\sqrt{s}\) and \(m_h\), and thus two different types of potentially dangerously large logarithms. The standard trick of a clever choice of the renormalization scale \(\mu\) is no longer effective; one can not set \emph{one} parameter \(\mu\) equal to two different mass scales simultaneously.

- Second, according to the Appelquist–Carrazone theorem \cite{Appelquist:1974tg} heavy particles should be eventually ‘decoupled’ from low-energy physics. However, a peculiarity of mass-independent renormalization schemes is that the decoupling theorem does not hold in its naive form for theories renormalized in such schemes: the effective QCD action to appear will not be canonically normalized. Even worse, large mass logarithms in general appear when one calculates a physical observable! (See the example below.)

Fortunately, both problems are controlled once a proper choice of the expansion parameter is made and the renormalization group improvement is performed \cite{Jegerlehner:2009ry,Jegerlehner:2011ti,Jegerlehner:2014kha}.

In order to be specific, consider QCD with \(n'_f = n_f -1\) light quarks \(\psi = \{\psi_l | l = 1 - n'_f\}\) with masses \(m = \{m_l | l = 1 - n'_f\}\) and one heavy quark \(h\) with mass \(m_h\).

The respective action \(I(\Psi, h, \bar{h}, g, m, m_h, \mu)\) is determined by integrating over the space–time the Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} (G^a_{\mu\nu})^2 + \sum_l \bar{\psi}_l (i \not{D} - m_l) \psi_l + \bar{h} (i \not{D} - m_h) h
\]

(87)

+ terms with ghost fields and the gauge-fixing term.

In the condensed notation of Section 2.2 the collection of fields \(\Phi\) is now decomposed as \(\Phi = \{\Psi, h, \bar{h}\}\) with \(\Psi = \{\psi, \bar{\psi}, A^a_\mu\}\). The (renormalized) generating functional of (connected) Green functions of light fields may now be written as

\[
Z^R (I; S, s) = R_{\text{MS}} \left\{ \int [d\Psi d\bar{h} \bar{h}] \exp \left( i I + \Psi \cdot S + J \cdot s \right) \right\}_c.
\]

(88)

Footnote:

\footnote{It should be stressed that the statement is literally valid only if power-suppressed corrections of order \((s/m_h^2)^n\) with \(n > 0\) are neglected. It is also understood that the effective Lagrangian without the heavy quark field remains renormalizable. Fortunately, the QCD Lagrangian (18) meets this demand. The standard model however, does not fulfil this requirement. This leads to the well-known deviations from the theorem such as the \(m_h^2\) effects in \(\Gamma(Z \to \bar{b}b)\) or the \(\rho\) parameter.}
Here $R_{\overline{MS}}$ is the ultraviolet $R$-operation in $\overline{MS}$-scheme. $S$ and $s$ are sources for the (light) elementary fields from $\Psi$ and for an external quark current $J = \overline{\psi}T\psi$ respectively. For the sake of notational simplicity we shall proceed in the Landau gauge and ignore the ghost field variables.

Integrating out the heavy quark should transform the generating functional (88) to that corresponding to the effective QCD with $n_f$ remaining quark flavours plus additional higher dimension interaction terms suppressed by powers of the inverse heavy mass. The current $J$ as well as any other composite operator will generically develop a non-trivial coefficient function even if one neglects all power-suppressed terms.

In a more formal language the result of integrating out the heavy quark may be summarized in the following master expansion for the generating functional (88):

$$Z_c^R(I; S, s) \to R_{\overline{MS}} \left\{ \int [d \Psi'] \exp \left[ i I^{\text{eff}}(\Psi, g) + \Psi' \cdot S' + \left( J' z_0 + \sum_n \frac{z_n}{m_{h_n}^{-3}} J_n \right) \cdot s_\mu \right] \right\},$$

where the sum is performed over operators constructed from the light fields, with the quantum numbers of those of the initial current $J$ and of mass dimension $\delta_n$. The effective action $I^{\text{eff}}(\Phi, g)$ can be written as

$$I^{\text{eff}}(\Phi, g) = I(\Psi', g')|_{h=0} + \sum_n \int \frac{z_n O_n(x)}{m_{h_n}^{d_n-4}} dx,$$

with

$$(g') = z_m g, \quad (m') = z_m m_q$$

and $\Psi' = \{ \psi' = z_2^{1/2} \psi, (A_\mu') = z_3^{1/2} A_\mu \}$, $J' = \overline{\psi'} T \psi'$. Here $\{O_n\}$ are Lorentz scalars of dimension $d_n > 4$, again constructed from the (primed) light fields only. At last,

$$S' = \{ S_\psi/\sqrt{z_2}, S_{\overline{\psi}}/\sqrt{z_2}, S_A/\sqrt{z_3} \},$$

and all ‘normalization constants’ $z$'s with various subscripts are series of the generic form

$$z_i = \begin{cases} 1 + \sum \limits_{i > 0} \alpha_i^? g^? & \text{if } ? = \psi, \overline{\psi}, 2, 3, g \text{ or } m, \\
\sum \limits_{i > 0} \alpha_i^? g^? & \text{if } ? = J, n. \end{cases}$$

with finite coefficients $\alpha_n$, which are polynomials in $\ln (\mu^2/m_{h_n}^2)$.

The master equation (89) requires some comments.

- The expansion (89) should be understood in the strictly perturbative sense; once it is performed it is necessary to utilize the usual renormalization group methods in order to resum all large logs of the heavy quark mass (see below).

- The master equation (89) governs the $m_h \to \infty$ asymptotic behaviour of all light Green functions: if one neglects power-suppressed terms and does not consider extra current insertions, then their asymptotic behaviour is completely determined by a few finite normalization constants. Even more: in the calculation of physical quantities, which do not depend on the normalization of quantum fields, only two constants remain, viz. $z_1$ and $z_m$. 

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• There exist several methods of computing the finite renormalization constants. The most advanced approach is based on the so-called heavy mass expansion algorithm and will be discussed in Section 3.2.

2.3.2 Matching Conditions for $\alpha_s$ and Masses

In this subsection we review the so-called matching conditions which allow the relating of the parameters of effective low-energy theory without a heavy quark to those of the full theory.

The master equation (89) states that the effective coupling constant $\alpha'_s$ and the (light) quark masses $m'_q$ are expressed in terms of those of the full theory, viz. $\alpha_s$ and $m_q, m_h$, via Eq. [37] — see (91):

$$\alpha'_s(\mu) = \alpha_s(\mu) C(\alpha_s(\mu), x),$$  \hspace{1cm} (93)

$$m'_q(\mu) = m_q(\mu) H(\alpha_s(\mu), x).$$  \hspace{1cm} (94)

Here $x = \ln(\frac{m^2_h}{\mu^2})$ and the functions $C$ and $H$ exhibit the following structure:

$$C(\alpha_s, x) = 1 + \sum_{k \geq 1} C_k \left(\frac{\alpha_s}{\pi}\right)^k, \quad C_k(x) = \sum_{0 \leq i \leq k} C_{ik} x^i,$$  \hspace{1cm} (95)

$$H(\alpha_s(\mu), x) = 1 + \sum_{k \geq 1} H_k \left(\frac{\alpha_s}{\pi}\right)^k, \quad H_k(x) = \sum_{0 \leq i \leq k} H_{ik} x^i,$$  \hspace{1cm} (96)

with $C_{ik}$ and $H_{ik}$ being pure numbers. At present, the functions $C$ and $H$ are known at two-loop level [37–39] and read\(^3\)

$$C_1 = \frac{1}{6} x, \quad C_2 = \frac{11}{72} + \frac{11}{24} x + \frac{1}{36} x^2,$$  \hspace{1cm} (97)

$$H_1 = 0, \quad H_2 = \frac{89}{432} + \frac{5}{36} x + \frac{1}{12} x^2.$$  \hspace{1cm} (98)

Another useful form of (93) is obtained after expressing its r.h.s. in terms of the pole mass $M_h$:

$$\alpha'_s(\mu) = \alpha_s(\mu) \left\{ 1 + \frac{X}{6} \frac{\alpha_s(\mu)}{\pi} + \left(-\frac{7}{24} + \frac{19X}{24} + \frac{X^2}{36}\right) \left[\frac{\alpha_s(\mu)}{\pi}\right]^2 \right\},$$  \hspace{1cm} (99)

with $X = \ln(\frac{M^2_h}{\mu^2})$.

The effective $\alpha'_s$ and the light quark masses evolve with $\mu$ according to their own effective RG equations [36]. It is important to stress that the master equation and hence (93) were derived under the requirement that the normalization scale $\mu$ is much less than $m_h$. However, once obtained, Eqs. (93, 94), present universal relations, valid order by order in perturbation theory.

This implies that on formal grounds one is free to choose the matching value of $\mu = \mu_0$ to determine the value of, say, $\alpha'_s$ in terms of the parameters of the full theory. The final

---

\(^3\)The constant term in $C_2$ is cited according to Ref. [39], where it has been recalculated using two different approaches.
result should not depend on $\mu_0$. However, in practice, some dependence remains from the truncation of higher-orders. Thus the problem is completely similar to that discussed in Section 2.2.2. The correct prescription, hence, is to solve the matching conditions (93, 94) with $\mu$ fixed somewhere in the vicinity of $m_h$ to suppress all mass logarithms. A popular particular choice is to set $\mu = \bar{m}_h(\mu)$ and thus nullify all mass logarithms. The mass $m_h = \bar{m}_h(m_h)$ is sometimes referred to as scale invariant mass of the quark $h$. Finally, one should run the effective coupling constant and quark masses to a lower normalization scale with the effective renormalization group equations.

2.3.3 Matching Equations for Effective Currents

From a fundamental point of view the treatment of effective currents does not differ significantly from the one discussed for the effective coupling constant and masses. Moreover, for the customary case of bilinear quark currents it is even easier: in many instances there exist some extra constraints like Ward identities which help to fix the constant $z_3$. Two cases are of particular interest.

Vector current: This is the most simple and well-known case. For $J = \bar{\psi}_q \gamma_\mu \psi_q$ one derives from the vector Ward identity\(^4\) that

$$z_v \equiv \begin{cases} 1 & \text{if } q \text{ is a light quark} \\ 0 & \text{if } q = h \end{cases} \quad (100)$$

Thus the functional form of a (light) vector quark current is unchanged after integrating out a heavy quark and rewriting it in terms of the effective (that is properly normalized) light quark fields.

Axial vector current: Here the situation is more complicated due to the famous axial vector anomaly. A statement similar to (100) may be proved only for non-singlet axial vector current constructed from light quark fields [41–43]. Explicitly, if $J_A = \sum_{L,\nu} a_{\mu \nu} \bar{\psi}_L \gamma_\nu \gamma_5 \psi_L$ with a traceless matrix $\{a_{\mu \nu}\}$ then the corresponding effective current reads

$$J'_A = \sum_{L,\nu} a_{\mu \nu} \bar{\psi}_L' \gamma_\nu \gamma_5 \psi_L'.$$ \quad (101)

It is understood in (101) that $\gamma_5$ is treated in a way which does not violate the (non-anomalous) chiral Ward identity. In fact this requirement is unmet if the axial vector currents are minimally renormalized with the 't Hooft–Veltman definition of $\gamma_5$. The necessary modifications are discussed in Section 3.3.

If, however, one has a non-singlet combination of light and heavy diagonal axial vector currents then there are no simple formulas like (100) and (101): the resulting effective current is in general not a non-singlet combination of some light axial vector currents. This case is discussed in Refs. [41, 42].

\(^4\)An explicit derivation may be found e.g. in Ref. [40].
2.3.4 Power Suppressed Corrections

The apparatus of the effective theory also allows the taking into account of power suppressed corrections. These can in turn be separated into the corrections to the effective Lagrangian and an effective current. Below we list for illustrative purpose some well-known results.

QCD Lagrangian

The least power suppressed contribution to the sum in (90) is given by a four-quark operator of dimension 6 (see Ref. [44]), viz.

\[- \frac{\alpha_s^2}{15m_h^2} \sum_i (\bar{\psi}_i \gamma_\mu t^a \psi_i) (\bar{\psi}_i \gamma_\mu t^a \psi_i) \]. (102)

Here the colour group generators \( t^a \) are normalized in the standard way \( \text{Tr}(t^a t^b) = \delta^{ab}/2 \).

Vector and axial vector currents

The formulae look almost identical for vector and axial vector (non-singlet) currents (if, of course, the “correct” treatment of \( \gamma_5 \) is employed, see above and Section 5). For the case most useful in practice, namely that of a massless light quark (axial) vector current, one obtains [40]

\[ \bar{\psi} \gamma_\mu (\gamma_\mu \gamma_5) \psi_1 \rightarrow \bar{\psi} \gamma_\mu (\gamma_\mu \gamma_5) \psi_1 \]

\[ + \left\{ \frac{1}{135} \ln \left( \frac{\mu^2}{m_h^2} \right) - \frac{56}{2025} \right\} \left( \frac{\alpha_s}{\pi} \right)^2 \frac{\partial^2}{m_h^2} [\bar{\psi} \gamma_\mu (\gamma_\mu \gamma_5) \psi_1] + O(\alpha_s^3) + O(1/m_h^4) \]. (103)

2.3.5 Example

To provide an example of a peculiar realization of the decoupling theorem in MS-like schemes we now discuss the evaluation of a ‘physical’ quantity — the pole mass \( M_l \) of a light quark \( q_l \) in the full and the effective theories.

First of all we recall that the pole mass is defined as the position of the pole of the quark propagator computed in perturbation theory. It is a renormalization scheme and gauge invariant object [27, 28] whose numerical value should obviously not depend on the theory — full or effective — in which it is evaluated in.

The result of the evaluation of \( M_l \) in the \( \overline{\text{MS}} \)-scheme at two-loop level in the QCD with a heavy quark reads — the formula below is in fact just an inversion of (47):

\[ M_l = m_l(\mu) \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{4}{3} + \ln \frac{\mu^2}{m_l^2} \right) \right\} + \left[ \frac{\alpha_s(\mu)}{\pi} \right]^2 \left[ K_l(m) - \frac{8}{3} + \left( \frac{173}{24} - \frac{13}{36} n_f \right) \ln \frac{\mu^2}{m_l^2} + \left( \frac{15}{8} - \frac{1}{12} n_f \right) \ln^2 \frac{\mu^2}{m_l^2} \right] \}. \] (104)

If \( m_h \rightarrow \infty \) then, according to (49), the function \( K_l(m) \) behaves as \( \ln^2(m_h/m_l)/3 \) and thus the r.h.s. of Eq. (105) is not well defined! This is, of course, a manifestation of
the fact that in this limit the initial parameters of the full theory are not adequate to construct the perturbative theory expansion for a low energy quantity.

However, using the relations (93) and (94) and expressing the r.h.s of (47) in terms of the effective \( \alpha'_s \) and \( m' \), the resulting expression becomes well-defined at the \( m_h \to \infty \) limit and reads

\[
M_t = m_t(\mu) \left[ 1 + \frac{\alpha_s}{\pi} \left[ \frac{4}{3} + \ln \left( \frac{\mu^2}{(m^l)^2} \right) \right] + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{3817}{288} + \frac{2}{3}(2 + \ln 2)\zeta(2) - \frac{1}{6}\zeta(3) - \frac{n'}{3} \left( \zeta(2) + \frac{71}{48} \right) \right. \\
+ \frac{4}{3} \sum_{1 \leq j \leq n'_f} \Delta \left( \frac{m_f}{m^l} \right) + \left( \frac{173}{24} - \frac{13}{36} n'_j \right) \ln \left( \frac{\mu^2}{(m^l)^2} \right) + \left. \left( \frac{15}{8} - \frac{1}{12} n'_j \right) \ln^2 \left( \frac{\mu^2}{(m^l)^2} \right) \right] \right] .
\]

(105)

Now it can be easily seen that (106) is nothing but (47) written in the effective theory with the decoupled heavy quark!

### 2.4 Quark Masses

In this section we briefly discuss the presently available numeric values of pole and running quark masses at different scales. The exposition below serves to explain and motivate the choice of the input quark masses in the numerical discussion of Part 7. It is not intended to provide a comprehensive review of this involved issue (for some recent reviews see, for example, Refs. [45, 46]).

#### 2.4.1 Light \( u, d \) and \( s \) Quarks

For a light quark \( q = u, d, s \) the concept of the pole mass \( M_q \) is clearly meaningless, at least in the framework of the perturbative definition given above. In contrast, the running mass \( m_q \) is well defined, provided the scale parameter \( \mu \) is not too small. Traditionally, the reference scale \( \mu \) is taken to be 1 GeV. The latest available values for these masses are

\[
\bar{m}_u(1 \, \text{GeV}) + \bar{m}_d(1 \, \text{GeV}) = (12 \pm 2.5) \, \text{MeV}, \quad \frac{\bar{m}_u}{\bar{m}_d} = 0.4 \pm 0.22, \quad (106)
\]

\[
\bar{m}_s(1 \, \text{GeV}) = 189 \pm 32 \, \text{MeV}. \quad (107)
\]

(These values (106) and (107) are cited according to Refs. [47–49]; some earlier determinations can be found in Refs. [50–53]).

In all of the applications considered in the present work, it is clearly more than legible to consider \( u \) and \( d \) quarks as massless. Also, the strange quark mass will be neglected everywhere except for small corrections induced by \( m_s \) in the relation between pole and running masses of \( c \) and \( b \) quarks, respectively, as discussed below.
2.4.2 Charm and Bottom

Within the effective four-quark theory the relation between the pole and the running masses of the charmed quark reads (it is, in fact, Eq. (105) with \( n_f = 4 \))

\[
M_c = \bar{m}_c(\mu) \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{4}{3} + \ln \frac{\mu^2}{\bar{m}_c^2} \right) \right. \\
+ \left[ \frac{\alpha_s(\mu)}{\pi} \right]^2 \left[ 10.319 + \frac{4}{3} \Delta \left( \frac{\bar{m}_s}{\bar{m}_c} \right) + \frac{415}{72} \ln \frac{\mu^2}{\bar{m}_c^2} + \frac{37}{24} \ln^2 \frac{\mu^2}{\bar{m}_c^2} \right] \right\}.
\]  

(108)

This equation may be used in two ways. First, if one is given a value of \( \bar{m}_c(\mu) \) then (108) may be used to construct \( M_c \) in the following way. One runs \( \bar{m}_c(\mu) \) (using RG equations in the \( n_f = 4 \) theory) to find the scale invariant mass \( m_c = \bar{m}_c(m_c) \), and then evaluates the r.h.s. of (108) with \( \mu = m_c \). Second, let us suppose that \( M_c \) is known and we would like to find the running mass \( \bar{m}_c(\mu) \) at some reference point \( \mu \). Even in this case the use of (108) is preferable to that of (47), as the latter would contain a contribution proportional to the ill-defined pole mass of the strange quark.

In the case of the b quark the relation (47) assumes the following form (all running masses and the coupling constant are now defined in the \( n_f = 5 \) effective QCD)

\[
M_b = \bar{m}_b(\mu) \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{4}{3} + \ln \frac{\mu^2}{\bar{m}_b^2} \right) \right. \\
+ \left[ \frac{\alpha_s(\mu)}{\pi} \right]^2 \left[ 9.278 + \frac{4}{3} \Delta \left( \frac{\bar{m}_s}{\bar{m}_b} \right) + \frac{4}{3} \Delta \left( \frac{\bar{m}_c}{\bar{m}_b} \right) + \frac{389}{72} \ln \frac{\mu^2}{\bar{m}_b^2} + \frac{35}{24} \ln^2 \frac{\mu^2}{\bar{m}_b^2} \right] \right\}.
\]

(109)

This equation is to be used in the same way as (108).

In the literature there is a variety of somewhat different results for the masses of b and c quarks. Also, there exist strong indications that the very concept of the pole mass is plagued with severe non-perturbative ambiguities [24]. It may well happen that eventually the most accurate and unambiguous mass parameter related to a quark will be its running mass taken at some convenient reference point. However, for illustrative purposes we will use the following ansatz for the pole masses \( M_c \) and \( M_b \):

\[
M_c = 1.6 \pm 0.10 \text{ GeV} \quad \text{and} \quad M_b = 4.7 \pm 0.2 \text{ GeV}.
\]

(110)

The central values and uncertainty bars in (110) are in broad agreement with Refs. [46, 54] and also with those used by the Electroweak Precision Calculation Working Group [55]. Table ?? shows the running masses obtained from (108,109), and RG equations at various relevant scales in dependence on \( \alpha_s(M_Z) \) with \( M_Z = 91.188 \text{ GeV} \).

2.4.3 Top

The top quark mass value as reported by the CDF collaboration [56] is

\[
M_t = 174 \pm 10^{13} \text{ GeV}.
\]

(111)

In our numerical discussions we shall use a conservative input value of \( M_t = 174 \pm 20 \text{ GeV} \).
Table 1

Values of \( \Lambda_{\text{MS}}^{(5)} \), \( \alpha_s^{(5)}(M_\tau) \), \( m_c^{(5)}(M_c) \), \( m_b^{(5)}(M_b) \), \( m_t^{(5)}(M_t) \) and \( m_b(m_b) \) (in GeVs) for different values of \( \alpha_s^{(5)}(M_Z) \), and the default values of \( M_c \) and \( M_b \) as in (110). (110)

<table>
<thead>
<tr>
<th>( \alpha_s^{(5)}(M_Z) )</th>
<th>( \Lambda_{\text{MS}}^{(5)} )</th>
<th>( \alpha_s^{(5)}(M_c) )</th>
<th>( m_c^{(5)}(M_c) )</th>
<th>( m_b^{(5)}(M_b) )</th>
<th>( m_t^{(5)}(M_t) )</th>
<th>( m_b^{(5)}(M_Z) )</th>
<th>( m_b(m_b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.11</td>
<td>0.129</td>
<td>0.188</td>
<td>1.27</td>
<td>1.03</td>
<td>4.13</td>
<td>3.01</td>
<td>4.20</td>
</tr>
<tr>
<td>0.115</td>
<td>0.175</td>
<td>0.248</td>
<td>1.21</td>
<td>0.953</td>
<td>4.07</td>
<td>2.89</td>
<td>4.15</td>
</tr>
<tr>
<td>0.12</td>
<td>0.233</td>
<td>0.32</td>
<td>1.12</td>
<td>0.855</td>
<td>3.99</td>
<td>2.77</td>
<td>4.10</td>
</tr>
<tr>
<td>0.125</td>
<td>0.302</td>
<td>0.403</td>
<td>1.01</td>
<td>0.734</td>
<td>3.91</td>
<td>2.64</td>
<td>4.04</td>
</tr>
<tr>
<td>0.13</td>
<td>0.383</td>
<td>0.499</td>
<td>0.853</td>
<td>0.583</td>
<td>3.82</td>
<td>2.5</td>
<td>3.97</td>
</tr>
</tbody>
</table>

In order to find the corresponding running mass we use the equation below obtained from (47) (we deal now with the fully-fledged \( n_f = 6 \) theory, and discard completely negligible terms caused by the masses of \( s \) and \( c \) quarks)

\[
\overline{m}_i(\mu) = M_i \left\{ 1 - \frac{\alpha_s(\mu)}{\pi} \left( \frac{4}{3} + \ln \frac{\mu^2}{M_i^2} \right) \right. \\
\left.- \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left[ 9.125 + \frac{4}{3} \Delta \left( \frac{M_b}{M_i} \right) + \frac{35}{8} \ln \frac{\mu^2}{M_i^2} + \frac{3}{8} \ln^2 \frac{\mu^2}{M_i^2} \right] \right\} .
\]  

(112)

After setting \( \mu = M_i \) and evaluating \( \alpha_s^{(6)}(M_i) \) one finds

\[ \alpha_s^{(6)}(M_i) = 0.109, \quad \overline{m}_i(M_i) = 164 \text{ GeV}, \]

for our default value \( \alpha_s^{(5)}(M_Z) = 0.120 \) corresponding to \( \Lambda_{\text{MS}}^{(6)} = 233 \text{ MeV} \).

3 Calculational Techniques

In this Part we discuss available calculational techniques to perform small and heavy mass expansions of two-point correlators as well as the problem of \( \gamma_\mu \) in dimensional regularization.

3.1 Current Correlators at Large Momentum

A lot of results on higher-order radiative corrections were derived after neglecting quark masses, originating from massless diagrams and resulting in a drastic simplification of
calculations. However, problems arise when quark masses are taken into account, at least in the form of power corrections. In the simplest cases the evaluation of, say, a quadratic quark mass correction may be reduced to the computation of massless diagrams which are obtained by naively expanding the massive propagators in the quark mass. However, this strategy fails in the general case starting from quartic mass terms. The so-called logarithmic mass singularities appear and render the simple Taylor expansion meaningless. In this section the general structure of non-leading mass corrections will be discussed as well as some approaches for their evaluation. The presentation is mainly based on Refs. [57-59].

In investigating the asymptotic behaviour of various correlators at large momentum transfer, it proves to be very useful to employ the Wilson expansion in the framework of the MS scheme. Consider vector current correlator

\[ i \int \langle 0 | T J_\mu(x) J_\nu(0) | 0 \rangle e^{i q x} dx = (g_{\mu\nu} - q_{\mu} q_{\nu} / q^2) \Pi(Q^2), \]  

(113)

with \( J_\mu = \bar{q} \gamma_\mu q \). Here \( q \) is a quark with mass \( m_q \equiv m \). To simplify the following discussion we will consider the second derivative \( \Pi''(Q^2) \equiv d^2 \Pi(Q^2) / d(Q^2)^2 \), which can be seen from (72) to satisfy a homogeneous RG equation,

\[ \mu^2 \frac{d}{d \mu^2} \Pi''(Q^2) = 0. \]  

(114)

The high energy behaviour of \( \Pi''(Q^2) \) in the deep Euclidean region may be reliably evaluated in QCD by employing the operator product expansion:

\[ Q^2 \Pi''(Q^2, \alpha_s, m, \mu) \xrightarrow{Q^2 \to \infty} K_0(Q^2, \alpha_s, m, \mu) 1 \]

\[ + \sum_n \frac{1}{(Q^2)^{n/2}} \sum_{\text{dim } O_i = n} K_i(Q^2, \alpha_s, m, \mu) \langle 0 | O_i(\mu) | 0 \rangle. \]  

(115)

We have explicitly separated the contribution of the unit operator from that of the operators with non-trivial dependence on the field variables. The coefficient functions \( K_0 \) and \( K_i \) depend upon the details of the renormalization prescription for the composite operators \( O_i \). The usual procedure of normal ordering for the composite operators appearing on the r.h.s. of Eq. (115) becomes physically inconvenient if quark mass corrections are to be included [58]. From the calculational viewpoint it also does not lead to any insight in computing power-suppressed mass corrections involving mass logarithms.

Indeed, the coefficient function in front of the unit operator in (115) represents the usual perturbative contributions and, if normal ordering is used, it contains in general mass and momentum logarithms of the form

\[ \left( \frac{m^2}{Q^2} \right)^n \left( \ln \frac{\mu^2}{Q^2} \right)^{n_1} \left( \ln \frac{\mu^2}{m^2} \right)^{n_2}, \]  

(116)

with \( n, n_1 \) and \( n_2 \) being non-negative integers. More specifically, one can write

\[ K_0^{NO}(Q^2, \alpha_s, m, \mu) \xrightarrow{Q^2 \to \infty} \sum_{n \geq 0, l \geq 0} \left( \frac{m^2}{Q^2} \right)^n \left( \frac{\alpha_s}{\pi} \right)^{l-1} F_{nl}(L, M), \]  

(117)
Figure 4: (a) Lowest order contribution to the correlator $\Pi''(Q^2)$. (b) Vacuum diagram contributing to the perturbative VEV of the operator $\bar{q}q$.

where $L = \ln(\mu^2/Q^2)$, $M = \ln(\mu^2/m^2)$, and the superscript NO is a reminder of the normal ordering prescription being used. The function $F_{nl}(L, M)$ corresponds to the contribution of the $l$-loop diagrams, and is a polynomial of degree not higher than $l$, in both $L$ and $M$. The contributions due to non-trivial operators — that is containing some dependence on field variables — are completely decoupled from those of the unit operator if the normal ordering is employed, since the vacuum expectation value vanishes for every non-trivial operator $O$:

$$\langle 0|O|0 \rangle \equiv 0.$$

The situation improves drastically if one abandons the normal ordering prescription. It was realized some time ago [57, 58] that all logarithms of quark masses may be completely shifted to the vacuum expectation values (VEV) of non-trivial composite operators appearing on the r.h.s. of (115) if the latter are minimally subtracted.

To give a simple example, let us consider the correlator (113) in the lowest order one-loop approximation. First, we use the normal ordering prescription for the composite operators which appear in the OPE of the time ordered product in (113). To determine the coefficients of the various operators, one possible method is to sandwich both sides of the OPE between appropriate external states. By choosing them to be the vacuum, only the unit operator $1$ will contribute on the r.h.s., if the normal ordering prescription is used. This means that the bare loop of Fig. 4a contributes entirely to the coefficient $K_0$ in (115). A simple calculation gives (in the sequel we neglect all terms of order $1/Q^6$ and higher):

$$Q^2\Pi''(Q^2) \xrightarrow{Q^2 \to \infty} K_0(Q^2)1 + \frac{4}{Q^4}\langle 0|m\bar{q}q|0 \rangle,$$

$$K_0^{NO}(Q^2, m, \mu) = Q^2\Pi''(Q^2, m, \mu)\big|_{\alpha_s=0} = -\frac{1}{16\pi^2} \left[ 1 + \frac{48m^4}{Q^4}(1 + L - M) \right].$$

29
The coefficient function $K_0^{NO}$ contains mass singularities (the M-term). On the other hand, if one does not follow the normal ordering prescription, then the operator $m \bar{q} q$ develops a non-trivial vacuum expectation value even if the quark gluon interaction is turned off by setting $\alpha_s = 0$. Indeed, after minimally removing its pole singularity, the one loop diagram of Fig. 4b leads to the following result [60]:

\[
\langle 0 | \bar{q} q | 0 \rangle^{PT} = \frac{3m^3}{4\pi^2} \left( \ln \frac{\mu^2}{m^2} + 1 \right).
\]  

(120)

By inserting this into (115), the new coefficient function $K_0$ can be extracted, with the result:

\[
K_0 = -\frac{1}{16\pi^2} \left[ 1 + \frac{48m^4}{Q^4}(2 + L) \right].
\]  

(121)

The mass logarithms are now completely transferred from the CF $K_0$ to the VEV of the quark operator (120)! The same phenomenon continues to hold even after the $\alpha_s$ corrections are taken into account for (pseudo)scalar and pseudovector correlators, independently of their flavour structure [59, 61].

The underlying reason for this was first found in Ref. [62]. There it was shown that no coefficient function can depend on mass logarithms in every order of perturbation theory if the minimal subtraction procedure is scrupulously observed\(^5\). This is true irrespective of the specific model and correlator under discussion. Three important observations may be made in this context:

- Prior knowledge of the fact that any conceivable correlator can be expanded in a series of the form (116) makes it possible to obtain without calculation important information on the structure of mass logs as they appear in various correlators. For example, in QCD any correlator should contain no mass logarithms in the quadratic in mass terms [57, 58]. This holds true because there does not exist a gauge-invariant non-trivial operator of (mass) dimension two in QCD.

- From the purely calculational point of view the problem of computing non-leading mass corrections to current correlators becomes much simpler. This is due to two facts. First, all coefficient functions are expressed in terms of massless Feynman integrals while VEV’s of composite operators are by definition represented in terms of some massive integrals without external momenta (tadpole diagrams). Second, methods have been elaborated for computing analytically both types of Feynman integrals.

- The abandonment of the normal ordering slightly complicates the renormalization properties of composite operators. An instructive example is provided by the ‘quark mass operator’ $O_2 = m \bar{q} q$. The textbook statement (see, for example, Ref. [6]) that this operator is RG invariant is no longer valid. Indeed, the vacuum diagram of Fig. 4b has a divergent part which has to be removed by a new counterterm proportional to the operator $m^4 1$. In other words, $m \bar{q} q$ begins to mix with the ‘operator’ $m^4 1$ [58].

\(^5\)In particular it also excludes the normal ordering, as the latter amounts to a specific non-minimal subtraction of diagrams contributing to VEV’s of composite operators.
To lowest order, the corresponding anomalous dimension matrix reads:

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} m_\tilde{q}q \\ m^4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{4\pi^2} \\ 0 & -4\alpha_s/\pi \end{pmatrix} \begin{pmatrix} m_\tilde{q}q \\ m^4 \end{pmatrix}. \quad (122)$$

The non-vanishing, off-diagonal matrix element describes the mixing of the two operators under renormalization and was obtained from the divergent part of the vacuum diagram in Fig. 4. The diagonal matrix elements are just the anomalous dimensions of the respective operators in the usual normal-ordering scheme. The lower one is equal to $4\gamma_m(\alpha_s)$. Note that the general structure of the anomalous dimension matrix of all gauge-invariant operators of dimension four has been established in Refs. [63, 58]. This information was used recently [64] to evaluate the corrections of order $m_\tilde{q}^4\alpha_s^2$ to the vector current correlator (see Section 5.4).

### 3.2 Top Mass Expansion in $s/m_t^2$

Our discussion of the dependence of cross-sections and decay rates on the quark masses has up to now dealt with five flavours light enough to be produced in $e^+e^-$ collisions. The top quark, on the other hand, is too heavy to be present in the final state, even at LEP energies. Nevertheless it constitutes a virtual particle. Virtual top loops appear for the first time in second order $\alpha_s^2$. Massive multi-loop integrals may conveniently be simplified considering the heavy top limit $m_t \to \infty$. In this effective field theory approach the top is integrated out from the theory. Then the Lagrangian of the effective theory contains only light particles. The effects of the top quark are accounted for through the introduction of additional operators in the effective Lagrangian. For the vector current correlator their contributions are suppressed by inverse powers of the heavy quark mass $s/m_t^2$. As we will also explicitly see, no decoupling is operative in the case of the axial vector correlator. A logarithmic top mass dependence signals the breakdown of anomaly cancellation if the top quark is removed from the theory.

The heavy mass expansion is constructed as follows (see Refs. [65–68]; a rigorous mathematical formulation can be found in Ref. [68]): Let the Feynman integral $\langle \Gamma \rangle$ of a Feynman graph $\Gamma$ depend on a heavy mass $M$ and some other 'light' masses and external momenta which we will generically denote as $m$ and $q$ respectively. In the limit $M \to \infty$ with $q$ and $m$ fixed $\langle \Gamma \rangle$ may be represented by the asymptotic expansion:

$$\langle \Gamma \rangle_{M \to \infty} = \sum_\gamma C_\gamma^{(t)} \times \langle \Gamma/\gamma \rangle^{\text{eff}}. \quad (123)$$

The diagrams $\langle \Gamma/\gamma \rangle^{\text{eff}}$ of the effective theory consist of light particles only, whereas the top mass is only present in the 'coefficient functions' $C_\gamma^{(t)}$. The notation $\langle \Gamma/\gamma \rangle^{\text{eff}}$ means that the hard subgraph $\gamma$ of the original diagram $\Gamma$ is contracted to a blob. By definition a hard subgraph contains at least all heavy quark lines and becomes one particle irreducible if each top quark propagator is contracted to a point. The Feynman integral of the hard subgraph is expanded in a formal (multidimensional) Taylor expansion with respect to the small parameters, namely the light masses and the external momenta of $\gamma$. It should
be noted that the set of external momenta for a subgraph $\gamma$ is defined \textit{with respect to} $\gamma$ and thus in general consists of some genuine external momenta (that is, those shared by $\gamma$ and the very diagram $\Gamma$) as well as momenta flowing through \textit{internal} lines of $\Gamma$, which are \textit{external ones} of $\gamma$ (see the example below). This Taylor series $C^{(t)}_{\gamma}$ is inserted in the effective blob and the resulting Feynman integral has to be calculated. All possible hard subgraphs have to be identified and the corresponding results must be added.

The prescription for the construction of the coefficient function $C^{(t)}_{\gamma}$ for a hard subgraph $\gamma$ can be formulated as follows: Suppose the Feynman integral $\langle \gamma \rangle (M, q^\gamma, m^\gamma, \mu)$ corresponds to a hard subgraph $\gamma$ and depends on external momenta $q^\gamma$ and light masses $m^\gamma$ in addition to the heavy mass $M$. Then

$$C^{(t)}_{\gamma} = t_{\{q^\gamma, m^\gamma\}} \langle \gamma \rangle (M, q^\gamma, m^\gamma, \mu), \quad (124)$$

where the operator $t_{\{x_1, x_2, \ldots\}}$ performs the formal Taylor expansion according to the rule:

$$t_{\{x_1, x_2, \ldots\}} = \sum_{n \geq 0} t_{\{x_1, x_2, \ldots\}}^{(n)}, \quad (125)$$

$$t_{\{x_1, x_2, \ldots\}}^{(n)} F(x_1, x_2 \ldots) = \frac{1}{n!} \left( \frac{d}{d \xi} \right)^n F(\xi x_1, \xi x_2 \ldots)_{\xi=0}. \quad (126)$$

Here several comments are in order.

- The differentiation with respect to $\xi$ in (126) may be carried out in two ways. One could simply differentiate the Feynman integral, which is a smooth function of $\xi$ at $\xi \neq 0$. A more practical way is to differentiate the corresponding \textit{integrand}.

- The operation of setting $\xi$ zero is to act on the differentiated integrand.

- It may be immediately seen that the expression

$$t_{\{q^\gamma, m^\gamma\}}^{(n)} \langle \gamma \rangle (M, q^\gamma, m^\gamma, \mu)$$

scales with $M$ as $M^{\omega(\gamma) - n}$ where $\omega(\gamma)$ is the (mass) dimension of the Feynman integral $\langle \gamma \rangle$ determined without counting any dimensionful coupling constant as well the 't Hooft mass $\mu$. Therefore, in every application of the hard mass expansion the terms with too high value of $n$ in (125) may be dropped.

- By construction the coefficient function $C^{(t)}_{\gamma}$ is a polynomial with respect to its external momenta $q^\gamma$ and the light masses $m^\gamma$.

As an example we consider the two-loop diagram $\Gamma$ depicted in Fig. 5 which contributes to the fermion propagator in QED. The heavy fermion of mass $M$ is contained in the virtual fermion loop, whereas the open fermion line corresponds to a propagating light fermion with mass $m$. The integral reads (in Feynman gauge):

$$\mu^{2\frac{i}{2}} \int \frac{d^D p}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Sp} \left[ \gamma_\alpha(\not{p} + M) \gamma_\beta(\not{k} + M) \right] \left[ \gamma_\alpha(\not{q} - \not{k} + m) \gamma_\beta \right]$$

$$\mu^{2\frac{i}{2}} \int \frac{d^D p}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} (-k^2)^2 (M^2 - p^2)(M^2 - (p - k)^2)[m^2 - (q - k)^2]. \quad (127)$$
Figure 5: Hard Mass Procedure.

The integration momenta are denoted as $k$ and $p$ for the outer and the inner loops respectively. Two different integration regions can be identified. In the first region is $k \ll M$, $p \simeq M$. The corresponding hard subgraph $\gamma_1$ is shown in Fig. 5 and $\langle \gamma_1 \rangle$ has to be expanded with respect to its only external momentum, $k$. The second region is characterized by $k, p \approx M$. The hard subgraph $\gamma_2$ coincides with $\Gamma$ and the Feynman integral $\langle \gamma_2/\gamma_2 \rangle$ reduces to unity. In this case the hard subgraph $\langle \Gamma \rangle$ must be expanded with respect to the external momentum $q$ and, in case of a non-vanishing light mass $m$, also with respect to $m$. The sum of all contributions results in a power series in the inverse top mass.

Working up to the power-suppressed terms of order $q^2/M^2$, one has

$$C_{\gamma_1}^{(t)} = (t_{[k]}^{[0]} + t_{[k]}^{[1]} + t_{[k]}^{[2]} )\langle \gamma_1 \rangle (M, k, \mu) \quad \text{and} \quad C_{\gamma_2}^{(t)} = (t_{[q,m]}^{[0]} + t_{[q,m]}^{[1]} )\langle \gamma_2 \rangle (M, q, m, \mu).$$

In explicit form these coefficient functions are given by the following Feynman integrals

$$C_{\gamma_1}^{(t)} = \mu^2 i \int \frac{d^D p}{(2\pi)^D} \text{Sp} \left[ \gamma_\alpha (\hat{p} + M) \gamma_\beta (\hat{\hat{k}} + M) \right] \left( 1 + r + r^2 \right),$$

with \( r = \frac{k^2 - 2pk}{M^2 - p^2} \) and

$$C_{\gamma_2}^{(t)} = \mu^4 i^2 \int \frac{d^D p}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \times \left[ \text{Sp} \left[ \gamma_\alpha (\hat{p} + M) \gamma_\beta (\hat{\hat{k}} + M) \right] \gamma_\alpha (\hat{q} - \hat{k} + m) \gamma_\beta \right].$$

It is of course understood that the terms of higher than second order in the expansion parameters are discarded in the integrand of (129).


3.3 $\gamma_5$ in $D$ Dimensions

Multi-loop calculations with dimensional regularization often encounter the question of how to treat $\gamma_5$ in $D$ dimensions. Occasionally the problem can be circumvented by exploiting chiral symmetry which allows, for example, the relating of the non-singlet axial correlator in the massless limit to the corresponding vector correlator. In general, however, a consistent definition must be formulated. A rigorous choice is based on the original definition by 't Hooft and Veltman [5b], and formalized by Breitenlohner and Maison [69] with a modification introduced in [70]. In this self-consistent approach $\gamma_5$ is defined as

$$
\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma ,
$$

with $\epsilon_{0123} = 1$. For our discussion we consider the cases for both the non-singlet axial current $j_{\mu\nu}^{[\mathrm{NS}]}$ and the singlet one $j_{\mu\nu}^{[\mathrm{S}]}$, which are defined with the help of the antisymmetrized combination $\gamma^{[\mu\nu\rho]} = (\gamma^\mu \gamma^\nu \gamma^\rho - \gamma^\nu \gamma^\mu \gamma^\rho)/2$ in order to guarantee Hermiticity for noncommuting $\gamma_5$.

$$
\begin{align*}
\langle j_{\mu\nu}^{[\mathrm{NS}]} \rangle &= \frac{1}{2} \gamma (\gamma_{\nu} \gamma_{\mu} - \gamma_{\mu} \gamma_{\nu}) t^{a} \Psi \\
&= \frac{i}{3!} \epsilon_{\mu\nu\rho\sigma} \gamma^{[\mu\nu\rho]} t^{a} \Psi = \frac{i}{3!} \epsilon_{\mu\nu\rho\sigma} A_{\mathrm{NS}}^{[\mu\nu\rho]} t^{a} , \\
\langle j_{\mu\nu}^{[\mathrm{S}]} \rangle &= \frac{1}{2} \gamma (\gamma_{\nu} \gamma_{\mu} - \gamma_{\mu} \gamma_{\nu}) \Psi \\
&= \frac{i}{3!} \epsilon_{\mu\nu\rho\sigma} \gamma^{[\mu\nu\rho]} \Psi = \frac{i}{3!} \epsilon_{\mu\nu\rho\sigma} A_{\mathrm{S}}^{[\mu\nu\rho]} .
\end{align*}
$$

Here $t^{a}$ are the generators of the $SU(n_f)$ flavour group.

The four-dimensional Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ is kept outside the renormalization procedure where all indices can be considered as four dimensional whereas the calculation is performed with the generalized currents $A_{\mathrm{NS}}^{[\mu\nu\rho]}, A_{\mathrm{S}}^{[\mu\nu\rho]}$ in $D$ dimensions. As a consequence of the lost anticommutativity of $\gamma_5$, standard properties of the axial current as well as the Ward identities are violated. In particular, it turns out that the renormalization constant $Z_{\mathrm{NS}}$ of the non-singlet current is not one any more. To restore the correctly renormalized non-singlet axial current an extra finite renormalization is introduced with corresponding finite renormalization constant $z_{\mathrm{NS}} [6, 71]$. One thus has for the renormalized non-singlet axial current the following expression:

$$
\langle j_{\mu\nu}^{[\mathrm{NS}]} \rangle_R = z_{\mathrm{NS}} Z_{\mathrm{NS}} \langle j_{\mu\nu}^{[\mathrm{NS}]} \rangle_B
$$

with [72, 73]

$$
Z_{\mathrm{NS}} = 1 + a^2 \frac{1}{\epsilon} \left[ \frac{11}{6} - \frac{1}{9} n_f \right] + a^3 \frac{1}{\epsilon^2} \left[ -\frac{121}{36} + \frac{11}{27} n_f - \frac{1}{81} n_f^2 + \epsilon \left( \frac{391}{72} - \frac{44}{81} n_f + \frac{1}{486} n_f^2 \right) \right] (134)
$$

and

$$
z_{\mathrm{NS}} = 1 - \frac{4}{3} a + a^2 \left( \frac{19}{36} + \frac{1}{54} n_f \right) . (135)
$$

34
The prescription described above and the use of the non-singlet axial current defined according to Eq. (133) lead to the same characteristics for nonanomalous amplitudes as would be obtained within a naive approach featuring completely anticommuting $\gamma_5$. First, the Ward identity is recovered. Second, the anomalous dimension of the non-singlet axial current vanishes. For diagrams with an even number of $\gamma_5$ connected to the external current it has been checked that the treatment based on an anticommutativity of $\gamma_5$ leads to the same answer [74].

Similar considerations may be carried out for the singlet axial vector current. However, in this case there is some freedom in defining the renormalized current. This is due to the fact that in any physical application the current never appears as it is but only in a (virtually non-singlet) combination with another axial vector current. A physically motivated definition has been suggested in Ref. [42], where the singlet axial vector current has been defined with the help of the following limiting procedure:

$$\left( j^{(S)}_{5\mu} \right)_R \xrightarrow{m_T \to \infty} Z^{NS} Z^{NS} \left[ j^{(S)}_{5\mu} - n_f \frac{i}{3!} \epsilon_{\mu \nu \rho \sigma} \Psi_T \gamma_5 \gamma^{[\nu \rho \sigma]} \Psi_T \right] B.$$ \quad (136)

Here, $\Psi_T$ is the field of an auxiliary quark $T$ and thus the combination in the squared brackets is a non-singlet one (in the extended QCD with $n_f + 1$ flavours!). Due to the asymptotic freedom, the large $m_T$ limit of (136) does exist and is naturally identified with the renormalized singlet axial current. Explicitly, the r.h.s. of (136) can be written without any auxiliary fields in the form (note that the renormalization constant $Z^S$ was first found in Ref. [73])

$$\left( j^{(S)}_{5\mu} \right)_R = z^S Z^S \left( j^{(S)}_{5\mu} \right)_B,$$ \quad (137)

with

$$Z^S = 1 + a \frac{1}{\epsilon} \left[ \frac{\log n_f}{6} + \frac{5}{36} n_f \right] + a^2 \frac{1}{\epsilon^2} \left[ \frac{121}{36} - \frac{11}{216} n_f + \frac{5}{324} n_f^2 + \epsilon \left( \frac{391}{72} + \frac{1296}{61} n_f + \frac{13}{1944} n_f^2 \right) \right]$$ \quad (138)

and

$$z^S = 1 + a \left( \frac{\frac{-5}{18} n_f - \frac{11}{3}}{\beta_0} \right) + a^2 \left[ \frac{1}{\beta_0^2} \left( \frac{185}{2592} n_f^2 + \frac{391}{864} n_f + \frac{2651}{144} \right) - \frac{1}{\beta_0^2} \left( \frac{13}{1296} n_f^2 + \frac{61}{864} n_f + \frac{391}{48} \right) \right],$$ \quad (139)

where $\beta_0 = (11 - \frac{2}{3} n_f)/4$. It should be noted that an equivalent definition of the singlet axial vector current is obtained by demanding that it have a vanishing anomalous dimension.

4 Exact Result of Order $O(\alpha_s)$

The exact QCD corrections for arbitrary quark masses are known in order $O(\alpha_s)$. The result is different for vector and axial current correlators. Whereas the former can be
taken directly from QED [75] the latter have been obtained in Ref. [76]. (For the non-diagonal current and arbitrary, different masses the result can be found in [77].) With \( v^2 = 1 - 4m^2/s \) they read:

\[
\begin{align*}
\mathcal{R}^{\text{NS}}_V &= v \frac{3 - v^2}{2} \left[ 1 + \frac{4}{3} \frac{\alpha_s(s)}{\pi} K_V \right], \\
\mathcal{R}^{\text{NS}}_A &= v^3 \left[ 1 + \frac{4}{3} \frac{\alpha_s(s)}{\pi} K_A \right].
\end{align*}
\]

(140)

\( K_V \) and \( K_A \) have been calculated in Refs. [76-78]. A compact form for the correction can be found in Ref. [78]:

\[
\begin{align*}
K_V &= \frac{1}{v} \left[ A(v) + \frac{P_V(v)}{(1 - v^2/3)} \ln \frac{1 + v}{1 - v} + \frac{Q_V(v)}{(1 - v^2/3)} \right], \\
K_A &= \frac{1}{v} \left[ A(v) + \frac{P_A(v)}{v^2} \ln \frac{1 + v}{1 - v} + \frac{Q_A(v)}{v^2} \right],
\end{align*}
\]

(141)

with

\[
A(v) = (1 + v^2) \left[ \text{Li}_2 \left( \frac{1 - v}{1 + v} \right) + 2 \text{Li}_2 \left( \frac{1 - v}{1 + v} \right) + \ln \frac{1 + v}{1 - v} \ln \frac{(1 + v)^2}{8v^2} \right] + 3v \ln \frac{1 - v^2}{4v} - v \ln v,
\]

\[
P_V(v) = \frac{33}{24} + \frac{22}{24} v^2 - \frac{7}{24} v^4, \quad Q_V(v) = \frac{5}{4} v - \frac{3}{4} v^3,
\]

\[
P_A(v) = \frac{21}{32} + \frac{59}{32} v^2 - \frac{19}{32} v^4 - \frac{3}{32} v^6, \quad Q_A(v) = -\frac{21}{16} v + \frac{30}{16} v^3 + \frac{3}{16} v^5.
\]

(143)

Convenient parametrizations are [79]:

\[
\begin{align*}
K_V &= \frac{\pi^2}{2v} - \frac{3 + v}{4} \left( \frac{\pi^2}{2} - \frac{3}{4} \right), \\
K_A &= \frac{\pi^2}{2v} - \left[ \frac{19}{10} - \frac{22}{5} v + \frac{7}{2} v^2 \right] \left( \frac{\pi^2}{2} - \frac{3}{4} \right).
\end{align*}
\]

(144)

Let us consider this result in the limit where \( s \) approaches the threshold region \( (v \to 0) \) as well as the high energy regime \( (v \to 1) \).

For \( v \to 0 \) the correction factors simplify to:

\[
1 + \frac{4}{3} \frac{\alpha_s}{\pi} K_V \xrightarrow{v \to 0} \frac{2\pi \alpha_s}{3v} + \left( 1 - \frac{16}{3} \frac{\alpha_s}{\pi} \right),
\]

\[
1 + \frac{4}{3} \frac{\alpha_s}{\pi} K_A \xrightarrow{v \to 0} \frac{2\pi \alpha_s}{3v} + \left( 1 - \frac{8}{3} \frac{\alpha_s}{\pi} \right).
\]

(145)

For very small \( v \) higher-order contributions must be taken into consideration. In QED these can be summed to yield the Sommerfeld rescattering factor:

\[
R_{\text{QED}} = \frac{\pi \alpha_s/v}{1 - e^{-\pi \alpha_s/v}}
\]

(146)
In QCD the coupling constant $\alpha$ would be replaced in this formula by $4\alpha_s/3$.

However, the scale of $\alpha_s(Q^2)$ cannot be fixed with certainty, since subleading logarithms have not yet been evaluated. It has been argued in Ref. [79] that the choice $\alpha_s(|P_t|)$, combined with Eq. (144) allows for an adequate description of $R$ in the threshold region and provides a smooth connection between resonances and continuum. For top quarks a new element enters through their large decay rate. Resonance and open $t\bar{t}$ production merge. An account of the resulting phenomena is beyond the scope of this paper and can be found in Refs. [81–86].

The behaviour of the result for large $s$ can easily be extracted from the analytic formulae [78, 86, 87]. In Born approximation the leading term of the vector and the axial vector correlators are of order $m^4/s^2$ and $m^2/s$ respectively:
Figure 6: Comparison between the complete $O(\alpha_s)$ correction function (solid line) and approximations of increasing order (dashed lines) in $m^2$ for vector (upper graph) and axial vector current (lower graph) induced rates.
\[ v^3 \rightarrow 1 - \frac{m^2}{s^2} + 6 \frac{m^4}{s^4} + O(m^6/s^6). \]

Including first-order QCD corrections leads to:

\begin{align*}
  r_{VNS} \overset{v \rightarrow 1 - 6 \frac{m^4}{s^4}}{\rightarrow} & \frac{\alpha_s}{\pi} \left[ 1 + 12 \frac{m^2}{s^2} + \frac{m^4}{s^4} \left( 10 - 24 \ln \frac{m^2}{s^2} \right) \right], \\
  r_{NS}^{A} \overset{v \rightarrow 1 - 6 \frac{m^2}{s^2} + 6 \frac{m^4}{s^4}}{\rightarrow} & \frac{\alpha_s}{\pi} \left[ 1 - \frac{m^2}{s^2} \left( 6 + 12 \ln \frac{m^2}{s^2} \right) + \frac{m^4}{s^4} \left( -22 + 24 \ln \frac{m^2}{s^2} \right) \right].
\end{align*}

The approximations to the correction functions for the vector and the axial vector current correlators (including successively higher-orders without the factor \( \alpha_s/\pi \)) are compared to the full result in Fig. 6. As can be seen in this figure, for high energies — say for \( 2m_b/\sqrt{s} \) below 0.3 — an excellent approximation is provided by the constant plus the \( m^2 \) term. In the region of \( 2m/\sqrt{s} \) above 0.3 the \( m^4 \) term becomes increasingly important. The inclusion of this term improves the agreement significantly and leads to an excellent approximation, even up to \( 2m/\sqrt{s} \approx 0.7 \) or 0.8. For the narrow region between 0.6 and 0.8 the agreement is further improved through the \( m^6 \) term.

The mass \( m \) in this formula is understood as the physical mass, defined through the location of the pole of the quark propagator in complete analogy with the treatment of the electron mass in QED. However, if one tried to control fully the \( m^2/s \) and \( m^4/s^2 \) terms one might worry about the logarithmically enhanced coefficient which could invalidate perturbation theory. These leading logarithmic terms may be summed through renormalization group techniques.

In order \( \alpha_s \), this can be trivially achieved by substituting

\[ m^2 = \overline{m}^2 \left[ 1 + \frac{\alpha_s}{\pi} (8/3 - 2) \ln (\overline{m}^2/s) \right] \]

which implies (for completeness also \( m^6/s^2 \) terms from Ref. [64] are included)

\begin{align*}
  r_{VNS} \overset{v \rightarrow 1 - 6 \frac{\overline{m}^4}{s^4} - 8 \frac{\overline{m}^6}{s^6}}{\rightarrow} & \frac{\alpha_s}{\pi} \left[ 1 + 12 \frac{\overline{m}^2}{s^2} - 22 \frac{\overline{m}^4}{s^4} - \frac{16}{27} \left( 6 \ln \frac{\overline{m}^2}{s^2} + 155 \right) \frac{\overline{m}^6}{s^6} \right], \\
  r_{NS}^{A} \overset{v \rightarrow 1 - 6 \frac{\overline{m}^2}{s^2} + 6 \frac{\overline{m}^4}{s^4} + 4 \frac{\overline{m}^6}{s^6}}{\rightarrow} & \frac{\alpha_s}{\pi} \left[ 1 - 22 \frac{\overline{m}^2}{s^2} + 10 \frac{\overline{m}^4}{s^4} + \frac{8}{27} \left( -39 \ln \frac{\overline{m}^2}{s^2} + 149 \right) \frac{\overline{m}^6}{s^6} \right].
\end{align*}

A systematic discussion of higher-order terms will be given in the subsequent sections.
5 Non-singlet Contributions

5.1 Massless Limit

This section will cover those results obtained in the limit of massless quarks. As discussed in the previous part, non-singlet contributions exhibit a universal charge factor which is given by the Born result and can be trivially factored. The first-order correction was derived in the context of QED some time ago \[75\]. The second order coefficient has been calculated by several groups \[88\]. The initial calculation of the \(O(\alpha_s^3)\) term described in \[89\] was later corrected by two groups \[90, 91\]. An implicit test of the results has recently been performed \[74\].

The full result for the non-singlet current reads as follows:

\[
\begin{align*}
\mathcal{R}_{NS}(s) & = 1 + \frac{\alpha_s(s)}{\pi} + \left[ \frac{\alpha_s(s)}{4\pi} \right]^2 \left\{ \frac{730}{3} - 176 \zeta(3) + \left[ -\frac{44}{3} + \frac{32}{3} \zeta(3) \right] n_f \right\} \\
& + \left[ \frac{\alpha_s(s)}{4\pi} \right]^3 \left\{ \frac{174058}{9} - 17648 \zeta(3) + \frac{8800}{3} \zeta(5) \right\} n_f \\
& + \left[ \frac{6276}{27} + \frac{16768}{9} \right] n_f \\
& + \left[ \frac{4832}{81} - \frac{1216}{27} \zeta(3) \right] n_f^2 - \left[ \frac{484}{3} - \frac{176}{9} n_f + \frac{16}{27} n_f^2 \right] \pi^2 \right\}.
\end{align*}
\]

This leads to the following numerical result:

\[
\begin{align*}
\mathcal{R}_{NS}(s) & = 1 + \frac{\alpha_s(s)}{\pi} + \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \left( 1.9857 - 0.1153 n_f \right) \\
& + \left[ \frac{\alpha_s(s)}{\pi} \right]^3 \left( -6.6369 - 1.2001 n_f - 0.0052 n_f^2 \right).
\end{align*}
\]

Those terms which depend on the number of quark flavours \(n_f\) are due to virtual fermion loops with light quarks. They appear for the first time at second order \(\alpha_s^2\).

Mixed QED and QCD corrections can be deduced from the QCD results in a straightforward manner \[92\]. One obtains

\[
\begin{align*}
\mathcal{R}_{\text{QED}}^{(0)} & = Q_f^2 \left[ \frac{3}{4} \frac{\alpha(s)}{\pi} \left( 1 - \frac{1}{3} \frac{\alpha_s(s)}{\pi} \right) \right].
\end{align*}
\]

Corrections of order \(\alpha^2\) are also given in Ref. \[92\]. They are small and will not be considered here.

5.2 Top Mass Corrections

The top quark is also present at second order through a virtual quark loop. The corrections in the corresponding double bubble diagram (Fig. 7) are known in analytical form, if the masses of the quarks in the external loop are neglected \[93, 94\]. The absorptive part from the cut through the two (massless) quark lines contributes for \(s > 0\) and is calculated in
Figure 7: Double Bubble Diagram.

Ref. [93]. The one from the cut through all four quark lines contributes for $s > 4m_t^2$ and can be found in Ref. [94]. Only the former is of relevance for the present discussion. Its contribution to $r_{NS}^{(0)}$ reads:

$$
 r_{NS}^{(0)} = \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \left\{ \frac{4}{9} \left( 1 - 6x^2 \right) \left[ \text{Li}_3(A^2) - \zeta(3) - 2\zeta(2) \ln A + \frac{2}{3} \ln^3 A \right] 
+ \frac{2}{27} (19 + 46x) \sqrt{1 + 4x} \left[ \text{Li}_2(A^2) - \zeta(2) + \ln^2 A \right] 
+ \frac{5}{54} \left( \frac{53}{3} + 44x \right) \ln x + \frac{3355}{648} + \frac{119}{9} \right\}, \tag{153}
$$

where $A = (\sqrt{1 + 4x} - 1)/\sqrt{4x}$ with $x = m_t^2/s$.

The leading term has also been determined [40] by employing the heavy mass expansion as described in Section 2.3. In the heavy top limit the correction reads:

$$
 r_{NS}^{(0)} = \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \frac{s}{m_t^2} \left( \frac{44}{675} + \frac{2}{135} \ln \frac{m_t^2}{s} \right). \tag{154}
$$

As shown in Fig. 8, the heavy mass expansion provides an excellent approximation to the full answer from $m_t \gg s$, even down to the threshold $4m_t^2 = s$. The result was derived in the theory with $n_f = 6$, whence $\alpha_s$ should be taken for the correction term accordingly. However, since $\alpha_s|_{n_f=5} = \alpha_s|_{n_f=6} + \mathcal{O}(\alpha_s^2)$, this distinction is irrelevant for the terms under consideration. Note that the diagrams of Fig. 7 were studied in Refs. [95, 96], where an exact double integral representation was obtained. The r.h.s. of (153) was numerically evaluated in Ref. [97].

It seems appropriate at this point to already here anticipate the mass corrections arising from internal loops of quarks with $m^2/s \ll 1$. Also, these corrections are universal. They will be derived in Sections 5.3 and 5.4. The leading $\alpha_s^2m^2/s$ term is absent. The
Figure 8: The function $\rho^V$ describing virtual corrections in the range $s/m^2 < 4$ (solid curve) and the approximation of the heavy mass expansion (dashed curve).

first non-vanishing terms are of order $\alpha_s^3 m^2/s$ and $\alpha_s^2 m^4/s^2$ and provide a correction,

$$ r_{NS}^{(0)} = \left[ \frac{\alpha_s(s)}{\pi} \right]^3 \left[ -15 + \frac{2}{3} n_f \right] \left[ \frac{16}{3} - 4\zeta(3) \right] \sum_f \frac{\bar{m}_f^2}{s} $$

$$ + \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \sum_f \frac{\bar{m}_f^2}{s^2} \left[ \frac{13}{3} - \ln \frac{\bar{m}_f^2}{s} - 4\zeta(3) \right]. $$

These corrections, as well as those from a heavy top, apply equally well to vector and axial correlators.

5.3 Mass Corrections of Order $m^2/s$

In view of the high precision reached in the cross-section measurements the large size of the first-order corrections made the knowledge of higher order QCD corrections desirable. Their exact computation for arbitrary quark masses would be a tremendous task. Fortunately for many considerations and experimental conditions, quark masses can be neglected in comparison with the characteristic energy of the problem, or are considered as small parameters. This holds true for the light u, d and s quarks, once the CMS energy exceeds a few GeV, and is equally valid for charm and bottom quarks at LEP energies of about 90 GeV. The problem may therefore be simplified by performing an expansion in the small parameter $m^2/s$, which reduces the calculational effort to massless propagator integrals. The leading quadratic terms $m^2/s$ and, for the case of lower energies, the quartic mass terms $m^4/s^2$, represent a very good approximation.
In first order this expansion is trivially obtained in Eq. (148) from the exact result. We already noticed the large logarithm \( \ln m^2/s \) which makes the reliability of perturbation theory questionable. Its occurrence is connected to the use of the pole mass as an expansion parameter. The problem may be overcome by employing RG techniques and is conveniently achieved in the \( \overline{\text{MS}} \)-scheme. In this calculational scheme leading logarithms \( \ln m^2/s \) are summed and absorbed in the \( \overline{\text{MS}} \) mass \( \overline{m}(\mu^2) \) with \( \mu \) being the renormalization scale. One can then write

\[
\begin{align*}
    r^V(f) & = r^{[0]} + \frac{\overline{m}^2}{s} r^{[1]} + \frac{\overline{m}^4}{s^2} r^{[2]} + \mathcal{O}(m^6/s^3), \\
    r^A(f) & = r^{[0]} + \frac{\overline{m}^2}{s} r^{[1]} + \frac{\overline{m}^4}{s^2} r^{[2]} + \mathcal{O}(m^6/s^3).
\end{align*}
\]

(156)

The massless results \( r^{[0]} \) are identical for the vector and the axial vector correlators, whereas the mass corrections \( r^{[n]} \) differ from \( r^{[n]} \) for \( n \geq 1 \).

It was found in Ref. [62] (discussed in some detail in Section 3.1) that the \( \overline{\text{MS}} \) scheme has the remarkable property of all coefficient functions for QCD operators being polynomial in masses and momenta. From this, and the fact that no non-trivial operators of mass dimension two exist in QCD, follows that no logarithms in \( m^2/s \) appear in \( r^{[1]} \) and \( r^{[1]} \). Therefore \( r^{V/A}[0] \) and \( r^{V/A}[1] \) can be written as a perturbation expansion to all orders in \( \alpha_s \) with pure numerical coefficients.

Since to first order \( \alpha_s \), only trivial operators (unit operator times a combination of quark masses) of mass dimension four exist, logarithms in \( r^{[2]} \) are absent in order \( \alpha_s \) and show up for the first time in second order \( \alpha_s^2 \).

### 5.3.1 Vector-Induced Corrections

In this section we demonstrate that the mass corrections of order \( \mathcal{O}(\alpha_s^3) \) to the flavour non-singlet contribution of the vector-induced decay rate \( \Gamma^V \) can be obtained from the three-loop vector current correlator [87] without an explicit four-loop calculation. The argument is based on the RG invariance of the Adler function

\[
D^V(Q^2) = -12\pi Q^2 \frac{d}{dQ^2} \left( \frac{\Pi^V}{Q^2} \right)
\]

\[
= 3 \left\{ \frac{1 + \alpha_s(\mu)}{\pi} + \ldots \right\} - \frac{\overline{m}^2(\mu)}{Q^2} \left[ b_{00}^{V} + (b_{01}^{V} + b_{11}^{V} \ell) \left( \frac{\alpha_s(\mu)}{\pi} \right) \right]
\]

\[
+ (b_{02}^{V} - a_{02}^{V} + b_{12}^{V} \ell + b_{22}^{V} \ell^2) \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 + (b_{03}^{V} + b_{13}^{V} \ell + b_{23}^{V} \ell^2 + b_{33}^{V} \ell^3) \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \right\},
\]

(157)

where \( \ell \equiv \ln (\mu^2/Q^2) \). The term \( a_{02}^{V} \) originates from the b quark propagating in an inner fermion loop. Mass corrections are therefore also present for the decay of the Z-boson into massless quarks.
The coefficients up to and including the second order $O(\alpha_s^2)$ were obtained in [98] (the $a_{02}^V$ term was first computed in [95]). They read\(^6\)

\begin{align}
    b_{00}^V &= 6, \\
    b_{01}^V &= 28, \quad b_{11}^V = 12, \\
    b_{02}^V &= [28799 + 992 \zeta(3) - 8360 \zeta(5) - 882 n_f]/72, \\
    b_{12}^V &= (3303 - 114 n_f)/18, \quad b_{22}^V = (513 - 18 n_f)/18, \\
    a_{02}^V &= [32 - 24 \zeta(3)]/3.
\end{align}

The crucial point for the subsequent calculation is the invariance of $D^V$ (157) under RG transformations

\[ \mu^2 \frac{d}{d\mu^2} D^V = 0, \] (159)
combined with the absence of $\ln \mu^2/m^2$ terms in Eq. (157). Recursion relations between the coefficients of the Adler function allow the calculation of the order $O(\alpha_s^3)$ coefficients $b_{13}^V, b_{23}^V, b_{33}^V$ from the lower order coefficients combined with those of anomalous mass dimension and the $\beta$-function:

\begin{align}
    b_{11}^V &= 2b_{00}^V \gamma_m, \\
    b_{12}^V &= (\beta_0 + 2\gamma_m) b_{01}^V + 2\gamma_m b_{11}^V, \\
    b_{22}^V &= \frac{1}{2}(\beta_0 + 2\gamma_m) b_{11}^V, \\
    b_{13}^V &= 2(\beta_0 + \gamma_m)(b_{02}^V - a_{02}^V) + (\beta_1 + 2\gamma_m) b_{01}^V + 2\gamma_m b_{11}^V, \\
    b_{23}^V &= (\beta_0 + \gamma_m) b_{12}^V + \frac{1}{2}(\beta_1 + 2\gamma_m) b_{11}^V, \\
    b_{33}^V &= \frac{2}{3}(\beta_0 + \gamma_m) b_{22}^V.
\end{align}

The coefficient $b_{03}^V$ cannot be obtained via this recursion method. However, the term proportional to this coefficient does not contribute to $R^V$. The vector contribution to the decay rate is then written in the form:

\[
    \frac{\bar{m}^2}{s} r_{V}^{(1)} = \frac{\bar{m}^2}{s} \left\{ \lambda_0^V + \frac{\alpha_s(\mu)}{\pi} \left[ \lambda_1^V + \lambda_2^V \ln \frac{s}{\mu^2} \right] \right. \\
    + \left[ \frac{\alpha_s(\mu)}{\pi} \right]^2 \left[ \lambda_3^V + \lambda_4^V \ln \frac{s}{\mu^2} + \lambda_5^V \ln^2 \frac{s}{\mu^2} \right] + \left[ \frac{\alpha_s(\mu)}{\pi} \right]^3 \left[ \lambda_6^V + \cdots \right] + \cdots \right\}.
\]

\(^6\)The result (158) has been recently confirmed in Ref. [99]. Hence, the correction of the originally published coefficient 992 of $\zeta(3)$ in $b_{02}^V$ to 1008 suggested in Ref. [100] and unfortunately used in Ref. [88] turned out to be an error. Numerically the use of the right result leads to only a slight increase (less than 0.6%) in the magnitude of $\lambda_6^V$ in comparison with that given in Ref. [87].
If we set the normalization point $\mu^2 = s$, the remaining logarithms of $s/\mu^2$ are absorbed in the running coupling constant and the running mass. The coefficients $\lambda$ can be obtained from the expansion coefficients of the Adler function by first integrating Eq. (157) to obtain $\Pi^V/Q^2$ and subsequently taking the imaginary part of $\Pi^V/Q^2$ to arrive at $r^V$:

$$\lambda_0^V = 0, \quad \lambda_1^V = b_1^V, \quad \lambda_2^V = 0,$$
$$\lambda_3^V = b_{12}^V - 2b_{22}^V, \quad \lambda_4^V = -2b_{22}^V, \quad \lambda_5^V = 0,$$
$$\lambda_6^V = b_{13}^V - 2b_{23}^V + (6 - \pi^2)b_{33}^V, \quad \lambda_7^V = -2b_{23}^V + 6b_{33}^V, \quad \lambda_8^V = 3b_{33}^V, \quad \lambda_9^V = 0. \quad (162)$$

The term $\pi^2$ in $\lambda_8^V$ is a consequence of the analytical continuation from space-like to time-like momenta and arises from the term $\ln^3 \mu^2/Q^2 \rightarrow (\ln \mu^2/|Q^2| \pm i\pi)^3$. Explicitly, non-zero entries above read:

$$\lambda_1^V = 12,$$
$$\lambda_3^V = -\frac{13}{3} n_f + \frac{253}{2},$$
$$\lambda_4^V = -57 + 2 n_f,$$
$$\lambda_6^V = -\frac{1}{9} n_f^2 \pi^2 + \frac{125}{54} n_f^2 \pi^2 + \frac{17}{3} n_f \pi^2 - \frac{466}{27} n_f \zeta(3) + \frac{1045}{27} n_f \zeta(5) \quad (163)$$
$$- \frac{4846}{27} n_f - \frac{285}{4} \pi^2 + \frac{490}{3} \zeta(3) - \frac{5225}{6} \zeta(5) + 2442,$$
$$\lambda_7^V = -\frac{13}{9} n_f^2 + \frac{175}{2} n_f - \frac{4505}{4},$$
$$\lambda_8^V = \frac{1}{3} n_f^2 - 17 n_f + \frac{855}{4}.$$

The $a_{02}$ contribution originates from the $b$ quark vacuum polarization graphs and is thus also present for final states with massless quarks. (More precisely, it originates in this case from QCD corrections to $q\bar{q}bb\bar{b}$ configurations.) The same correction would arise in $r^A$. This term has been anticipated in Eq. (155). The final answer can still be interpreted as an incoherent sum of the contributions from different quark species. In particular this implies that contributions from three gluon intermediate final states (singlet contributions) are absent in the $O(\alpha_s^3)$ mass terms. This contrasts with the corrections for $m = 0$, which receive third-order contributions precisely from this configuration — see Eq. (180) below.

Numerically, one finds a quite decent decrease in the terms of successively higher-orders, which supports confidence in the applicability of these results for predictions of the rate. This will be studied in more detail in Part 7.

### 5.3.2 Axial Vector-Induced Corrections

The situation is more involved if one wants to apply similar RG arguments to the axial vector-induced rate in order to again compute the corresponding mass corrections from
non-singlet diagrams. The comparison of the expansion of the Adler function

\[
D^A = -12\pi^2 Q^2 \frac{d}{dQ^2} \left( \frac{\Pi^A_i}{Q^2} \right) \\
= 3 \left\{ \left( 1 + \frac{\alpha_s(\mu)}{\pi} + \ldots \right) - \frac{m^2(\mu)}{Q^2} \left[ \sum_{i,j \geq 0} b_{ij}^A \ell^i \left( \frac{\alpha_s(\mu)}{\pi} \right)^j \right] + \mathcal{O}(m^4) \right\},
\]

(164)

with Eq. (157) shows that in the axial case the highest order term of the power series in \( \ell \) within a given order \( \mathcal{O}(\alpha_s^j) \) is proportional to \( \ell^{j+1} \), whereas in the vector case the \( \ell \)-expansion terminated at \( \ell^j \). This structure is dictated by the anomalous dimension \( \gamma^{AA}_m \), which vanishes in the vector case.

The expansion of \( D^V \) contained the second order coefficient \( a_{V2} \), originating from an inner b-quark loop. The same \( \mathcal{O}(\alpha_s^3) \) term is also present in the axial case for massless as well as massive external quark lines. The mass correction of \( \mathcal{O}(\alpha_s^3) \) from external quark loops has not yet been calculated.

The coefficients of the expansion (164) read:

\[
\begin{align*}
 b_{00}^A &= -12, & b_{10}^A &= -6, \\
 b_{01}^A &= -\frac{151}{2} + 24\zeta(3), & b_{11}^A &= -34, & b_{21}^A &= -6.
\end{align*}
\]

(165)

The complication in deriving from these the second-order coefficients of the logarithmic terms arises from the fact that the mass dependent part of \( D^A \) obeys the inhomogeneous RG equation:

\[
\mu^2 \frac{d}{d\mu^2} D^A = \frac{m^2(\mu)}{Q^2} \gamma^{AA}_m = \frac{m^2(\mu)}{Q^2} \sum_{i \geq 0} \left( \gamma^{AA}_m \right)_i \left[ \frac{\alpha_s(\mu)}{\pi} \right]^i.
\]

(166)

Therefore, recursion relations can be set up again, although in this case order \( \mathcal{O}(\alpha_s^n) \) coefficients \( b_{kn}^A \) \((k > 0)\) are not only expressed through the \( \{ b_{ij} \} \) with \( 0 \leq i \leq j + 1 \leq n \), but also through the expansion coefficients of the anomalous dimension \( \gamma^{AA}_m \) \((r \leq n)\). In fact, the second-order coefficients satisfy the relations

\[
\begin{align*}
 b_{10}^A &= - \left( \gamma^{AA}_m \right)_0, \\
 b_{11}^A &= - \left( \gamma^{AA}_m \right)_1 + 2b_{00}^A \gamma_m^0, \\
 b_{21}^A &= b_{10}^A \gamma_m^0, \\
 b_{12}^A &= - \left( \gamma^{AA}_m \right)_2 + b_{01}^A (\beta_0 + 2\gamma_m^0) + 2b_{00}^A \gamma_m^1, \\
 b_{22}^A &= \frac{1}{2} b_{11}^A (\beta_0 + 2\gamma_m^0) + b_{10}^A \gamma_m^1, \\
 b_{32}^A &= \frac{1}{3} b_{21}^A (\beta_0 + 2\gamma_m^0).
\end{align*}
\]

(167)
[Note: for the vanishing anomalous dimension the Eqs. (160) and (167) coincide.] Therefore the anomalous dimension \( \gamma_m^{\text{AA}} \) must be known to the same order to which the decay rate is computed. The calculation of \( \gamma_m^{\text{AA}} \) is sketched in Section 2.2.5 and leads to the following result [32]

\[
\gamma_m^{\text{AA}} = 6 \left\{ 1 + \frac{5}{3} \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{455}{12} - \frac{1}{3} n_f - \frac{1}{2} \zeta(3) \right] \right\}. \tag{168}
\]

As for the vector case we write a general expansion for the axial vector-induced rate:

\[
\frac{m^2}{s} \tau^{(1)}_{\text{AA}} = \frac{m^2(\mu)}{s} \left\{ \lambda^A_0 + \frac{\alpha_s(\mu)}{\pi} \left[ \lambda^A_1 + \lambda^A_2 \ln \frac{s}{\mu^2} \right] \right. \\
+ \left. \left[ \frac{\alpha_s(\mu)}{\pi} \right]^2 \left[ \lambda^A_3 + \lambda^A_4 \ln \frac{s}{\mu^2} + \lambda^A_5 \ln^2 \frac{s}{\mu^2} \right] \right\}. \tag{169}
\]

The coefficients read as follows:

\[
\begin{align*}
\lambda^A_0 &= b^A_{10} , & \lambda^A_1 &= b^A_{11} - 2b^A_{21} , & \lambda^A_2 &= -2b^A_{21} , \\
\lambda^A_3 &= b^A_{12} - 2b^A_{22} + (6 - \pi^2)b^A_{32} , & \lambda^A_4 &= -2b^A_{22} + 6b^A_{32} , & \lambda^A_5 &= 3b^A_{32} .
\end{align*}
\]

Or, explicitly,

\[
\begin{align*}
\lambda^A_0 &= -6 , \\
\lambda^A_1 &= -22 , \\
\lambda^A_2 &= 12 , \\
\lambda^A_3 &= -\frac{1}{3} n_f \pi^2 - 4 n_f \zeta(3) + \frac{151}{12} n_f + \frac{19}{2} \pi^2 + 117 \zeta(3) - \frac{8221}{24} , \\
\lambda^A_4 &= -\frac{16}{3} n_f + 155 , \\
\lambda^A_5 &= n_f - \frac{57}{2} .
\end{align*}
\]

The discussion in this and the previous section is tailored for an external current coupled to \( b \bar{b} \) and includes mass corrections from internal \( b \) quark loops as well as from the loops coupled to the external current. A slightly different situation occurs for a non-singlet correlator arising from massless quarks. Internal bottom quarks as indicated in the double bubble graph still induce \( m_b^2/s \) corrections. However, a slight generalization of the arguments presented above demonstrates that these terms are again absent in order \( \mathcal{O}(\alpha_s^2) \). From corrections of the diagrams in Fig. 7 one obtains the terms of order \( \mathcal{O}(\alpha_s^3) \),
which should for convenience be incorporated into $r_{NS}^{[0]}$ and summed over all massive quark species, adding the term

$$r_{NS}^{[0]} \rightarrow r_{NS}^{[0]} + \left( \frac{\alpha_s}{\pi} \right)^3 \sum_f \frac{m_f^2}{s} (-2)(\beta_0 + \gamma_m) a_{02}$$

$$= r_{NS}^{[0]} - \left[ \frac{\alpha_s(s)}{\pi} \right]^3 \left( 15 - \frac{2}{3} n_f \right) \left[ \frac{16}{3} - 4\zeta(3) \right] \sum_f \frac{m_f^2(s)}{s} .$$  

\hspace{10cm} (172)

5.4 Mass Corrections of Order $m^4/s^2$

Terms of higher-order in $m^2/s$ are quite unimportant as far as $Z$ decays into b-quarks are concerned. However, at lower energies these should be taken into account in order to arrive at an adequate description of the cross-section. In fact, as shown in Fig. 6, the order $\alpha_s$ correction functions $K_V$ and $K_A$ introduced in Eq. (140) are well described by the first few terms of the expansion in $m^2/s$, not only at high energies, but even fairly close to threshold. Hence one should arrive at a reliable result to $O(\alpha_s^2)$ near the threshold through the incorporation of the first terms of the expansion in $\alpha_s^2(m^2/s)^n$. The second-order calculation of quartic mass corrections presented below is based on Ref. [64]. The calculation was performed for vector and axial vector current non-singlet correlators. The first is of course relevant for electron–positron annihilation into heavy quarks at arbitrary energies, the second for $Z$ decays into b quarks and for top production at a future linear collider.

Quartic mass corrections were already presented to order $\alpha_s$ in Part 4, expressed in terms of the pole mass. In this section the result in the $\overline{\text{MS}}$ scheme is given and the second order $\alpha_s^2$ contribution is discussed. The calculation is based on the operator product expansion of the $T$-product of two vector currents, $J_\mu = \bar{u}\gamma_\mu d$ and $J_\nu^+ = \bar{d}\gamma_\nu u$. Here $u$ and $d$ are simply two generically different quarks with masses $m_u$ and $m_d$. The operator product expansion includes power law suppressed terms up to operators of dimension four induced by non-vanishing quark masses. Renormalization group arguments similar to those already employed in the previous section allowed a reduction in the $\alpha_s^2 m^4$ terms. Quarks which are not coupled to the external current will influence the result in order $\alpha_s^2$ through their coupling to the gluon field. The result may be immediately transformed to the case of the electromagnetic current of a heavy, say, $t$ (or $b$) quark.

The asymptotic behaviour of the transverse part of this (operator valued) function for $Q^2 = -q^2 \rightarrow \infty$ is given by an OPE of the following form (different powers of $Q^2$ may be studied separately and only operators of dimension 4 are displayed):

$$i \int T(J_\mu(x)J_\nu^+(0))e^{iqx} dx = \frac{1}{Q^4} \sum_n (q_\mu q_\nu - g_{\mu\nu} q^2) \bar{C}_n (Q^2, \mu^2, \alpha_s) O_n + \ldots$$

\hspace{10cm} (173)

Only the gauge invariant operators $G_{\mu\nu}^2, m_i q_j q_j$ and a polynomial of fourth order in the masses contributes to physical matrix elements. Employing renormalization group arguments the vacuum expectation value of $\sum_n C_n O_n$ is under control up to terms of order $\alpha_s$ as far as the constant terms are concerned and even up to $\alpha_s^2$ for the logarithmic terms proportional to $\ln Q^2/\mu^2$. Only these logarithmic terms contribute to the absorptive
Figure 9: Contributions to $R^V$ from $m^4$ terms including successively higher orders in $\alpha_s$ (order $\alpha_s^0$, $\alpha_s^1$, $\alpha_s^2$ corresponding to dotted/ dashed/ solid lines) as functions of $2m_{\text{pole}}/\sqrt{s}$.

part. Hence one arrives at the full answer for $\alpha_s^2 m^4/s^2$ corrections. Internal quark loops contribute in this order, giving rise to the terms proportional to $m^2 m_i^2$ and $m_i^4$ below.

The result reads [below we set for brevity the $\overline{\text{MS}}$ normalization scale $\mu = \sqrt{s}$ and $\bar{m}_u(s) = \bar{m}_d(s) = \bar{m}$]:

$$\frac{m^4}{s^2} r_V^{(2)} = \frac{m^4}{s^2} \left\{ -6 - 22 \frac{\alpha_s(s)}{\pi} \right. $$
$$+ \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \left[ n_f \left( \frac{1}{3} \ln \frac{m^2}{s} - \frac{2}{3} \pi^2 - \frac{8}{3} \zeta(3) + \frac{143}{18} \right) $$
$$- \frac{11}{2} \ln \frac{m^2}{s} + 27 \pi^2 + 112 \zeta(3) - \frac{3173}{12} + 12 \sum_i \frac{m_i^2}{m^2} $$
$$+ \left( \frac{13}{3} - 4\zeta(3) \right) \sum_i \frac{m_i^4}{m^4} - \sum_i \frac{m_i^4}{m^4} \ln \frac{m_i^2}{s} \right\},$$

(174)

$$\frac{m^4}{s^2} r_A^{(2)} = \frac{m^4}{s^2} \left\{ 6 + 10 \frac{\alpha_s(s)}{\pi} $$
$$+ \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \left[ n_f \left( -\frac{7}{3} \ln \frac{m^2}{s} + \frac{2}{3} \pi^2 + \frac{16}{3} \zeta(3) - \frac{41}{6} \right) $$

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\[
\begin{eqnarray}
&+& \frac{77}{2} \ln \frac{m^2}{s} - 27\pi^2 - 220 \zeta(3) + \frac{3533}{12} - 12 \sum_i \frac{\bar{m}_i^2}{m^2} \\
&+& \left( \frac{13}{3} - 4 \zeta(3) \right) \sum_i \frac{\bar{m}_i^4}{m^4} - \sum_i \frac{\bar{m}_i^4}{m^4} \ln \frac{\bar{m}_i^2}{s} \right) \right] \right). 
\end{eqnarray}
\]

Note that the sum over \( i \) includes also the quark coupled to the external current and with mass denoted by \( m \). Hence in the case of one heavy quark \( u \) of mass \( m \) one should set \( \sum_i \frac{\bar{m}_i^4}{m^4} = 1 \) and \( \sum_i \frac{\bar{m}_i^2}{m^2} = 1 \). In the opposite case, when one considers the correlator of light (massless) quarks the heavy quarks appear only through their coupling to gluons. There one finds for the correction term:

\[ r_V = r_A = \left[ \frac{\alpha_s(s)}{\pi} \right]^2 \sum_J \frac{\bar{m}_J^4(s)}{s^2} \left[ \frac{13}{3} - \ln \frac{\bar{m}_J^2(s)}{s} - 4 \zeta(3) \right], \]

as anticipated in Eq. (155).

The \( Z \) decay rate is hardly affected by the \( m^4 \) contributions. The lowest order term in Eq. (174) evaluated with \( \bar{m} = 2.6 \) GeV amounts to \( \pm 6 \bar{m}/s^2 = \pm 5 \times 10^{-6} \) for the vector (axial vector) current induced \( Z \to b\bar{b} \) rate. Terms of increasing order in \( \alpha_s \) become successively smaller. The \( m_b^4 \) correction to \( \Gamma(Z \to q\bar{q}) \), which starts in order \( \alpha_s^2 \), is evidently even smaller. It is worth noting, however, that the corresponding series, evaluated in the onshell scheme, leads to terms which are larger by about one order of magnitude and of oscillatory signs. From these considerations it is clear that \( m^4 \) corrections to the \( Z \) decay rate are well under control — despite the still missing singlet piece — and that they can be neglected for all practical purposes.

The situation is different in the low energy region — say several GeV above the charm or the bottom threshold. For definiteness the second case will be considered and for simplicity all other masses will be put to zero. The contributions to \( R^V \) from \( m^4 \) terms are presented in Fig. 9 as functions of \( 2m/\sqrt{s} \) in the range from 0.05 to 1. The input parameters \( M_{\text{pole}} = 4.70 \) GeV and \( \alpha_s(m_b^2) = 0.12 \) have been chosen. Corrections of higher-orders are added successively. The prediction is fairly stable with increasing order in \( \alpha_s \), as a consequence of the fact that most large logarithms were absorbed in the running mass. The relative magnitude of the sequence of terms from the \( m^8 \) expansion is displayed in Fig. 10. The curves for \( m^0 \) and \( m^2 \) are based on corrections up to third order in \( \alpha_s \), with the \( m^2 \) term starting at first order. The \( m^4 \) curve receives corrections from order zero to two.

Of course, very close to threshold — say above 0.75 (corresponding to \( \sqrt{s} \) below 13 GeV) — the approximation is expected to break down, as indicated already in Fig. 6. Below the \( b\bar{b} \) threshold, however, one may decouple the bottom quark and consider mass corrections from the charmed quark within the same formalism.

### 5.5 Partial Rates

The formulae for the QCD corrections to the total rate \( \Gamma_{\text{had}} \) have a simple, unambiguous meaning. The theoretical predictions for individual \( q\bar{q} \) channels, however, require additional interpretation. In fact, starting from order \( \mathcal{O}(\alpha_s^2) \) it is no longer possible to assign
Figure 10: Predictions for $R^V$ including successively higher orders in $m^2$.

all hadronic final states to well specified $q\bar{q}$ channels in a unique manner. The vector- and axial vector-induced rates receive (non-singlet) contributions from the diagrams, where the heavy quark pair is radiated off a light $q\bar{q}$ system (see Fig. 11a). The analytical result for this contribution for arbitrary $m^2/s$ can be found in Ref. [94] and is reproduced in the Appendix. The rate for this specific contribution to the $q\bar{q}b\bar{b}$ final state in the limit of small $m^2/s$ is given by

$$R_{q\bar{q}b\bar{b}}^{NS} = \frac{\Gamma_{q\bar{q}b\bar{b}}^{NS}}{\Gamma_{q\bar{q}}^{\text{Born}}} = \left(\frac{\alpha_s}{\pi}\right)^2 \frac{1}{27} \left\{ \ln^3 \frac{s}{m_b^2} - \frac{19}{2} \ln^2 \frac{s}{m_b^2} + \left[ \frac{146}{3} - 12\zeta(2) \right] \ln \frac{s}{m_b^2} \right\} + O(m_b^2/s).$$

(177)

Numerically one obtains

$$R_{q\bar{q}b\bar{b}}^{NS} = \left(\frac{\alpha_s}{\pi}\right)^2 \{0.922/0.987/1.059\} \quad \text{for } m_b = 4.9/4.7/4.5 \text{ GeV}. \quad \text{(178)}$$

The contributions from this configuration to the total rate (in particular the logarithmic mass singularities) are nearly cancelled by those from the corresponding virtual corrections (see Fig. 12b).

Despite the fact that $b$ quarks are present in the four-fermion final state, the natural prescription is to assign these events to the $q\bar{q}$ channel. They must be subtracted experimentally from the partial rate $\Gamma_{b\bar{b}}$. This should be possible, since their signature is characterized by a large invariant mass of the light quark pair and a small invariant mass of the bottom system, which is emitted collinear to the light quark momentum. If
As shown in Fig. 12a the leading contribution well matches the full calculation (for bottom quarks it is about 10% above the exact answer) leading to the analytic result presented in the Appendix. A slightly different approach to the evaluation of heavy quark multiplicities, which attempts the resummation of leading logarithms, can be found in Ref. [101].

6 Singlet Contribution

6.1 Massless Final State

6.1.1 Vector Currents

Singlet contributions to the Z decay rate or to the total cross-section originate from diagrams that can be split into two parts by cutting gluon lines only. In the vector case the first of these contributions arises in order $\mathcal{O}(\alpha_s^3)$ and is induced by 'light-by-light' scattering diagrams (see Fig. 3). The charge structure of this contribution differs from the non-singlet terms. Hence the lowest order singlet contribution is UV finite. In the notation introduced in Section 2.1 one obtains:

$$r_s^V = \frac{1}{3} \left( \frac{\alpha_s(s)}{4\pi} \right)^3 \left( \frac{d_{abc}}{4} \right)^2 \left[ \frac{176}{3} - 128\zeta(3) \right]$$

$$\approx \frac{1}{3} \left( \frac{\alpha_s(s)}{\pi} \right)^3 (-1.240).$$

At this point a brief comment concerning mass-dependent terms is in order. As discussed in Section 5.3.1, $m_b^2/s$ terms from diagrams depicted in Fig. 3 are absent. This
Figure 12: a) The function $\rho^R$ describing the production of four fermions in the region $0 < x = m^2/s < 1/4$. Solid line: exact result; dashed-dotted line: logarithmic and constant terms only; dotted line: including $m^2/s$ corrections; dashed line: including $m^2/s$ and $m^4/s^2$ corrections. b) Corresponding curves for $\rho^V$ describing virtual corrections.
leaves potential contributions with heavy top quarks from the same diagram. However, these are suppressed by a factor $s/m_t^2$ and asymptotically decoupled. In Section 5.2 the corrections of order $O(\alpha_s^2 s/m_t^2)$ from non-singlet diagrams were calculated and shown to be small. Corrections of order $O(\alpha_s^3 s/m_t^2)$ will therefore be ignored throughout. Hence no mass corrections from singlet diagrams will be considered in the vector case.

It should be stressed again that the knowledge of the two functions $R_{VNS}^V$ and $R_{VNS}^V$ and hence of $r_{VNS}^V$ is sufficient to evaluate all possible vector current correlators like $R^{em}, R^V$ and $R^{em}$. 

6.1.2 Axial Case

In the (fictitious) case of mass degenerate isospin doublets, singlet contributions from up and down quarks compensate exactly, since $a_u = -a_d$. Interesting enough, individual contributions to four-fermion final states are nevertheless present. The individual contributions from these cut double triangle diagrams to the Z decay rate for a massless (u and d) doublet are given as follows:

$$\Gamma_{uuuu} = \Gamma_{dddd} = -\frac{1}{2} \Gamma_{uudddd}. \quad (181)$$

For the top and bottom quark this cancellation is no longer operative as a consequence of the large mass splitting within the multiplet. For $m_t \to \infty$ one recovers the predictions based on an anomalous axial current. The $O(\alpha_s^2)$ calculation has been performed in Refs. [78, 102] for arbitrary $m_t^2/s$. The additional term in the Z decay rate can be decomposed into a term from two- (bb), three- (bbg) and four-parton (bbbg) configurations. The two gluon cut is forbidden according to the Landau-Yang selection rules [103], which forbid the decay of a parity odd particle with non-even integer total angular momentum into two massless vector bosons.

The respective diagrams contribute an additional singlet piece,

$$r_{S}^A = d_{S}^2 \left( \frac{\alpha_s}{\pi} \right)^2 + d_{S}^3 \left( \frac{\alpha_s}{\pi} \right)^3. \quad (182)$$

The first term can be decomposed into contributions from two-, three- and four-particle intermediate states:

$$d_{S}^2 = \frac{1}{3}(\text{Re} I_2 + \Delta I_3 + I_4) = \frac{1}{3} I. \quad (183)$$

The functions $I$ are well approximated by

$$\text{Re} I_2 = -7.210 + 1.481 \frac{s}{4m_t^2} + 1.363 \left( \frac{s}{4m_t^2} \right)^2 + 3 \ln \frac{s}{m_t^2},$$

$$\Delta I_3 = -1.580 - 0.444 \frac{s}{4m_t^2} - 0.731 \left( \frac{s}{4m_t^2} \right)^2,$$

$$I_4 = -0.460,$$

$$I = -9.250 + 1.037 \frac{s}{4m_t^2} + 0.632 \left( \frac{s}{4m_t^2} \right)^2 + 3 \ln \frac{s}{m_t^2}. \quad (184)$$

\footnote{Recently these corrections have been evaluated in Ref. [39] and proved to be quite small.}

\footnote{The exact formula in terms of Clausens functions can be found in Ref. [78].}

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The asymptotic behaviour for large $m_t^2$ is particularly simple\footnote{Note that the above result for $\bar{I}$ at large $m_t$ has been checked through completely independent calculations in Ref. [104] (up to terms of order $s/m_t^2$) and in Ref. [39] (up to terms of order $s^2/m_t^2$).}:

$$\Re I_2 \Rightarrow 2\zeta(2) - \frac{21}{2} + \frac{10}{27} \frac{s}{m_t^2} + 3 \ln \frac{s}{m_t^2},$$

$$\Delta I_3 \Rightarrow -4\zeta(2) + 5 - \frac{s}{9m_t^2},$$

$$I_4 \Rightarrow 2\zeta(2) - \frac{15}{4},$$

$$I \Rightarrow -\frac{37}{4} + \frac{7}{27} \frac{s}{m_t^2} + 3 \ln \frac{s}{m_t^2}. \quad (185)$$

At this point the scale in $\alpha_s$ is still ambiguous, as is the precise definition of $m_t$. In fact, since two different mass scales, $M_Z$ and $m_t$, are present in the problem, the asymptotic behaviour for large $m_t$ cannot be directly derived from this result. The evaluation of the leading logarithms in Ref. [43] allows the resolution of the ambiguity between the choice of $\mu = m_t$ and $\mu = M_Z$. The remaining constant term has been evaluated in Ref. [105]. For the renormalization scale $\mu^2 = s$, and with $m_t^2 = \bar{m}_t^2(s)$, one gets in the limit of large $m_t$:

$$d_s^2 = \frac{1}{3} \left\{ -\frac{37}{4} + 3 \ln \frac{s}{\bar{m}_t^2(s)} \right\},$$

$$d_s^3 = \frac{1}{3} \left\{ -\frac{5651}{72} + 3\zeta(3) + \frac{23}{12} \pi^2 + \frac{31}{6} \ln \frac{s}{\bar{m}_t^2(s)} + \frac{23}{4} \ln^2 \frac{s}{\bar{m}_t^2(s)} \right\}. \quad (186)$$

For practical purposes it is more convenient to employ the on-shell mass as the input parameter. Relating the $\overline{\text{MS}}$ mass at scale $\mu^2 = s$ to the on-shell mass through Eq. (47), one arrives at

$$d_s^2 = \frac{1}{3} \left\{ -\frac{37}{4} + 3 \ln \frac{s}{M_t^2} \right\},$$

$$d_s^3 = \frac{1}{3} \left\{ -\frac{5075}{72} + 3\zeta(3) + \frac{23}{12} \pi^2 + \frac{67}{6} \ln \frac{s}{M_t^2} + \frac{23}{4} \ln^2 \frac{s}{M_t^2} \right\}. \quad (187)$$

For $d_s^3$ one should in fact include the subleading terms $\sim 1/m_t^2$ or, even more appropriately, employ the complete answer, or at least the approximation Eq. (184).

### 6.2 Bottom Mass Corrections in the Singlet Term

Bottom quark mass effects have been neglected in the previous section. Employing the techniques of light/heavy mass expansions discussed in Part 3, one may derive terms of order $m_b^2/s$ as well as of order $m_b^2/m_t^2$. Although the former are significantly more important for realistic top masses than the latter, we list both contributions for completeness.
The corresponding results are \[104, 106\]:

\[
\tau^A_S = -6 \frac{m_b^2}{s} \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -3 + \ln \frac{s}{m_t^2} \right] - 10 \frac{m_b^2}{m_t^2} \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{81}{81} - \frac{1}{54} \ln \frac{s}{m_t^2} \right].
\] (188)

We would like to conclude this section with a brief comment on \(O(\alpha_s^3 s^2/m_b^2)\) singlet terms. As stated in Section 5.3.1, these are absent in the case of vector-current correlators. Hence, for a complete treatment of order \(O(\alpha_s^3)\), including mass corrections, only axial singlet and non-singlet contributions of order \(O(\alpha_s^3 m_b^2/s)\) are missing. In fact, only corrections of this order to the currents \(\bar{b}\gamma\mu\gamma_5 b\) and \(\bar{t}\gamma\mu\gamma_5 t\) are not yet available [see the discussion in Section 5.2, Eq. (155)]. The Born contribution of these currents amounts to 20% of the \(Z\) decay rate only, and the missing corrections to these terms of \(O(\alpha_s^3 m_b^2/s)\) can be safely neglected both for the moment and the foreseeable future.

6.3 Partial Rates

Assigning of singlet terms \(\Gamma^S\) to partial decay rates into individual \(q\bar{q}\) final states \(\Gamma_{q\bar{q}}\), which in turn are attributed to individual quark currents, is evidently not possible in a straightforward manner. Singlet contributions arise from the interference between different quark amplitudes. Nevertheless, at the present level of precision where experiments are only able to identify heavy flavour rates with a relative precision of about 1%, pragmatic prescriptions for this separation can be found.

6.3.1 Axial Rate

Let us start with the \(O(\alpha_s^3)\) singlet term induced by the interference of axial top and bottom currents. It can be decomposed into two-, three- and four-particle cuts, corresponding to the interference terms in Figs. 13 and 14. For final states with bottom quarks only one has [78] [see also Eq. (184)]:

56
\[
R_{bb}^s = \frac{1}{3} \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ -7.210 + 1.481 \frac{s}{4m_t^2} + 1.363 \left( \frac{s}{4m_t^2} \right)^2 + 3 \ln \frac{s}{m_t^2} \right\} \\
= - \left( \frac{\alpha_s}{\pi} \right)^2 (3.52 \pm 0.25), \\
R_{b\bar{b}g}^s = \frac{1}{3} \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ -1.580 - 0.444 \frac{s}{4m_t^2} - 0.731 \left( \frac{s}{4m_t^2} \right)^2 \right\} \\
= - \left( \frac{\alpha_s}{\pi} \right)^2 (0.60 \pm 0.01), \\
R_{b\bar{b}b\bar{b}}^s = - \frac{1}{3} \left( \frac{\alpha_s}{\pi} \right)^2 0.460 \\
= - \left( \frac{\alpha_s}{\pi} \right)^2 0.15,
\]

where \( m_t = 174 \pm 20 \) GeV has been assumed in the numerical evaluation. The first, logarithmically enhanced term dominates and can reasonably be assigned to \( \Gamma_{bb} \). The same holds true for the three particle contribution from \( b\bar{b}g \). At this point it should be stressed that other \( q\bar{q} \) final states are affected by this mechanism:

\[
\begin{align*}
R_{dd}^s &= -R_{uu}^s = R_{ss}^s = -R_{cc}^s = R_{bb}^s, \\
R_{d\bar{d}g}^s &= -R_{u\bar{u}g}^s = R_{s\bar{s}g}^s = -R_{c\bar{c}g}^s = R_{b\bar{b}g}^s.
\end{align*}
\]

These terms are proportional to the isospin of the quark, whence contributions from the massless doublet cancel. The assignment of singlet \( q\bar{q} \) and \( q\bar{q}g \) terms can therefore be performed in a convincing manner.

The situation is more intricate for the four-fermion final states, say \( q\bar{q}b\bar{b} \), which originate from the interference between \( q\bar{q} \) and \( b\bar{b} \) induced amplitudes (Fig. 14). Neglecting masses one obtains

\[
R_{b\bar{b}b\bar{b}}^s = -\frac{1}{2} R_{b\bar{b}u\bar{u}}^s = \frac{1}{2} R_{b\bar{b}d\bar{d}}^s = -\frac{1}{2} R_{b\bar{b}c\bar{c}}^s = \frac{1}{2} R_{b\bar{b}g}^s.
\]

Among the final states from these diagrams with at least one \( b\bar{b} \) pair, only the \( b\bar{b}b\bar{b} \) term remains after all compensations have been taken into account. Numerically it is tiny, about a factor 25 below the \( b\bar{b} + b\bar{b}g \) rate, and can therefore be ignored at present. The four-fermion contribution is present also for other light quarks. Within one complete doublet the cancellation occurs between mixed and pure configurations:

\[
R_{u\bar{u}d\bar{d}}^s = R_{d\bar{d}d\bar{d}}^s = -\frac{1}{2} R_{u\bar{u}d\bar{d}}^s = R_{b\bar{b}b\bar{b}}^s.
\]

If one were to insist on distributing the four-fermion singlet part to specific partial rates, the separation could only be performed in an analysis tailored to the specific experimental cuts. Mixed configurations like \( b\bar{b}c\bar{c} \) could either be assigned to \( \Gamma_{bb} \) (‘hierarchical’...
Figure 13: Singlet contributions with final states $b\bar{b}$ and $b\bar{b}g$.

Figure 14: Singlet contribution with four-fermion final state $q\bar{q}b\bar{b}$.

assignments) or with equal weight to $\Gamma_{b\bar{b}}$ and $\Gamma_{c\bar{c}}$ ('democratic' assignment). Furthermore, four fermion events with secondary radiation (which exhibit mass singularities — see Section 5.5) lead to similar signatures and must be subtracted from the singlet parts with the help of Monte Carlo simulations. However, as stated above, the issue can be ignored at present and the assignment of the axial singlet terms to partial rates according to the quark isospin seems adequate.

The $\alpha_s^3$ term has not been decomposed into individual cuts, except for the three gluon final state discussed below. Nevertheless it is evident from the structure of the calculation that the leading logarithmically enhanced terms can be interpreted as a correction to the $b\bar{b}$ configuration and hence are again proportional to the weak isospin. Therefore the complete axial singlet rate equations (184) and (187) can be assigned to $\Gamma_{q\bar{q}}$ with a weight proportional to the weak isospin $I_3^q$.

As noted above, the three-gluon rate induced by the axial current has been calculated with the truly tiny result [107],

$$
\Gamma^A_{ggg} = \frac{G_F M_Z^3}{24\sqrt{2}\pi} \left( \frac{\alpha_s}{\pi} \right)^3 \frac{1}{16} \left[ \frac{2981}{3} - 58\pi^2 - \frac{21}{5} \pi^4 - 8\zeta(3) \right] = 0.00072 \text{ MeV}. \quad (193)
$$
6.3.2 Vector Rate

In order \( \alpha_s^2 \) there are no singlet terms as a consequence of charge conjugation (Furry's Theorem). The \( \alpha_s^3 \) term given in Section 6.1.1 receives contributions from two to five parton configurations. Again, only the three-gluon contribution has been calculated separately [107]:

\[
\Gamma^V_{ggg} = \frac{G_F M_Z^3}{24 \sqrt{2} \pi} \left( \frac{\alpha_s}{\pi} \right)^3 \left( \sum_f v_f \right)^2 \frac{5}{144} \left[ -124 + \frac{41}{3} \pi^2 + \frac{7}{15} \pi^4 \right. \\
-128\zeta(3) + 200\zeta(5) - 8\pi^2\zeta(3) \right] = 0.0041 \text{ MeV}. \quad (194)
\]

The prediction is again far below the level of detectability.

7 Numerical Discussion

7.1 Z Decays

One of the central tasks at LEP is the extraction of a precise value for \( \alpha_s \) from the hadronic \( Z \) decay rate (or from derived quantities such as \( R_{\text{had}} \) or \( \sigma \)). Another quantity of interest is the ration \( \Gamma_{\overline{b}b}/\Gamma_{\text{had}} \), which provides important limits on the mass of the top quark and, indirectly, on new physics. It is therefore mandatory to explore the sensitivity of these predictions with respect to uncertainties in the input parameters, such as quark masses or \( \alpha_s \), and to deduce estimates on the uncertainties from as yet uncalculated higher-orders.

For the convenience of the reader we shall now extract from the previous parts a summary of the main results combined with a numerical evaluation that is particularly tailored for the energy regime around the \( Z \). If not stated otherwise, \( \alpha_s \) will denote the QCD coupling \( \alpha_s(s) \) in the \( \overline{\text{MS}} \)-scheme evaluated for five flavours at the scale \( s \). As input for \( \Lambda_{\overline{\text{MS}}} \) we shall use the value \( \Lambda_{\overline{\text{MS}}} = 233 \text{ MeV} \) corresponding to \( \alpha_s(M_Z^2) = 0.1200 \). The b-mass \( \overline{m}_b(s) \) will be taken as \( \overline{\text{MS}} \)-mass in a five-flavour theory at the mass scale \( s \). For the bottom pole mass we shall use the value of \( M = M_b = (4.7 \pm 0.2) \text{ GeV} \). It is related to \( \overline{m}_b(M_b) \) through Eq. (47), evaluated at \( \mu = M_b \). The running mass is therefore dependent on \( \alpha_s \). A few typical values are given in Table 2, where we also anticipate the values relevant for the subsequent discussion at lower energies. Our default value corresponds to a running mass at the \( Z \) peak of \( m = \overline{m}_b(M_Z^2) = 2.77 \text{ GeV} \). The running charm mass is about a factor of five smaller than \( \overline{m}_b \). Corrections from \( \alpha_s^2/s \) terms are hence entirely negligible for \( Z \) decays. For \( \sin^2 \theta_W \) the value 0.2321 is adopted. We also use the value \( 1/\alpha(M_Z) = 127.9 \pm 0.1 \) [108] for the running fine structure constant at the scale of \( M_Z \). For the top (pole) mass we choose \( M_t = 174 \pm 20 \text{ GeV} \).

7.1.1 The Total Hadronic Decay Rate \( \Gamma_{\text{had}} \)

The hadronic decay rate can be cast into following form:

\[
\Gamma_{\text{had}} = \sum_{i=1}^{12} \Gamma_i = \sum_{i=1}^{12} \Gamma_0 R_i, \quad (195)
\]
Table 2
Table of bottom masses.

<table>
<thead>
<tr>
<th>$\Lambda_{\overline{MS}}$</th>
<th>$\alpha_s(M_Z)$</th>
<th>$\alpha_s(10 \text{ GeV})$</th>
<th>$M$</th>
<th>$\overline{m}(M)$</th>
<th>$\overline{m}(M_Z)$</th>
<th>$\overline{m}(10 \text{ GeV})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.150</td>
<td>0.112</td>
<td>0.165</td>
<td>4.70</td>
<td>4.10</td>
<td>2.95</td>
<td>3.69</td>
</tr>
<tr>
<td>0.200</td>
<td>0.117</td>
<td>0.176</td>
<td>4.70</td>
<td>4.04</td>
<td>2.84</td>
<td>3.59</td>
</tr>
<tr>
<td>0.233</td>
<td>0.120</td>
<td>0.183</td>
<td>4.70</td>
<td>3.99</td>
<td>2.77</td>
<td>3.54</td>
</tr>
<tr>
<td>0.300</td>
<td>0.125</td>
<td>0.195</td>
<td>4.70</td>
<td>3.91</td>
<td>2.64</td>
<td>3.43</td>
</tr>
<tr>
<td>0.400</td>
<td>0.131</td>
<td>0.210</td>
<td>4.70</td>
<td>3.80</td>
<td>2.48</td>
<td>3.28</td>
</tr>
</tbody>
</table>

with $\Gamma_0 = G_F M_Z^2 / 24 \sqrt{2}\pi = 82.94 \text{ MeV}$, $v_f = 2 I_3^f - 4 Q_f \sin^2 \theta_w$, $a_3^f = 2 I_3^f$, and the following separate contributions:

Massless non-singlet corrections:

\[
R_1 = 3 \sum_f (v_f^2 + a_f^2) r_1 \\
= 3 \sum_f (v_f^2 + a_f^2) \left[ 1 + \frac{\alpha_s}{\pi} + 1.40932 \left( \frac{\alpha_s}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s}{\pi} \right)^3 \right];
\]  \hspace{1cm} (196)

Massive universal corrections (‘double bubble’):

\[
R_2 = 3 \sum_f (v_f^2 + a_f^2) r_2 \\
= 3 \sum_f (v_f^2 + a_f^2)(-6.126) \left( \frac{\alpha_s}{\pi} \right)^3 \sum_{f'=b} \frac{m_{f'}}{s} \\
R_3 = 3 \sum_f (v_f^2 + a_f^2) r_3 \\
= 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right)^2 \sum_{f'=b} \frac{m_{f'}}{s^2} \left[ -0.4749 - \ln \frac{m_{f'}}{s} \right] \] \hspace{1cm} (197)

\[
R_4 = 3 \sum_f (v_f^2 + a_f^2) r_4 \\
= 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right)^2 \frac{s}{M_0^2} \left[ 0.0652 + 0.0148 \ln \frac{M_0^2}{s} \right];
\]
Massive non-singlet corrections (vector):

\[ R_5 = 3v_b^2 r_5 \]
\[ = 3v_b^2 \frac{m_b}{s} \left[ \frac{\alpha_s}{\pi} + 8.736 \left( \frac{\alpha_s}{\pi} \right)^2 + 45.657 \left( \frac{\alpha_s}{\pi} \right)^3 \right] \]  

(198)

\[ R_6 = 3v_b^2 r_6 \]
\[ = 3v_b^2 \frac{m_b^4}{s^4} \left[ -6 - 22 \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left( 139.489 - 3.8333 \ln \frac{m_b^2}{s} \right) \right]; \]

Massive non-singlet corrections (axial):

\[ R_7 = 3r_7 \]
\[ = 3 \frac{m_b^2}{s} (-6) \left[ 1 + 3.667 \frac{\alpha_s}{\pi} + 14.286 \left( \frac{\alpha_s}{\pi} \right)^2 \right] \]  

(199)

\[ R_8 = 3r_8 \]
\[ = 3 \frac{m_b^4}{s^2} \left[ 6 + 10 \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left( -217.728 + 26.833 \ln \frac{m_b^2}{s} \right) \right]; \]

Singlet corrections (axial):

\[ R_9 = 3r_9 \]
\[ = 3 \left\{ \left( \frac{\alpha_s}{\pi} \right)^2 \frac{1}{3} \right\} \left[ -9.250 + 1.037 \frac{s}{4M_t^2} + 0.632 \left( \frac{s}{4M_t^2} \right)^2 + 3 \ln \frac{s}{M_t^2} \right] \]
\[ + \left( \frac{\alpha_s}{\pi} \right)^3 \frac{1}{3} \left[ -47.963 + 11.167 \ln \frac{s}{M_t^2} + 5.75 \ln^2 \frac{s}{M_t^2} \right] \];  

(200)

\[ R_{10} = 3r_{10} \]
\[ = 3 \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ -6 \frac{m_b^2}{s} \left[ -3 + \ln \frac{s}{M_t^2} \right] - 10 \frac{m_b^2}{M_t^2} \left[ 0.0988 - 0.0185 \ln \frac{s}{M_t^2} \right] \right\}; \]

Singlet corrections (vector):

\[ R_{11} = 3 \left( \sum_f v_f \right)^2 r_{11} \]  

(201)

\[ = \left( \sum_f v_f \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 (-1.2395); \]

\[ O(\alpha \alpha_s) \] corrections:

\[ R_{12} = 3 \sum_f (v_f^2 + a_f^2) Q_f^2 r_{12} \]  

(202)

\[ = 3 \sum_f (v_f^2 + a_f^2) Q_f^2 \left( \frac{\alpha_s}{\pi} \right)^3 \frac{1}{4} \left[ 1 - \frac{1}{3} \frac{\alpha_s}{\pi} \right]. \]
The various contributions are listed in Table 3, where the terms of order $\alpha_s$, $\alpha_s^2$ and $\alpha_s^3$ are also separately displayed.

Electroweak corrections are not incorporated. To predict precise numerical results for the width, the formulas should be used only in conjunction with electroweak corrections. Estimates of the theoretical error from as yet uncalculated higher-orders are to some extent subjective. They are frequently based on the stability of the prediction against a variation of the renormalization scale or the comparison of predictions in different schemes. Alternatively, one may simply identify the last calculated term with an upper limit on the uncertainty.

Let us start with the discussion of the mass corrections. In Fig. 15 mass corrections of successively higher-orders in the $\overline{\text{MS}}$-scheme are compared with those in the on-shell scheme. The poor convergence of the OS prediction, which results from the large logarithms in the coefficients, is evident. The $\overline{\text{MS}}$ prediction, however, is fairly stable. $R^V$
and \( R^A \) are calculated to order \( \mathcal{O}(\alpha_s^3) \) and \( \mathcal{O}(\alpha_s^4) \) respectively, whence all error estimates can be limited to the axial contribution. The size of the \( \alpha_s^2 \) term in the sum \( R_7 + R_8 \), resulting from the non-singlet contribution, amounts to 0.03 MeV, which can be taken as a safe error estimate. (Including the corresponding singlet term would reduce it to 0.02 MeV.)

Figure 15: Mass Corrections from \( r_V^{(1)} \) (upper graph) and \( r_A^{(1)} \) (lower graph). The left-hand bars represent the result in the on-shell scheme, the right-hand ones are obtained in the MS-scheme.

From the study of the stability of the prediction with respect to a variation of the renormalization scale an alternative error estimate can be deduced. It is obtained by using an equation equivalent to Eq. (199), but with arbitrary scale \( \mu^2 \) (see Appendix). In Fig. 16 the scale is varied between \( \mu^2 = s/4 \) and \( \mu^2 = 4s \). The corresponding error in the prediction for the decay rate amounts to \( \delta \Gamma^{[m]} = (^{+0.022}_{-0.006}) \) MeV.

The uncertainty in the prediction from the input mass is essentially proportional to the relative error in \( m^2 \). Adopting \( M_b = 4.7 \pm 0.2 \) GeV, one is lead to \( \delta m^2/m^2 \approx \pm 0.11 \) and hence to a variation of the \( m^2 \) terms by \( \pm 11\% \). It is clear that this contribution leads
to the dominant error in the corrections corresponding to $\pm 0.17$ MeV. The combined uncertainty from mass term is therefore below $\delta \Gamma^{(m)} = \pm 0.21$ MeV.

Figure 16: Renormalization scale dependence of the massive axial QCD corrections; $[\alpha_s(M_Z) = 0.12]$.

We now move to the massless limit. As discussed in Section 6.1 the reliability of the singlet terms is significantly improved through the inclusion of the $\alpha_s^3$ corrections. This is illustrated in Fig. 17, where the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ predictions as functions of the renormalization scale are compared. From the variation of the full prediction an uncertainty $\delta \Gamma_S = (^{+0.12}_{-0.05})$ MeV can be deduced. (Taking the last calculated term of order $\alpha_s^3$ as an error estimate would lead to $\delta \Gamma_S = \pm 0.25$ MeV.) The singlet terms furthermore depend on the top mass. A change of our default value $M_t = 174$ GeV by $\pm 20$ GeV leads to a variation of $\delta \Gamma_S$ by $(^{+0.10}_{-0.08})$ MeV.
Figure 17: Renormalization scale dependence of the axial singlet massless QCD corrections; $\alpha_s(M_Z) = 0.12$. 

Figure 18: Renormalization scale dependence of the non-singlet massless QCD corrections. 
$[\alpha_s(M_Z) = 0.12]$
Clearly, even the combining linearly of the modulus of the three dominant errors (from the bottom mass, the singlet contribution and $M_t$) leads to an uncertainty in the nonuniversal corrections of only $\left(\pm 0.4^3\right)\text{ MeV}$. This corresponds to an uncertainty in the value of $\alpha_s$ extracted from $\Gamma_{\text{had}}$ of $\left(10^{-4}\right)$ and a relative error in $\Gamma_{b\bar{b}}/\Gamma_{\text{had}}$ of $\left(10^{-4}\right)$, significantly below the anticipated experimental precision.

The remaining uncertainty in the $\alpha_s$ determination results from the unknown terms of $O(\alpha_s^4)$ in the non-singlet correction $r_{NS}^{(0)}$. In Fig. 18 the variation of $r_{NS}^{(0)} = r_1$ with $\mu^2$ is displayed. Evidently one arrives at a fairly stable answer in order $O(\alpha_s^4)$. The variation of $\delta r_{NS}^{(0)} = \left(\pm 0.4\right)\times 10^{-4}$ may be interpreted as an error estimate. It translates into $\delta \Gamma^{(0)} = \left(\pm 0.07\right)\text{ MeV}$ and corresponds to an uncertainty in $\alpha_s$ of $\delta \alpha_s = \left(\pm 0.07\right)\times 10^{-3}$. This result is of course dependent on the input for $\alpha_s$. For central values of $\alpha_s = 0.115$ and $\alpha_s = 0.125$ one obtains $\delta \alpha_s = \left(\pm 0.11\right)\times 10^{-3}$ and $\delta \alpha_s = \left(\pm 0.16\right)\times 10^{-3}$ respectively.

An alternative approach has been advocated in Ref. [109] (see also Ref. [110]), where an attempt is made to actually arrive at an estimate for the $\left(\alpha_s/\pi\right)^4$ term. Adopting their value for the coefficient of $-97$ one obtains a shift of $\delta r_{NS}^{(0)} = -2.1 \times 10^{-4}$ and hence of $\alpha_s$ by $-6.3 \times 10^{-4}$, quite comparable to the error estimate presented above. As a third option one may again take the last calculated term in $\delta r_{NS}^{(0)}$ for an error estimate, resulting in $\delta r_{NS}^{(0)} = \pm 7.1 \times 10^{-4}$ and $\delta \alpha_s = \pm 21.7 \times 10^{-4}$. The choice is left to the reader.

### 7.1.2 The Partial Rate $\Gamma_{b\bar{b}}$

Another quantity of interest is the ratio $\Gamma_{b\bar{b}}/\Gamma_{\text{had}}$. Assuming that $q\bar{q}$ events with secondary radiation of b quarks (see Section 5.5) are assigned to $\Gamma_{q\bar{q}}$, the universal QCD corrections $\Gamma_i$, $i = 1 \ldots 4$ cancel to a large extent and the nonuniversal parts dominate. Their contribution is small and the resulting uncertainty hence even smaller. The ratio can thus be predicted quite unambiguously. The near independence of the prediction on $\alpha_s$ is shown in Fig. 19. The flatness of the solid curve is the result of two (accidental) cancellations. With increasing $\alpha_s(M_Z)$ the mass correction is lowered through the reduction of the running b mass (for fixed pole mass) the singlet correction, however, essentially increases proportional to $\alpha_s^2$. This is illustrated by the dashed curve, where the running mass has been kept fixed to the default value.

The deviation from the parton model prediction (with $m_b = 0$) amounts to 0.7% with a negligible error from higher-orders in $\alpha_s$. The uncertainties from the input values for $M_b$ and $M_t$ amount to $\pm 3 \times 10^{-4}$ and $\pm 2 \times 10^{-4}$ respectively.
Figure 19: The ratio \( \left\{ \frac{\Gamma_{b5}/v_b^2 + a_b^2}{\Gamma_{\text{had}}/\sum(v_j^2 + a_j^2)} \right\} \) versus \( \alpha_s \). The dashed curve corresponds to a case of a fixed value of \( m_b(M_Z) = 2.77 \) GeV, while the continuous one takes into account the implicit \( \alpha_s \) dependence of \( m_b(M_Z) \).

### 7.1.3 Quick Estimates

The previous formulae display the full dependence of the QCD corrections on the input parameters \( m_b \) and \( M_t \) as well as their effect on the vector and axial vector rate of the various quark species separately. These are the formulae most suited as building blocks for detailed fitting programs. It seems, however, also useful to provide a short numerical formula suited for quick estimates. For this purpose we set \( m = m_b(M_Z) = 2.77 \pm 0.15 \) GeV (corresponding approximately to \( M = m_b|_{\text{pole}} = 4.7 \pm 0.20 \) GeV) and \( M_t|_{\text{pole}} = 174 \pm 20 \) GeV. One obtains:

\[
\frac{\Gamma_{\text{had}}}{\Gamma_0} = 3 \sum_f (v_f^2 + a_f^2) \left\{ 1 + \frac{\alpha_s}{\pi} + 1.409 \left( \frac{\alpha_s}{\pi} \right)^2 - 12.767 \left( \frac{\alpha_s}{\pi} \right)^3 + (0.023 \pm 0.005) \left( \frac{\alpha_s}{\pi} \right)^2 + (-0.006 \pm 0.001) \left( \frac{\alpha_s}{\pi} \right)^3 \right\} \\
+ 3v_b^2 \left\{ (-5 \times 10^{-6} \pm 1 \times 10^{-6}) + (0.011 \pm 0.001) \frac{\alpha_s}{\pi} \right\} \\
+ (0.097 \pm 0.01) \left( \frac{\alpha_s}{\pi} \right)^2 + (0.51 \pm 0.05) \left( \frac{\alpha_s}{\pi} \right)^3 \right\} \\
+ 3 \left\{ (-0.0055 \pm 0.0006) + (-0.020 \pm 0.002) \frac{\alpha_s}{\pi} \right\} \\
+ (-4.41 \pm 0.26) \left( \frac{\alpha_s}{\pi} \right)^2 + (-17.60 \pm 0.30) \left( \frac{\alpha_s}{\pi} \right)^3 \right\} \tag{203}
\]
+ 3 \left( \sum f v_f \right)^2 \left\{ -0.413 \left( \frac{\alpha_s}{\pi} \right)^3 \right\} \\
+ 3 \sum_f \left( v_f^2 + a_f^2 \right) Q_f^2 \left\{ 0.001867 - 0.000622 \frac{\alpha_s}{\pi} \right\}.

The origin of the terms is still evident from the structure of the couplings. For a simple evaluation of QCD corrections to the total rate one may now combine the correction coefficients with the numerically evaluated weights $v_f^2 / \sum_f (v_f^2 + a_f^2)$ etc. and arrive at

$$\Gamma_{\text{had}} = \Gamma_{\text{had}} \bigg|_{m_b = 0, \alpha_s = 0} \left\{ 1 + \frac{\alpha_s}{\pi} + 1.40932 \left( \frac{\alpha_s}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s}{\pi} \right)^3 \\
- (0.00040 \pm 0.00008) - (0.0023 \pm 0.0002) \frac{\alpha_s}{\pi} \\
- (0.63 \pm 0.04) \left( \frac{\alpha_s}{\pi} \right)^2 \\
- (2.69 \pm 0.06) \left( \frac{\alpha_s}{\pi} \right)^3 \right\}.$$ (204)

A similar treatment of $\Gamma_{b^5}$ implies

$$\Gamma_{b^5} = \Gamma_{b^5} \bigg|_{m_b = 0, \alpha_s = 0} \left\{ 1 + \frac{\alpha_s}{\pi} + 1.40932 \left( \frac{\alpha_s}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s}{\pi} \right)^3 \\
- (0.0035 \pm 0.0004) - (0.010 \pm 0.001) \frac{\alpha_s}{\pi} \\
- (2.95 \pm 0.17) \left( \frac{\alpha_s}{\pi} \right)^2 - (11.9 \pm 0.3) \left( \frac{\alpha_s}{\pi} \right)^3 \right\}.$$ (205)

and

$$\frac{\Gamma_{b^5}}{\Gamma_{\text{had}}} = \left( \frac{\Gamma_{b^5}}{\Gamma_{\text{had}}} \right) \bigg|_{m_b = 0, \alpha_s = 0} \left\{ 1 - (0.0031 \pm 0.0003) - (0.005 \pm 0.0005) \frac{\alpha_s}{\pi} \\
- (2.3 \pm 0.1) \left( \frac{\alpha_s}{\pi} \right)^2 - (6.9 \pm 0.3) \left( \frac{\alpha_s}{\pi} \right)^3 \right\}.$$ (206)

From these formulae it is evident that the coefficients of the $\alpha_s^2$ and the $\alpha_s^3$ terms are entirely different from the massless non-singlet case as a consequence of the bottom mass effects and virtual top loops discussed in this work. The deviation of this from 1 can be traced to two sources: the bottom mass term, responsible for the term independent of $\alpha_s$, and the singlet term mainly responsible for the $\alpha_s^2$ (and $\alpha_s^3$) contribution.

### 7.2 The Low-Energy Region

**Near the Bottom Threshold**

The previous discussion has dealt mainly with the applications of the theoretical results to the high energy region. However, as indicated already in Section 4.4 the results are also
applicable for energies relatively close to the threshold of heavy quarks, if \( m^2/s, m^4/s^2 \) and \( m^6/s^3 \) terms are included. This was demonstrated for the Born and the \( O(\alpha_s) \) formulae in Section 4.4, where it was shown that these leading terms provide an excellent approximation even if the ratio \( 4m^2/s \) approaches 0.8. With this justification a detailed analysis of \( R_{\text{had}} \) can be performed for the region above the charmonium resonances and below the bottom threshold (excluding, of course, the narrow \( T \) resonances). Furthermore the region above the \( \bar{B}b \) resonances — say, above 11.5 or 12 GeV — can be treated in the same approximation.

With increasing statistics and precision at LEP the uncertainty in \( \alpha_s \) from the measurement of the hadronic decay rate of the \( Z \) can be reduced to \( \pm 0.002 \). It would be highly desirable to test the evolution of the strong coupling as predicted by the beta function through a determination of \( \alpha_s \) from essentially the same observable — at lower energy, however. The region from several GeV above charm threshold (corresponding to the maximal energy of BEPC around 5.0 GeV) to just below the \( B \) meson threshold at around 10.5 GeV (corresponding to the 'off resonance' measurements of CESR) seems particularly suited for this purpose. As a consequence of the favourable error propagation, the accuracy in the measurement (compared to 91 GeV) may decrease by a factor of about three or even four at 10 and 5.6 GeV respectively, to achieve comparable precision in \( \Lambda_{\overline{MS}} \):

\[
\delta \alpha_s(s) = \frac{\alpha_s^2(s)}{\alpha_s^2(M_Z^2)} \delta \alpha_s(M_Z^2).
\]

Most of the results discussed above for massless quarks are applicable also for the case under consideration. However, two additional complications arise:

i) Charm quark effects cannot be ignored completely and should be taken into consideration through an expansion in the ratio \( m_c^2/s \), employing the results of Sections 5.3 and 5.4 for terms of order \( m_c^2/s \) and \( m_c^4/s^2 \).

ii) Contributions involving virtual bottom quarks are present, starting from order \( \alpha_s^2 \). Their contribution depends in a nontrivial manner on \( m_b^2/s \). In order \( \alpha_s^2 \) these are discussed in Section 5.2 and are shown to be small. Estimates for the corresponding contributions of order \( \alpha_s^3 \) indicate that they are under control and can be safely neglected, provided that one works within the correctly defined effective four-quark theory.

The results presented below are formulated for a theory with \( n_f = 4 \) effective flavours and with the corresponding definitions of the coupling constant and the quark mass. The relation to a formulation with \( n_f = 5 \) appropriate for the measurements above the \( \bar{b}b \) threshold was discussed in Section 2.3 and will be given at the end of the paper.

We shall now list the independent contributions and their relative importance. Neglecting for the moment the masses of the charmed quark and \textit{a fortiori} of the \( u, d \) and \( s \) quarks one predicts in order \( \alpha_s^3 \):

\[
R_{\text{NS}} = \sum_{f=u,d,s,c} 3Q_f^2 \left[ 1 + \frac{\alpha_s}{\pi} + 1.5245 \left( \frac{\alpha_s}{\pi} \right)^2 - 11.52033 \left( \frac{\alpha_s}{\pi} \right)^3 \right]
\]

(207)
for the non-singlet contribution. The second and the third order coefficients are evaluated with \( n_f = 4 \), which means that the bottom quark loops are absent. In order \( \alpha_s^2 \) the bottom quark loops can be taken into consideration with their full mass dependence given in Section 5.2. However, the leading term of order \( \alpha_s^2 m_b^2 \) provides a fairly accurate description even up to the very threshold \( s = 4m_b^2 \). Hence one has to add a correction

\[
\delta R_{mb} = \sum_{f=u,d,s,c} 3Q_f^2 \left( \frac{\alpha_s}{\pi} \right)^2 \frac{s}{m_b^2} \left[ \frac{44}{675} + \frac{2}{135} \ln \frac{m_b^2}{s} \right]. \tag{208}
\]

For the singlet term one obtains

\[
R_S = - \left( \frac{\alpha_s}{\pi} \right)^3 \left( \sum_{u,d,s,c} Q_f \right)^2 1.239 = -0.55091 \left( \frac{\alpha_s}{\pi} \right)^3. \tag{209}
\]

The bottom quark is absent in this sum. In view of the smallness of the \( \alpha_s^2 s/m_b^2 \) correction (even close to the \( b \bar{b} \) threshold!) all other terms of \( \mathcal{O}(\alpha_s^3) \) from virtual \( b \) quarks are also neglected. This can be justified with the results of Ref. [39] where \( s/m^2 \) terms are evaluated. In the same spirit it is legitimate to use the scale invariant value of the \( b \) quark mass \( \tilde{m}_b = \tilde{m}_b(\tilde{m}_b^2) \) as defined in the five-quark theory.

In contrast to the bottom mass effects of the charmed mass can be incorporated through an expansion in \( m_c^2/s \). Quadratic mass corrections are included up to order \( \alpha_s^3 \), quartic mass terms up to order \( \alpha_s^2 \). Since \( m_c^2/s \) is in itself a small expansion parameter, the order \( \alpha_s^2 m_c^4/s^2 \) terms should be sufficient for the present purpose.

The charmed mass corrections are therefore given by

\[
\delta R_{mc} = 3Q_c^2 12 \frac{m_c^2}{s} \frac{\alpha_s}{\pi} \left[ 1 + 9.097 \frac{\alpha_s}{\pi} + 53.453 \left( \frac{\alpha_s}{\pi} \right)^2 \right] - 3 \sum_{f=u,d,s,c} Q_f^2 \frac{m_c^2}{s} \left( \frac{\alpha_s}{\pi} \right)^3 6.476
\]

\[
+ 3Q_c^2 \frac{m_c^4}{s^2} \left\{ -6 - 22 \frac{\alpha_s}{\pi} + \left[ 141.329 - \frac{25}{6} \ln \left( \frac{m_c^2}{s} \right) \right] \left( \frac{\alpha_s}{\pi} \right)^2 \right\}
\]

\[
+ 3 \sum_{f=u,d,s,c} Q_f^2 \frac{m_c^4}{s^2} \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -0.4749 - \ln \left( \frac{m_c^2}{s} \right) \right]
\]

\[
- 3Q_c^2 \frac{m_c^6}{s^3} \left\{ 8 + 16 \frac{\alpha_s}{27} \left[ 6 \ln \left( \frac{m_c^2}{s} \right) + 155 \right] \right\}, \tag{210}
\]

where \( n_f = 4 \) has been adopted everywhere. Note that terms of order \( \alpha_s^2 m_c^2/s \) are more important than those of order \( \alpha_s^2 m_c^4/s^2 \) in the whole energy region under consideration. For completeness, \( m_c^6/s^3 \) and \( \alpha_s m_c^6/s^3 \) terms are also listed, which, however, are insignificant and will be ignored in the numerical analysis.

The charm quark mass is to be taken as \( m_c = \tilde{m}_c^{[4]}(s) \) and is to be evaluated in the four-flavour theory via the standard RG equation with the initial value \( \tilde{m}_c(\tilde{m}_c) = 1.12 \) GeV, corresponding to a pole mass of 1.6 GeV in the case of \( \alpha_s^{[5]}(M_Z) = 0.120 \) (see Section 2.4.2). A similar line of reasoning can be pursued for bottom mass terms in the region several
above the $B$ meson threshold. The formula given below is expected to provide a
reliable answer for $\sqrt{s}$ around 15 GeV and perhaps even down to 13 GeV:

$$
\delta R_m = 3 \left( Q_c^2 \frac{m_c^2}{s} + Q_b^2 \frac{m_b^2}{s} \right) 12 \frac{\alpha_s^{[5]}}{\pi} \left[ 1 + 8.736 \frac{\alpha_s^{[5]}}{\pi} + 45.657 \left( \frac{\alpha_s^{[5]}}{\pi} \right)^2 \right] 
- 3 \sum_{f=u, d, s, c, b} Q_f^2 \left( \frac{m_c^2}{s} + \frac{m_b^2}{s} \right) \left( \frac{\alpha_s^{[5]}}{\pi} \right)^3 6.126 
+ 3 Q_c^2 \frac{m_c^4}{s^2} \left\{ -6 - 22 \frac{\alpha_s^{[5]}}{\pi} + \left[ 139.489 - \frac{23}{6} \ln \left( \frac{m_c^2}{s} \right) + 12 \frac{m_b^2}{m_c^2} \right] \left( \frac{\alpha_s^{[5]}}{\pi} \right)^2 \right\} 
+ 3 Q_b^2 \frac{m_b^4}{s^2} \left\{ -6 - 22 \frac{\alpha_s^{[5]}}{\pi} + \left[ 139.489 - \frac{23}{6} \ln \left( \frac{m_b^2}{s} \right) + 12 \frac{m_c^2}{m_b^2} \right] \left( \frac{\alpha_s^{[5]}}{\pi} \right)^2 \right\} 
+ 3 \sum_{f=u, d, s, c, b} Q_f^2 \frac{m_f^4}{s^2} \left( \frac{\alpha_s^{[5]}}{\pi} \right)^2 \left[ -0.4749 - \ln \left( \frac{m_f^2}{s} \right) \right] 
+ 3 \sum_{f=u, d, s, c, b} Q_f^2 \frac{m_f^4}{s^2} \left( \frac{\alpha_s^{[5]}}{\pi} \right)^2 \left[ -0.4749 - \ln \left( \frac{m_c^2}{s} \right) \right] 
- 3 Q_b^2 \frac{m_b^4}{s^3} \left\{ 8 + 16 \frac{\alpha_s^{[5]}}{27} \left[ 6 \ln \left( \frac{m_b^2}{s} \right) + 155 \right] \right\}. 
$$

(211)

Above the $B$ meson threshold it is more convenient to express all quantities for $n_f = 5$
theory and thus in (7.2) all the coupling constant and quark masses are evaluated in the
five-flavour theory at the scale $\mu = \sqrt{s}$.

The transition from four- to five-flavour theory is performed as follows: The charm
mass is naturally defined in the $n_f = 4$ theory. In order to obtain the value of $m_c = \bar{m}_c^{[5]}(s)$
the initial value $\bar{m}_c^{[4]}(1 \text{ GeV})$ is evolved via the $n_f = 4$ RG equation to the point $\mu^2 = M_b^2$
and from there up to $\mu^2 = s$, now, however, with the $n_f = 5$ RG equation. The bottom
mass, on the other hand, is naturally defined in the $n_f = 5$ theory irrespective of the
characteristic momentum scale of the problem under consideration. Hence we take $\bar{m}_b(s)$
obtained from the scale invariant mass $\bar{m}_b(\bar{m}_b)$ after running the latter with the help of
the $n_f = 5$ RG equation. Finally, $\alpha_s^{[4]}$ and $\alpha_s^{[6]}$ are related through the matching Eq. (99).

In Tables 4–7 the predictions for $R$ at 10.5 and 13 GeV are listed for different values
of $\alpha_s^{[5]}(M_Z)$ together with the values of $\alpha_s(s)$ and the running masses\footnote{The results for the 5 GeV region and for a larger variety of values for $\alpha_s$ are given in Ref. [111]. Some slight differences between the numbers in Tables 4–7 and those in Ref. [111] stem from different input values for $M_c$.}. Note that our predictions are presented without QED corrections from the running of $\alpha_s$ and from initial
state radiation. Figure 20 shows the behaviour of the ratio $R(s)$ as a function of energy
below and above the bottom threshold, for $\alpha_s(M_Z) = 0.120, 0.125$ and 0.130. The light
quark $u, d, s, c, c)$ contribution is also displayed separately above 10.5 GeV. It is evident
that the predictions from the four- and five-flavour theories join smoothly. The additional
contribution from the $b\bar{b}$ channel is presented down to 11.5 GeV, where resonances start
to contribute and the perturbative treatment necessarily ceases to apply. Evidently the
Table 4

Values of $\Lambda^{(5)}_{\overline{\text{MS}}}$, $\Lambda^{(4)}_{\overline{\text{MS}}}$, $\alpha_s^{(4)}(s)$, $m_c^{(4)}(s)$ and $\overline{m}_b(\overline{m}_b)$ at $\sqrt{s} = 10.5$ GeV for different values of $\alpha_s^{(5)}(M_Z^2)$.

<table>
<thead>
<tr>
<th>$\alpha_s^{(5)}(M_Z^2)$</th>
<th>$\Lambda^{(5)}_{\overline{\text{MS}}}$ (MeV)</th>
<th>$\Lambda^{(4)}_{\overline{\text{MS}}}$ (MeV)</th>
<th>$\alpha_s^{(4)}(s)$</th>
<th>$m_c^{(4)}(s)$ (GeV)</th>
<th>$\overline{m}_b(\overline{m}_b)$ (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1200</td>
<td>233</td>
<td>320</td>
<td>0.177</td>
<td>0.751</td>
<td>4.09</td>
</tr>
<tr>
<td>0.1250</td>
<td>302</td>
<td>403</td>
<td>0.188</td>
<td>0.637</td>
<td>4.03</td>
</tr>
<tr>
<td>0.1300</td>
<td>383</td>
<td>498</td>
<td>0.199</td>
<td>0.500</td>
<td>3.96</td>
</tr>
</tbody>
</table>

$b\bar{b}$ channel is present with full strength down to the resonance region — an important consequence of QCD corrections. From this discussion it should be evident that the theoretical prediction is well under control. Mass effects are small below the $b\bar{b}$ threshold as well as a few GeV above. An experimental test is of prime importance.

7.3 Conclusions

In this report we have tried to present, in a comprehensive form, the theoretical framework and all formulae presently available that are required to predict the QCD corrected total cross-section of $e^+e^-$ annihilation and the $Z$ decay rate into hadrons, with optimal accuracy. The presentation is supposed to be self-contained — and hopefully self-consistent — such that all formulae relevant for the prediction of experimental quantities can be deduced from this work without the need to combine results from different publications. Particular emphasis has been put on the influence of the non-vanishing bottom quark mass and on contributions from virtual top quarks, which are of particular importance for the so-called singlet contributions. Much of the discussion has been tailored for the 90 GeV region, where experiments at LEP provide highly accurate data, but, applications to ‘low energies’, around 10 GeV or even lower, have been mentioned whenever appropriate.

The topic is approached from three different viewpoints: from the purely theoretical, laying the ground for the discussion, from the calculational, providing the formulae, and from the practical viewpoint, discussing the relative importance of the various contributions and associated uncertainties. In the first two parts the basis of the subsequent calculations is presented. They contain, essentially, a brief review of the field theoretical ingredients: $\beta$ function and anomalous dimensions, the relations between various definitions of the mass, the decoupling of heavy quarks, and the corresponding transitions between different ‘effective theories’. In many circumstances quarks are either light ($m^2 \ll s$) or heavy ($m^2 \gg s$). The corresponding expansions provide powerful tools. They are described in Part 2.

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Table 5

Predictions for $R(s)$ at $\sqrt{s} = 10.5$ GeV; the contributions to $\delta R_{mc}$ are shown separately for every power of the quark mass.

<table>
<thead>
<tr>
<th>$\alpha_s^{(5)}(M^2_Z)$</th>
<th>$R_{NS}$</th>
<th>$R_S$</th>
<th>$\delta R_{m^2}$</th>
<th>$\delta R_{m^2}$</th>
<th>$\delta R_{m_b}$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1200</td>
<td>3.530</td>
<td>-0.000098</td>
<td>0.0077</td>
<td>-0.00023</td>
<td>0.0026</td>
<td>3.540</td>
</tr>
<tr>
<td>0.1250</td>
<td>3.543</td>
<td>-0.00012</td>
<td>0.0061</td>
<td>-0.000121</td>
<td>0.0030</td>
<td>3.551</td>
</tr>
<tr>
<td>0.1300</td>
<td>3.556</td>
<td>-0.00014</td>
<td>0.0041</td>
<td>-0.000046</td>
<td>0.0034</td>
<td>3.563</td>
</tr>
</tbody>
</table>

Table 6

Values of $\Lambda^{(5)}_{\overline{MS}}$, $\alpha_s^{(5)}(s)$, $m_c^{(5)}(s)$ and $m_b^{(5)}(s)$ at $\sqrt{s} = 13$ GeV for different values of $\alpha_s^{(5)}(M^2_Z)$.

<table>
<thead>
<tr>
<th>$\alpha_s^{(5)}(M_Z)$</th>
<th>$\Lambda^{(5)}_{\overline{MS}}$</th>
<th>$\alpha_s^{(5)}(s)$</th>
<th>$m_c^{(5)}(s)$</th>
<th>$m_b^{(5)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1200</td>
<td>233 MeV</td>
<td>0.172</td>
<td>0.729 GeV</td>
<td>3.41 GeV</td>
</tr>
<tr>
<td>0.1250</td>
<td>302 MeV</td>
<td>0.183</td>
<td>0.617 GeV</td>
<td>3.29 GeV</td>
</tr>
<tr>
<td>0.1300</td>
<td>383 MeV</td>
<td>0.194</td>
<td>0.483 GeV</td>
<td>3.17 GeV</td>
</tr>
</tbody>
</table>
Table 7

Predictions for $R(s)$ at $\sqrt{s} = 13$ GeV; the contributions to $\delta R_m$ are shown separately for every power of the quark masses.

<table>
<thead>
<tr>
<th>$\alpha_s(\frac{M_Z^2}{M})$</th>
<th>$R_{NS}$</th>
<th>$R_S$</th>
<th>$\delta R_m^2$</th>
<th>$\delta R_m^4$</th>
<th>$\delta R_m^6$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1200</td>
<td>3.875</td>
<td>-0.000023</td>
<td>0.023</td>
<td>-0.011</td>
<td>-0.0014</td>
<td>3.877</td>
</tr>
<tr>
<td>0.1250</td>
<td>3.888</td>
<td>-0.000027</td>
<td>0.023</td>
<td>-0.0092</td>
<td>-0.0011</td>
<td>3.901</td>
</tr>
<tr>
<td>0.1300</td>
<td>3.901</td>
<td>-0.000032</td>
<td>0.022</td>
<td>-0.0079</td>
<td>-0.00091</td>
<td>3.914</td>
</tr>
</tbody>
</table>

Figure 20: The ratio $R(s)$ below and above the $b$ quark production threshold at 10.5 GeV for $\alpha_s(M_Z) = 0.120, 0.125$ and 0.130. The contributions from light quarks are displayed separately.
The results of higher-order calculations are scattered in numerous original publications, often with conflicting conventions and notations. In Parts 3, 5 and 6 the formulae are collected and presented in a uniform way. The brief presentation of exact results of $O(\alpha_s)$ (i.e. with arbitrary $m^2/s$) in Part 4 leads quickly to Parts 5 and 6 where $m^2/s$ or $s/m^2$ are treated as expansion parameters, and results of up to order $\alpha_s^3$ are collected. An important classification of amplitudes originates from the distinction between singlet and non-singlet diagrams, with markedly different behaviour in the limit $m^2/s \gg 1$, and our presentation follows this classification. Whilst most of the discussion is concerned with predictions for the total cross-section or the decay rate, occasionally also results for partial rates, for example into four fermion states or for the inclusive rate into $b\bar{b}$ quarks, are presented. The formulae are displayed in two different forms: first in analytical form with the relevant coefficients given by fractions and Riemann's Zeta function, as functions of $n_f$, and then entirely numerically (Part 7 and the Appendices) in terms of decimal fractions.

The final Part is most relevant to practical applications. The numerical relevance (or irrelevance) of the various contributions is clarified. Their stability with respect to variations of the renormalization scale $\mu$ is studied. Estimates are presented for the errors from the truncation of the perturbation series, the ‘theoretical error’, and for the error induced by the uncertainty in the input parameters $M_b$ and $M_t$. A safe upper limit on the combined uncertainty of $\Gamma(Z \to \text{hadrons})$ from quark mass ($M_b$ and $M_t$) dependent corrections amounts to about 0.4 MeV. The uncertainty from the truncation of the perturbative expansion in the massless limit in $O(\alpha_s^3)$ is highly subjective, with estimates ranging from 0.6 MeV up to 1.2 MeV. Fortunately enough, in the foreseeable future, both sources of errors will not affect the precision of $\alpha_s$ as determined at LEP through total cross-section measurements.
For the convenience of the reader, simple numerical formulae are presented which allow a quick estimate of $\Gamma_{\text{had}}$, $\Gamma_{\text{bq}}$, and their ratio. A similar discussion for the energy region around the bottom threshold concludes this Part. For easy access the formulas most frequently used in practical applications are collected and rewritten in the Appendices.

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A.1 Some Useful Formulae

Zeta function

The Riemann Zeta function is defined by

\[ \zeta(s) = \sum_{k=1}^{\infty} k^{-s} . \]  \hspace{1cm} (212)

Some particular values are:

\[ \zeta(0) = -\frac{1}{2} , \quad \zeta(2) = \frac{\pi^2}{6} = 1.6449341 , \]

\[ \zeta(3) = 1.2020569 , \quad \zeta(4) = \frac{\pi^4}{90} = 1.0823232 , \]  \hspace{1cm} (213)

\[ \zeta(5) = 1.0369278 . \]

Dilogarithm

\[ \text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} , \quad (|x| < 1) \]  \hspace{1cm} (214)

or

\[ \text{Li}_2(x) = -\int_0^x \frac{\ln (1-t)}{t} \, dt . \]  \hspace{1cm} (215)
A.2 Renormalization Group Functions

The RG equation for the quark mass reads:

$$
\mu^2 \frac{d}{d\mu^2} \tilde{m}(\mu) = \tilde{m}(\mu) \gamma_m \equiv -\tilde{m}(\mu) \sum_{i > 0} \gamma_m^i \left( \frac{\alpha_s}{\pi} \right)^{i+1}.
$$

(216)

It is solved by

$$
\tilde{m}(\mu) = \tilde{m}(\mu_0) \exp \left\{ \frac{1}{\pi} \int_{\alpha_s(\mu)}^{\alpha_s(\mu_0)} \frac{\gamma_m(x)}{\beta(x)} \, dx \right\}
$$

$$
= \tilde{m}(\mu_0) \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\gamma_m^0/\beta_0} \left\{ 1 + \left( \frac{\gamma_m^1}{\beta_0} - \frac{\beta_1^0 \gamma_m^0}{\beta_0^2} \right) \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(\mu_0)}{\pi} \right] \right.
$$

$$
+ \frac{1}{2} \left( \frac{\gamma_m^1}{\beta_0} - \frac{\beta_1^0 \gamma_m^0}{\beta_0^2} \right)^2 \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(\mu_0)}{\pi} \right]^2
$$

$$
+ \frac{1}{2} \left( \frac{\gamma_m^2}{\beta_0} - \frac{\beta_2^0 \gamma_m^0}{\beta_0^2} + \frac{\beta_2^0 \gamma_m^0}{\beta_0^2} \right) \left[ \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 - \left( \frac{\alpha_s(\mu_0)}{\pi} \right)^2 \right] \right\}.
$$

(217)

with

$$
\gamma_m^0 = 1, \quad \gamma_m^1 = \left( \frac{202}{3} - \frac{20}{9} n_f \right) / 16,
$$

$$
\gamma_m^2 = \left\{ 1249 - \left[ 2216 \frac{1}{27} + \frac{160}{3} \zeta(3) \right] n_f - \frac{140}{81} n_f^2 \right\} / 64.
$$

Similarly, one has for the strong coupling constant ($L \equiv \ln \mu^2 / \Lambda_{\overline{MS}}^2$):

$$
\mu^2 \frac{d}{d\mu^2} \left[ \frac{\alpha_s(\mu)}{\pi} \right] = \beta \equiv -\sum_{i > 0} \beta_i \left( \frac{\alpha_s}{\pi} \right)^{i+2},
$$

(219)

Integration gives

$$
\frac{\alpha_s(\mu)}{\pi} = \frac{1}{\beta_0 L} \left\{ 1 - \frac{1}{\beta_0 L} \frac{\beta_1 \ln L}{\beta_0} + \frac{1}{\beta_0^2 L^2} \left[ \frac{\beta_1^2}{\beta_0^2} (\ln^2 L - \ln L - 1) + \frac{\beta_2}{\beta_0^2} \right] \right\},
$$

(220)

with

$$
\beta_0 = \left( 11 - \frac{2}{3} n_f \right) / 4, \quad \beta_1 = \left( 102 - \frac{38}{3} n_f \right) / 16,
$$

$$
\beta_2 = \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right) / 64.
$$

(221)
A.3 List of Radiative Corrections

In this section all contributions to the total hadronic $Z$ decay rate are collected. Complete order $\alpha_s$ prediction with full mass dependence ($v^2 = 1 - 4m^2/s$):

$$\Gamma_{\text{had}} = \Gamma_0 3 \left\{ \sum_f v f' v^2 - \frac{3 - v^2}{2} \left[ 1 + \frac{4}{3} \frac{\alpha_s(s)}{\pi} K_V \right] \right.$$ 

$$+ \sum_f a^2_f v^3 \left[ 1 + \frac{4}{3} \frac{\alpha_s(s)}{\pi} K_A \right] \right\},$$

(222)

with

$$K_V = \frac{1}{v} \left[ A(v) + \frac{P_V(v)}{(1 - v^2/3)} \ln \frac{1 + v}{1 - v} + \frac{Q_V(v)}{(1 - v^2/3)} \right]$$

$$K_A = \frac{1}{v} \left[ A(v) + \frac{P_A(v)}{v^2} \ln \frac{1 + v}{1 - v} + \frac{Q_A(v)}{v^2} \right],$$

(223)

and

$$A(v) = (1 + v^2) \left\{ \ln_2 \left[ \frac{(1 - v)^2}{1 + v} \right] + 2 \ln_2 \left( \frac{1 - v}{1 + v} \right) + \ln \frac{1 + v}{1 - v} \ln \left( \frac{(1 + v)^3}{8v^2} \right) \right.$$ 

$$+ 3v \ln \frac{1 - v^2}{4v} - v \ln v, \right.$$ 

(224)

$$P_V(v) = \frac{33}{24} + \frac{22}{24} v^2 - \frac{7}{24} v^4, \right.$$ 

$$Q_V(v) = \frac{5}{4} v - \frac{3}{4} v^3,$$

$$P_A(v) = \frac{21}{32} + \frac{59}{32} v^2 - \frac{19}{32} v^4 - \frac{3}{32} v^6, \right.$$ 

$$Q_A(v) = - \frac{21}{16} v + \frac{30}{16} v^3 + \frac{3}{16} v^5,$$

(225)

where $\Gamma_0 = G_F M_Z^2/24\sqrt{2\pi} = 82.94$ MeV.

Including higher-order corrections the decay rate can be written in the following form:

$$\Gamma_{\text{had}} = \sum_{i=1}^{12} \Gamma_i = \sum_{i=1}^{12} \Gamma_0 R_i .$$

(226)

The separate contributions are given by the following expressions. [Below in Eqs. (6-15) $\alpha_s$, $m_f$ and $m_b$ stand for $\alpha_s^{(s)}(\mu)$, $m_f^{(s)}(\mu)$ and $m_b^{(s)}(\mu)$ respectively.]

Massless non-singlet corrections:

$$R_1 = 3 \sum_f (v_f^2 + a_f^2) \left\{ 1 + \frac{\alpha_s}{\pi} \right.$$ 

$$+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{365}{24} - 11 \zeta(3) + n_f \left( \frac{11}{12} + 2 \zeta(3) \right) + \left( \frac{11}{4} + \frac{1}{6} n_f \right) \ln \frac{s}{\mu^2} \right] \right.$$ 

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\begin{align*}
+ \left( \frac{\alpha_s}{\pi} \right)^3 & \left[ \frac{87029}{288} - \frac{121}{48} \pi^2 - \frac{1103}{4} \zeta(3) + \frac{275}{6} \zeta(5) \right] \\
+ n_f \left( - \frac{7847}{216} + \frac{11}{36} \pi^2 + \frac{262}{9} \zeta(3) - \frac{25}{9} \zeta(5) \right) & + n_f^2 \left( \frac{151}{162} - \frac{1}{108} \pi^2 - \frac{19}{27} \zeta(3) \right) \\
+ & \left( \frac{121}{48} - \frac{11}{36} \right) n_f \ln \frac{s}{\mu^2} \\
+ & \left\{ \alpha_s \right\}^3 \left\{ \frac{7.5625 - 0.9167 n_f + 0.0278 n_f^2}{\ln \frac{s}{\mu^2}} \right\} ;
\end{align*}

Massive universal corrections ('double bubble'):

\begin{align*}
R_2 & = 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right)^3 \sum_{f'=b} \frac{m_{f'}}{s} \left\{ - 80 + 60 \zeta(3) + n_f \left[ \frac{32}{9} - \frac{8}{3} \zeta(3) \right] \right\} \\
& = 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right)^3 \sum_{f'=b} \frac{m_{f'}}{s} \left\{ - 7.8766 + 0.3501 n_f \right\} \\
R_3 & = 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right)^2 \sum_{f'=b} \frac{m_{f'}}{s^2} \left\{ \frac{13}{3} - 4 \zeta(3) - \ln \frac{\bar{m}_{f'}}{s} \right\} \\
& = 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right)^2 \sum_{f'=b} \frac{m_{f'}}{s^2} \left\{ - 0.4749 - \ln \frac{\bar{m}_{f'}}{s} \right\} \\
R_4 & = 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right) \frac{s}{M^2} \left\{ \frac{44}{675} + \frac{2}{135} \ln \frac{M^2}{s} \right\} \\
& = 3 \sum_f (v_f^2 + a_f^2) \left( \frac{\alpha_s}{\pi} \right) \frac{s}{M^2} \left\{ 0.0652 + 0.0148 \ln \frac{M^2}{s} \right\} ;
\end{align*}

Massive non-singlet corrections (vector):

\begin{align*}
R_5 & = 3 v_b^2 \frac{\bar{m}_b}{s} \left\{ 12 \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{253}{2} - \frac{13}{3} n_f + (-57 + 2 n_f) \ln \frac{s}{\mu^2} \right] \right\} \\
& + \left( \frac{\alpha_s}{\pi} \right)^3 \left[ \frac{2522}{4} - \frac{285}{3} \pi^2 + \frac{310}{3} \zeta(3) - \frac{5225}{6} \zeta(5) \right] \\
& + n_f \left( - \frac{4942}{27} + \frac{17}{3} \pi^2 - \frac{394}{27} \zeta(3) + \frac{1045}{27} \zeta(5) \right) + n_f^2 \left( \frac{125}{54} - \frac{\pi^2}{6} \right) \pi^2 - \frac{1}{9} \pi^2 \right\}
\end{align*}
$$\begin{align*}
R_6 &= 3v_b \frac{m_b^4}{s^2} \left\{ -6 + \frac{\alpha_s}{\pi} \left[-22 + 24 \ln \frac{s}{\mu^2} \right] \\
&+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -\frac{3029}{12} + 27 \pi^2 + 112 \zeta(3) - \frac{11}{2} \ln \frac{m_b^2}{s} - 225 \ln \frac{s}{\mu^2} - 81 \ln^2 \frac{s}{\mu^2} \\
&+ n_f \left( \frac{143}{18} - \frac{1}{3} \pi^2 - \frac{8}{3} \zeta(3) + \frac{1}{3} \ln \frac{m_b^2}{s} - \frac{22}{3} \ln \frac{s}{\mu^2} + 2 \ln^2 \frac{s}{\mu^2} \right) \right] \right\} \\
&= 3v_b \frac{m_b^4}{s^2} \left\{ -6 + \frac{\alpha_s}{\pi} \left[-22 + 24 \ln \frac{s}{\mu^2} \right] \\
&+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 148.693 - 5.5 \ln \frac{m_b^2}{s} + 255 \ln \frac{s}{\mu^2} - 81 \ln^2 \frac{s}{\mu^2} \\
&+ n_f \left( -1.8408 + 0.3333 \ln \frac{m_b^2}{s} - 7.3333 \ln \frac{s}{\mu^2} + 2 \ln^2 \frac{s}{\mu^2} \right) \right] \right\} ; \quad (230)
\end{align*}$$

Massive non-singlet corrections (axial):

$$\begin{align*}
R_7 &= 3 \frac{m_b^2}{s} \left\{ -6 + \frac{\alpha_s}{\pi} \left[-22 + 12 \ln \frac{s}{\mu^2} \right] \\
&+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -\frac{8221}{24} + \frac{19}{2} \pi^2 + 117 \zeta(3) + n_f \left( \frac{151}{12} - \frac{1}{3} \pi^2 - 4 \zeta(3) \right) \\
&+ \left( 155 - \frac{16}{3} n_f \right) \ln \frac{s}{\mu^2} + \left( -\frac{57}{2} + n_f \right) \ln^2 \frac{s}{\mu^2} \right] \right\} \\
&= 3 \frac{m_b^2}{s} \left\{ -6 + \frac{\alpha_s}{\pi} \left[-22 + 12 \ln \frac{s}{\mu^2} \right] \\
&+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -108.14 + 4.4852 n_f + (155 - 5.3333 n_f) \ln \frac{s}{\mu^2} + (-28.5 + n_f) \ln^2 \frac{s}{\mu^2} \right] \right\} \quad (231)
\end{align*}$$

$$\begin{align*}
R_8 &= 3 \frac{m_b^4}{s^2} \left\{ 6 + \frac{\alpha_s}{\pi} \left[ 10 - 24 \ln \frac{s}{\mu^2} \right] \\
&+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{3389}{12} - 27 \pi^2 - 220 \zeta(3) + \frac{77}{2} \ln \frac{m_b^2}{s} - 207 \ln \frac{s}{\mu^2} + 81 \ln^2 \frac{s}{\mu^2} \right] \right\} \\
&= 3 \frac{m_b^4}{s^2} \left\{ 6 + \frac{\alpha_s}{\pi} \left[ 10 - 24 \ln \frac{s}{\mu^2} \right] \\
&+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{3389}{12} - 27 \pi^2 - 220 \zeta(3) + \frac{77}{2} \ln \frac{m_b^2}{s} - 207 \ln \frac{s}{\mu^2} + 81 \ln^2 \frac{s}{\mu^2} \right] \right\} 
\end{align*}$$
\[ + n_f \left( -\frac{41}{6} + \frac{2}{3} \pi^2 + \frac{16}{3} \zeta(3) - \frac{7}{3} \ln \frac{m_b^2}{s} + \frac{22}{3} \ln \frac{s}{\mu^2} - 2 \ln^2 \frac{s}{\mu^2} \right) \right] \}
\[ = 3 \left( \frac{m_b^4}{s^2} \right) \left\{ 6 \frac{\alpha_s}{\pi} \left[ 10 - 24 \ln \frac{s}{\mu^2} \right] \right. \\
+ \left. \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -248.515 + 38.5 \ln \frac{m_b^2}{s} - 207 \ln \frac{s}{\mu^2} + 81 \ln^2 \frac{s}{\mu^2} \right. \\
\left. + n_f \left( 6.1574 - 2.3333 \ln \frac{m_b^2}{s} + 7.3333 \ln \frac{s}{\mu^2} - 2 \ln^2 \frac{s}{\mu^2} \right) \right\} ; \tag{232} \]

**Singlet corrections (axial):**

\[ R_\theta \ = \ \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ -9.250 + 1.037 \frac{s}{4M_t^2} + 0.632 \left( \frac{s}{4M_t^2} \right)^2 + 3 \ln \frac{s}{M_t^2} \right\} \\
\ + \left. \left( \frac{\alpha_s}{\pi} \right)^3 \left\{ -\frac{5075}{72} + \frac{23}{12} \pi^2 + 3 \zeta(3) + \frac{67}{6} \ln \frac{s}{M_t^2} \right. \\
\left. + \frac{23}{4} \ln^2 \frac{s}{M_t^2} + \frac{373}{8} \ln \frac{s}{\mu^2} - \frac{23}{4} \ln^2 \frac{s}{\mu^2} \right\} \right. \\
\left. + n_f \left( 9.250 - 1.037 \frac{s}{4M_t^2} - 0.632 \left( \frac{s}{4M_t^2} \right)^2 - 3 \ln \frac{s}{M_t^2} \right) \right\} \right. \\
\left. + \left( \frac{\alpha_s}{\pi} \right)^3 \left\{ -47.963 + 11.167 \ln \frac{s}{M_t^2} \right. \\
\left. + 5.75 \ln^2 \frac{s}{M_t^2} + 46.625 \ln \frac{s}{\mu^2} - 5.75 \ln^2 \frac{s}{\mu^2} \right\} \right\} ; \tag{233} \]

\[ R_{10} \ = \ 3 \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ -6 \frac{m_b}{s} \left[ -3 + \ln \frac{s}{M_t^2} \right] - 10 \frac{m_b}{M_t^2} \left[ \frac{8}{51} - \frac{1}{54} \ln \frac{s}{M_t^2} \right] \right\} ; \tag{234} \]

**Singlet corrections (vector):**

\[ R_{11} \ = \ 3 \left( \sum_{f} v_f \right)^2 r_{11} \tag{235} \]

\[ = \left( \sum_{f} v_f \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 (-1.2395) ; \]

**O(\alpha_s) corrections:**

\[ R_{12} \ = \ 3 \sum_{f} (v_f^2 + a_f^2) Q_f^2 r_{12} \tag{236} \]

\[ = 3 \sum_{f} (v_f^2 + a_f^2) Q_f^2 \left[ \frac{3}{4} \frac{\alpha}{\pi} \left[ 1 - \frac{1}{3} \frac{\alpha_s}{\pi} \right] \right] ; \]
$R_{\text{had}}$ from virtual photons:

The prediction for the ‘classical’ production through a value $R^{m} = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma_{\text{point}}$ is obtained from the above equations by setting $a_f = 0$ and $R^{m} = \sum_i R_i$ after setting $v_f \rightarrow Q_f$ and $a_f \rightarrow 0$;

Secondary radiation of heavy quarks:

The rate for secondary radiation of a pair of quarks with mass $m$ is given by

$$\frac{\Gamma_{Q\overline{Q}e\overline{e}}}{\Gamma_{q\overline{q}}} = \frac{2}{3} \left( \frac{\alpha_s}{\pi} \right)^2 \varrho^R \left( \frac{m^2}{s} \right),$$

(237)

where

$$\varrho^R(x) = \frac{4}{3} \left(1 - 6x^2\right) \left\{ \frac{1}{2} \text{Li}_3 \left(\frac{1-w}{2}\right) - \frac{1}{2} \text{Li}_3 \left(\frac{1+w}{2}\right) + \text{Li}_3 \left(\frac{1+w}{1+a}\right) - \text{Li}_3 \left(\frac{1-w}{1+a}\right) + \frac{1}{2} \ln \left(\frac{1+w}{1-w}\right) \left[ \zeta(2) - \frac{1}{12} \ln^2 \left(\frac{1+w}{1-w}\right) \right] + \frac{1}{2} \ln^2 \left(\frac{a-1}{a+1}\right) - \frac{1}{2} \ln \left(\frac{1+w}{2}\right) \ln \left(\frac{1-w}{2}\right) \right\}$$

$$+ \frac{1}{9} a (19 + 46x) \left[ \text{Li}_2 \left(\frac{1+w}{1+a}\right) + \text{Li}_2 \left(\frac{1-w}{1+a}\right) - \text{Li}_2 \left(\frac{1+w}{1-a}\right) - \text{Li}_2 \left(\frac{1-w}{1-a}\right) + \ln \left(\frac{a-1}{a+1}\right) \ln \left(\frac{1+w}{1-w}\right) \right]$$

$$+ 4 \left(\frac{19}{72} + x + x^2\right) \left[ \text{Li}_2 \left(-\frac{1+w}{1-w}\right) - \text{Li}_2 \left(-\frac{1-w}{1+w}\right) - \ln x \ln \left(\frac{1+w}{1-w}\right) \right]$$

$$+ 7 \left(\frac{73}{189} + \frac{74}{63} x + x^2\right) \ln \left(\frac{1+w}{1-w}\right) - \frac{1}{3} \left(\frac{2123}{108} + \frac{2489}{54} x\right) w,$$

(238)

with

$$a = \sqrt{1+4x}, \quad w = \sqrt{1-4x}.$$  

(239)
References


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The Large Quark Mass Expansion of
\( \Gamma(Z^0 \rightarrow \text{hadrons}) \) in the Order \( \alpha_s^3 \).

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Abstract

We present the analytical \( \alpha_s^3 \) correction to the \( Z^0 \) decay rate into hadrons. We have calculated this correction up to (and including) terms of the order \( (M_Z^2/m_{\text{top}}^2)^3 \) in the large top-quark-mass expansion.

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1 Introduction

Precision measurements of the $Z^0$ decay rate into hadrons at LEP [1] provide precise means of extracting the QCD coupling constant from the experiment. This is a very clean process from a theoretical point of view since its calculation can be reduced to the calculation of the $Z$ boson propagator within the standard model. The calculational techniques of Feynman diagrams have advanced to the point where the calculation of the $\alpha_s^3$ order (4 loop approximation of the $Z$ boson propagator) is feasible. The $\alpha_s^3$ approximation to the $Z^0$ decay rate into hadrons is important for an accurate determination of the QCD coupling constant $\alpha_s$ or, equivalently, the fundamental scale of QCD, $\Lambda_{\text{QCD}}$.

The hadronic $Z^0$ decay rate is a sum of vector and axial vector contributions of which the vector contribution is known to order $\alpha_s^3$ from the calculation of $\sigma_{\text{tot}}(e^+e^- \rightarrow \gamma \rightarrow \text{hadrons})$ [2]. This calculation was performed in the approximation of effective QCD with five massless quarks involving the calculation of massless diagrams only. The correctness of this calculation is strongly supported by [3], where the non-trivial connection between the result [2] and the $\alpha_s^3$ approximation [4] to deep inelastic sum rules was established.

The calculation of the axial vector part of the hadronic $Z^0$ decay rate is more involved than that of the vector part. This is because the heavy quark does not decouple in the axial vector part and one cannot avoid calculating massive diagrams, even in the leading order of the large mass expansion. The axial vector part was calculated to order $\alpha_s^2$ in [5] and confirmed in [6], where the operator-product-expansion technique was used to sum up the massive logarithmic terms. The $Z^0$ decay into three gluons in order $\alpha_s^3$ has been calculated in [7]. The $\alpha_s^3$ correction to the axial vector part of the hadronic $Z^0$ decay rate in the leading order of the large top-mass expansion was presented in [8, 9]. The extension of the large mass expansion of both the vector and axial vector parts to the order $(M_Z^2/m_{\text{top}}^2)^3$ was given in Ref. [10]. That calculation involves the decoupling mechanism at the next-to-next-to-leading (NNL) order. In this chapter we give a concise account of the hadronic $Z^0$ decay rate, including top-quark-mass-suppressed contributions in effective QCD with five active massless quark flavours.

2 Preliminaries

For the $Z^0$ decay rate into hadrons, the quantity to be determined is the squared matrix element summed over all final hadronic states. One can express this quantity as the imaginary part of a current correlator in the standard way

$$\sum_n \langle 0 | J^\mu | n \rangle \langle n | J^\nu | 0 \rangle = 2 \text{Im} \Pi^{\mu\nu},$$

(1)

$$\Pi^{\mu\nu} = i \int d^4 z e^{i q \cdot z} \langle 0 | T [J^\nu(z)J^\mu(0)] | 0 \rangle = -g^{\mu\nu} q^2 \Pi_1(q^2) - g^{\mu\nu} q \cdot q \Pi_2(q^2).$$

(2)

Here $J^\mu = g/(2 \cos \theta_W) \sum_{i=1}^6 \bar{\psi}_i \gamma^\mu (g'_V - g'_A \gamma^5) \psi_i$ is the neutral weak quark current coupled to the $Z^0$ boson in the Lagrangian of the standard model, where we use the notations as given in Ref. [11] $g'_V = t_{3L}(\bar{t}) - 2q_i \sin^2 \theta_W$ and $g'_A = t_{3L}(\bar{t})$.

The hadronic $Z^0$ decay width is expressed as

$$\Gamma_{\text{had}} \equiv \Gamma^V_{\text{had}} + \Gamma^A_{\text{had}} = M_Z \text{Im} \Pi_1(M_Z^2 + i\epsilon),$$

(3)

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with the indicated decomposition into vector and axial vector parts imposed by the structure of the neutral current. The calculation of $\text{Im} \Pi_1$ in the order $g^2\alpha_s^2$ is a calculation within perturbative QCD except for two weak-current vertex insertions (i.e. the weak current is considered as an external current for QCD).

Throughout this chapter we use dimensional regularization [12] in $D = 4 - 2\varepsilon$ space-time dimensions and the standard modification of the minimal subtraction scheme [13], the $\overline{MS}$ scheme [14]. For the treatment of the $\gamma_5$ matrix in dimensional regularization we use the technique described in [15], which is based on the original definition of $\gamma_5$ in [12]. We work in the approximation of five massless quark flavours and the top quark mass is large compared to the $Z^0$ mass. We should stress that the top quark does not decouple [16] from the axial vector part, due to diagrams of the axial anomaly type.

It is convenient to split the vector and axial vector contribution into non-singlet and singlet parts

$$\Gamma_{\text{had}}^V = \Gamma_{\text{had}}^{V,NS} + \Gamma_{\text{had}}^{V,S}, \quad \Gamma_{\text{had}}^A = \Gamma_{\text{had}}^{A,NS} + \Gamma_{\text{had}}^{A,S}. \tag{4}$$

The non-singlet parts come from Feynman diagrams where both weak current vertices are located in one fermion loop. The singlet contributions come from diagrams where each weak current vertex is located in a separate fermion loop. For the $\alpha_s^3$ approximation the massive non-singlet diagrams are presented in Fig. 1, the singlet diagrams are presented in Figs. 2, 3.

The $\alpha_s^3$ approximation for the vector part in effective QCD with five active massless quark flavours in the $\overline{MS}$ scheme was calculated in [2] (in the leading order of the large top-quark mass expansion). This calculation used the fact that the top quark decouples for the vector part in the leading order of the large quark mass expansion. Therefore within the effective five-flavour QCD this calculation involved only massless diagrams, the result reading:

$$\Gamma_{\text{had}}^{V,NS} = \frac{G_F M_Z^3}{2\pi \sqrt{2}} \sum_{i=1}^{5} (g_Y^i)^2 \left[ 1 + \frac{\alpha_s^{[5]}}{\pi} + 1.40923 \left( \frac{\alpha_s^{[5]}}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s^{[5]}}{\pi} \right)^3 \right]$$

$$\Gamma_{\text{had}}^{V,S} = \frac{G_F M_Z^3}{2\pi \sqrt{2}} \left( \sum_{i=1}^{5} g_Y^i \right)^2 \left[ -0.41318 \left( \frac{\alpha_s^{[5]}}{\pi} \right)^3 \right], \tag{5}$$

with the Fermi constant $G_F = g^2 \sqrt{2}/[8 \cos^2(\theta_W) M_Z^2]$. Here $\alpha_s^{[5]}(M_Z)$ is the coupling constant in the effective QCD with five active flavours. The coupling constant of effective five-flavour QCD, $\alpha_s^{[5]}$, and the coupling constant of full six-flavour QCD, $\alpha_s^{[6]}$, both obey the renormalization group equation [with $n_f = 5$ for $\alpha_s^{[5]}$ and $n_f = 6$ for $\alpha_s^{[6]}$]

$$\frac{\partial \alpha_s/\pi}{\partial \ln Q^2} = \beta \left( \frac{\alpha_s}{\pi} \right)$$

$$= -\beta_0 \left( \frac{\alpha_s}{\pi} \right)^2 - \beta_1 \left( \frac{\alpha_s}{\pi} \right)^3 - \beta_2 \left( \frac{\alpha_s}{\pi} \right)^4 + O(\alpha_s)^5, \tag{6}$$

where

$$\beta_0 = \frac{1}{4} \left( \frac{11}{3} C_A - \frac{4}{3} T_F n_f \right)$$
\[ \beta_1 = \frac{1}{16} \left( \frac{34}{3} C_A^2 - 4 C_F T_F n_f - \frac{20}{3} C_A T_F n_f \right) \]
\[ \beta_2 = \frac{1}{64} \left( \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f - \frac{1415}{27} C_A^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f + \frac{158}{27} C_A T_F^2 n_f \right). \] (7)

The three-loop QCD beta function in the \( \overline{\text{MS}} \)-scheme was calculated in Ref. [17]. \( C_F = 4/3 \) and \( C_A = 3 \) are the Casimir operators of the fundamental and adjoint representation of the colour group \( SU(3) \), \( T_F = 1/2 \) is the trace normalization of the fundamental representation.

The solution of Eq. (6) in the NNL order has the standard form

\[ \frac{\alpha_s}{\pi} = \frac{1}{\beta_0 \ln \left( \frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2} \right)} + \frac{\beta_1}{\beta_0^2 \ln^2 \left( \frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2} \right)} \left[ \beta_2 \ln^2 \ln \left( \frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2} \right) - \beta_1^2 \ln \ln \left( \frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2} \right) + \beta_2 \beta_0 - \beta_1^2 \right], \] (8)

where it is understood that the scale \( \Lambda_{\overline{\text{MS}}} \) also depends on the number of active flavours.

The (top-quark mass-dependent) relation between \( \alpha_s^{(6)} \) and \( \alpha_s^{(5)} \) is called the decoupling relation and will be discussed in the next section [see Eq. (11)]. This relation gives, together with the expression (8) for \( \alpha_s^{(5)} \) and \( \alpha_s^{(6)} \), the connection between \( \Lambda_{\overline{\text{MS}}}^{(6)} \) and \( \Lambda_{\overline{\text{MS}}}^{(5)} \) via the top quark mass \( m_t \).

Let us turn to the singlet axial vector part. In the standard model, quarks in a weak doublet couple with opposite sign to the \( Z^0 \) boson in the axial vector part of the neutral current. That is why the contributions from light doublets add up to zero in the massless limit for axial vector singlet diagrams. The only non-zero contribution comes from the top–bottom doublet due to the large-mass difference between top and bottom quarks.

The axial vector singlet part in the leading order of the large top-quark mass expansion has recently been calculated \([8, 9]\). The result in the effective theory with five active massless quark flavours is

\[ \Gamma^{A,NS}_{\text{had}} = \frac{G_F M_Z^3}{2 \pi \sqrt{2}} \sum_{i=1}^{5} (g_A^i)^2 \left[ 1 + \frac{\alpha_s^{(5)}}{\pi} + 1.40923 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \right]. \] (9)
\[ \Gamma_{\text{had}}^{4,S} = \frac{G_F M_Z^3}{2\pi \sqrt{2}} \left( g_A^{\text{bot}} \right)^2 \left\{ \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 \left( - \frac{17}{6} \right) \right. \\
+ \left. \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \left[ - \frac{4673}{144} + \frac{67}{12} \xi_3 + \frac{23}{36} \pi^2 - \frac{1}{36} \ln \left( \frac{M_Z^2}{m_t^2} \right) - \frac{1}{6} \ln^2 \left( \frac{M_Z^2}{m_t^2} \right) \right] \right\} \]

\[ + \frac{G_F M_Z^3}{2\pi \sqrt{2}} \left( g_A^{\text{bot}} g_A^{\text{top}} \right) \left\{ \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 \left[ \frac{1}{4} - \ln \left( \frac{M_Z^2}{m_t^2} \right) \right] \right. \\
+ \left. \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \left[ - \frac{2717}{432} + \frac{55}{12} \xi_3 - \frac{7}{4} \ln \left( \frac{M_Z^2}{m_t^2} \right) - \frac{25}{12} \ln \left( \frac{M_Z^2}{m_t^2} \right) \right] \right\} \]

\[ = \frac{G_F M_Z^3}{2\pi \sqrt{2}} \left( \frac{1}{4} \right) \left\{ \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 \left[ - \frac{37}{12} + \ln \left( \frac{M_Z^2}{m_t^2} \right) \right] \right. \\
+ \left. \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \left[ -18.6544 + \frac{31}{18} \ln \left( \frac{M_Z^2}{m_t^2} \right) + \frac{23}{12} \ln \left( \frac{M_Z^2}{m_t^2} \right) \right] \right\}, \quad (10) \]

where we separated the two weak coupling structures (not present in Refs. (8, 9)) and used the notation \( g_A^{\text{bot}} = g_A^5 \) and \( g_A^{\text{top}} = g_A^6 \). Here and below \( m_t = m_t(M_Z) \) is the \( \overline{\text{MS}} \) top mass at the scale \( M_Z \). One may relate it to the pole mass through the expression

\[ m_t(M_Z) = m_{\text{pole}} \left\{ 1 - \frac{\alpha_s(M_Z)}{\pi} \left[ \ln \left( \frac{M_Z^2}{m_{\text{pole}}^2} \right) + \frac{4}{3} \right] + O(\alpha_s^2) \right\}, \]

which is known in the NNL approximation [18], or relate it to \( m_t(m_t) \) through the expression

\[ m_t(M_Z) = m_t(m_t) \left\{ 1 - \frac{\alpha_s(M_Z)}{\pi} \ln \left[ \frac{M_Z^2}{m_t^2} \right] + O(\alpha_s^2) \right\}. \]

This would correspondingly modify the coefficients of the \( \alpha_s^3 \) term in Eq. (10) but we prefer to use the \( \overline{\text{MS}} \) top-quark mass, \( m_t(\mu) \) (at \( \mu = M_Z \)), which is the original mass from the QCD Lagrangian.

In the present chapter we present the power-suppressed top-quark mass corrections for both the vector and axial vector contributions. The Feynman diagrams that we have to calculate to obtain the power-suppressed top-quark mass corrections are given in Figs. 1, 2 and 3.
Figure 1. Massive diagrams (= with top-quark loops) contributing to the vector non-singlet part, $\Gamma_{\text{had}}^{V,NS}$ and axial non-singlet part, $\Gamma_{\text{had}}^{A,NS}$. The symbol $\otimes$ is used to indicate an external vertex of the neutral weak vector current for $\Gamma_{\text{had}}^{V,NS}$ and an axial vector vertex for $\Gamma_{\text{had}}^{A,NS}$. It is understood that for each diagram at least one fermion loop has to be a massive top-quark loop, and that a loop containing the external vertices is always a massless quark loop.

Figure 2. Massive diagrams (= with top-quark loops) contributing to the vector singlet part, $\Gamma_{\text{had}}^{V,S}$. The symbol $\otimes$ is used to indicate a vector current vertex. It is understood that for each diagram at least one fermion loop has to be a massive top-quark loop.
3 \( \Gamma_{\text{had}} \) in the order \( \alpha_s^3 \left( \frac{M_Z^2}{m_{\text{top}}^2} \right)^3 \)

It is known that the Appelquist–Carazzone theorem \cite{19} about the decoupling of a heavy particle in quantum field theory does not work in its naive form for the \( \overline{\text{MS}} \) renormalization scheme, and one needs to make an extra shift in the coupling constant \cite{19} to make the decoupling explicit (i.e. to kill the large-mass logarithms) \cite{20, 21}. However, in the presence of an axial vector current the decoupling does not work (even after the shift in the coupling constant) due to the presence of axial anomaly type diagrams — as is the case for the axial vector singlet contribution to \( \Gamma_{\text{had}} \).

Since we have explicitly calculated the top-quark-mass terms in the order \( \alpha_s^3 \) for \( \Gamma_{\text{had}} \) in Ref.\cite{10}, we can derive the decoupling relation for the QCD coupling constant in the NNL order. Because the vector contribution to \( \Gamma_{\text{had}} \) should obey the decoupling mechanism, the use of the decoupling relation should convert the full six-flavour result for \( \Gamma_{\text{had}}^{V,NS} \) to the previously known result in effective massless five-flavour QCD \cite{19}. The NNL order decoupling relation obtained in this way reads

\[
\frac{\alpha_s^{(6)}(\mu)}{\pi} = \frac{\alpha_s^{(5)}(\mu)}{\pi} + \left[ \frac{\alpha_s^{(5)}(\mu)}{\pi} \right]^2 \frac{T_F}{3} \ln \frac{\mu^2}{m_t^2(\mu)} + \left[ \frac{\alpha_s^{(5)}(\mu)}{\pi} \right]^3 \times \left\{ \frac{T_F^2}{9} \ln^2 \frac{\mu^2}{m_t^2(\mu)} + \frac{5C_A T_F - 3C_F T_F}{12} \ln \frac{\mu^2}{m_t^2(\mu)} + \frac{13}{48} T_F C_F - \frac{2}{9} T_F C_A \right\} \quad (11)
\]

where \( \mu \) is the renormalization scale and \( m_t(\mu) \) is the top-quark mass in the \( \overline{\text{MS}} \) scheme. Please note that the term \( 13/48 \ T_F C_F \) that we found is slightly different from the one in Ref.\cite{21}. We therefore checked Eq.\ (11) in an independent way in Ref.\cite{10}.

We will now give the results for \( \Gamma_{\text{had}} \) in effective QCD with five active massless flavours at the renormalization scale \( \mu = M_Z \). These results are obtained by substitution of the decoupling relation, Eq.\ (11), into the results for six-flavour QCD that were obtained in Ref.\cite{10}. We will use the notation \( x \equiv M_Z^2/m_t^2 \).
\[ \Gamma_{\text{had}}^{V,NS} + \Gamma_{\text{had}}^{A,NS} + \Gamma_{\text{had}}^{V,S} + \Gamma_{\text{had}}^{A,S}. \]

\[ \Gamma_{\text{had}}^{V(A),NS} = \frac{G_F M_Z^2}{2\pi \sqrt{2}} \sum_{i=1}^{5} (g_{V(A)})^2 \left[ 1 + \left( \frac{\alpha_s^{(5)}}{\pi} \right) b_1 + \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 b_2 + \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 b_3 + O(\alpha_s^4) \right], \quad (12) \]

\[ b_1 = 1, \]
\[ b_2 = 1.4092 \]
\[ + [0.065185 - 0.014815 \ln(x)] x \]
\[ + [-0.0012311 + 0.00039683 \ln(x)] x^2 \]
\[ + [0.000061327 - 0.000023516 \ln(x)] x^3 + O(x^4), \]

\[ b_3 = -12.767 \]
\[ + [-0.17374 + 0.21242 \ln(x) - 0.037243 \ln^2(x)] x \]
\[ + [-0.0075218 - 0.00058859 \ln(x) + 0.00038305 \ln^2(x)] x^2 \]
\[ + [0.00050411 - 0.00012099 \ln(x) + 0.000031419 \ln^2(x)] x^3 + O(x^4). \]

\[ \Gamma_{\text{had}}^{V,S} = \frac{G_F M_Z^2}{2\pi \sqrt{2}} \left[ \left( \sum_{i=1}^{5} g_{V} \right)^2 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 c_3 + g_{V}^{\text{top}} \left( \sum_{i=1}^{5} g_{V} \right) \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 c_3^{\text{top}} + O(\alpha_s^4) \right], \quad (13) \]

\[ c_3 = -0.41318, \]
\[ c_3^{\text{top}} = 0.027033 x + 0.0036355 x^2 + 0.00058874 x^3 + O(x^4). \]

\[ \Gamma_{\text{had}}^{A,S} = \frac{G_F M_Z^2}{2\pi \sqrt{2}} \left( \frac{1}{4} \right) \left[ \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 d_2 + \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 d_3 + O(\alpha_s^4) \right], \quad (14) \]

\[ d_2 = -3.0833 + \ln(x) + 0.086420 x + 0.0058333 x^2 + 0.00062887 x^3 + O(x^4), \]

\[ d_3 = -18.654 + 1.7222 \ln(x) + 1.9167 \ln^2(x) \]
\[ + [-0.12585 + 0.28646 \ln(x) - 0.011111 \ln^2(x)] x \]
\[ + [-0.0031322 + 0.012117 \ln(x) - 0.0011905 \ln^2(x)] x^2 \]
\[ + [-0.00088827 + 0.00047262 \ln(x) - 0.00017637 \ln^2(x)] x^3 + O(x^4). \]

We want to stress that the results show an excellent convergence of the large top-quark mass expansion, which can be seen from the fast decrease of the coefficients, with the order of the expansion parameter \( x = M_Z^2/m_t^2 \).

Note that the massive logarithms \( \ln \left( \frac{M_Z^2}{m_t^2} \right) \) are present in the leading order of the large-mass expansion in \( \Gamma_{\text{had}}^{A,S} \) only (the violation of decoupling of the top quark). These
logarithms can be summed up using the operator-product-expansion technique to produce a result that is finite in the limit of an infinitely large top-quark mass — as was done in Ref. [6] for the order $\alpha_s^2$. However, for realistic values of $m_t$ the above expressions can be trusted and are quite stable with respect to a change in the renormalization parameter $\mu$ around the natural scale for this process $\mu = M_Z$ (see, for example, Ref. [10]).

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Non-Universal Corrections to the Decay Rate $\Gamma(Z \rightarrow b\bar{b})$

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Abstract

The partial decay rate $\Gamma(Z \rightarrow b\bar{b})$ is significantly influenced by the mass of the top-quark due to electroweak radiative corrections. The leading $\sim m_t^2$ and the next-to-leading contribution $\sim \ln m_t^2$ are known to be numerically of similar size. In this work we calculate the QCD corrections to the logarithmic correction using the heavy top-mass expansion. The $\mathcal{O}(\alpha_s, \ln m_t^2)$ corrections are of the same order as the QCD corrections to the quadratic top-mass term, but of different sign.

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1 Introduction

Although the direct observation of the top quark is out of the range of the experiments at LEP, several observables are affected by the top quark through virtual states in higher-order radiative corrections.

High-precision measurements and the comparison of these quantities with the theoretical predictions allow the extraction of bounds on the top mass. Present analysis of $e^+e^-$ collisions at the $Z$ peak estimates the top mass in the range $m_t = 173^{+12}_{-18}$ GeV [1], which is in agreement with the top masses of $m_t = 174 \pm 10_{-12}^{+13}$ GeV from $\bar{p}p$ collisions at TEVATRON [2]. Of particular interest for deducing the limits on $m_t$ from LEP data is the partial decay rate $\Gamma(Z \rightarrow b\bar{b})$. On the one hand, this quantity exhibits a strong sensitivity on the top mass [3], as the leading term of the electroweak corrections is quadratic in $m_t$ and the next-to-leading logarithmic contribution $\ln m_t^2$ is numerically of the same size. On the other hand, the already small uncertainty of the measurement of $R_{b\bar{b}} = \Gamma(Z \rightarrow b\bar{b})/\Gamma(Z \rightarrow \text{hadrons}) = 0.2192 \pm 0.0018$ [1] is expected to be further reduced in the future. As a consequence, the determination of QCD corrections to the electroweak one-loop result became increasingly important.

For the quadratic top-mass contribution these $\mathcal{O}(\alpha_\alpha, m_t^2)$ corrections were calculated by four independent groups [4, 5, 6, 7]. Here, the QCD corrections of order $\mathcal{O}(\alpha_\alpha, \ln m_t^2)$ to the next-to-leading top-mass contribution are discussed.

Our calculation is performed using dimensional regularization in the $\overline{\text{MS}}$-scheme with anticommuting $\gamma_5$ in the 't Hooft–Feynman gauge. As in our previous work [7], we employ the hard-mass procedure [8, 9, 10, 11] to derive the next-to-leading order in the inverse top-mass expansion. In order to handle large expressions during this expansion we use the algebraic manipulation program FORM [12]. Massless multiloop integrals are evaluated with the help of the software package MINCER [13].

If we consider the properly resummed propagator of the $Z$ boson we get, in the case $q^2 \approx M_Z^2$, for the transverse part,

$$D_{T}^{\mu\nu}(q^2) = -\frac{1}{i} \frac{1}{1 + \Pi(M_Z^2)} \frac{g_{\mu\nu}}{M_Z^2 - q^2 - [i \text{Im} \Pi(M_Z^2) M_Z] / [1 + \Pi(M_Z^2)]},$$

where $M_Z$ is the renormalized $Z$ boson mass and $\Pi(q^2)$ is the transverse part of the polarization tensor $\Pi_{\mu\nu}(q^2)$, defining $\Pi'(M_Z^2)$ by $\text{d Re} \, \Pi(q^2)/dq^2$ evaluated at $q^2 = M_Z^2$. Thus we can write the partial decay rate $\Gamma(Z \rightarrow b\bar{b})$ in the following form

$$\Gamma(Z \rightarrow b\bar{b}) = \left[\Gamma_0 \left(v^2 + a^2\right) + \Delta \Gamma\right] \frac{1}{1 + \Pi(M_Z^2)} ,$$

with $v = -1 + \frac{4}{3} s_W^2$, $a = -1$ and $\Gamma_0 = \alpha M_Z/16 \, c_W^2 s_W^2$. All vertex corrections and the $\gamma Z$-mixing are contained in $\Delta \Gamma$. Expressing the result in terms of $G_F$ (instead of $\alpha$) introduces a factor $(1 - \Delta r)$. The above equation can be written in the form

$$\Gamma(Z \rightarrow b\bar{b}) = \left[\frac{G_F M_Z^3}{8\sqrt{2} \pi} \left(v^2 + a^2\right) + \Delta \Gamma\right] \frac{1 - \Delta r}{1 + \Pi(M_Z^2)} .$$

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The top-mass dependence of the last term is of universal nature. This contribution — as well as the universal part of $\Delta \Gamma$ — can be expressed through the universal $\rho$-parameter \cite{14, 15}. Here, we calculate the non-universal part of order $\mathcal{O}(\alpha_s \ln m_t^2)$.

The outline is as follows: Section 2 contains further details of the calculation and in Section 3 we present the results in the \overline{MS} and OS scheme.

## 2 Calculation of the Order $\mathcal{O}(\alpha_s \ln m_t^2)$

In the decay rate $\Gamma(Z \to b\bar{b})$ the top-mass dependence is due to the appearance of the top-quark as a virtual particle, i.e. it is induced by $t$ to $b$-transitions due to the exchange of charged Higgs ghosts $\Phi^\pm$ or $W$ bosons. One can distinguish between seven classes of diagrams, as listed in Fig. 1, of which the imaginary part has to be calculated. QCD corrections are obtained by attaching gluon lines in all possible ways. This results in 58 topologically-different diagrams.

We perform our calculation in the large top-mass limit $m_t \to \infty$, i.e. we apply an asymptotic expansion in the inverse heavy mass $1/m_t$ \cite{8, 9, 10, 11}. In practice one isolates all possible hard subgraphs and expands them w.r.t. the small masses and (external) momenta. Afterwards this expansion is inserted as an effective vertex into the remaining diagram.

The hard-mass procedure was already used in our previous work \cite{7} to evaluate the $\mathcal{O}(\alpha_s m_t^2)$ corrections. In this order only diagrams with Higgs ghost exchange contributed. The calculation of the next-to-leading $\mathcal{O}(\alpha_s \ln m_t^2)$ corrections is extended in three different ways. First, the diagrams with $W$ boson exchange need to be taken into account. Second, additional hard subgraphs of the Higgs ghost diagrams contribute to the considered order. Last, the subgraphs already considered in \cite{7} must be expanded to the next higher order.

We find that the next-to-leading order results are separately finite for each class of diagrams containing a Higgs ghost. Third-order poles $1/\epsilon^3$ compensate in the sum of all possible hard subgraphs of a diagram and second-order poles $1/\epsilon^2$ cancel after replacing the bare top-mass by the \overline{MS} renormalized one

$$m_t^{\text{bare}} = m_t \left(1 - \frac{\alpha_s}{\pi} \frac{1}{\epsilon}\right).$$

(4)

The remaining simple divergence drops out in the imaginary part of the diagram.

The class of $W$ diagrams and their renormalized contribution $R\Pi^{(W)}(q^2)$ remains to be calculated. Here $R$ denotes the so-called Bogoliubov–Parasiuk $R$-operation \cite{16} adopted to the minimal subtraction scheme \cite{17}. This procedure determines the finite renormalized value of a given regularized Feynman integral. It works on a graph-by-graph basis and removes all subdivergencies together with the overall UV divergence in a way compatible with adding local counterterms to the Lagrangian. We apply this prescription to subtract
all subdivergencies of each $W$ diagram, whereas the overall divergence is again eliminated by taking the imaginary part.

There are several checks for the correctness of our result. The first is the fact that after renormalization the seven classes of diagrams are separately finite.

The second check is the gauge invariance with respect to QCD. The calculation is done with an arbitrary QCD gauge parameter $\xi_S$, introduced through the gluon propagator. As expected, each class is separately independent of $\xi_S$ if the sum of all diagrams is taken.

The third check is the invariance of the gauge parameter $\xi_W$ from the elektroweak theory. $\xi_W$ appears through the propagator of the $W$ boson and the $\Phi^\pm$. The total sum of all diagrams is indeed independent of $\xi_W$.

Figure 1: The seven distinct classes of elektroweak diagrams, which contain the top-quark. Dashed lines: Higgs ghost; wavy (internal) lines: $W$ boson; thin lines: $b$ quark; thick lines: $t$ quark.
3 Results and Discussion

In order to explicitly calculate the $O(\alpha_s, \ln m_t^2)$ correction to the $Z \to b\bar{b}$ vertex, we employ the hard-mass procedure as an expansion up to the next-to-leading order in the $1/m_t$ series. As a consequence, the leading-order result is automatically reproduced. Furthermore, it became necessary to repeat the calculation up to next-to-leading order for the case without QCD corrections (see Ref. [18]) since they induce corrections in first order $\alpha_s$ through top-mass renormalization.

Let us first recall this purely electroweak result for the non-universal decay rate.

$$\Delta \Gamma_{Z\to b\bar{b}}^{\text{non-univ.}} = -\Gamma_0 \frac{G_F}{\sqrt{2} \pi^2} \left(1 - \frac{2}{3} s_W^2 \right) \left\{ \tilde{m}_t^2 + M_Z^2 \ln \frac{M_W^2}{\tilde{m}_t^2} \left( -\frac{17}{6} + \frac{8}{3} s_W^2 \right) \right\}$$

In Eq. (5) the choice $\mu^2 = M_W^2$ was made for the renormalization scale and the result of [3] is reproduced.

The QCD correction for the leading $m_t^2$ term was calculated in [4, 5, 6, 7] and will be given below. For the next-to-leading $m_t$ contribution to the non-universal decay rate we find the $O(\alpha_s)$ correction in the $\overline{\text{MS}}$ scheme

$$\Delta \Gamma_{Z\to b\bar{b}}^{\text{non-univ.}} = -\Gamma_0 \frac{G_F}{\sqrt{2} \pi^2} \left(1 - \frac{2}{3} s_W^2 \right) \frac{\alpha_s}{\pi}$$

$$M_Z^2 \left\{ \ln \frac{M_W^2}{\tilde{m}_t^2} \left( -\frac{17}{6} + \frac{8}{3} s_W^2 \right) + \frac{7}{81} \ln \frac{M_Z^2}{\tilde{m}_t^2} \left( -1 + \frac{2}{3} s_W^2 \right) + \Delta \Gamma_{\text{rem}}^{\overline{\text{MS}}} \right\}.$$  

$\Delta \Gamma_{\text{rem}}$ contains the constant terms of the non-universal diagrams containing the top quark and also terms from graphs without the top quark that we have not calculated.

Let us add some comments concerning the choice of scale as an argument for the running mass and coupling constant.

The leading $O(\alpha_s, m_t^2)$ calculation exhibits a remarkable structure. All integrals factorize and can be classified into two categories, both of which are separately finite and gauge-invariant. The first is characterized by a factorization into a two-loop massive tadpole integral and a massless one-loop p-integral, where the gluon is part of the former. Since this integration is affected by only one mass scale — the top-quark mass — it is instructive to evaluate $\alpha_s$ at the scale $m_t^2$ for these contributions. Contrary to this, the second category comprises all corrections that factorize into a one-loop massive tadpole and a two-loop massless p-integral, with the gluon lines contained in the latter. The only scale in the massless p-integral is the energy regime of the process under consideration, and it seems to be appropriate to calculate $\alpha_s$ at this scale. Finally the scale of the renormalized $\overline{\text{MS}}$ top mass is chosen $\mu^2 = m_t^2$, because the overall $m_t^2$ factor always results from the tadpole integrals.
The leading $m_t^2$ correction can then be written in a factorized form, where both scales are separated according to the above classification and higher orders are neglected. We have not found a similar factorization in the $\ln m_t^2$ term, mainly because the two classes are not separately finite. The only suggestive choice for this contribution therefore is $\mu^2 = M_W^2$, as already employed in Eq. (6). The fact that different choices for the scale can enter the same formula is possible as long as the renormalization group guarantees compensating terms for the change of scale to be of higher and therefore negligible order.

We include also the pure electroweak corrections in our result and transform it via

$$\tilde{m}(\mu^2) = m_{OS} \left[ 1 - \frac{\alpha_s}{\pi} \left( \frac{4}{3} + \ln \frac{\mu^2}{m_{OS}^2} \right) \right] \quad (7)$$

into the OS scheme. In view of the preceding discussion it can be cast into the following form:

$$\Delta \Gamma_{Z \rightarrow b \bar{b}}^{\text{non-univ.}} = -\Gamma_0 \frac{G_F}{\sqrt{2}\pi^2} \left( 1 - \frac{2}{3} s_W^2 \right) \left\{ m_t^2 \left[ 1 - \frac{\pi^2}{3} \frac{\alpha_s(m_t^2)}{\pi} \right] \left[ 1 + \frac{\alpha_s(M_Z^2)}{\pi} \right] + M_Z^2 \ln \frac{M_W^2}{m_t^2} \left[ -\frac{17}{6} + \frac{8}{3} s_W^2 \right] \left[ 1 + \frac{\alpha_s(M_W^2)}{\pi} \right] + M_Z^2 \frac{7}{81} \ln \frac{M_Z^2}{m_t^2} \left[ -1 + \frac{2}{3} s_W^2 \right] \frac{\alpha_s(M_W^2)}{\pi} + \Delta \Gamma_{\text{rem}}^{OS} + O(\alpha_s^2) \right\} \quad (8)$$

The factorized leading $m_t^2$ contribution contains the term $\left[ 1 + \alpha_s(s)/\pi \right]$ which is identical with the familiar pure QCD correction to $\Gamma(Z \rightarrow b \bar{b})$. The following interpretation is at hand. The virtual top quark is of purely electroweak origin and its effect can be accounted for by effective vertices. QCD corrections enter in two ways. On the one hand, they may be part of the effective vertex with the relevant scale being $m_t^2$. On the other, these QCD-corrected vertices are dressed by the gluon, thus being multiplied by the correction factor $\left[ 1 + \alpha_s(s)/\pi \right]$.

The leading $m_t^2$ QCD correction is positive in the OS and negative in the $\overline{\text{MS}}$ scheme. In both cases the corresponding $\ln m_t^2$ term has the opposite sign. The leading order $OS$ result is reduced by about a half. The modulus of the next-to-leading term in the $\overline{\text{MS}}$ scheme is slightly bigger than the $m_t^2$ term, so that the sum is small and positive. In Fig. 2 the leading order and the sum of the leading and next-to-leading order corrections are plotted against the top mass. Our numbers are based on $\Lambda_{QCD} = 235$ MeV corresponding to $\alpha_s(m_t^2 = (174 \text{ GeV})^2) = 0.109$, $\alpha_s(M_Z^2) = 0.120$, $\alpha_s(M_W^2) = 0.123$. As an exemplary value for the $\overline{\text{MS}}$ top mass we have $\tilde{m}_t[m_t^2 = (174 \text{ GeV})^2] = 172.7$ GeV.

It is easy to see that the inclusion of the next-to-leading correction reduces the difference in the predictions between both schemes considerably.

Figure 3 compares the QCD corrections with the pure electroweak result, both in the $\overline{\text{MS}}$ and in the OS scheme. One can recognize that in the $\overline{\text{MS}}$ scheme the corrections3 are numerically tiny ($2.5 \times 10^{-4}$).
The radiative corrections described in this work can be discussed from a different, though equivalent, point of view. They may be summarized by parametrizing the partial $Z$ into $b$ quarks through $\rho = 1 + \Delta \rho$ and $\kappa = 1 + \Delta \kappa$:

$$\Gamma(Z \to b\bar{b}) = \frac{G_F M_Z^2}{24\sqrt{2}\pi} N_C \rho \left[ (1 - \frac{4}{3} \kappa \tilde{s}_W^2)^2 R_V + R_A \right]$$

$$\Delta \rho_{\text{non-univ.}} = -\frac{G_F}{2\sqrt{2}\pi^2} \left\{ m_t^2 \left[ 1 - \frac{\pi^2}{3} \alpha_s(m_t^2) \right] \right. + M_Z^2 \ln \frac{M_W^2}{m_t^2} \left( -\frac{17}{6} + \frac{8}{3} \tilde{s}_W^2 \right) \right. + \frac{\alpha_s(M_W^2)}{\pi} \left. \frac{M_Z^2}{m_t^2} \frac{7}{81} \left( -1 + \frac{2}{3} \tilde{s}_W^2 \right) \right\}$$

$$\Delta \kappa_{\text{non-univ.}} = -\frac{1}{2} \Delta \rho_{\text{non-univ.}}$$

The master Eq. (9) is organized in such a form that pure QCD corrections are easily incorporated. With first-order corrections reading $R_V = R_A = [1 + \alpha_s(M_Z^2)/\pi]$ and apart from neglected higher orders $\mathcal{O}(\alpha_s^3)$, the parameters $\Delta \rho_{\text{non-univ.}}$ and $\Delta \kappa_{\text{non-univ.}}$ correspond to the contributions of Eq. (8), where the overall factor $[1 + \alpha_s(M_Z^2)/\pi]$ has been taken out. Numerical evaluation shows that now the next-to-leading order term $\propto \alpha_s(M_W^2) \ln(M_Z^2/m_t^2)$ of these remaining corrections is only small ($-0.9\%$ for $m_t = 174$ GeV) as compared to the leading order $\propto \alpha_s(m_t^2)m_t^2$.

To conclude, we discussed QCD corrections to the electroweak non-universal result for the partial width $\Gamma(Z \to b\bar{b})$ in the limit of a heavy top quark. We used an expansion in the inverse top mass to calculate the next-to-leading term of order $\mathcal{O}(\alpha_s \ln m_t^2)$. 

Figure 2: Leading order and sum of leading and next-to-leading order QCD corrections.
Figure 3:

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On the QCD Corrections to $\Delta \rho$

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Abstract

We discuss some recent developments in the evaluation of the QCD corrections to $\Delta \rho$, their interpretation, an estimate of the theoretical error, and its effect on electroweak physics.

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The $\rho$ parameter is conventionally defined as the ratio of the effective neutral and charged-current couplings at $q^2 = 0$. Although in the Standard Model (SM) the associated radiative correction $\Delta\rho \equiv 1 - \rho^{-1}$ is process-dependent, its fermionic component, $(\Delta\rho)_f$, is universal. We recall that its contribution, $\rho_f = [1 - (\Delta\rho)_f]^{-1}$, is frequently separated, as an overall renormalization factor, when neutral currents amplitudes are expressed in terms of $G_\mu$, even when one considers amplitudes with $q^2 \sim M^2_Z$. Furthermore, $(\Delta\rho)_f$ is related to the leading asymptotic corrections, for large $m_t$, of the basic radiative corrections $\Delta r$, $\Delta^r$, and $\Delta^r \rho$ [1-2]. For these reasons, it is highly desirable to evaluate $(\Delta\rho)_f$ as accurately as possible, and to obtain an estimate of the theoretical error caused by the unknown higher-order corrections. In particular, the study of the QCD corrections to $(\Delta\rho)_f$ has recently been the subject of considerable attention.

Neglecting higher-order electroweak effects $\sim (\alpha_s^2 s^2)(m_t^2/M_W^2)^2$, but retaining QCD corrections, $(\Delta\rho)_f$ can be written as

$$ (\Delta\rho)_f = \frac{3G_\mu m_t^2}{8\sqrt{2}\pi^2} [1 + \delta_{\text{QCD}}] , $$

where $m_t$ is the pole mass, the first factor is the one-loop result [3], and $\delta_{\text{QCD}}$ represents the relevant QCD correction. The $O(\alpha_s)$ contribution was obtained some time ago by Djouadi and Verzegnassi [4]

$$ \delta_{\text{QCD}} = -\frac{2}{9}(\pi^2 + 3)a(m_t) + ... = -2.860 a(m_t) + ... $$

where $a(\mu) \equiv \alpha_s(\mu)/\pi$. Attempts to go beyond this result have been carried out via two different methods: the dispersive approach, pioneered by Kniehl, Kuhn, and Stuart [5], and the direct examination of the relevant Feynman amplitudes at $q^2 = 0$. The dispersive approach and its comparison with other calculations is reviewed in the contribution of B.A. Kniehl to these Proceedings and in Ref. [6]. The present contribution only discusses recent developments in the Feynman diagram approach, their interpretation, and an estimate of the theoretical error.

Very recently, Avdeev, Fleischer, Mikhailov, and Tarasov [7] carried out a complete three-loop calculation of $O(\alpha_s^2 \alpha_s^2)$. Their result, obtained in the limit $m_b \to 0$, can be expressed to a good accuracy as

$$ \delta_{\text{QCD}} = c_1 a(m_t) + c_2 a^2(m_t) + ... , $$

where $c_1 = -2.860...$ is the Djouadi–Verzegnassi result, and

$$ c_2 = -21.271 + 1.786 \ N_f = -10.55... . $$

In Eq. (4) $N_f = 6$ is the total number of quarks contributing via the vacuum polarization loops (5 massless quarks and the top). The contribution of the top quark is very different from that of the massless quarks and has been split into two parts: one, corresponding to a ‘massless top’ (hence the factor $N_f = 6$) and the remainder, which is included in the first term. As pointed out in Ref. [7], it is also interesting to note that about 40%
of $c_2$ arises from the ‘anomaly-type diagrams’, where the $\ell \gamma^\mu \gamma_5 t$ currents are attached to triangle diagrams linked by two virtual quarks [8].

In order to discuss the evaluation of $\alpha_s(\mu)$ in these expressions, we recall that this parameter conventionally evolves with five active flavours for $m_b < \mu < m_t$, and with six for $\mu > m_t$. At $\mu = m_t$, $\alpha_s(\mu)$ is continuous through $O(\alpha_s^3)$. There is a very small discontinuity, $-(25/72)a^3(m_t)$ [9], but that occurs beyond the order of current $(\Delta \rho)_f$ calculations and, moreover, is negligibly small. For the purpose of our discussion, we can therefore treat $\alpha_s(\mu)$ as being continuous at $\mu = m_t$. From these observations it follows that $\alpha_s(m_t)$ in Eq. (2) should be evaluated conventionally, evolving $\alpha_s(\mu)$ from $\alpha_s(M_Z)$ with a three-loop $\beta$ function and five active flavours. As $(\Delta \rho)_f$ and $m_t$ are physical observables, they are $\mu$-independent. Therefore the same is true of $\delta_{QCD}$. However, because Eq. (3) is truncated in $O(\alpha_s^2)$, its evaluation depends somewhat on the chosen scale. The $\mu$-dependence of Eq. (3) can be studied by using the simple relations

$$
\alpha_s(m_t) = \alpha_s^{(5)}(\mu) \left[ 1 - \beta_1^{(5)} \frac{\alpha_s^{(5)}(\mu)}{\pi} \ln \left( \frac{\mu}{m_t} \right) \right] + \ldots
= \alpha_s^{(6)}(\mu) \left[ 1 - \beta_1^{(6)} \frac{\alpha_s^{(6)}(\mu)}{\pi} \ln \left( \frac{\mu}{m_t} \right) \right] + \ldots ,
$$

(5)

where $\beta_1^{(n_f)} = -\frac{1}{2}(11 - \frac{2n_f}{3})$, the ellipses stand for higher-order terms, and the superscript $n_f$ in $\alpha_s^{(n_f)}$ and $\beta_1^{(n_f)}$ represents the number of active flavours in the evaluation of $\alpha_s(\mu)$. Equation (5) follows from the fact that $\alpha_s^{(5)}(\mu)$ and $\alpha_s^{(6)}(\mu)$ satisfy the RG equations with $n_f = 5$ and $n_f = 6$, respectively, and from the above-mentioned continuity at $\mu = m_t$. Some authors employ $\alpha_s^{(6)}(\mu)$ for $\mu < m_t$ as well as for $\mu > m_t$. That choice, although theoretically acceptable, is inconvenient because the experimental value of $\alpha_s(M_Z)$ should be identified with $\alpha_s^{(5)}(M_Z)$ rather than $\alpha_s^{(6)}(M_Z)$. Therefore, it is natural to use the expression involving $\alpha_s^{(5)}(\mu)$ in Eq. (5) when $m_b < \mu < m_t$, and that in terms of $\alpha_s^{(6)}(\mu)$ for $\mu > m_t$. For clarity, the following related observations are relevant: i) The suggestion has been made by some theorists that, when replacing $\alpha_s(m_t)$ by the r.h.s. of Eq. (5) with $n_f = 5$ and $\mu < m_t$, one should also change $N_f$ from 6 to 5 in the $\mu$-independent coefficient $c_2$. We see, however, that such a procedure is theoretically inconsistent. While the shift from $n_f = 6$ to $n_f = 5$ in Eq. (5) is re-absorbed in a redefinition of $\alpha_s(\mu)$ in a $\mu$-independent manner, a change of $N_f$ in Eq. (4) for $\mu < m_t$ makes $\delta_{QCD}$ $\mu$-dependent in $O(\alpha_s^2)$. This contradicts the basic requirement that $\delta_{QCD}$ should be $\mu$-independent through the order of the calculation. Or, putting this in a slightly different language, $m_t$ is $\mu$-independent and, unlike $m_t(\mu)$, it cannot be adjusted to absorb a $\mu$-dependent contribution from $\delta_{QCD}$. ii) Some theorists evaluate $\alpha_s(m_t)$ by evolving $\alpha_s^{(6)}(\mu)$ from $\alpha_s(M_Z)$. As $\alpha_s(M_Z)$ should be identified with $\alpha_s^{(5)}(M_Z)$, we see from Eq. (5) that such an approach introduces a small but unnecessary error.

In this report we follow the above observations: in the range $m_b < \mu < m_t$, $\alpha_s(\mu)$ is evaluated by evolving $\alpha_s(\mu)$ from $\alpha_s(M_Z)$ with a three-loop $\beta$ function and five active flavours. For definiteness, we take $\alpha_s(M_Z) = 0.118$ and $M_Z = 91.19$ GeV, and adjust $\Lambda_{\overline{MS}}^{(6)}$ accordingly. In order to discuss the $\mu$-dependence of the truncated series or to optimize

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[2] See, for example, Eq. (A.2) of Ref. [10].
the perturbative expansions, we employ Eq. (5), choosing the five-flavours expression in the domain \( m_b < \mu < m_t \). As an illustration, for \( m_t = 200 \text{ GeV} \), Eq. (3) leads to \( \delta_{\text{QCD}} = -0.0961 - 0.0119 = -0.1080 \), so that the \( O(\alpha_s^3) \) term implies an enhancement \( 0.0119/0.0961 = 12.4\% \) of the leading QCD result. If \( \alpha_s(m_t) \) in Eq. (3) is expressed in terms of \( \alpha_s(\mu) \), and the resulting series truncated in \( O(\alpha_s^2) \), for \( 0.1 \leq \mu/m_t \leq 1 \) we find a variation \( \leq 5.8 \times 10^{-3} \), a bound that amounts to 5.4\% of the total QCD correction or 49\% of the \( O(\alpha_s^2) \) term. It is difficult, however, to make a precise statement of the error based on these considerations because the \( \mu \)-dependence of the truncated series depends on the chosen interval. For example, in this case the variation is significantly smaller in the interval \( 0.15 < \mu/m_t < 1 \), and significantly larger in the range \( 0.075 < \mu/m_t < 1 \). Furthermore, the mildness of the \( \mu \)-dependence of the first \( N \) terms of a QCD expansion over an interval is consistent with but does not imply the smallness of the higher-order terms. The point is that, for example, the \( \mu \)-dependence induced by a significantly large \( O(\alpha_s^{N+1}) \) term is of \( O(\alpha_s^{N+2}) \) and may be cancelled by sizable contributions of that and higher orders. The application of the Brodsky–Lepage–Mackenzie (BLM) [11], Principle of Minimal Sensitivity (PMS) [12], and Fastest Apparent Convergence (FAC) [13] methods to optimize the scale in Eq. (3) is discussed later on.

On the other hand, we note that the expansion in Eq. (3) involves rather large and increasing coefficients, a feature that frequently indicates significant higher-order terms. Furthermore, arguments advanced in Ref. [14] suggest that there are at least two scales in \( \delta_{\text{QCD}} \): one, of \( O(m_t) \), associated with the intrinsic corrections to the electroweak amplitude, and another one, much smaller, related to contributions to the pole mass \( m_t \) involving small gluon momenta. It is therefore a good idea to find alternative expressions for \( \delta_{\text{QCD}} \) that separate the two scales, and at the same time involve terms of \( O(\alpha_s^2) \) with coefficients of \( O(1) \) rather than \( O(10) \). The advantage of this strategy is explained later on, when we discuss the theoretical error. A simple way of implementing this idea has been outlined in Ref. [15]: one expresses first \((\Delta \rho)_f \) in terms of \( \hat{m}_t(m_t) \), the running \( \overline{\text{MS}} \) mass evaluated at the pole mass, and then relates \( \hat{m}_t(m_t) \) to \( m_t \) by optimizing the expansion for \( m_t/\hat{m}_t(m_t) \), which is known through \( O(\alpha_s^2) \). Recently these arguments have been considerably refined [16] using the new results of Ref. [7].

Calling \( \mu_t \) the solution of \( \hat{m}_t(\mu) = \mu \) and using Eq. (19) of Ref. [7], we have

\[
(\Delta \rho)_f = \frac{3G_{\mu t}^2}{8\sqrt{2\pi^2}} [1 + \delta_{\text{QCD}}^{\overline{\text{MS}}}] ,
\]

\[
\delta_{\text{QCD}}^{\overline{\text{MS}}} = -0.19325 a(\mu_t) + 0.07111 a^2(\mu_t) .
\]

We note that the convergence pattern of Eq. (7) is very nice, with very small leading and next-to-leading coefficients. For this reason we will assume that the terms of \( O(\alpha_s^2) \) and higher are negligible in Eq. (6) and evaluate Eq. (7) with \( \mu_t \rightarrow m_t \) as this introduces only a small change of \( O(\alpha_s^3) \).

We can now express \((\Delta \rho)_f \) in terms of \( \hat{m}_t(m_t) \), by using the NLO expansion [16]

\[
\frac{\mu_t}{\hat{m}_t(m_t)} = 1 + \frac{8}{3} a^2(m_t) + [35.96 - 2.45 n_f] a^3(m_t) ,
\]

\[
\mu_t = \hat{m}_t(m_t) .
\]
where \( n_f = 5 \) is the number of light flavours. Defining \( \Delta_{QCD} \) by

\[
(\Delta \rho)_t = \frac{3G_F \hat{m}_t^2(m_t)}{8\sqrt{2\pi}^2} \left[ 1 + \Delta_{QCD} \right],
\]

we have

\[
1 + \Delta_{QCD} = \left[ \frac{\mu_t}{\hat{m}_t(m_t)} \right]^2 \left[ 1 + \delta_{\overline{MS}}^{QCD} \right].
\]

Neglecting terms of \( O(\alpha_s^3) \), we can combine Eqs. (6)–(10) in a single expansion:

\[
\Delta_{QCD} = -0.19325 \ a(m_t) + C \ a^2(m_t).
\]

In Ref. [15], Eq. (11) was proposed before the results of Ref. [7] became known, and it was argued, on the basis of convergence assumptions, that \( |C| \leq 6 \). From Eqs. (7)–(8) we see that \( C = 16/3 + 0.071 = 5.40 \), consistent with the arguments of Ref. [15]. On the other hand, with \( C = 5.40 \) the two terms in Eq. (11) nearly cancel and the error estimate is unnecessarily large. It is clearly better to insert in Eq. (10) the separate expansions in Eqs. (7) and (8), as the magnitude of the last terms in these expressions is significantly smaller than in Eq. (11). In the case of Eq. (8) we retain the relatively large \( O(\alpha_s^3) \) contribution in order to control the scale of the leading term. As its coefficient, 23.71, is rather large, we apply the BLM optimization procedure [15] to Eq. (8), and obtain

\[
\frac{\mu_t}{\hat{m}_t(m_t)} = 1 + \frac{8}{3} \ a^2(0.252 \ m_t) - 4.47 \ a^3(0.252 \ m_t) \quad \text{[BLM]}.\]

As an illustration, for \( m_t = 200 \) GeV, Eqs. (8) and (12) give \( \mu_t/\hat{m}_t(m_t) = 1.00391 \) and 1.00423, respectively. The optimization of Eq. (8) using the PMS [12] and the FAC [13] methods is very close numerically to Eq. (12) (the difference is \( \leq 5 \times 10^{-6} \)).

Values for \( \delta_{\overline{MS}}^{QCD} \) [Eq. (7) with \( \mu_t \rightarrow m_t \)] and \( \Delta_{QCD} \) [evaluated via Eqs. (10), (7) and (12)] are shown in Table 1. We see that these are indeed small effects. In particular, \( \Delta_{QCD} = (2 - 3) \times 10^{-3} \), depending on \( m_t \), a remarkably small correction. Comparing Eqs. (1) and (9), we find

\[
1 + \delta_{QCD} = \left[ \frac{\hat{m}_t(m_t)}{m_t} \right]^2 \left[ 1 + \Delta_{QCD} \right].
\]

For \( m_t/\hat{m}_t(m_t) \) we have the well-known expansion [17]

\[
\frac{m_t}{\hat{m}_t(m_t)} = 1 + \frac{4}{3} \ a(m_t) + K \ a^2(m_t).
\]

When neglecting the masses of the first five flavours, \( K \) is given by [9], [18]

\[
K = 16.0065 - n_f \ 1.0414 + 0.1036,
\]

\(^3\text{In Ref. [18] } C \text{ was also estimated to be } \approx +3 \text{ by optimization arguments, but the more conservative value } C = 0.36 \text{ was employed in the final analysis.}\)
where the first term corresponds to the quenched approximation, and the second and third are the vacuum polarization contributions of the massless quarks ($n_f = 5$) and the top quark, respectively. As expected, the latter is much smaller. References [7], [17] include the contribution of all flavours to the vacuum polarization and, therefore, in those calculations $\hat{m}_t(\mu)\equiv m_t$ evolves with six active quarks in the $\beta$ and $\gamma$ coefficients. The same is true of $\hat{m}_t(\mu_i) = m_t$ in Eq. (6) and $\hat{m}_t(m_i)$ in Eq. (14). Equivalently one can express these relations in terms of $\hat{m}_t(\mu)$ and $\hat{m}_t(m_i)$ evaluated with five active flavours [9]. Through terms of $O(\alpha_s^2)$ this is simply done by ‘decoupling’, i.e. subtracting the small top contribution 0.1036 from $K$, and $2 \times 0.1036 \approx 0.21$ from the coefficient of $a^2$ in Eq. (7). These changes are very small and, moreover, they are not necessary for our purposes. As our aim is to combine Eqs. (6) and (14), both evaluated with six flavours, we have defined in the same way $\mu_t$ and $\hat{m}_t(m_i)$ in Eq. (8) et seq., without decoupling the small top quark contribution.

When Eq. (14) is inserted in Eq. (13), it induces a contribution $\approx -0.11$ to $\delta_{QCD}$ while, as we saw before, $\Delta_{QCD} \approx 0.002$. Thus, we find the intriguing result that, when $(\Delta\rho)_t$ is expressed in terms of $m_t$, the correction is almost entirely contained in $[\hat{m}_t(m_t)/m_t]^2$, a pure QCD effect that can be studied in isolation from electroweak physics. Once this is recognized, it becomes clear that the magnitude and error of $\delta_{QCD}$ are largely controlled by the value of $[\hat{m}_t(m_t)/m_t]^2$ and the accuracy with which it can be calculated. As $K$ is quite large (10.90 for $n_f = 5$), it is natural to apply the three well-known approaches [11]--[13] to optimize the scale in Eq. (14). Given an $O(\alpha_s^2)$ expansion of the form $S = 1 + A a(m_t) + (B - C n_f) a^2(m_t)$, where $A$, $B$, and $C$ are numbers, the application of these methods lead to the optimized expansions

$$S = 1 + A a(\mu^*) + \left[ B - \frac{33}{2} C \right] a^2(\mu^*) \quad \text{[BLM]},$$

$$S = 1 + A a(\mu^{**}) - \frac{A\beta_2}{2\beta_1} a^2(\mu^{**}) \quad \text{[PMS]},$$

$$S = 1 + A a(\mu^{***}) \quad \text{[FAC]},$$

where $\beta_2 = -[51 - 19 n_f/3]/4$, $\mu^* = e^{-3C/A} m_t$, $\mu^{**} = e^{(B-C n_f)/A\beta_1} m_t$, and $\mu^{***} = e^{(\beta_2/2\beta_1)} \mu^{**}$. In deriving these expressions we have assumed that $\alpha_s$ evolves with $n_f$ active flavours between $m_t$ and the optimized scales. To obtain Eq. (17) we have employed the two-loop RG differential operator to find out, to a good accuracy, the stationary point. This can be checked by inserting Eq. (5) in the r.h.s. of Eq. (14) and evaluating numerically the series, truncated in $O(\alpha_s^2)$, as a function of $\mu$. Applying Eqs. (16)--(18) to Eq. (14),

$$\frac{m_t}{\hat{m}_t(m_t)} = 1 + \frac{4}{3} a(\mu^*) - 1.07 a^2(\mu^*) \quad \text{[BLM]},$$

\[^4\]It is worth emphasizing that even this small decoupling subtraction should not be made when discussing the $\mu$-dependence of Eq. (1) because, unlike $\hat{m}_t(\mu)$, $m_t$ is $\mu$-independent.

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where \( \mu^* = 0.0960 \) \( m_t \), \( \mu^{**} = 0.1005 \) \( m_t \), and \( \mu^{***} = 0.1185 \) \( m_t \). In these expressions \( \alpha_i \) is evaluated with \( n_f = 5 \). As the optimization scales are in the \( m_b-m_t \) range, this is consistent with the discussion after Eq. (5). For \( m_t = 200 \) GeV, Eqs. (19)–(21) give \( m_t/\hat{m}_t(m_t) = 1.06304, 1.06303, \) and 1.06297, respectively. For \( m_t = 174 \) GeV, the corresponding values are 1.06477, 1.06477, and 1.06470. Finally, for \( m_t = 130 \) GeV, we have 1.06875, 1.06875, and 1.06867. Thus, the three approaches give expansions with similar scales, coefficients of \( O(1) \), and remarkably close values. In contrast, the expansion in Eq. (14), which involves a large second-order coefficient of \( O(10) \), gives 1.0571, 1.0584, and 1.0614 for \( m_t = (200, 174, 130) \) GeV, which are (0.6–0.7)% smaller. One must conclude that the coefficients of the unknown terms of \( O(a^3) \) and higher in Eq. (14) and/or Eqs. (19)–(21) are large. For instance, if Eq. (19) were exact, the coefficient of the \( a^3(m_t) \) and \( a^4(m_t) \) terms in Eq. (14) would be \( \approx 104 \) and 1041, respectively.\(^5\) In the following we employ the optimized expression for \( m_t/\hat{m}_t(m_t) \), which, for definiteness, we identify with the BLM expansion [Eq. (19)]. The advantage of this procedure will be explained later on, when we discuss the theoretical error.

Table 2 displays the values of \( \delta_{\text{QCD}} \) obtained from Eq. (13), using Eq. (19) and our previous determination of \( \Delta_{\text{QCD}} \) [Eq. (10)], and compares them with those derived from Eqs. (3) and (4). In order to show the effect of the higher-order contributions (H.O.C.), we also exhibit the fractional enhancement of the total QCD correction over the conventional \( O(a) \) result [Eq. (2)]. From Table 2 we see that, for \( 130 \) GeV \( \leq m_t \leq 220 \) GeV, the evaluation of \( [\delta_{\text{QCD}}] \) from Eq. (13) leads to a (18–20)% enhancement over Eq. (2) and is \( (5.2–6.6) \times 10^{-3} \) larger than the results from Eq. (3). As a percentage of the total QCD correction this difference amounts to \( \approx 5\% \), a reasonably small effect. However, the last two columns in Table 2 show that the H.O.C. are \( \approx 45\% \) larger in the evaluation based on Eq. (13). Most of this divergence can be traced to different ways of treating the dominant factor \( [\hat{m}_t(m_t)/m_t]^2 \) in Eq. (13). For example, it is amusing to note that the expansion \( [\hat{m}_t(m_t)/m_t]^2 - 1 = -8/3a(m_t) - (2K - 16/3)a^2(m_t) \), obtained from Eq. (14), is quite close numerically to Eqs. (3) and (4). For \( m_t = (200, 174, 130) \) GeV it gives \(-0.1081, -0.1106, \) and \(-0.1160, \) to be compared with \(-0.1080, -0.1102, \) and \(-0.1154 \) from Eqs. (3) and (4) (see Table 2). This conforms with our observation that the bulk of the QCD correction to \( (\Delta \rho) \) is contained in \( [\hat{m}_t(m_t)/m_t]^2 \), but at the same time shows the main source of the difference between the two calculations. In fact, in the evaluation of Eq. (13) we have employed Eq. (19), the optimized version of Eq. (14), rather than the expansion mentioned above.

As \( c_2 \) in Eq. (4) is \( O(10) \), in analogy with our discussion of Eq. (14), we may directly attempt to optimize Eqs. (3) and (4). Applying Eqs. (19)–(21) to Eqs. (3) and (4) with \( N_f = n_f + 1 = 6 \), we have

\[
\delta_{\text{QCD}} = -2.860 \, a(0.154 \, m_t) + 9.99 \, a(0.154 \, m_t) \quad [\text{Eq. (3); BML}],
\]

\(^5\)For recent applications of the BLM, PMS, and FAC approaches to estimate higher-order coefficients in other cases, see Refs. [19,20].
\[ \delta_{\text{QCD}} = -2.860 \, a(0.324 \, m_t) + 1.80 \, a^2(0.324 \, m_t) \quad \text{[Eq. (3); PMS]}, \quad (23) \]

\[ \delta_{\text{QCD}} = -2.860 \, a(0.382 \, m_t) \quad \text{[Eq. (3); FAC]}. \quad (24) \]

Once more, in Eqs. (22)–(24) \( \alpha \) is evaluated with 5 active flavours, in accordance with the discussion after Eq. (5). For \( m_t = 200 \, \text{GeV} \), Eqs. (22)–(24) give \(-0.1084\), \(-0.11045\), and \(-0.11038\), respectively. The difference between Eqs. (23), (24) and Eq. (13) is now only \( \approx 2.9 \times 10^{-3} \), which is \( \approx 45\% \) smaller than that between Eq. (3) and Eq. (13). On the other hand, although the \( a^2 \) coefficient in Eq. (23) is roughly of \( O(1) \), that in Eq. (22) is \( O(10) \). It has been pointed out that in many NLO QCD calculations, it is a good approximation to retain only the leading term, evaluated at the BLM scale \([11,14]\). From Eq. (22) we see that this is not the case in the expansion of Eqs. (3) and (4). Thus, in contrast to Eqs. (19)–(21), when applied to Eqs. (3) and (4) the three optimization procedures do not uniformly lead to coefficients of \( O(1) \) and similar scales.

A frequently used method to estimate the theoretical error of QCD expansions is to consider the magnitude of the last included or known term \([21]\). By this criterion the dominant error in Eq. (13) is contained in the first factor. We employ Eq. (19) as it is the optimized expansion with the largest second-order term. The corresponding error is

\[ \delta \left[ \frac{\hat{m}_t(m_t)}{m_t} \right]^2 \approx \pm \frac{2 \times 1.07 \, a^2(\mu^*)}{[m_t/\hat{m}_t(m_t)]^3} \approx \pm 1.77 \, a^2(\mu^*). \quad (25) \]

For \( m_t = (200,174,130) \, \text{GeV} \), this amounts to \( \pm(4.3,4.5,5.1) \times 10^{-3} \). The calculation of the small correction \( \Delta_{\text{QCD}} \) involves in turn two uncertainties. One, associated with \( [\mu_t/\hat{m}_t(m_t)]^2 \), can be estimated from Eq. (12) as

\[ \delta \left[ [\mu_t/\hat{m}_t(m_t)]^2 \right] \approx \pm 8.9 \, a^3(0.252 \, m_t), \quad (26) \]

and leads to \( \pm(6,7,8) \times 10^{-4} \). The other, involving \( \delta_{\text{QCD}}^{\text{MS}} \), is given by

\[ \delta(\delta_{\text{QCD}}^{\text{MS}}) \approx \pm 0.071 \, a^2( \, m_t), \quad (27) \]

and amounts to \( \pm(8,8,9) \times 10^{-5} \). In Eq. (13) these two errors are decreased by a factor \( [\hat{m}_t(m_t)/m_t]^2 \approx 0.88 \). Adding the three uncertainties linearly we get an overall error estimate for Eq. (13) due to higher-order corrections:

\[ \delta(\delta_{\text{QCD}}) = \pm(4.9,5.2,5.9) \times 10^{-3} \quad \text{[Eq. (13)]}, \quad (28) \]

for \( m_t = 200,174, \) and \( 130 \, \text{GeV} \), respectively. It could be argued that the smallness of the leading and NLO coefficients in Eq. (7) is fortuitous. However, in order to lead to a contribution \( \approx 2.5 \times 10^{-3} \) for \( m_t = 200 \, \text{GeV} \), which would modify our error estimate by 50\%, the coefficient of \( a^3(m_t) \) should be \( \approx 66 \), or 930 times the coefficient of \( a^2(m_t) \). If the same ‘last term’ criterion were applied to Eq. (3), the error estimate would be

\[ \text{Ref. [7]} \] the FAC scale is reported to be \( 0.348 \, m_t \). The apparent difference with Eq. (24) is due to the fact that \( \alpha_s^{(6)} \) is employed in that paper. In fact, both results are equivalent through \( O(\alpha_s^2) \).
The consideration of the optimized expansions is very ambiguous in the case of Eqs. (3, 4); while the PMS method [Eq. (23)] leads to a small error estimate, the BLM approach [Eq. (22)] gives a very large uncertainty \( \approx \pm (2.0-2.4) \times 10^{-2} \). This ambiguity may perhaps be related to the observation that the expansion in Eq. (3) involves more than one scale.

We note that the two different evaluations of \( \delta_{\text{QCD}} \), by means of Eqs. (3), (4) and Eqs. (13), (10) and (19), respectively, coincide through \( O(\alpha_s^2(m_t)) \). The numerical difference between the two means that at least in one of these calculations there are significant contributions of \( O(a^3) \) and higher. Although the present author prefers the latter approach, on the grounds that it involves expansions with coefficients of \( O(1) \) and leads by the ‘last term criterion’ to a significantly smaller error estimate, it seems impossible to rigorously decide at present which is more accurate. This could be clarified, in principle, by evaluating the \( O(\alpha_s^3) \) terms in Eq. (3) and/or Eq. (14). Prospects for achieving this, however, appear to be quite remote [18]. On the other hand, we believe that Eq. (28) is a reasonable estimate of the theoretical error due to unknown higher-order corrections. In particular, we note that Eq. (28) is also very close to the difference between the two \( \delta_{\text{QCD}} \) evaluations, which can also be used as an estimate of the theoretical error [16], and to the scale variation of Eq. (3) in the interval \( 0.1 < \mu/m_t < 1 \). Finally, one should remember that there is at present an additional 5\% error in \( \delta_{\text{QCD}} \) associated with the \( \pm 0.006 \) uncertainty in \( \alpha_s(M_Z) \). This leads to an additional contribution of \( \pm (5.8, 5.9, 6.2) \times 10^{-3} \) to \( \delta(\delta_{\text{QCD}}) \), for \( m_t = (200, 174, 130) \) GeV. In contrast to the theoretical error in Eq. (28), it is likely that this uncertainty can be decreased in the near future.

As pointed out in Ref. [16], the values for \( \delta_{\text{QCD}} \) given in Table 2 can accurately be represented by simple empirical formulae. We find that the calculation based on Eq. (13) (third column of Table 2) and the error estimate of Eq. (28) due to higher-order corrections can be conveniently expressed as

\[
\delta_{\text{QCD}} = -2.860 \, a(\xi m_t) ,
\]

\[
\xi = 0.321^{+0.110}_{-0.072} \quad \text{[Eqs. (13), (28)]} ,
\]

while Eq. (3) (second column of Table 2) corresponds [16] to \( \xi = 0.444 \). We emphasize that Eqs. (29) and (30) are not the result of a FAC optimization. They are simply heuristic formulae that reproduce the values in the tables with errors of at most \( \pm 1 \times 10^{-4} \) for \( \xi = 0.321 \) and \( \xi = 0.431 \), and of at most \( 3 \times 10^{-4} \) for \( \xi = 0.248 \). We also emphasize that in these expressions \( \alpha_s \) is evaluated with five active flavours in the manner explained before. We see that Eq. (30) is somewhat smaller than the effective scale associated with Eq. (3); however, both evaluations are roughly consistent within our theoretical error estimate. By a numerical coincidence, the central value in Eq. (30) is very close to the PMS scale in Eq. (23); however, Eqs. (29) and (23) differ somewhat because of the presence of the \( O(a^2) \) term in the latter.

In summary, in this report we have emphasized a number of results and observations: i) Working in the NLO approximation, we have pointed out that \( \Delta_{\text{QCD}} \), the QCD correction when \( (\Delta \rho)_f \) is expressed in terms of \( m_t(m_t) \), is remarkably small,
\[
\approx (2-3) \times 10^{-3}. \text{ This means that precision electroweak physics essentially measures this parameter, an important input for GUTs studies. The evaluation of } \Delta_{\text{QCD}} \text{ and } \delta_{\text{QCD}}^{\overline{\text{MS}}} \text{ shows that } \hat{m}_t(m_t) \text{ and } \mu_t \equiv \hat{m}_t(m_t) \text{ are very good expansion parameters as they absorb the bulk of the QCD corrections} \text{[22]. Arguments for these desirable properties from effective field theory have been given by Peris \text{[23]. If the smallness of } \Delta_{\text{QCD}} \text{ persists beyond the NLO approximation, it also means that } \delta_{\text{QCD}}, \text{ the QCD correction when } (\Delta \rho)_t \text{ is expressed in terms of } m_t, \text{ is almost entirely contained in } [\hat{m}_t(m_t)/m_t]^2, \text{ a pure QCD effect that can be studied in isolation from electroweak physics. ii) We have also pointed out that when the BLM, PMS, and FAC optimization methods are applied to the expansion for } m_t/\hat{m}_t(m_t), \text{ they lead to similar scales, coefficients of } O(1), \text{ and remarkably close values. Combining these optimized expansions with } \Delta_{\text{QCD}}, \text{ one obtains an evaluation of } \delta_{\text{QCD}} \text{ which shows an } (18-20)\% \text{ enhancement over the two-loop calculation, depending on } m_t. \text{ Using the magnitude of the last included terms as an estimate of the theoretical error due to higher-order corrections, this analysis leads to } \delta(\delta_{\text{QCD}}) \approx \pm(5-6) \times 10^{-3}, \text{ depending on } m_t. \text{ This estimate seems reasonable, as it is also close to the difference between the } \delta_{\text{QCD}} \text{ evaluation mentioned above and that obtained from the expansion proposed by Avdeev et al. If the theoretical error is combined quadratically with that arising from } \delta_{\alpha}, \text{ one obtains an overall uncertainty in } \delta_{\text{QCD}} + \delta_{\alpha} \text{ of } \pm(7.6, 7.9, 8.6) \times 10^{-3} \text{ for } m_t = (200, 174, 130) \text{ GeV. For } m_t = 200 \text{ GeV and } M_H = 300 \text{ GeV, this induces errors } \pm 9.5 \times 10^{-5}, \pm 3.1 \times 10^{-4} \text{ in } \Delta \rho, \pm 3.2 \times 10^{-5} \text{ in the } \overline{\text{MS}} \text{ parameter } \sin^2 \theta_W(M_Z), \text{ and } \pm 840 \text{ MeV and } \pm 5.5 \text{ MeV in the predicted masses of } m_t \text{ and } M_W, \text{ respectively. We also recall that these results scale approximately as } m_t^2, \text{ so that they are actually smaller for } m_t < 200 \text{ GeV. Thus the effect of these errors in electroweak physics is rather mild. It should also be remembered that the concept of pole mass has an intrinsic uncertainty } \sim \Lambda_{\text{QCD}} \text{ and, for the top quark, this may amount to } 200-300 \text{ MeV} \text{[24].}
\]

**Acknowledgements**

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**Note added in proof**

After this analysis was completed, two different groups (see Refs. [25, 26]), reported a significant modification of Eq. (4). The new value is \( c_2 = -23.525 + 1.7862 n_f = -14.594 \). This implies that \( 0.07111 - 3.970 \text{ in Eq. (7) and } C = 1.364 \text{ in Eq. (11), rather than 5.40. Here, we briefly update our analysis and comment on the main implications. In our approach, the correction } \Delta_{\text{QCD}} \text{ is now given by}

\[
\Delta_{\text{QCD}} = -0.19325 a(m_t) - 3.970a^2(m_t) + \frac{16}{3}a^2(0.252m_t) - 9.97a^3(0.252m_t), \tag{31}
\]

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which follows by combining Eq. (12) with the modified Eq. (7). The smaller scale in
the last two terms reflect the incorporation of large $O(a^3)$ terms which arise from the
occurrence of the pole mass in $m_t(m_t)$ — compare with Eq.(8). The new values of $\Delta_{QCD}$
are still very small, but of the opposite sign to those reported in Table 1. As an example,
for $m_t = (220, 174, 130)$GeV they amount to $(-2.64, -2.50, -2.27) \times 10^{-3}$. Inserting the
previous values of $m_t/m_t$ from Eq. (19) and the revised values of $\Delta_{QCD}$ into Eq. (13),
we obtain the new values of $\delta_{QCD}$ that replace those reported in the third column of Table
2. To illustrate: for the previous $m_t$ entries, they amount to $-0.1155, -0.1202, -0.1265.
Our error estimates, based on the magnitude of the last terms in Eqs. (19) and (31),
happen to coincide numerically with those previously reported in Eq. (28). The empirical
formula that summarizes our approach is now given by

$$\delta_{QCD} = -2.8599a(\xi m_t),$$

with $\xi = 0.260^{+0.079}_{-0.059}$. This is to be compared with $\xi = 0.339$, a scale that reproduces the
numerical values from Eq. (3) with the revised $c_2$ coefficient.

It is interesting to point out that: i) the results of the revised $a(m_t)$ expansion —
Eq.(3) with $c_2 = -14.594$ — lie almost exactly at the lower boundary (in absolute value)
of the band derived from our central values and error estimates; ii) the results obtained by
directly applying the three optimization methods to the revised expansion in Eq.(3) are
now close among themselves, and also lie near our central values. As an illustration, for
$m_t = 174$GeV, the three optimizations of Eq. (3) give $-0.1193, -0.1199, -0.1199$, while
our central value is $-0.1202$. On the other hand, the revised Eq.(3) gives $-0.1150$. The
new value $C = 1.364$ in Eq. (11) and correspondingly $c_2 = -14.594$ in Eq. (3) is very
close to the central value of the estimate $C = 0 \pm 6$ (correspondingly, $c_2 = -15.958 \pm 6$),
given in Ref. [15] before the complete $O(\alpha/\alpha_s^2)$ corrections became available.

References


at the Tennessee International Symposium on ‘Radiative Corrections: Status and


ph/9401357.


[18] Private communication by D.J. Broadhurst.


(1991) 144, and Ref. [19].

[22] The observation, on the basis of $O(\alpha_s)$ calculations, that $m_t(m_t)$ and $\tilde{m}_t(\mu_t)$ are
very good expansion parameters has been made by a number of authors. See,
for example, F. Jegerlehner, in ‘Progress in Particle and Nuclear Physics’, edited
by A. Faessler (Pergamon, Oxford, 1991); W.J. Marciano, BNL-49236 (1993);
MPI-Ph/94-11, TUM-T31-60/94.


The corrections $\delta_{QCD}^{\text{MS}}$ and $\Delta_{QCD}$. The first one is given by Eq. (7) with $\mu_t \to m_t$, while the second is obtained from Eq. (10), with $\mu_t/m_t(m_t)$ evaluated according to Eq. (12) [$\alpha_s(M_Z) = 0.118$ is employed].

<table>
<thead>
<tr>
<th>$m_t$ [GeV]</th>
<th>$10^3 \delta_{QCD}^{\text{MS}}$</th>
<th>$10^3 \Delta_{QCD}$</th>
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</thead>
<tbody>
<tr>
<td>130</td>
<td>-6.80</td>
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</tr>
<tr>
<td>150</td>
<td>-6.67</td>
<td>2.59</td>
</tr>
<tr>
<td>174</td>
<td>-6.53</td>
<td>2.28</td>
</tr>
<tr>
<td>200</td>
<td>-6.41</td>
<td>2.01</td>
</tr>
<tr>
<td>220</td>
<td>-6.33</td>
<td>1.84</td>
</tr>
</tbody>
</table>

The corrections $\delta_{QCD}$ and $\Delta_{QCD}$. The second column is based on Eq. (3) [7]. The third column is based on Eq. (13) with $m_t/m_t(m_t)$ obtained from Eq. (19) and $\Delta_{QCD}$ evaluated according to Table 1. The fourth and fifth columns give the fractional enhancement over the conventional $O(\alpha_s)$ result [Eq. (2)] due to the inclusion of higher-order contributions (H.O.C.).

<table>
<thead>
<tr>
<th>$m_t$ [GeV]</th>
<th>$\delta_{QCD}$ [Eq. (3)]</th>
<th>$\delta_{QCD}$ [Eq. (13)]</th>
<th>H.O.C. [Eq. (3)]%</th>
<th>H.O.C. [Eq. (13)]%</th>
</tr>
</thead>
<tbody>
<tr>
<td>130</td>
<td>-0.1154</td>
<td>-0.1220</td>
<td>13.2</td>
<td>19.6</td>
</tr>
<tr>
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<td>-0.1128</td>
<td>-0.1189</td>
<td>12.9</td>
<td>19.0</td>
</tr>
<tr>
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<td>-0.1160</td>
<td>12.6</td>
<td>18.5</td>
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<tr>
<td>200</td>
<td>-0.1080</td>
<td>-0.1133</td>
<td>12.4</td>
<td>18.0</td>
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<tr>
<td>220</td>
<td>-0.1064</td>
<td>-0.1116</td>
<td>12.2</td>
<td>17.7</td>
</tr>
</tbody>
</table>
Estimation of Higher-Order QCD Effects on Electroweak Parameters

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Abstract

We review recent developments in the estimation of higher-order QCD effects on universal electroweak parameters. Special attention is paid to the assessment of the theoretical errors.

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\section{Introduction}

Precision tests of the standard electroweak theory have reached a mature stage since their beginnings more than two decades ago. QCD corrections to electroweak parameters have been identified as important ingredients on the theoretical side of this enterprise \cite{1-3}, and oblique parameters to leading order in QCD involve the evaluation of vacuum-polarization functions to $O(\alpha_s \alpha_s)$. The usual $O(\alpha_s \alpha_s)$ perturbative contributions to the intermediate-boson self-energies, entailing quark loops with gluon interchange, have been investigated with two approaches: (i) direct calculation of the diagrams employing dimensional regularization \cite{4}, and (ii) application of dispersion relations (DRs) \cite{5-8}. These corrections are dominated by their leading high-$m_t$ terms, of $O(\alpha_s G_F m_t^2)$, which are contained in $\Delta \rho$ \cite{9}. Neglecting $m_b$, one has \cite{4,6}

\begin{equation}
\Delta \rho = N_c x_t \left\{ 1 - C_F \left[ \zeta(2) + \frac{1}{2} \frac{\alpha_s(\mu)}{\pi} \right] \right\} \approx 3 x_t \left[ 1 - 2.85991 \frac{\alpha_s(\mu)}{\pi} \right],
\end{equation}

where $\zeta(2) = \pi^2/6$, $N_c = 3$, $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, $x_t = \left( G_F m_t^2 / 8\pi^2 \sqrt{2} \right)$, $m_t$ is the top-quark pole mass, and $\mu$ is the renormalization scale of the strong coupling constant. We may write $\mu = \xi m_t$, with $\xi = O(1)$, since $m_t$ is the only mass scale of the problem. The observation that the $O(\alpha_s G_F m_t^2)$ term in Eq. (1) is so sizeable and sensitive to the choice of $\xi$ has triggered the quest for contributions beyond $O(\alpha_s \alpha_s)$.

In the absence of a complete $O(\alpha_s^2 \alpha_s^2)$ calculation, one approach has been to concentrate on the class of QCD corrections that are connected with multi-gluon exchange in the virtual $t\bar{t}$ system, since these may be expected to constitute the bulk of the QCD corrections beyond $O(\alpha_s \alpha_s)$. The effect of multi-gluon exchange (and the finite lifetime of the top quark) on the excitation curve of $e^+e^- \rightarrow t\bar{t}$, which by the optical theorem is related to the absorptive parts of the photon and $Z$-boson self-energies, has been studied in great detail in the literature \cite{10}. It is well known that these diagrams can be resummed in the ladder approximation and that their combined contribution is chiefly accumulated in the $t\bar{t}$ threshold region, leading to an enhancement of the production cross section along with a lowering of its onset \cite{10}. For low values of $m_t$, this phenomenon is incarnated by the appearance of toponium resonances, which are, however, smeared out to a coherent structure for the $m_t$ values currently favoured by the CDF \cite{11}. On the other hand, the absorptive parts of the photon and $Z$-boson self-energies in $O(\alpha_s \alpha_s)$ correspond to a steplike onset of the excitation curve at $\sqrt{s} = 2m_t$, which is quite unrealistic. The excess of the resummed result for the absorptive part relative to the $O(\alpha_s \alpha_s)$ evaluation induces a shift in the real part, which enters the electroweak parameters. The use of a properly subtracted DR is the method of choice to extract the real part from the imaginary part \cite{7,8}.

Recently, the $O(\alpha_s^2 G_F m_t^2)$ correction to $\Delta \rho$ has been computed \cite{12}. The new term to be included within the square brackets of Eq. (1) reads $(-21.27063+1.78621 N_F)\left[ \alpha_s(\mu)/\pi \right]^2$, where $N_F$ is the number of active quark flavours \cite{12}. This provides the welcome opportunity to check the predictions of Refs. \cite{3,8} concerning the sign and size of the dominant
corrections to $\Delta \rho$ beyond $O(\alpha_s G_F m_t^2)$. Such a comparison may also aid in assessing the theoretical error due to the lack of knowledge of contributions beyond $O(\alpha_s^2 G_F m_t^2)$. This error may then be used in connection with the fixed-order result of Ref. [12].

This report is organized as follows. In Section 2, we outline the approach and the results of Refs. [3,8]. Section 3 contains an error estimation for this approach in the case of $\Delta \rho$. Alternative approaches of estimating higher-order QCD effects on $\Delta \rho$ are reviewed in Section 4. Section 5 concludes.

2 $t\bar{t}$ Threshold Effects

Although loop amplitudes involving the top quark are mathematically well behaved, it is evident that interesting and possibly significant features connected with the $t\bar{t}$ threshold cannot be accommodated when the perturbation series is truncated at finite order. In fact, perturbation theory up to $O(\alpha \alpha_s)$ predicts a discontinuous steplike threshold behaviour for $\sigma(e^+e^- \to t\bar{t})$. More realistic description includes the formation of toponium resonances by multi-gluon exchange. For $m_t > 130$ GeV, the revolution period of a $t\bar{t}$ bound state exceeds its lifetime, and the individual resonances are smeared out to a coherent structure. Then a continuous description in terms of a Green function is more appropriate [10].

By Cutkosky’s rule [13], $\sigma(e^+e^- \to t\bar{t})$ corresponds to the absorptive parts of the photon and $Z$-boson vacuum polarizations, and its enhancement at threshold induces additional contributions in the corresponding real parts, which can be computed via dispersive techniques. Decomposing the vacuum-polarization tensor generated by the insertion of a top-quark loop into a gauge-boson line as

$$\Pi_{\mu\nu}^{V,A}(q) = \Pi_{\mu\nu}^{V,A}(q^2) g_{\mu\nu} + \lambda_{\mu\nu}^{V,A}(q^2) q_{\mu} q_{\nu},$$

where $V$ and $A$ label the vector and axial-vector components and $q$ is the external four-momentum, and imposing Ward identities, one derives the following set of dispersion relations [7,8]:

$$\Pi^V(q^2) = \frac{q^2}{\pi} \int \frac{ds}{s} \frac{\text{Im} \Pi^V(s)}{q^2 - s - i\epsilon},$$

$$\Pi^A(q^2) = \frac{1}{\pi} \int \frac{ds}{s} \left[ \frac{\text{Im} \Pi^A(s)}{q^2 - s - i\epsilon} + \text{Im} \lambda^A(s) \right].$$

In the threshold region, only $\text{Im} \Pi^V(q^2)$ and $\text{Im} \lambda^A(q^2)$ receive significant contributions and are related by $\text{Im} \lambda^A(q^2) \approx -\text{Im} \Pi^V(q^2)/q^2$, while $\text{Im} \Pi^A(q^2)$ is strongly suppressed due to centrifugal barrier effects [8]. Of course, $\lambda^V(q^2) = -\Pi^V(q^2)/q^2$ by the relevant Ward identity [7]. These contributions in turn lead to shifts in $\Delta \rho$, $\Delta r$, and $\Delta \kappa$. A crude estimation may be obtained by setting $\text{Im} \Pi^V(q^2) = \text{Im} \Pi^V(4m_t^2) = \alpha_s m_t^2$ in the interval $(2m_t - \Delta)^2 \leq q^2 \leq 4m_t^2$, where $\Delta$ may be regarded as the binding energy of the 1S state. This yields

$$\Delta \rho = -\frac{G_F}{2\sqrt{2}} \frac{\alpha_s}{\pi} m_t \Delta,$$
\[ \Delta r = \frac{c_w^2}{\sin^2 \theta_W} \Delta \rho \left[ 1 - \left( 1 - \frac{8}{3} \sin^2 \theta_W \right)^2 \frac{M_W^2}{4m_t^2 - M_Z^2} + \frac{16}{9} s_w^4 \frac{M_Z^2}{m_t^2} \right], \]
\[ \Delta \kappa = \frac{c_w^2}{\sin^2 \theta_W} \Delta \rho \left[ 1 - \left( 1 - \frac{8}{3} \sin^2 \theta_W \right) \frac{M_Z^2}{4m_t^2 - M_Z^2} \right]. \] (4)

Obviously, the threshold effects have the same sign as the \( O(\alpha_s G_f m_t^2) \) corrections. For realistic quark potentials, one has approximately \( \Delta \sim m_t \), so that the threshold contributions scale like \( m_t^2 \). Again, \( \Delta \rho \) is most strongly affected, while the corrections to \( \Delta r_{rem} = \Delta r + c_w^2/\sin^2 \theta_W \Delta \rho \) and \( \Delta \kappa_{rem} = \Delta \kappa - c_w^2/\sin^2 \theta_W \Delta \rho \) are suppressed by \( M_Z^2/m_t^2 \). A comprehensive numerical analysis may be found in Refs. \([3,8]\). For \( 150 \text{ GeV} \leq m_t \leq 200 \text{ GeV} \), the threshold effects enhance the QCD corrections by roughly 30%.

3 Error estimation

We emphasize that the above QCD corrections come with both experimental and theoretical errors. In the following, we shall estimate them for \( \Delta \rho \) assuming \( m_t = 174 \text{ GeV} \) \([11]\); the results are similar for other universal electroweak parameters. The experimental errors are governed by the \( \alpha_s \) measurement, \( \alpha_s(M_Z) = 0.118 \pm 0.006 \) \([14]\). This amounts to errors of \( \pm 5\% \) and \( \pm 18\% \) on the continuum and threshold contributions to \( \Delta \rho \), respectively. This reflects the fact that the \( \alpha_s \) dependence is linear in the continuum, while that of the 1S peak height is approximately cubic. Since these errors are strongly correlated, we add them linearly to obtain an overall experimental error of \( \pm 8\% \) on the QCD correction to \( \Delta \rho \). Theoretical errors are due to unknown higher-order corrections. In the continuum, they are usually estimated by varying the renormalization scale, \( \mu \), of \( \alpha_s(\mu) \) in the range \( m_t/2 \leq \mu \leq 2m_t \), which amounts to \( \pm 11\% \). The theoretical error on the threshold contribution is mainly due to model dependence and is estimated to be \( \pm 20\% \) by comparing conventional quark potentials. For \( m_t \) as large as \( 174 \text{ GeV} \), there is also a minor uncertainty connected with the choice of the lower bound of the dispersion integral, which, in Ref. \([8]\), was chosen to be \( s = (2m_t - \Delta)^2 \) with \( \Delta = \sqrt{m_t \Gamma_t} \), which is \( 15.6 \text{ GeV} \) for \( m_t = 174 \text{ GeV} \). The variation of \( \Delta \) in the wide range between \( \sqrt{m_t \Gamma_t}/2 \) and \( 2\sqrt{m_t \Gamma_t} \) induces a shift of \( \pm 9\% \) in the threshold contribution. Combining the common experimental error and the theoretical errors quadratically, we obtain a total error of \( \pm 12\% \) on the QCD correction to \( \Delta \rho \). This corresponds to an absolute QCD-related uncertainty in \( \Delta \rho \) of \( \pm 1.5 \times 10^{-4} \). Owing to the magnification factor \( c_w^2/\sin^2 \theta_W \), the corresponding error on \( \Delta r \) and \( \Delta \kappa \) is \( \pm 5.0 \times 10^{-4} \). We would like to stress that, in the case of \( \Delta r \) and thus the \( M_W \) prediction from the muon lifetime, this error is almost as large as the one from hadronic sources introduced via \( \Delta \alpha \). For higher \( m_t \) values, it may even be larger.

4 Alternative approaches

During the last year, a number of papers \([12,15-21]\) have appeared which are directly or indirectly related to the issues addressed here. Most of them estimate QCD corrections to \( \Delta \rho \) beyond the leading-order QCD term of Eq. (1) with \( \alpha_s(\mu) \) evaluated at \( \mu = m_t \). To facilitate the comparison between the various results, it is convenient to absorb these
additional corrections into the \( \mathcal{O}(\alpha_s) \) term of Eq. (1) by evaluating the latter with an appropriately adjusted scale, \( \mu = \xi m_t \). Using again the reference values \( m_t = 174 \text{ GeV} \) and \( \alpha_s(m_Z) = 0.118 \), our Green-function approach [8] yields \( \xi = 0.190^{+0.007}_{-0.005} \), where we have included the estimated error of \( \pm 30\% \) [8] on the \( \bar{t}\bar{t} \) threshold contribution.

Alternative, conceptually very different approaches of scale setting [15–17] yield results in the same ballpark. In Ref. [15], it is suggested that long-distance effects lower the renormalization point for \( \alpha_s(\mu) \) in Eq. (1) through the contributions of the near-mass-shell region to the evolution of the quark mass from the mass shell to distances of order \( 1/m_t \). To estimate these effects, the authors of Ref. [15] apply the Brodsky–Lepage–Mackenzie (BLM) criterion [22] to Eq. (1) and find \( \xi = 0.154 \). The author of Ref. [16] first expresses the fermionic contribution to \( \Delta \rho \) [see Eq. (1)] in terms of \( \overline{m}_t(m_t) \), where \( \overline{m}_t(\mu) \) is the top-quark MS mass at renormalization scale \( \mu \), and then relates \( \overline{m}_t(m_t) \) to \( m_t \) by optimizing the expansion of \( m_t/\overline{m}_t(m_t) \), which is known through \( \mathcal{O}(\alpha_s^2) \) [23], according to the BLM criterion [22]. In Ref. [17], he refines this argument by using the new results of Ref. [12] and an expansion of \( m_t/\overline{m}_t(m_t) \), where \( \mu_t = \overline{m}_t(\mu) \), and obtains \( \xi = 0.323 \). He argues that an important advantage of this approach is that the coefficients of \( \alpha_s/\pi^n \) \((n = 1, 2)\) in the final expressions are either very small or of \( \mathcal{O}(1) \). In particular, when \( \Delta \rho \) is expressed in terms of \( \overline{m}_t(m_t) \), the QCD correction is only \( (2–3) \times 10^{-3} \) in the next-to-leading-order approximation. As a consequence, when \( \Delta \rho \) is rewritten in terms of \( m_t^2 \), the QCD correction to \( \Delta \rho \) is almost entirely contained in \( \overline{m}_t(m_t)/m_t^2 \), i.e., it is a pure QCD effect.

More details on the approach of Refs. [16,17] are reported elsewhere in these proceedings [24].

We recall that the purpose of our analysis [3,8] has been to predict the dominant higher-order QCD corrections to \( \Delta \rho \) and other universal electroweak parameters prior to their direct diagrammatical computation order by order. Therefore, the advent of the \( \mathcal{O}(\alpha_s^2 G_F m_t^2) \) correction to \( \Delta \rho \) [12] provides the welcome opportunity for a first non-trivial check of our method. We observe that this term has indeed the very sign predicted by our study [3,8] of \( \bar{t}\bar{t} \) threshold effects. Moreover, it accounts for the bulk of their size, too. In fact, this term may be absorbed into the \( \mathcal{O}(\alpha_s G_F m_t^2) \) term by choosing \( \xi = 0.348 \) for \( n_f = 6 \) flavours [12]. Arguing that \( n_f = 5 \) is more appropriate for \( \mu < m_t \), this value comes down to \( \xi = 0.324 \) [25], which is not far outside the range \( 0.133 \leq \xi \leq 0.287 \) predicted in the \( \bar{t}\bar{t} \) threshold analysis. The residual difference may be understood by observing that the ladder diagrams of \( \mathcal{O}(\alpha_s^2 G_F m_t^2) \), with \( n \geq 3 \), and the effect of the finite top-quark decay width are not included in the fixed-order calculation of Ref. [12]. On the other hand, the result of Refs. [3,8] does not account for those \( \mathcal{O}(\alpha_s^2 G_F m_t^2) \) diagrams which are not of the ladder type; their contribution is not actually small [12].

At this point, we should address some seeming controversies in the literature. Criticism concerning the subtraction prescription used in our dispersion relations [7,8] has been voiced [18,19]. In Ref. [18], it has been argued, by appealing to the operator product expansion, that our prescription and the alternative one proposed in Ref. [5] are equivalent. From the observation [7] that the threshold contributions evaluated with the two sets of dispersion relations are similar in magnitude but differ in sign, the authors of Ref. [18] have concluded that the difference between the two results should be interpreted as the intrinsic error of the approach. To meet this criticism, we remark that our prescription derives its origin from the Ward identities among the various vacuum-polarization amplitudes [7,8],
which is not the case for Ref. [5]. Although the physical results of Ref. [5] are correct, the dispersion relations used for their derivation do not, in general, yield correct results, i.e., results that agree with those derived using dimensional regularization. This has been demonstrated in Ref. [26] by establishing a perturbative counterexample, namely the two-loop $O(\alpha_s G_F m_t^2)$ correction to the leptonic decay width of the Higgs boson. In fact, in this particular case, the dispersive calculation according to Refs. [7,8] agrees with dimensional regularization, while the dispersion relations of Ref. [5] lead to a wrong answer. We should mention that, in a subsequent preprint [20], the authors of Ref. [18] suggested, again by means of the operator product expansion, that this argument may be extended to all orders in $\alpha_s$. This example clearly shows that the dispersion relations of Refs. [7,8] and Ref. [5] are not equivalent. To prevent possible confusion, we stress that, similarly to dimensional or Pauli-Villars regularization, the use of dispersion relations is a well-defined method of regularizing the ultraviolet divergences of bare Feynman amplitudes, e.g., vacuum polarizations [7,8], vertex corrections [27], etc. The regularized divergences — whether extracted by dimensional, Pauli-Villars, dispersive, or some other kind of regularization — are an intrinsic property of the amplitudes and have, as a matter of principle, no connection with the physical observables that one intends to calculate with them. In other words, the subtraction prescription used in the dispersive regularization of bare amplitudes is independent of the particular application, i.e., of whether corrections to electroweak parameters, partial decay widths, etc., are to be calculated.

The claim [21] that the $t\bar{t}$ threshold effects are greatly overestimated in Refs. [3,8] is based on a simplified analysis, which demonstrably [28] suffers from a number of severe analytical and numerical errors. Speculations [19] that the dispersive computation of $t\bar{t}$ threshold effects is unstable are quite obviously unfounded, since they arise from uncorrelated and unjustifiably extreme variations of the continuum and threshold contributions. For instance, the authors of Ref. [19] vary the value of $\Lambda_{\text{QCD}}^{[4]}$ in the QCD potential between 100 and 500 MeV, which corresponds to a variation of $\alpha_s(m_Z)$ between 0.0997 and 0.1292, i.e., to an error in $\alpha_s(m_Z)$ that exceeds the present experimental error, $\pm 0.006$ [14], by a factor of 2.5. At the same time, they do not, however, adjust accordingly the value of $\Lambda_{\text{QCD}}$ appearing in the formula for $\alpha_s$ used in the continuum calculation, which causes a spurious mismatch of the threshold and continuum contributions. Furthermore, they vary the $t\bar{t}$ kinetic energy at which the threshold and continuum calculations are matched, i.e., the upper bound of the dispersion integral, quite extremely, between $5 \times 10^{-3}$ and 1.3 units of $2m_t$. A major source of confusion is probably the point that they ascribe the unavoidable scale dependence of the $O(\alpha_s G_F m_t^2)$ continuum result [cf. Eq. (1)], which they estimate by choosing $m_t/4 \leq \mu \leq 4m_t$, to the uncertainty in the much smaller threshold contribution, which artificially amplifies this uncertainty. One should bear in mind that the threshold contribution is not an absolute quantity but is defined relative to a certain standard convention for the fixed-order calculation of the continuum contribution, such as the choice of the renormalization scheme and scale, and the order in $\alpha_s$. As stated explicitly in Ref. [8], we have chosen the $O(\alpha_s G_F m_t^2)$ evaluation of the continuum contribution in the on-shell scheme at $\mu = m_t$ [cf. Eq. (1)] as the reference for the definition of the threshold contribution. The salient point is that the sum of the continuum and threshold contributions, which is the physically relevant quantity, is in fact very insensitive to this convention, considerably less than the continuum calculation.
alone. This is nicely demonstrated for the $\mu$ dependence in Table II of Ref. [28]. Finally, since the authors of Ref. [19] find that the uncertainty in the threshold contribution mainly arises from the intermediate-energy region above threshold, where the transition from the non-relativistic regime to the relativistic one takes place, it is worthwhile to point out that, in our Green-function calculation of $\Delta \rho$ with $m_t = 174$ GeV, 76% of the effect originates at $\sqrt{s} < 2m_t + 800$ MeV. Consequently, the questionable contribution from $\sqrt{s} > 2m_t + 800$ MeV amounts to $24\% \times 30\% \approx 7\%$ of the $\mathcal{O}(\alpha_s G_F m_t^2)$ continuum result [the total threshold contribution to $\Delta \rho$ evaluated for $m_t = 174$ GeV in the Green-function approach is about 30% of the $\mathcal{O}(\alpha_s G_F m_t^2)$ result]. We find it hard to understand how the uncertainty in this 7% fraction can possibly be as large as the $\mathcal{O}(\alpha_s G_F m_t^2)$ term itself as is suggested in Ref. [19].

5 Conclusions

Historically, the study of $t\bar{t}$ threshold effects on $\Delta \rho$ and other electroweak parameters [3,6-8,28] has been an attempt to estimate the dominant beyond-leading-order QCD corrections to these parameters prior to their explicit diagrammatical computation. This approach is based on the assumption that the dominant effects arise from multi-gluon exchange in the $t\bar{t}$ system close to threshold. As for the imaginary parts of the loop amplitudes, these diagrams can be resummed in the non-relativistic approximation [10]. This treatment naturally takes into account the finite lifetime of the top quark as well [10]. The real parts of the loop amplitudes may be found via properly subtracted dispersion relations. The subtractions may be derived from Ward identities [7,8]. This approach has both advantages and disadvantages. Its virtues are (i) that it resums the class of higher-order corrections which are believed to be dominant, and (ii) that it accounts for the rapid decay of the heavy top quark. Both features are necessarily absent in the $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ calculation [12]. The drawbacks of this approach are (i) that it misses $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ non-ladder diagrams, (ii) that it uses a non-relativistic approximation which is reliable only close to the $t\bar{t}$ threshold region and leaves a certain freedom as to how the threshold and continuum contributions are to be matched and as to where the onset of the $t\bar{t}$ excitation curve is to be taken, and (iii) that the results depend to a certain extent on the specific form of the inter-quark potential used in the Schrödinger equation.

It is important to notice that the size of the threshold contribution is not an absolute quantity. In Refs. [3,8], it has been estimated relative to the $\mathcal{O}(\alpha_s G_F m_t^2)$ result in the on-shell scheme of quark-mass renormalization and with $\alpha_s$ evaluated at scale $\mu = m_t$. Since the $t\bar{t}$ threshold contribution and the $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ term have the same sign and are comparable in size, it is obvious that the $t\bar{t}$ threshold contribution would be greatly reduced if it was expressed relative to the $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ result. However, this would require knowledge of the imaginary parts of the $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ neutral-gauge-boson self-energies for $q^2 \geq 4m_t^2$, which is not available at the present time.

In view of this situation, a conservative approach would be to rely on the $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ result [12] and to assign to it a sufficiently ample error. Leaving aside the experimental error on $\alpha_s(M_Z)$, there are two obvious sources of theoretical uncertainty in the result of Ref. [12], namely the scheme and scale dependences. In Fig. 1, we show the leading-order
[4,6] and next-to-leading-order [12] QCD correction to $\Delta \rho$ in the on-shell and $\overline{\text{MS}}$ schemes of top-quark-mass renormalization relative to the one-loop term, $3x_t$, as a function of the scale parameter, $\xi$. We have used the pole-mass value $m_t = 174$ GeV, and the three-loop formula for $\alpha_s$ with $\alpha_s(M_Z) = 0.118$ [14]. In the $\overline{\text{MS}}$ scheme, the mass and charge renormalization scales are identified. As expected, the scheme and scale dependences are appreciably reduced as we pass from two loops to three loops. Looking at the $\xi$ values advocated by the various approaches discussed in Section 4, it seems plausible to estimate the scale dependence by considering the interval $1/4 \leq \xi \leq 4$. Also, taking into account the scheme dependence, we read off from Fig. 1 that the QCD correction to $\Delta \rho$ is bounded by $-10.06\%$ and $-11.37\%$. It should be noted that these bounds both originate from the lower part of the $\xi$ interval. Taking the three-loop on-shell result with $\xi = 1$, which is $-11.13\%$, as the central value and taking the error to be symmetric around this value, we obtain a theoretical uncertainty of roughly $\pm 10\%$ on the QCD correction to $\Delta \rho$. Combining this quadratically with the experimental uncertainty of $\pm 5\%$ due to the error $\pm 0.006$ on $\alpha_s(M_Z)$ [14], we obtain an overall uncertainty of $\pm 11\%$. This is quite similar to the value $\pm 12\%$ obtained in Section 3 for the analysis of Refs. [3,8]. Finally, we remark that scheme and scale dependences are also important issues in connection with the $\mathcal{O}(G_F^2 m_t^4)$ correction to $\Delta \rho$ [29]; for a recent discussion of this point, see Ref. [30].
Figure 1: Leading-order and next-to-leading-order QCD correction to $\Delta \rho$ in the on-shell and $\overline{\text{MS}}$ schemes relative to the one-loop term, $3x_t$, as a function of the scale parameter, $\xi$. In the $\overline{\text{MS}}$ scheme, the mass- and charge-renormalization scales are identified.

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Note added in proof

After this contribution had been submitted, an error in the $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ analysis of $\Delta \rho$ [12] was detected [31]. This error has been confirmed by the authors of Ref. [12]. The correct term to be included within the square brackets of Eq. (1) should read \((-25.31131 + 1.78621 N_F)\alpha_s(\mu)/\pi\)^2, where $N_F$ is the number of active quark flavours. This affects some of our observations.

We remark that this revised result nicely agrees with our estimate [3,8] based on the analysis of the $t\bar{t}$ threshold effects. In fact, the new $\mathcal{O}(\alpha_s^2 G_F m_t^2)$ term may be absorbed into the $\mathcal{O}(\alpha_s G_F m_t)$ term by choosing $\xi = 0.233$ ($\xi = 0.224$) for $N_F = 6$ (5) flavours. Both values lie comfortably within the range $0.133 \leq \xi \leq 0.287$ predicted in the $t\bar{t}$ threshold analysis.

The revised version of Fig. 1 is presented here. Estimating the scheme and scale ($1/4 \leq \xi \leq 4$) dependences as described in Section 5, we find that the relative QCD correction to $\Delta \rho$ is bounded by $-10.86\%$ and $-12.07\%$. This corresponds to a theoretical error of $\pm7\%$ on the central value, $-11.61\%$, obtained for $\xi = 1$ at three loops in the on-shell scheme. Combining this with the experimental uncertainty of $\pm5\%$, we obtain an overall uncertainty of $\pm9\%$. 
Figure 1: Leading-order and next-to-leading-order QCD correction to $\Delta \rho$ in the on-shell and $\overline{\text{MS}}$ schemes relative to the one-loop term, $3x_t$, as a function of the scale parameter, $\xi$. In the $\overline{\text{MS}}$ scheme, the mass- and charge-renormalization scales are identified.
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Theoretical Ambiguities of QCD Predictions at the $Z^0$ Peak

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Abstract

We discuss uncertainties of QCD predictions for the hadronic width of the $Z^0$ boson. Emphasis is put on quantitative estimates, taking into account the current precision of experimental data.

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1 Introduction

The hadronic width of the $Z^0$ boson, $\Gamma_h \equiv \Gamma(Z^0 \rightarrow \text{hadrons})$, or the ratio

$$R_Z \equiv \frac{\Gamma_h}{\Gamma_0}, \quad \Gamma_0 \equiv \frac{G_F M_Z^3}{2 \pi \sqrt{2}}, \quad (1)$$

which is closely related to the familiar ratio $R$ in $e^+e^-$ annihilations into hadrons, provides theoretically very clear conditions for the verification of perturbative QCD. This is due to several favourable circumstances:

- nonperturbative power corrections are expected to be negligible at the scale $M_Z$
- perturbative corrections are calculated up to next-to-next-to-leading order (NNLO)
- the dependence of perturbative expansion for (1) on the choice of the renormalization scheme (RS) is weak
- in calculating QCD corrections to (1) the running $\alpha_s$ can be taken as that corresponding to five effectively massless quarks, $\alpha^{(5)}_s$
- the effects of the finite bottom quark mass on the expansion coefficients of $\Gamma_h$ in this couplant $\alpha^{(5)}_s$ have been calculated up to the next-to-leading order (NLO)
- the explicit dependence of $\Gamma_h$ on the top quark mass $m_t$ has been calculated in the large $m_t$ expansion up to the NNLO
- the statistics of the data is large, and thus the experimental accuracy rather high, in particular compared with the closely related process of $e^+e^-$ annihilations into hadrons.

On the other hand, at the scale $M_Z$ the magnitude of QCD corrections to the basic electroweak decay mechanism is smaller than for quantities at lower energy scales, such as the $\tau$-lepton semileptonic decay width and therefore more difficult to pin down. In this part of the paper we give quantitative estimates of some of the above-mentioned uncertainties and address the related question: assuming the validity of QCD and taking into account these uncertainties, how accurately can the basic QCD parameter $\alpha_s$ be determined?

As any meaningful discussion of the quantitative importance of higher order QCD corrections depends on the ability of experiments to 'see' them, we start by recalling the relevant experimental data [1]:

$$M_Z = 91.187 \pm 0.007 \text{ GeV} \quad (2)$$
$$\Gamma_0 = 0.99528 \pm 0.00023 \text{ GeV} \quad (3)$$
$$\Gamma_h = 1.7407 \pm 0.0059 \text{ GeV}, \quad (4)$$

which imply the following relative errors, relevant for further discussion:

$$\frac{\Delta \Gamma_0}{\Gamma_0} = 2.3 \times 10^{-4}, \quad \frac{\Delta \Gamma_h}{\Gamma_h} = \frac{\Delta R_Z}{R_Z} = 3.5 \times 10^{-3}. \quad (5)$$
Note that the precision $\Delta \Gamma_0$ of the determination of $\Gamma_0$ is more that an order of magnitude better than that of $\Gamma_h$ and can thus be neglected with respect to $\Delta \Gamma_h$.

Because the dominant part of perturbative QCD predictions for $\Gamma_h$ has the same generic form as does $R_Z$, i.e.,

$$A \left[ 1 + \frac{\alpha_s}{\pi} \left( 1 + r_1 \frac{\alpha_s}{\pi} + \cdots \right) \right],$$

and $\alpha_s(M_Z)/\pi \approx 0.037$, these errors allow $\alpha_s(M_Z)$ to be determined to within about 8.5% accuracy. Translated into the sensitivity to $\Lambda$, this amounts to a factor of 1.9 uncertainty. Improving further the accuracy of the data by a factor of two would allow it to be extracted with an error of only 36%.

2 The RS dependence: general considerations

Over the last 15 years the problem of the renormalization scheme dependence of finite-order approximants to perturbation expansions in QCD (and other theories) has been the subject of lively and sometimes even heated debate. From time to time a 'resolution' of this problem is announced, but invariably it turns out that these 'solutions' contain the original ambiguity in some guise or another. We intend to provide a concise and balanced review of all the various approaches to this problem, but emphasize at the very beginning that, in our view, there is no clear winner. Nevertheless, as the dependence of finite-order perturbation expansions on the choice of RS is a very real phenomenological problem, which cannot be ignored, we think the right question in this context is: How sensitive are these approximations to the choice of RS? But even in this question there hides a catch, as to give it a concrete meaning we first have to define the set of 'allowed' RS. The point is that without some restriction on the considered RS we could get essentially any result we want. But again, as the selection of the 'allowed' RS is inevitably a subjective matter and may, moreover, depend on the quantity in question, the best we can do is choose a couple of approaches which are sufficiently general, have some rationale behind them and define the theoretical 'error' for this RS set.

Whether the theoretical error of some quantity should be considered large or small is, of course, not given a priori, but depends on the accuracy of experimental data to which it is compared. These experimental errors for the quantities related to the $Z^0$ decay were estimated in the preceding section.

In this section only the RS dependence of physical quantities will be discussed. For unphysical quantities, such as the Green functions with anomalous dimensions, the situation is more complicated and some of the approaches are not directly applicable. This, however, is not a serious limitation, as what we are actually interested in are clearly only the physical quantities.

Furthermore, we shall consider only the case of QCD with $n_f$ massless quark flavours. The reasons for these restrictions are twofold. First, the relations resulting from the renormalization group (RG) considerations and expressing the internal consistency of the renormalized perturbation theory have yet to be worked out for the general massive case. The lack of such relations precludes the general quantitative discussion of the RS
dependence problem, possible in the massless case. In the $Z^0$ mass range, however, we can with great accuracy consider QCD with five effectively massless quark flavours. This statement is quantified in Section 3.

In the following subsections we first discuss the quantitative description of the freedom connected with the choice of the RS ('kinematics' of the RS dependence problem) and then briefly review several of the approaches to choosing one of these RS ('dynamics' of the RS problem). We emphasize this distinction, as the two aspects are frequently mixed up. Only the latter aspect is really of substance, the former being merely a matter of convention and bookkeeping.

2.1 The description of the RS dependence

Consider the generic perturbation expansion for the physical quantity of the form

$$r(Q) = a(RS) \left[ r_0 + r_1(Q, RS)a(RS) + r_2(Q, RS)a^2(RS) + \cdots \right]; \quad r_0 = 1,$$

(7)

which appears in the expression for $\Gamma_h$ and (1). $Q$ in (7) denotes generically some external momentum on which $r$ depends$^1$ and $a(RS)$ is the renormalized constant $a \equiv \alpha_s/\pi$ (the adjective renormalized will be dropped in the following). There are many different ways to quantify the dependence of such physical quantities on the RS. As a matter of convention, we shall adopt the one suggested in Ref. [2]. First, we should define the meaning of the renormalization scheme itself. In massless QCD there must be some parameter with the dimension of the mass, for the moment loosely denoted as $\Lambda$, that sets the basic scale of the theory. Once this parameter is given, any quantity can in principle be calculated as a concrete number. For a given $\Lambda$, fixing the RS means specifying the values of all perturbative coefficients, $r_h(RS)$, as well as the value of the expansion parameter $a(RS)$ itself.

The labelling of the RS suggested in Ref. [2] starts with the familiar equation

$$\frac{da(\mu, RS)}{d\ln \mu} = \beta(a) = -b a^2(\mu, RS) \left( 1 + c a(\mu, RS) + c_2 a^2(\mu, RS) + \cdots \right),$$

(8)

expressing the dependence of $a$ on the scale $\mu$, which inevitably appears in the theory during the process of renormalization. The first two coefficients on the r.h.s. of (8), i.e., $b, c$ are unique functions of the number $n_f$ of massless quarks

$$b = \frac{11 N_c - 2 n_f}{6}; \quad c = \frac{51 N_c - 19 n_f}{11 N_c - 4 n_f},$$

(9)

but all the higher-order ones are completely arbitrary. Once they are given and some initial condition on $a$ is specified, (8) can be solved. The way of specifying the boundary condition is ambiguous, but its choice is a matter of convention only. One way of doing this is via the scale parameter $\tilde{\Lambda}$ introduced in the following implicit equation for the solution of (8) [2]:

$$b \ln \frac{\mu}{\Lambda} = \frac{1}{a} + c \ln \frac{c a}{1 + c a} + \int_0^a dx \left[ -\frac{1}{x^2 B^2(x)} + \frac{1}{x^2(1 + c x)} \right],$$

(10)

$^1$In view of the application to $\Gamma_h$ we restrict our considerations to quantities depending on a single external momentum $Q$.  

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where
\[ B^{(n)}(x) \equiv (1 + cx + c_2 x^2 + \cdots + c_{n-1} x^{n-1}). \]  
(11)

Note that as the integral on the r.h.s. of (10) behaves like \( O(\alpha_s) \) the higher-order coefficients \( c_k; k \geq 2 \) have no influence on the value of \( \Lambda \). Note also that the parameter \( \Lambda \) introduced in (10) differs from the \( \Lambda \) used in most phenomenological analyses by a factor close to unity: \( \Lambda = \bar{\Lambda} (2c/b)^{c/b} \). At the next-to-leading order (NLO) — i.e., keeping only the first two terms in (10) — the solution of (10) is often approximated by the first two terms of its expansion in powers of inverse logarithms \( \ln(\mu/\Lambda) \):

\[ a(\mu/\Lambda) = \frac{1}{b \ln (\mu/\Lambda)} - c \frac{\ln [\ln (\mu^2/\Lambda^2)]}{b^2 \ln^2 (\mu/\Lambda)} + \cdots \]  
(12)

The dependence of the couplant \( a \) on the parameters \( c_i; i \geq 2 \) is determined by equations similar to (8) [2]:

\[ \frac{\mathrm{d}a(\mu, c_i)}{\mathrm{d}c_i} = \beta_i = -\beta(a) \int_0^a \frac{b x^{i+2}}{[\beta(x)]^2} \mathrm{d}x, \]  
(13)

which are uniquely determined by the basic \( \beta \)-function in (8) and thus introduce no additional ambiguity.

It is obvious that a unique definition of \( a(\mu) \) at some \( \mu \) requires, as well as the specification of the coefficients \( c_i; i \geq 2 \), the specification of the boundary condition — i.e., for instance, the value of \( \bar{\Lambda} \). It is convenient to introduce the concept of the renormalization convention (RC), which is associated with a fully defined solution of (8): RC \( \equiv \{\bar{\Lambda}, c_i; i \geq 2\} \). As, however, \( \mu \) always enters this solution in the ratio \( \mu/\bar{\Lambda} \), we can either:

- select one of the solutions to (8), which we call referential renormalization convention (RRC) and vary \( \mu \) only or

- fix \( \mu \) by identifying it with some external momentum — for instance, \( Q \) — and vary the solution of (8), i.e., for fixed coefficients \( c_i \) the value of \( \bar{\Lambda} \) instead.

Both these options are completely equivalent and it is merely a matter of taste as to which one to use. We prefer the former. To vary both simultaneously is legal, but obviously redundant.

As the choice of the RRC is a matter of convention only, we cannot associate any physical meaning to the scale \( \mu \) itself. It serves to label the RS, but only in a given RRC. In two different RRCs the same \( \mu \) may correspond to different values of the couplant \( a \) as well as the coefficients \( r_k \). We emphasize this point as in many papers the RS is chosen by identifying \( \mu \) with some ‘natural’ physical scale of the process, such as the external momentum \( Q \). Although such a natural scale can usually be identified, its mere existence does not help fix the arbitrary scale \( \mu \), as to get a unique RS, the RCC also has to be specified. This is usually tacitly assumed to be the \( \overline{\text{MS}} \), but there is no theoretical argument for this choice, except that in this RCC the coefficients \( r_k \) are often small. If, however, the magnitude of the coefficients of the perturbative series for physical quantities were the criterion, we would be naturally drawn to the effective charges approach, described
below, where they actually vanish. In other words, because the choice of the \( \overline{\text{MS}} \) as the 

RRC is merely a convention, there is no reason to set \( \mu = Q \).

The above relation (10) allows the expression of \( \mu / \Lambda \) in terms of \( a \) and \( c_i \) and thus 

the labelling of the RS by means of the set of parameters \( a, c_i; i \geq 2 \). Using this way of 

labelling of the RS is very convenient as there is then no need to introduce the RRC and 

also no possibility of referring to the ‘natural’ scale to fix the RS.

In the NNLO order — i.e., taking into account also the first nonunique coefficient \( c_2 \) — we have the equation 

\[
\beta \ln \frac{\mu}{\Lambda} = \frac{1}{a} + c \ln \frac{ca}{\sqrt{1 + ca + c^2}} + f(a, c_2) ,
\]

where

\[
f(a, c_2) = \frac{2c_2 - c^2}{d} \left( \arctan \frac{2c_2 a + c}{d} - \arctan \frac{c}{d} \right) ; \quad d \equiv \sqrt{4c_2 - c^2} ; \quad 4c_2 > c^2
\]

\[
= \frac{2c_2 - c^2}{d} \left( \ln \frac{2c_2 a + c - d}{2c_2 a + c + d} - \ln \frac{c - d}{c + d} \right) ; \quad d \equiv \sqrt{c^2 - 4c_2} ; \quad 4c_2 < c^2 .
\]

Its solution depends on the value of \( c_2 \). We distinguish three different cases:

- \( c_2 = 0 \) (resp. \( c_i = 0, i \geq 2 \)), defining the so called 't Hooft RC [3]
- \( c_2 > 0 \), when \( \beta(a) < 0 \) is a monotonously decreasing function of \( a \) and the situation 
  is therefore qualitatively the same as for \( c_2 = 0 \)
- \( c_2 < 0 \), when \( \beta(a) \) has the infrared fixed point at \( a^*(c_2) \), given by the equation 
  \( \beta[a^*(c_2)] = 0 \). The corresponding solution of (8) then approaches finite value at 
  \( \mu = 0 \) and consequently

\[
\lim_{Q \to 0} r^{(a)}(Q) = a^*(c_2) \left[ 1 + r_1(\mu = Q)a^*(c_2) + r_2(c_2, \mu = Q)a^*(c_2) + \cdots \right] \]

has a finite infrared limit at the NNLO. This case is discussed in detail in Refs. [4, 5].

Note that the possibility of an infrared stable limit of finite-order approximants, which 

starts at the NNLO, does not have to survive the incorporation of still higher-order 

corrections and its physical relevance is therefore questionable.

While the explicit dependence of the couplant on \( c_i \) is given in (14) and (15), the 

dependence of the coefficients \( r_k \) on them is determined by the requirements of internal 

consistency from the perturbation theory. They imply that any finite-order approximant,

\[
r^{(N)}(Q) \equiv \sum_{k=0}^{N-1} r_k a^{k+1} = \mathcal{F}(\mu, c_i, \rho; i \leq N - 1) ,
\]

must satisfy the following consistency conditions:

\[
\frac{dr^{(N)}}{d \ln \mu} = \mathcal{O}(a^{N+1}) , \quad \frac{dr^{(N)}}{d c_i} = \mathcal{O}(a^{N+1}) .
\]
Iterating these equations we find:

\[ r_1(Q/\mu) = b \ln \frac{\mu}{Q} + r_1(\mu = Q) = b \ln \frac{\mu}{\Lambda} - \rho(Q/\Lambda) \]
\[ r_2(Q/\mu, c_2) = \rho_2 - c_2 + r_1^2 + cr_1, \quad (19) \]

and similarly for still higher orders. In the above relations the quantities \( \rho, \rho_2 \) etc., are RG invariants — i.e., contrary to the coefficients \( r_k \), they are independent of the choice of the RS. Note that all the dependence of the perturbative approximants on \( Q \) comes exclusively through the invariant \( \rho(Q/\Lambda) \), which can be written as

\[ \rho = b \ln \left( \frac{Q}{\bar{\Lambda}_{\text{RRC}}} \right) - r_1(\mu = Q, \text{RRC}) , \quad (20) \]

where the apparent dependence on the chosen RRC actually cancels between the two terms in (20).

A nontrivial part of any perturbative calculation really boils down to the evaluation of these invariants, the rest being essentially a straightforward exploitation of the RG considerations based on (18). Substituting for the term \( b \ln(\mu/\Lambda) \) in \( r_1 \) the expression (10), using (19) and inserting the resulting \( r_k \) into (17), any finite order approximant \( r^{(N)}(Q) \) can be expressed as an explicit function of the parameters specifying the RS, i.e., \( a, c_i; i \geq 2 \), and the invariants \( \rho_i \):

\[ r^{(N)}(Q) = f(\rho_j; j < N - 1; a, c_i; i \leq N - 1). \quad (21) \]

In this representation the RS dependence of NLO and NNLO approximants is quantitatively described by one- and two-dimensional manifolds, respectively, and the problem of choosing the RS is equivalent to selecting one particular point on these manifolds. Considered as a geometrical exercise we identify certain special points on these manifolds, corresponding to stationary points, where the variation of the approximants with respect to the free parameters vanishes locally.

In the next subsection we briefly describe some of the criteria for choosing the RS, which will define the RS set for which we shall later estimate the theoretical uncertainty of perturbative calculations of the quantities of interest. We shall discuss in some more detail the approaches described in subsections 2.5 and 2.6, as there have recently been some new developments in them.

### 2.2 Fixed RS calculations

Because of the computational simplicity and explicit gauge invariance, all the multi-loop calculations are nowadays done using the dimensional regularization technique. Within this technique the \( \overline{\text{MS}} \) renormalization prescription\(^2\) is often preferred on the grounds that it absorbs in the definition of the renormalized coupling the terms proportional to \( \ln 4\pi - \gamma_E \), which are considered to be artefacts of the dimensional regularization technique.

\(^2\)By 'prescription' we mean the specification of the scale \( \mu \) (by identifying it with some natural scale \( Q \)) as well as of finite parts of all counter-terms necessary to cancel the UV divergencies. Specifying the prescription implies the specification of the RS in the above-defined sense, but not vice versa.
In our way of labelling the RS, \( \overline{\text{MS}} \) corresponds to definite values of all the coefficients \( r_k, c_k \) and a fixed, but numerically undetermined, value of \( a \), which must be extracted from comparison with experimental data. This choice of the RS is very commonly used in phenomenological analyses, but there is no obvious reason why it should be preferred to, for instance, the MOM-like RS or any of the choices discussed in the following subsections. In geometrical terms this is reflected in the fact that the corresponding point on the hypersurfaces defined in (21) occupies no special position.

### 2.3 Principle of Minimal sensitivity (PMS)

In this approach, suggested in Ref. [2], the RS is fixed by demanding that

\[
\frac{d r^{(N)}}{d a} = \frac{d r^{(N)}}{d c_i} = 0 ,
\]

i.e., the \( N \)-th order partial sum has locally the property that the full expansion must satisfy globally. Though there is in general no guarantee that such a stationary point is unique or exists at all, in practical applications to lowest-order QCD quantities it works. At the NLO, when only \( a \) labels the RS, (22) reduces to

\[
2 - 2 \rho a + 2 c a \ln \frac{ca}{1 + ca} + c a \left( \frac{ca}{1 + ca} \right) = 0 ,
\]

and its solution has the form \( a \text{PMS} = (1/\rho)\left[1 + O(ca \text{PMS})\right] \).

At the NNLO we have two coupled equations for derivatives of \( r^{(2)}(a, c_2) \), with respect to \( a \) and \( c_2 \), which must be solved numerically. For the quantity (7) such a stationary point exists for any \( \rho \) if \( \rho_2 < 0 \) and for \( \rho > \rho_{\text{min}}(\rho_2) \) if \( \rho_2 > 0 \) [4].

Note that, contrary to the fixed RS approach, the PMS selects the RS which depends on the type and kinematics of the process under study. The same holds for the methods discussed in the next three subsections.

### 2.4 The method of effective charges (ECH)

The basic idea of this approach [6] is to choose the RS is such a way that the relation between the physical quantity and the couplant is the simplest possible one. For the quantity (7) it means:

\[
r(Q) = a \text{ECH} .
\]

In this approach there is no problem with the convergence of the perturbation expansion (7) itself, but it reappears in the perturbation expansion of the corresponding \( \beta \)-function (see below).

The conditions under which the parameters \( a, c_i \), or \( \mu, c_i \), can be chosen in such a way that (24) holds can be read directly off the consistency conditions (18). At the NLO, (24) implies the following equation for \( a \text{ECH} \):

\[
\frac{1}{a} + c a \ln \frac{ca}{1 + ca} = \rho ,
\]

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which has a solution \( a_{\text{ECH}} = (1/\rho)[1 + \mathcal{O}(c a_{\text{ECH}})] \), differing from \( a_{\text{PMS}} \) merely by the term of the order \( \mathcal{O}(c a_{\text{ECH}}) \). At the NLO the values of \( a_{\text{ECH}} \) thus correspond to intersections of the curves defined as

\[
\mathcal{r}^{(2)}(a, c_2) = a \left( 2 - \rho a + c a \ln \frac{c a}{1 + c a} \right),
\]

with the straight line \( \mathcal{r}^{(2)} = a \). For \( \rho > 0 \) there is always just one such intersection in the physically relevant range \( a > 0 \), while for \( \rho < 0 \) there is none in this range.

At the NNLO the situation is more complicated, as the condition (24) does not by itself determine uniquely \( a_{\text{ECH}} \), but merely implies the equation

\[
\mathcal{r}_1(a) + \mathcal{r}_2(a, c_2) = 0,
\]

which has, depending on the value of \( c_2 \), either no solution, or one or two solutions giving different \( a_{\text{ECH}} \). Going to still-higher orders the ambiguity of this approach grows even further as new free parameters, \( c_i \), crop up. In [6] this ambiguity is avoided by demanding that each of the coefficients \( r_i \) vanishes individually. Assuming this restricted version of the ECH method — i.e., demanding \( r_i = 0, i \geq 1 \) — we get the following expression for the associated \( \beta \)-function:

\[
\frac{d a_{\text{ECH}}}{d \ln \mu} \equiv \beta_{\text{ECH}}(a) = -b a_{\text{ECH}}^2 \left[ 1 + \rho_1 a_{\text{ECH}} + \rho_2 a_{\text{ECH}}^2 + \cdots \right],
\]

where \( \rho_1 = c \). The coefficients \( a_{\text{ECH}}, i \) of the ECH \( \beta \)-function \( \beta_{\text{ECH}} \) thus coincide with the RG invariants \( \rho_i \), introduced in (19). To express \( a_{\text{ECH}} \) as a function of the external momentum \( Q \), it is convenient to write it as a solution of the equation

\[
\frac{1}{a_{\text{ECH}}} + c \ln \frac{c a_{\text{ECH}}}{1 + c a_{\text{ECH}}} = b \ln \frac{Q}{\Lambda_{\text{ECH}}},
\]

where \( \Lambda_{\text{ECH}} \) defines the 'effective' \( \Lambda \) parameter, associated with the quantity under study. It is related to \( \Lambda_{\text{RS}} \) in any fixed RS simply as

\[
\Lambda_{\text{ECH}} = \Lambda_{\text{RS}} \exp \left[ r_1(\mu = Q, \text{RS})/b \right].
\]

As the ECH approach seems to offer a very simple and natural 'solution' to the RS problem one might naturally ask: where has all the ambiguity discussed in subsection 2.1 actually gone? In fact it has not disappeared entirely and reemerges, as discussed in Ref. [7], in a somewhat disguised form, even within the ECH approach.

### 2.5 The method of Brodsky, Lepage and MacKenzie

This method [8] borrows its basic idea from QED, where the renormalized electric charge is fully given by the vacuum polarization due to charged fermion–antifermion pairs. In QCD the authors of this method suggest fixing the scale \( \mu \) with the requirement that all the effects of quark pairs be absorbed in the definition of the renormalized couplant itself,
leaving nothing in the expansion coefficients. In the case of the quantity (7) and up to the NNLO,

$$ r(Q) = a(\mu, \text{RRC}) \left\{ 1 + \left[ r_{10} \left( \frac{\mu}{Q}, \text{RRC} \right) + n_f r_{11} \left( \frac{\mu}{Q}, \text{RRC} \right) \right] a(\mu, \text{RRC}) \right\}, $$

where we have now written the $n_f$ dependence of the coefficients $r_k$ explicitly, it amounts to the requirement that $\mu$ be chosen in such a way that $r_{11}(\mu/Q, \text{RRC}) = 0$. This implies

$$ \mu_{\text{BLM}} \equiv \mu \exp (-3r_{11}) $$

and

$$ r_{\text{BLM}}(Q) = a(\mu_{\text{BLM}}, \text{RRC}) \left\{ 1 + \left[ r_{10} \left( \frac{\mu}{Q}, \text{RRC} \right) + \frac{33}{2} r_{11} \left( \frac{\mu}{Q}, \text{RRC} \right) \right] a(\mu_{\text{BLM}}, \text{RRC}) \right\}. $$

The problem with this ‘scale-setting’ method is that the resulting scale as well as $r_{\text{BLM}}$ depend on the choice of the RRC! This is due to the fact that for a given $\mu$ the separation of the coefficient $r_1$ into the two parts $A + n_f B$ is not unique, but depends on the RRC used. This problem exists in principle QED as well, but there the quark loop effects can be rather unambiguously absorbed in the renormalized electric charge via the MOM RRC, which for massless quarks gives the same $B$ as the MS-like ones. This is no longer true in QCD, where various types of MOM-like RRC in general give different values of $B$, different again from that of MS-like one [9].

As emphasized in the general discussion above, fixing the scale without also simultaneously fixing the RRC does not, however, determine the RS, because the choice of the RRC is equally important as that of the scale $\mu$. The resulting ambiguity of the BLM approach, pointed out a long time ago [9], has not yet been satisfactorily resolved. In practical applications one usually starts from the MS RRC.

It is claimed in Ref. [8] that the BLM-improved expressions for the physical quantities have small NLO coefficients. However, as shown in Ref. [10], this is not necessarily the case when the BLM approach is generalized to higher orders. Let us first recall the main steps of the generalization suggested in Ref. [10].

Within the class of the so-called ‘regular’ RC the $n_f$ dependence of the expansion as well as $\beta$-function coefficients is polynomial in $n_f$:

$$ r_1 = r_{10} + r_{11} n_f $$

$$ r_2 = r_{20} + r_{21} n_f + r_{22} n_f^2 $$

$$ \beta_0 \equiv b = b_0 + b_1 n_f $$

$$ \beta_1 \equiv b_c = \beta_{10} + \beta_{11} n_f $$

$$ \beta_2 \equiv b_{c2} = \beta_{20} + \beta_{21} n_f + \beta_{22} n_f^2 + \beta_{23} n_f^3 $$

$$ \bar{\beta}_2 \equiv b_{\rho_2} = \bar{\beta}_{20} + \bar{\beta}_{21} n_f + \bar{\beta}_{22} n_f^2 + \bar{\beta}_{23} n_f^3. $$

Note that the scheme-invariant coefficient $\bar{\beta}_2$ contains the $n_f^3$ term as observed in Ref. [11].
The generalization of the BLM approach suggested in [10] assumes that the chosen scale \( \mu \) is determined by the following perturbative expansion:

\[
\mu^2 = \mu^2_{\text{BLM}} \left[ 1 + \gamma_1(n_\ell)a(\mu_{\text{BLM}}) + \ldots \right],
\]

(35)

where \( \mu^2_{\text{BLM}} \) is given in (32) and \( \gamma_1 = \gamma_{10} + \gamma_{11} n_\ell \). The parameters \( \gamma_{10} \) and \( \gamma_{11} \) are process dependent and can be determined from the following system of equations\(^3\):

\[
\begin{align*}
\tilde{\beta}_{23} - \beta_{23} &= -2 \beta_{10}^2 \gamma_{11} \\
\tilde{\beta}_{22} - \beta_{22} &= -2 \beta_{10} \gamma_{10} - 4 \beta_{00} \beta_{11} \gamma_{11} \\
\beta_{21} - \beta_{21} &= \beta_{01} \left( r_1^a - r_1^{a^2} \right) - \beta_{11} r_1^a - 4 \beta_{00} \beta_{01} \gamma_{11} - 2 \beta_{00}^2 \gamma_{11} \\
\tilde{\beta}_{20} - \beta_{20} &= \beta_{00} \left( r_1^a - r_1^{a^2} \right) - \beta_{10} r_1^a - 2 \beta_{00}^2 \gamma_{10},
\end{align*}
\]

(36)

which follows from the general expression for the scheme invariant \( \tilde{\beta}_2 \):

\[
\tilde{\beta}_2 = \beta_2 + b r_2 - \beta_1 r_1 - b r_1^2.
\]

(37)

In the above equations, \( r_1^a \) and \( r_1^{a^2} \) are the \( n_\ell \)-independent coefficients in the generalized BLM procedure. We have already mentioned that in practice the BLM approach is applied to the initial series with the coefficients defined in the \( \overline{\text{MS}} \)-scheme. Therefore, it is necessary to put \( \beta_{23} = 0 \). Now consider as an example perturbation expansion for the familiar quantity:

\[
R(s) \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \left( 3 \sum_{i=1}^{n_f} Q_i^2 \right) \left[ 1 + \sum_{k=0}^{\infty} t_k a^{k+1} \right].
\]

(38)

Applying the above generalization to (38) we get [10]

\[
\mu^2_{\text{BLM}} = \mu^2_{\text{MS}} \exp (0.69), \quad \gamma_{01} = 0.11, \quad \gamma_{11} \approx 3,
\]

(39)

which implies

\[
R(s) = \left( 3 \sum_{i=1}^{n_f} Q_i^2 \right) \left[ 1 + a_s + 0.08 a_s^2 - 23.3 a_s^3 \right] - \left( \sum_{i=1}^{n_f} Q_i \right)^2 1.24 a_s^2,
\]

(40)

where

\[
a_s = a \left[ \mu^2_{\text{BLM}} (1 + \gamma_1(n_\ell)a(\mu^2_{\text{BLM}}) \right].
\]

(41)

Notice that the coefficient of the NNLO correction is, indeed, not small. Therefore it is not true, as conjectured in Ref. [8], that the BLM-improved perturbative series have in general significantly smaller coefficients than the expansions in the \( \overline{\text{MS}} \).

\(^3\)Note the factor-of-two difference between our definition (8) of the \( \beta \)-function and that used in Ref. [10].
2.6 RS invariant perturbation theory

The basic problem of RS dependence can be traced back to the fact that the expansion parameter, the renormalized coupling $a$, is not a physical quantity, but rather an intermediate variable, allowing us to correlate different physical quantities. As such, it is inevitably ambiguous. This problem can be circumvented — at least in part — by expressing one physical quantity directly as power expansion in terms of the other. Consider, for instance, two physical quantities admitting the following perturbation expansions in some RS$^4$

$$R^{(1)} = a(RS) \left[ 1 + r_1^{(1)}(RS)a(RS) + \cdots \right]$$
$$R^{(2)} = a(RS) \left[ 1 + r_1^{(2)}(RS)a(RS) + \cdots \right]$$

Expressing $a(RS)$ from the first in terms of $R^{(1)}$ and substituting into the second equation we get

$$R^{(2)} = R^{(1)} \left[ 1 + \Delta^{(2,1)}R^{(1)} + \cdots \right],$$

where $\Delta^{(2,1)} = r_1^{(2)}(RS) - r_1^{(1)}(RS)$, as well as all the other coefficients $\Delta^{(2,j)}, j \geq 2$ of this expansion, are unique. This kind of expansion has already been discussed within the so-called ‘scheme invariant perturbation theory’ in Ref. [12] and recently resurrected within the so called ‘commensurate scale relations’ in Ref. [13]. The essence, however, remains the same.

As a special and interesting example of such a relation, consider the one between the derivative of $\gamma$ with respect to the external momentum $Q$ and the quantity $r(Q)$ itself. It is a straightforward exercise to show that this relation reads:

$$\frac{d\Gamma(Q)}{d\ln Q} = -b[r(Q)]^2 \left\{ 1 + cr(Q) + \rho_2[r(Q)]^2 \cdots \right\} = \beta EC [r(Q)],$$

where the r.h.s. is nothing else than the ‘effective’ $\beta$–function introduced above, evaluated at $r(Q)$!

In (43) as well as in (44) there is no trace of any RS ambiguity. It is nice to be able to show explicitly, as is done in Refs. [12–14], that all the coefficients in these expansions are, indeed, RS invariants, but it cannot be otherwise, as they relate two physical variables and there is no way their eventual dependence on the RS could be cancelled. Perturbation theory serves here merely as an intermediate, but vital, tool for evaluating coefficients such as $\Delta^{(2,j)}$ and $\rho_i$.

If relations such as (43) or (44) are truly unique, do they not solve the whole RS–dependence problem? The answer depends on what we expect from the perturbation theory. If we are interested merely in relating pairs of physical quantities admitting purely perturbative expansions the answer is positive. If, however, we ask the question: what is, on the basis of analyses of available experimental data, the QCD prediction for, say, the $Z^0$ width, then the answer is definitely negative. The point is that in predicting the value of $\Gamma^a$ from the relations between this quantity and some other physical quantity $R$, we find that the resulting predictions depend on the choice of the quantity $R$! What

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$^4$There is no problem generalizing this analysis to the case of different powers of the leading terms.
has been thrown out the door in the form of the RS ambiguity, comes back through the window as the ‘initial condition’ ambiguity [15]. Moreover, this new one is even more difficult to handle than the original one!

We therefore believe that the better way of incorporating the results of QCD analyses of many different physical quantities measured in different kinematical ranges is to introduce some intermediate, non-unique and thus unphysical, variable, which can then be used for QCD predictions of other physical quantities. The renormalized coupling serves just this purpose.

## 3 Quark mass thresholds in the running $\alpha_s$

The proper treatment of quark mass effects in the QCD running coupling constant $\alpha_s$ has so far not been in the forefront of interest of theorists and phenomenologists. This is now changing. Although the quark mass effects are not large, steady improvement in the precision of experimental data, coming in particular from a new generation of experiments at CERN and Fermilab, combined with significant progress in higher-order QCD calculations, has led to a renewed interest in quantitative aspects of these effects [16, 17, 18]. There are two basic reasons for this, both of which are relevant to the subject of this article.

The first concerns the exploitation of the complete NNLO QCD calculations that have recently become available for quantities such as (38) [19] or $\Gamma_h$ [20, 21] and which exist basically for massless quarks only. The NNLO corrections are tiny effects and to include them makes sense only if they are large compared to errors resulting from the approximate treatment of the quark mass thresholds in massless QCD. To quantify the importance of the NNLO correction to the coupling $\alpha_s$, consider the difference between the values of $\alpha_s(M_{\overline{MS}}, M_Z)$ in the NLO and NNLO approximations, assuming five massless quarks and taking, as an example, $\Lambda_{\overline{MS}}^{(6)} = 0.2$ GeV. At the NLO the couplant $\alpha \equiv \alpha_s/\pi$ is given as a solution to Eq. (14) for $c_2 = 0$ and we get $a_{NLO}^{MS, M_Z} = 0.03742$. At the NNLO and in $\overline{MS}$ RS, where $c_2(\overline{MS}, n_f = 5) = 1.475$, we find $a_{NNLO}^{MS, M_Z} = 0.03665$. The relative difference between these two approximations,

$$\frac{a_{NLO}^{MS, M_Z} - a_{NNLO}^{MS, M_Z}}{a_{NLO}^{MS, M_Z}} = 0.02,$$

thus amounts to about 2%. The NNLO correction should therefore be included only if the neglected effects can be expected to be smaller than this number. However, as we shall see, quark mass threshold effects can in some circumstances be of just this magnitude!\footnote{In general, as the magnitude of higher-order corrections to $\alpha_s$ depends on the renormalization scheme employed, as does also the estimate (45). However, if defined as the relative difference between the NLO and NNLO approximations, this dependence is not strong (see the concluding paragraph of Section 4).}

The second reason is related to the problem of comparing the values of $\alpha_s$, determined from different quantities characterized by vastly different momentum scales. As recently emphasized in an extensive review of $\alpha_s$ determinations [22], there is a small, but non-negligible discrepancy between the value of $\alpha_s(M_{\overline{MS}}, M_Z)$ obtained by extrapolation from
some of the low-energy quantities, and $\alpha_s(\overline{\mathrm{MS}}, M_Z)$ determined directly at the scale $M_Z$ at LEP — the latter giving the value higher by about 5-10%. Simultaneously, it has been noted in Ref. [22] that there is an exception to this behaviour in the case of the ratio $R_e$, which, when extrapolated from $m_\tau$ to $M_Z$, gives values of $\alpha_s$ close to those measured directly at LEP ($0.120 \pm 0.005$ [23]). The physical relevance of the discrepancy between the low-energy extrapolations and direct measurements of $\alpha_s$ at LEP has very recently been emphasized in Ref. [24]. In particular, its author argues that the extrapolation of $\alpha_s(m_\tau)$ to $\alpha_s(M_Z)$ is unreliable due to limited control of the power corrections. The question of estimating the theoretical uncertainty in the extraction of $\alpha_s(m_\tau)$ from data on $R_e$ is also discussed in Ref. [25]. As shown in Ref. [18], a part of this overestimate of the extrapolated value of $\alpha_s(M_Z)$ pointed out in Ref. [24] may in fact be due to the approximate treatment of the $c$ and $b$ quark thresholds.

We shall now analyze the quantitative consequences of the exact treatment of quark mass thresholds at the LO and formulate the conventional matching procedure [26] for massless quarks in such a way that its results are so close to those which are exact that the available NNLO calculation can be consistently included. We shall describe in detail the approximation in which the $u, d$ and $s$ quarks are considered massless while the $c, b$ and $t$ quarks remain massive.

As complete multiloop calculations with massive quarks are very complicated and available only at the leading order, all higher-order phenomenological analyses use the calculations with a fixed effective number $n_f$ of massless quarks, depending on the characteristic scale of the quantity. For relating two regions of different effective numbers of massless quarks the approximate matching procedure developed in Ref. [26] is commonly used. It should be emphasized that this procedure concerns only those mass effects that can be absorbed in the renormalized couplant. At higher orders there are, however, mass effects that remain in the expansion coefficients even after the effects of heavy quarks have been absorbed in a suitably defined running couplant.

In QCD with massive quarks the renormalization group equation for the couplant $a(\mu)$ formally looks the same as in massless QCD. The only, but important, difference concerns the two lowest order $\beta$-function coefficients, $b, \beta_1 = bc$, which are no longer unique as in massless QCD, but may depend on the scale $\mu$. While in the class of $\overline{\mathrm{MS}}$-like renormalization conventions $b = 11/2 - n_f/3$, exactly as in massless QCD, in MOM-like ones it becomes a nontrivial function of the scale $\mu$ [27]:

$$b(\mu/m_i) = \frac{11}{2} - \frac{1}{3} \sum_i h_i(x_i) , \quad x_i \equiv \frac{\mu}{m_i} ,$$  \hspace{1cm} (46)$$

where the sum runs over all the quarks considered, $m_i$ are the corresponding renormalized quark masses and the threshold function $h(x)$ is given in Ref. [27]. The shape of the function $h(x)$ is actually not quite unique and depends on the vertex chosen for the definition of the renormalized couplant. This fact was first noted in Ref. [28] and subsequently explained in Ref. [29]. The form of $h(x)$ used below corresponds to quark-gluon-quark vertex with massless quarks, using any momentum configuration and any invariant decomposition in MOM-like RS [29]. It is appropriate for most of the extrapolations from

\[\text{footnote}\]

\[\text{footnote text}\]
low energy quantities to LEP energy range. The same form is valid for the ghost-gluon-ghost vertex. The three gluon vertex gives somewhat different form of \( h(x) \), though its behaviour for small and large \( x \) is the same. According to [27]:

\[
    h(x) \equiv 6x^2 \int_0^1 \frac{z^2(1-z)^2}{1+z^2z(1-z)} = 1 - \frac{6}{x^2} + \frac{12}{x^3\sqrt{4+x^2 + x}} \ln \frac{\sqrt{4+x^2 + x}}{\sqrt{4+x^2 - x}} \simeq \frac{x^2}{5 + x^2}. \tag{47}
\]

The last, approximate, equality is a very accurate approximation of the exact form of \( h(x) \) in the whole range \( x \in (0, \infty) \). This allows a simple treatment of the quark mass thresholds at the LO. There is, unfortunately, no analogous calculation of the next \( \beta \)-function coefficient, \( b_c \), for massive quarks. This is one of the reasons why most of the phenomenological analyses use the so-called ‘step’ approximation, in which at any value of \( \mu \) one works with a finite effective number of massless quarks, and which changes discontinuously at some matching points \( \mu_i \). Consequently, \( b(n_f) \) effectively becomes a function of \( \mu \), discontinuous at these matching points, as shown in Fig. 1a.

![Figure 1](image)

Figure 1: a) \( b(\mu/m_i) \) together with three step approximations, corresponding to matching at the points \( \mu_i = \kappa m_i; i = c, b, t \) with \( \kappa = 1, 1.58, 2.24 \); b) the ratio \( R_a \) for the same three approximations, made to coincide at \( \mu_0 = 1 \text{ GeV} \).

The matching points are assumed to be proportional to the masses of the corresponding quarks, \( \mu_i \equiv \kappa m_i \). In principle, a different quark threshold could be associated with a different \( \kappa \), but for simplicity’s sake we take them equal. The free parameter \( \kappa \), allowing for the variation of the proportionality factor, turns out to be quite important for the accuracy of the step approximation. At the LO the matching procedure then consists of
the following relations at the matching points $\mu_i$ (the numbers in superscript define the corresponding effective number of massless quarks)\textsuperscript{7}:

\[
\begin{align*}
\delta_{\text{app}}^{\text{LO},3} (\kappa m_c / \Lambda^{[3]}) &= \delta_{\text{app}}^{\text{LO},4} (\kappa m_c / \Lambda^{[4]}) \Rightarrow \Lambda^{[4]} = \Lambda^{[3]} \left( \frac{\Lambda^{[3]}}{\kappa m_c} \right)^{1/3\delta(4)} \quad (48) \\
\delta_{\text{app}}^{\text{LO},4} (\kappa m_b / \Lambda^{[4]}) &= \delta_{\text{app}}^{\text{LO},5} (\kappa m_b / \Lambda^{[5]}) \Rightarrow \Lambda^{[5]} = \Lambda^{[4]} \left( \frac{\Lambda^{[4]}}{\kappa m_b} \right)^{1/3\delta(5)} \quad (49) \\
\delta_{\text{app}}^{\text{LO},5} (\kappa m_t / \Lambda^{[5]}) &= \delta_{\text{app}}^{\text{LO},6} (\kappa m_t / \Lambda^{[6]}) \Rightarrow \Lambda^{[6]} = \Lambda^{[5]} \left( \frac{\Lambda^{[5]}}{\kappa m_t} \right)^{1/3\delta(6)} \quad (50).
\end{align*}
\]

Note that each of the intervals of fixed $n_f$ is associated with a different value of the $\Lambda$-parameter, $\Lambda^{[n_f]}$. The resulting dependence of $a(\mu/m_i)$ on $\mu$ is thus continuous at each of the matching points, but its derivatives at these points are discontinuous, reflecting the discontinuity of the step approximations to $b(\mu/m_i)$. This procedure can easily be extended to any finite order. Let us point out that the more sophisticated procedure for matching the couplings corresponding to different effective $n_f$ developed in Ref. [30] coincides in the LO with the above relations (48–50). To estimate the errors in $\alpha$, resulting from the above-defined approximate treatment of quark thresholds we merely need solve the LO equation with exact explicit mass dependence as given in (46):

\[
\frac{da(\mu)}{d\ln \mu} = -a^2 \left( \frac{11}{2} - \frac{1}{3} \sum_{i=1}^{6} h(x_i) \right). \quad (51)
\]

For our purposes the approximation $h(x) \doteq x^2/(5 + x^2)$ is entirely adequate and yields

\[
a(\mu) = \frac{1}{\left( \frac{11}{2} - \frac{3}{3} \right) \ln \frac{\mu}{\Lambda^{[3]}} - \frac{1}{3} \sum_{i=c,b,t} \ln \frac{\sqrt{\mu^2 + 5m_i^2}}{(\Lambda^{[3]})^2 + 5m_i^2}} , \quad (52)
\]

where the fraction $\frac{3}{3}$ comes from the sum over the three massless quarks $u, d$ and $s$ and $\Lambda^{[3]}$ is the corresponding $\Lambda$-parameter appropriate to three massless quarks. For the heavy quarks $c, b$ and $t$ we take in the following $m_c = 1.5$ GeV, $m_b = 5$ GeV, $m_t = 170$ GeV. The distinction between the ‘light’ and ‘heavy’ quarks is given by the relative magnitude of $m_i$ and $\Lambda$, the latter being defined by the condition $5m_i^2 \gg \Lambda$. For the above values of $m_c, m_b$ and $m_t$ this condition is very well satisfied. Consequently, for $\mu \ll m_i, i = c, b, t$ (52) approaches smoothly $a^{\text{LO}}$ for $n_f = 3$, while for $\mu \gg m_i$, and neglecting $\Lambda^{[3]}$ with respect to $5m_i^2$, it goes to

\[
a(\mu) = \frac{1}{b(6) \ln \frac{\mu}{\Lambda^{[3]}} + \frac{1}{3} \ln \left( \frac{\sqrt{5m_c}}{\Lambda^{[3]}} \frac{\sqrt{5m_b}}{\Lambda^{[3]}} \frac{\sqrt{5m_t}}{\Lambda^{[3]}} \right)} = \frac{1}{b(6) \ln \frac{\mu}{\Lambda^{[6]}(\sqrt{5})}} , \quad (53)
\]

\textsuperscript{7}The parameter $\Lambda$ appearing in this as well as all the other formulae in this section is the leading order $\Lambda$ parameter, $\Lambda^{\text{LO}}$, which cannot be associated with any well-defined RS. As there is no danger of confusion, we shall drop the superscript ‘LO’ for the remainder of this section.
where the parameter $\Lambda^{(6)}(\kappa)$ depends in general on $\kappa$ and

$$\Lambda^{(6)}(\sqrt{5}) = \Lambda^{(3)} \left( \frac{\Lambda^{(3)]}}{\sqrt{5}m_c} \frac{\Lambda^{(3)]}}{\sqrt{5}m_b} \frac{\Lambda^{(3)]}}{\sqrt{5}m_t} \right)^{\frac{1}{\sqrt{5}}} = \left( \frac{1}{\sqrt{5}} \right)^{\frac{1}{\sqrt{5}}} \Lambda^{(6)}(1) \quad (54)$$

coincides with $\Lambda^{(6)}$ defined via the subsequent application of the matching relations (48–50) for $\kappa = \sqrt{5} \approx 2.24$. Even though from the point of view of the matching procedure, $\kappa$ is not exactly fixed, the value $\kappa = \sqrt{5}$ will be shown to be in a certain sense the best choice. The relation between $\Lambda^{(6)}$ and $\Lambda^{(3)}$ depends nontrivially on $\kappa$.

In Fig. 1a, $b(\mu)$ is plotted as a function of $\mu$ for the above-mentioned masses of $c$, $b$ and $t$ quarks, together with its step approximations and corresponding to three different values of $\kappa = 1, \sqrt{5}, \sqrt{5}/2$. There is hardly any sign of the steplike behaviour of the function $h(x)$ in the region of the $c$ and $b$ quark thresholds and only a very unpronounced indication of the plateau between the $b$ and $t$ quark thresholds. The step approximations are poor representations of the exact $h(x)$, primarily due to the rather slow approach of $h(x)$ to unity as $x \to \infty$. However, there is a marked difference between the three approximations. While the step approximation with the conventional choice $\kappa = 1$ underestimates the true $h(x)$ in the whole interval displayed, and would do so even when some smoothing were applied, $\kappa = \sqrt{5}$ gives clearly much better approximation as the corresponding curve is intersected by the exact $h(x)$ at about the middle of each step.

In Fig. 1b the $\mu$ dependence of the ratio

$$R_a \equiv \frac{a_{ex}^{L_0}(\mu)}{a_{ex}^{L_0}(\mu)} \quad (55)$$

between the above exact solution (53) and the approximate expressions for the above mentioned values of $\kappa$, is plotted assuming $\Lambda^{(3)} = 200$ MeV. As we basically want to compare the results of different extrapolations starting from the same initial $\mu_0$, $\Lambda^{(3)}$ used in the approximate solutions has been rescaled by the factor 1.004 with respect to $\Lambda^{(3)}$ in (52), thereby guaranteeing that all expressions coincide at $\mu_0 = 1$ GeV. Any deviation from unity in Fig. 1b is then entirely the effect of an approximate treatment of the heavy quark thresholds. Figure 1 contains several simple messages.

The approximate solutions based on the matching procedure defined in (48–50) are generally much better immediately below the matching point than above it, and worst at about $5m_{\text{match}}$. This reflects the fact that the function $h(x)$ vanishes fast (like $x^2$) at zero but approaches unity for $x \to \infty$ only very slowly. Moreover, in the $M_Z$ range the effect of the $c$ quark threshold is essentially the same as that of the $b$ quark and both are much more important than that of the top quark, although $M_Z/m_c \approx 60$, $M_Z/m_b \approx 18$, while $M_Z/m_t \approx 1/2$!

The effect of varying $\kappa$ is quite important, in particular with respect to the $c$ and $b$ quark thresholds. In general, $\kappa > 1$ improves the approximation above, but worsens it immediately below the matching point. The choice $\kappa = \sqrt{5}$, suggested by the asymptotic

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8The resulting ratio $R_a(M_Z)$ depends only weakly on $\Lambda^{(3)}$. Note, however, that the current analysis of the LEP data gives the value of $\Lambda^{(3)}$ at about 500–700 MeV, in disagreement with the QCD sum rules analysis [24].
behaviour of (53), is clearly superior in practically the whole displayed interval $\mu \in (1, 10^4)$ GeV and leads to an excellent (on the level of 0.1%) agreement with the exact solution in this interval. On the contrary, the conventional choice $\kappa = 1$ leads to a much larger deviation from the exact result, which exceeds 2% in most of this region. This discrepancy is of the same magnitude as the effects of NNLO corrections to the couplant itself. It thus turns out that the effect of an exact treatment of the quark mass thresholds for the extrapolation of $\alpha_s$ from the scales around $\mu \approx 2$ GeV up to $\mu = M_Z$ is as important as that of the NNLO correction to $\alpha_s$ and must therefore be taken into account whenever the latter is considered and compared with $\alpha_s$ determined at these vastly different scales.

On the other hand, the preceding discussion tells us little about the accuracy of the approximation of five massless quarks directly at the scale $M_Z$, for instance when calculating $\Gamma_h$. This question will be addressed in Section 5.

The analysis of quark mass effects in $\alpha_s$, presented above strictly speaking holds only for the LO. Nevertheless, as both the mass effects and the higher order perturbative corrections are small effects, it seems reasonable to expect that the conclusions drawn in this section will have more general validity.

4 Application to $\Gamma_h$

We now come to the quantitative estimate of the theoretical uncertainties of perturbative QCD predictions for $\Gamma_h$. In the preceding section we discussed the approximation in which $\alpha_s$ is given by an expression corresponding to five massless quarks. In this section we quantify the uncertainties resulting from the RS ambiguity, discussed in Section 2. The non trivial dependence of $\Gamma_h$ on $m_b$ and $m_t$, coming from effects not included in the running couplant corresponding to five massless quarks, is discussed in the next section. Nevertheless in order to avoid unnecessary repetition we include the dependence on the ratio $M_Z/m_t$ already in the formulae quoted below and obtained recently in Refs. [19, 20, 21, 31, 32]. The dependence of the expansion coefficients on the bottom quark mass has extensively been studied in Ref. [33]. We do not write it out here explicitly, as this would further complicate the structure of (56), each of the coefficients in (56) becoming a different function of the ratio $m_b/M_Z$. At the end of this section we shall merely recall the leading contributions to $R_Z$ coming from $m_b/M_Z$ terms and discuss their numerical importance.

In order to quantify the theoretical uncertainty related to the choice of the RS, we first define the set of ‘allowed’ RS. As emphasized above, this is to large extent a subjective matter. Based on our previous experience we define as a measure of this uncertainty the difference between the results obtained (for the same $\Lambda_{\overline{MS}}$) in the three principal methods set out in Section 3: PMS, ECH and $\overline{MS}$. This choice is, to large extent of course, arbitrary, but as the $\overline{MS}$ RS is used in most phenomenological analyses, we adopt it for the lack of anything better. The formulae quoted below are taken from Ref. [21].

The basic quantity of interest, $R_Z$, defined in (1), has a non trivial structure which mixes the effects of electroweak interactions with those of pure QCD. It can be written in the following decoupled form (i.e., for five massless flavours and the explicit $m_t$ dependence of the expansion coefficients) as the sum of three terms with different electroweak factors.
and separated further into four possible combinations of vector, axial vector and singlet, non-singlet contributions:

\[
R_Z = \left( R_{V,NS} + R_{A,NS} \right) + R_{V,S} + R_{A,S} \\
= \sum_{i=1}^{5} \left( g_{V,i}^2 + g_{A,i}^2 \right) \left[ 1 + a^{(5)} + (a^{(5)})^2 r_1 + (a^{(5)})^3 r_2 \right] \\
+ \left( \sum_{i=1}^{5} g_{V,i} \right)^2 \left[ (a^{(5)})^3 s_3 + (a^{(5)})^3 s_{3^{top}} \right] \\
+ \left( \frac{1}{4} \right) \left[ (a^{(5)})^2 t_2 + (a^{(5)})^3 t_3 \right],
\]

(56)

where \( g_{V,i} = t_{3,i} - 2Q_i \sin^2 \theta_W \), \( g_{A,i} = t_{3,i} \), \( t_3 \) is the third component of the weak isospin and the sums over the electroweak coupling constants equal

\[
\Gamma_1 \equiv \sum_{i=1}^{5} \left( g_{V,i}^2 + g_{A,i}^2 \right) = \frac{5}{2} + \frac{44}{9} \sin^4 \theta_W - \frac{14}{3} \sin^2 \theta_W = 1.6807 \pm 0.0012
\]

(57)

\[
\Gamma_2 \equiv \left( \sum_{i=1}^{5} g_{V,i} \right)^2 = \left( \frac{1}{2} + \frac{2}{3} \sin^2 \theta_W \right)^2 = 0.42850 \pm 0.00028.
\]

(58)

The last equalities in the above relations correspond to the world average \( \sin^2 \theta_W = 0.2329 \pm 0.0005 \). Note that (57) implies

\[
\frac{\Delta \Gamma_2}{\Gamma_2} \approx 6.5 \times 10^{-4} < \frac{\Delta \Gamma_1}{\Gamma_1} \approx 7.1 \times 10^{-4} \ll \frac{\Delta \Gamma_h}{\Gamma_h},
\]

(59)

which, combined with (5), means that the theoretical uncertainties of QCD predictions — in the square brackets in (56) — should be compared to the error of \( R_Z \) itself.

The expansion coefficients entering the above formula can be expressed as functions of the ratio \( x \equiv M_Z/m_t(M_Z) \), where \( m_t(M_Z) \) is the renormalized, ‘running’ mass of the top quark, taken at the scale \( M_Z \). For \( n_f = 5 \) and in small \( x \) expansions we have [21]:

\[
r_1 = 1.409 \\
+ 0.065185 - 0.014815 \ln x \times x \\
+ [-0.0012311 + 0.00039683 \ln x] \times x^2 \\
+ [0.000061327 - 0.000023516 \ln x] \times x^3 + \mathcal{O}(x^4)
\]

(60)

\[
r_2 = -12.767 \\
+ [-0.17374 + 0.21242 \ln x - 0.037243 \ln^2 x] \times x \\
+ [-0.0075218 - 0.00098589 \ln x + 0.0003805 \ln^2 x] \times x^2 \\
+ [0.00050411 - 0.00012099 \ln x + 0.00031449 \ln^2 x] \times x^3 + \mathcal{O}(x^4)
\]

(61)

\[
s_3 = -0.41318
\]

\[
s_{3^{top}} = 0.027033 x + 0.036355 x^2 + 0.00058874 x^3 + \mathcal{O}(x^4)
\]

(62)

\[
t_2 = -3.0833 + \ln x + 0.086420 x + 0.0058333 x^2 + 0.0006288 7x^3 + \mathcal{O}(x^4)
\]

(63)
Because of different electroweak factors in front of them, each of the expressions in the square brackets of (56) is separately, from the point of view of QCD, RS invariant. As the optimization according to either the PMS or ECH methods does not commute with the operation of addition, the first question we have to answer is the order of these operations. In the absence of uncertainties in the values of the electroweak factors, the proper way would be first to sum all three terms in (56) and then to fix the RS. In reality, however, the errors of electroweak factors induce uncertainties in values of the coefficient multiplying powers of the QCD couplant. To optimize, in one way or another, the resulting QCD perturbative expansion (56) in such circumstances is not a well-defined exercise and we therefore have chosen to follow the opposite route. We believe that to get an estimate of the RS dependence of QCD calculations this second route is adequate. In this section we shall discuss the numerical importance of the first two terms in (56) for $m_t = 0$. The effects of non-zero $m_t$, as well as the third term in (56), the existence of which is also closely related to non-zero $m_t$, are dealt with in the next section. The second term of (56), multiplied by (58), is given at the LO only and no optimization is possible. Fortunately it contributes, in $\overline{\text{MS}}$ RS, a mere $10^{-5}$ to $R_Z$ and is thus clearly negligible.

The dominant contribution to $R_Z$ comes from the first term in (56). The term in the square brackets can be written as $1 + r(M_Z)$, where $r(Q)$ has exactly the form of (7) and, moreover, for massless quarks coincides with the above expression for (38) [19]. For $n_f = 5$ the crucial RG invariant $\rho_2 = -15.45$, which implies (for detailed discussion see Ref. [4]) that the saddle point of the NNLO approximant $r^{[3]}(\rho)$ will occur at the point $(a_\text{PMS}, c_2^{\text{PMS}})$, where $c_2^{\text{PMS}}(\rho) \rightarrow 1.5 \rho_2 \div -23.2$. In Fig. 2 $r^{[3]}(\rho = 27)$ is plotted as a function of $a, c_2$ near this saddle point, together with the contours of the constant $r^{[3]}$. 

$$t_3 = 18.654 + 1.7222 \ln x + 1.9167 \ln^2 x$$
$$+ [-0.12585 + 0.28646 \ln x - 0.011111 \ln^2 x] x$$
$$+ [-0.003132 + 0.012117 \ln x - 0.0011905 \ln^2 x] x^2$$
$$+ [-0.00088827 + 0.00047262 \ln x - 0.00017637 \ln^2 x] x^3 + O(x^4).$$

Because of different electroweak factors in front of them, each of the expressions in the vicinity of the saddle point for $\rho = 23$. 

Figure 2: The NNLO approximant $r^{[3]}(a, c_2)$ in the vicinity of the saddle point for $\rho = 23$. 

$$n_f = 5, \, \rho = 27, \, \rho_2 = -15.45$$
We have calculated the NLO as well as the NNLO approximants of the quantity in the first square bracket of (56), with the unity subtracted, in the three chosen RSs and in the interval $\rho \in (18, 28)$, which corresponds to the measured value of $M_Z \approx 91.4$ GeV and $\Lambda_{\text{MS}}(5)$ in the interval $\Lambda_{\text{MS}}(5) \in (50, 500)$ MeV. As the differences are tiny we normalize all our results to the NLO result in the conventional $\overline{\text{MS}}$ RS, and plot the relative difference,

$$r_\rho = \frac{r^{(i)}(\text{RS})}{r^{\text{NLO}}(\Lambda_{\overline{\text{MS}}})} - 1,$$

where $i = \text{NLO, NNLO and RS} = \text{PMS, ECH or } \overline{\text{MS}}$. We draw the following conclusion from Fig. 3:

1. The differences between PMS and ECH approaches are minuscule, about 0.1% at the NLO and totally negligible at the NNLO.

2. The difference between the PMS (or ECH) and $\overline{\text{MS}}$ approaches is
   - about 0.7% at the NLO and
   - about 0.3% at the NNLO (this comes from the ratio of the dotted and dash–dotted curves in Fig. 3). This documents the trend, observed in earlier works, that inclusion of the NNLO corrections diminishes the RS dependences and thus decreases the theoretical uncertainty.

3. The differences between the NLO and NNLO approximations amount to about 2% in the $\overline{\text{MS}}$ RS (as already estimated in Section 3) and to about 3% for PMS (or ECH).

Given the current precision of the data (5), which translates into 8.5% accuracy on the couplant $a(M_Z)$, this implies that none of the mentioned differences is discernible. Moreover, only the last difference, namely the effect of including the NNLO correction, has any chance of being seen in the data, but even this would require at least a factor three improvement in the precision of $\Gamma_h$. 

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5 The dependence of $R_Z$ on $m_b$ and $m_t$

As already pointed out, massive quarks complicate the consistency conditions, discussed in Section 2, as their renormalized masses also 'run' and there is so far no generalization of the relations (19) available. We therefore estimate the effects of non-zero top and bottom quark masses within the \( \overline{\text{MS}} \) RS. Recall that (56) are expansions in \( a^{(6)} \) corresponding to five massless quarks, and thus all effects of finite \( m_b \) and \( m_t \) are contained in the expansion coefficients.

First we deal with the effects of the finite \( x = M_Z/m_t \) in the first and third term of (56). In the first case the \( x \)-dependent terms of \( r_1 \) and \( r_2 \) vanish in the limit \( x \to 0 \) and the leading, constant, ones correspond to \( n_f = 5 \) in accordance with the decoupling theorem [35]. For \( x = 0.4225 \), corresponding to \( M_Z = 91.187 \) and \( m_t(M_Z) = 140 \text{ GeV} \), as suggested in Ref. [21], \( R^{V,N_S} + R^{A,N_S} \) changes by the factor \( 3.7 \times 10^{-5} \), which is an effect two orders of magnitude smaller than the experimental error of \( R_Z \), estimated in (5), and one order of magnitude smaller than the effects discussed in the preceding section. It can therefore be safely neglected.

The contribution of the last term in (56), coming from the axial vector, flavour singlet channel, and where the familiar axial anomaly operates, is numerically non-negligible. Despite the fact that the leading terms in \( t_2 \) and \( t_3 \) are \( x \)-independent, the non-vanishing of \( R^{A,S} \) is due to non-zero \( m_t \), or, more precisely, to the effect of the difference \( m_t - m_b \).\(^9\) This comes from the fact that axial couplings of quarks in weak doublets are opposite and, apart from mass effects, their contributions cancel. Consequently, for \( m_t = 0 \) both \( t_2 \) and \( t_3 \), and thereby also \( R^{A,S} \), must vanish. The fact that this vanishing is not obvious in the small \( x \) expansions of (56) is not surprising, as \( m_t \to 0 \) corresponds on the contrary to \( x \to \infty \). Note that the presence of powers of \( \ln x \) in the coefficients \( t_2 \) and \( t_3 \) implies that a resummation of these logarithms is necessary [32] even in the limit \( x \to 0 \). As the contribution \( R^{A,S} \) can be interpreted as the top quark mass effect, we again determine its contribution in the \( \overline{\text{MS}} \) RS only. For \( m_t(M_Z) = 140 \text{ GeV} \), \( R^{A,S} = -0.00160 \).

The effects of finite value of \( m_b/M_Z \) have been studied in detail in Ref. [33]. Here we merely recall the form and numerical values of the leading contributions in the vector and axial vector, non-singlet \( b \bar{b} \) channels resulting from these effects:

\[
R^{V,N_S}_{b} = \frac{g^2_{V,b}}{1 + \left[ 1 + 12 \frac{m^2_b(M_Z)}{M^2_Z} \right] a^{(5)}(M_Z) + \mathcal{O}(a^{(5)})},
\tag{66}
\]

\[
R^{A,N_S}_{b} = \frac{g^2_{A,b}}{1 - 6 \frac{m^2_b(M_Z)}{M^2_Z} + \mathcal{O}(a^{(5)})}.
\tag{67}
\]

For \( m_b(M_Z) = 4.8 \) [33] and \( a^{(5)}(M_Z) = 0.037 \) the leading \( m_b/M_Z \) contribution to \( R_Z \) coming from the axial channel thus amounts to \( 4.2 \times 10^{-3} \), whereas in the vector channel we get a mere \( 1.5 \times 10^{-4} \).

\(^9\) An alternative interpretation of this effect is discussed in Ref. [36].
6 The estimate of the still-higher-order terms

As we saw in the preceding two sections, the NNLO corrections to $\Gamma_h$ are, compared to experimental accuracy of its measurement, tiny effects, and so it seems reasonably safe to stop at this order. On the other hand it is generally accepted that perturbative series in QCD do not converge, but represent merely asymptotic expansions to the full result. In such situation it is certainly useful to have at least some estimate of the magnitude of the so-far uncalculated (and in the near future uncalculable) higher orders.

An attempt in this direction has recently been made [37], using the so-called ‘improvement formula’ of Ref. [2], which represents an approximation of the PMS optimization discussed in Subsection 2.3. Its essence is to re-expand the PMS result optimized to the Nth order in powers of the couplant in any fixed RS\(^10\) and take the coefficient of this expansion at the (N+1)th order as an estimate of its true value. Instead of the PMS approach one of Subsection 2.4 can be equally well used for this purpose. The resulting estimates are only slightly different.

Here we outline the main steps of this method for the quantity (38), closely related to $\Gamma_h$. Consider first the Nth order partial sum of the perturbative expansion for a physical quantity $D^{11}$:

$$D_N = \sum_{i=0}^{N-1} d_i a^{i+1} .$$

Carrying out the optimization of (68) according to either the PMS or ECH approaches leads to the optimized result, denoted below as $D_N^{\text{opt}}(a_{\text{opt}})$. If we now re-expand $D_N^{\text{opt}}(a_{\text{opt}})$ in terms of the couplant $a(RS)$ in a chosen RS\(^{12}\) we find

$$D_N^{\text{opt}}(a_{\text{opt}}) = D_N(a) + \delta D_N^{\text{opt}} a^{N+1} ,$$

where

$$\delta D_N^{\text{opt}} = \Omega_N(d_i, c_i) - \Omega_N(d_i^{\text{opt}}, c_i^{\text{opt}})$$

give, according to Ref. [37], the estimate of the coefficient $d_N$ in the chosen RS. For the three lowest orders the functions $\Omega_N(d_i, c_i)$ are given as [37]:

$$\Omega_2 = d_0 d_1 (c_1 + d_1)$$
$$\Omega_3 = d_0 d_1 (c_2 - \frac{1}{2} c_1 d_1 - 2d_1^2 + 3d_2)$$
$$\Omega_4 = \frac{d_0}{3} (3c_3 d_1 + c_2 d_2 - 4c_2 d_1^2 + 2c_1 d_1 d_2 - c_1 d_3 + 14d_1^3 - 28d_1^2 d_2 + 5d_2^2 + 12d_1 d_3).$$

These formulae can be derived from the following exact equations relating $\Omega_j$ to the coefficients $d_j, c_j$ and the RG invariants $\rho_j$:

$$d_j = \frac{\rho_j}{j-1} - \frac{c_j}{j-1} + \frac{\Omega_j}{d_0} .$$

\(^{10}\)The optimized result is, of course, independent of the choice of this RS and is constructed from quantities up to the Nth order only.

\(^{11}\)In this section we drop the specification of the number of quark flavours in the couplant $a$.

\(^{12}\)The magnitude and therefore also the estimate of higher order coefficients depends on the choice of RS. We drop the argument ‘RS’ of $a(RS)$ here.
Table 1

The estimate of the so-far uncalculated higher-order coefficients for the quantity $R(s)$ in the $\overline{\text{MS}}$ RS and using the ECH optimization procedure.

<table>
<thead>
<tr>
<th>n</th>
<th>$r_2^{\text{exact}}$</th>
<th>$r_2^{\text{estimate}}$</th>
<th>$r_3^{\text{estimate}}$</th>
<th>$r_4^{\text{estimate}} - c_3 r_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-7.84</td>
<td>-14.41</td>
<td>-166</td>
<td>-1750</td>
</tr>
<tr>
<td>2</td>
<td>-9.04</td>
<td>-12.65</td>
<td>-147</td>
<td>-1161</td>
</tr>
<tr>
<td>3</td>
<td>-10.27</td>
<td>-11.04</td>
<td>-128</td>
<td>-668</td>
</tr>
<tr>
<td>4</td>
<td>-11.52</td>
<td>-9.59</td>
<td>-112</td>
<td>-263</td>
</tr>
<tr>
<td>5</td>
<td>-12.76</td>
<td>-8.32</td>
<td>-97</td>
<td>67</td>
</tr>
<tr>
<td>6</td>
<td>-14.01</td>
<td>-7.19</td>
<td>-83</td>
<td>330</td>
</tr>
</tbody>
</table>

Note that in order to evaluate $\Omega_j$ only $d_i, c_i$ at lower orders $i \leq j - 1$ are actually needed!

If the ECH approach is used for the optimization, the formulae (70) is particularly simple as $d_i^{\text{opt}} = 0$ by definition and thus

$$
\delta D_2^{\text{ECH}} = \Omega_2(d_1, c_1) \\
\delta D_3^{\text{ECH}} = \Omega_3(d_1, d_2, c_1, c_2) \\
\delta D_4^{\text{ECH}} = \Omega_4(d_1, d_2, d_3, c_1, c_2, c_3). \tag{73}
$$

The estimate (73) is thus equivalent to the assumption that $d_N$ is dominated by the last term in (72). Extensive discussion of this assumption and its consequences is given in Ref. [37] and is also related to Ref. [38].

Using the PMS approach, the resulting estimate of $d_N$ differs from (73) by the presence of the second term in (70), which does not vanish as in the ECH approach. However, it was shown in Ref. [37] that $\Omega_2(c_i^{\text{PMS}}, c_i^{\text{PMS}})$ and $\Omega_4(c_i^{\text{PMS}}, c_i^{\text{PMS}})$ are small and $\Omega_3(c_i^{\text{PMS}}, c_i^{\text{PMS}}) = 0$. In the following numerical estimates only the ECH-based results are therefore presented.

There is one subtle point in the derivation of estimates for higher-order coefficients $r_k$ of time-like quantities like (1) or (38). For instance, $R(s)$ of (38) is related to the so-called $D$-function $D(Q^2)$, defined primarily in the Euclidean region, via the dispersion relation

$$
D(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s + Q^2)^2} ds. \tag{74}
$$

The knowledge of $R(s)$ is in principle equivalent to that of $D(Q^2)$, but as most of the optimization procedures or methods of higher-order estimates do not commute with the functional on the r.h.s. of (74), we face the question as to which quantity to apply Eq. (71). This nontrivial problem is discussed in Ref. [37], the conclusion being that they should be applied to the quantities in the Euclidean region — for instance, $D(Q^2)$. Having obtained the estimates for higher-order coefficients $d_j$, the corresponding estimates for the coefficients $r_j$ of $R(s)$ follow from the relations

336
\[ r_1 = d_1, \]
\[ r_2 = d_2 - \frac{\pi^2 b^2}{12}, \]
\[ r_3 = d_3 - \frac{\pi^2 b^2}{4}, \]
\[ r_4 = d_4 - \frac{\pi^2 b^2}{4} \left( 2d_2 + \frac{7}{3} c_1 d_1 + \frac{1}{2} c_1^2 + c_2 \right) + \frac{\pi^4 b^4}{80}. \]  

The terms, proportional to powers of \( \pi^2 \), come from the analytical continuation of the couplant \( a(\mu) \) from the Euclidean region, where \( \mu^2 < 0 \) to the Minkowskean one, where \( \mu^2 > 0 \). Taking into account the fact that in the \( \overline{\text{MS}} \) RS we have \([39]\):

\[ d_1(\overline{\text{MS}}) \approx 1.986 - 0.115 n_f, \]
\[ d_2(\overline{\text{MS}}) \approx 18.244 - 4.216 n_f + 0.086 n_f^2, \]  
\[ c_2(\overline{\text{MS}}) = \frac{77139 - 15099 n_f + 325 n_f^2}{9504 - 576 n_f}, \]

and using (73) we get the estimates, obtained originally in Ref. \([37]\), summarized in Table 2 and valid for the \( \overline{\text{MS}} \) RS\(^3\). In order to get some feeling of the possible accuracy of these estimates the above table also includes the results for the NNLO coefficient \( r_2 \), for which the exact calculations are available. In the case of the coefficient \( r_4 \), only the estimate for the combination \( r_4 - r_1 c_3 \) is presented, as the four-loop \( \beta \)-function coefficient \( c_3(\overline{\text{MS}}) \) is so far unknown.

As the dominant contribution to (1) comes from the non-singlet channel — first term of (56) — the above estimates are relevant for this quantity as well. For \( a(\overline{\text{MS}}, M_Z) = 0.037 \), \( n_f = 5 \) and using the estimates of Table 1, we find that the terms \( r_3 a^4 \) and \( r_4 a^5 \) contribute approximately \(-3 \times 10^{-4}\) and \(8 \times 10^{-6}\) respectively\(^4\). Note that while the latter contribution is entirely negligible, the former is of the same order as the RS uncertainty of the NNLO contribution.

### 7 Summary and conclusions

In the preceding sections we have analysed various contributions to, and theoretical uncertainties of, the quantity (1). The results of these analyses are summarized in Table 2. All these numbers should be contrasted with the current experimental error of \( R_Z \), which is \(5.9 \times 10^{-3}\). We see that there is a number of effects comparable to the current experimental accuracy of \( R_Z \), the most important of them being the NLO perturbative correction and, interestingly, the effects of finite \( b \) quark mass correction to the Born term in the axial channel. On the other hand, the data are not yet sufficiently precise to be sensitive to, for instance, the NNLO perturbative correction. Further improvement in the measurement of \( \Gamma_h \) is clearly very desirable.

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\(^3\)Neglecting the terms of the light-by-light type, which violate the structure of (38).

\(^4\)For the latter contribution the additional assumption, concerning \( c_3(\overline{\text{MS}}) \), had been made: \( c_3 = c_3^2/c \).
Table 2

Summary of various contributions to, and uncertainties of, $R_Z$. All numbers, except items 8, 11, 12 are correspond to $\overline{\text{MS}}$ RS.

<table>
<thead>
<tr>
<th>Type of contribution</th>
<th>Contributes to $R_Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 LO non-singlet channel</td>
<td>$+62.2 \times 10^{-3}$</td>
</tr>
<tr>
<td>2 NLO non-singlet channel</td>
<td>$-3.24 \times 10^{-3}$</td>
</tr>
<tr>
<td>3 NNLO non-singlet channel</td>
<td>$-1.08 \times 10^{-3}$</td>
</tr>
<tr>
<td>4 $N^3$LO non-singlet channel</td>
<td>$-5.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>5 $N^4$LO non-singlet channel</td>
<td>$+1.3 \times 10^{-5}$</td>
</tr>
<tr>
<td>6 $m_t$ in non-singlet channel</td>
<td>$+3.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>7 $m_t$ in singlet channel</td>
<td>$-1.6 \times 10^{-3}$</td>
</tr>
<tr>
<td>8 smooth thresholds in $\alpha_s$</td>
<td>$+1.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>9 $m_b$ effects in Born term, axial channel</td>
<td>$-4.2 \times 10^{-3}$</td>
</tr>
<tr>
<td>10 $m_b$ effects in LO term, vector channel</td>
<td>$+1.5 \times 10^{-4}$</td>
</tr>
<tr>
<td>11 RS uncertainty at NLO</td>
<td>$-4.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>12 RS uncertainty at NNLO</td>
<td>$-2.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>13 experimental error</td>
<td>$\pm 5.9 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

8 Acknowledgements

We are grateful to our colleagues at CERN, where part of this work was done, for their hospitality. In its final stage the work of one of us (A.I.K.) was supported by the Russian Fund for Fundamental Research under Grant No. 94-02-04548-a.
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[18] J. Chýla, PRA-HEP/95-1


Introduction

The cross-section of any process is obtained from the ratio $\sigma = N/L$, where $N$ is the number of events and $L$ is the luminosity. The luminosity depends on machine and beam parameters. In LEP/SLC its most precise determination comes from the measurement of the event rate of the known process through the inverse relation $L = N/\sigma_{\text{known}}$. The known process should be calculable from well-established theory, and the low-angle Bhabha (LABH) $e^+e^- \rightarrow e^+e^-$ scattering process fulfils this role. Why? Because it is dominated by $t$-channel photon exchange and, in principle, can be calculated within perturbative Quantum Electrodynamics (QED) with arbitrary precision. It is also important that at sufficiently low angles the event rate for the LABH process is higher than the $Z$ event rate at the top of $Z$ resonance. In fact, the systematic component of the experimental error is now dominant. The experimental precision of the luminosity measurement at LEP has increased dramatically from the beginning of its operation. From the initial 1.0% (already a big improvement on the 2–3% of PETRA/PEP) it has now gone below 0.1%, with a level of 0.05% not out of the question in the future! The ‘theoretical uncertainty’ in the calculation of the LABH process for realistic acceptance/cuts ($\sim 0.25\%$) is now dominant (see contributions by Jadach et al., Caffo et al. and Cacciari et al.). There is an urgent need to either decrease it to the level of the purely experimental one or better it. Is this possible? Most probably, yes.

In this brief introduction we will characterize the main features of the theoretical calculation of the LABH process. We will also touch on the question of the significance of the precise measurement of the luminosity for precision tests of the standard model.

The LABH cross-section is calculated from QED with the usual perturbative techniques. The matrix element is calculated from Feynman diagrams, and the integration over the phase space with cuts/acceptance corresponding to realistic experimental conditions is done numerically. The QED perturbative series is truncated to a certain order. Higher orders can also be estimated and summed up by certain standard techniques such as exponentiation. At PETRA/PEP $O(\alpha^4)$ calculation was sufficient (see following sections for more information). Already at that time it was concluded that the phase-space integration for complicated realistic cuts/acceptance could only be done with the help of the Monte Carlo method — or more precisely, only by using a Monte Carlo event generator.

Let us now briefly characterize the basic features of the QED perturbative calculations of the LABH cross-section. Let us start with ‘photonic’ corrections in which the Born diagram with a single $t$-channel photon exchange is dressed with any number of virtual and real photon lines. First of all, the smallness of the electron mass squared compared with the typical $t$-channel transfer ($|t| \sim 1$ GeV$^2$) has to be stressed. As a result, the QED corrections are often magnified by the so-called big logarithm $L = \ln(|t|/m_e^2)$. At LEP, even for a scattering angle as low as $\theta = 0.025$, we have $L = 15$. At the same time, relevant formulae and kinematics simplify a lot because terms proportional to $m_e^2/|t|$ may be neglected. Terms of order $\theta^2 \sim 4|t|/s$ are also often small enough to be neglected, especially at angles below 50 mrad. They usually have to be kept at the Born cross-section and in kinematics. Another useful feature of QED corrections to the LABH process is the possibility of neglecting an entire class of corrections named as up–down interference.
This is a subset of QED real and virtual corrections in which, in addition to the usual t-channel photon, other photon lines connect the upper e\(^-\) line with the lower e\(^+\) fermion line. At angles small enough (\(\theta < 100\) mrad) and for the usual experimental cuts the total —virtual plus real— contribution from all corresponding Feynman diagrams is negligible (see contributions by Jadach et al. and Caffo et al.).

In general, pure 'photonic' corrections are difficult to calculate but can be made as small as possible by adding higher orders in the calculations. The other 'non-photonic' corrections are dominated by the vacuum polarization and \(\gamma Z\) interference. Also, light fermion pair production and multiperipheral-type diagrams enter at the \(\mathcal{O}(\alpha^2)\) (see, for instance, the contribution by Arbuzov et al.). The vacuum polarization on the t-channel photon line is as sizeable as the bulk of 'photonic' corrections. It is inherently uncertain because of the QCD component, calculated using a dispersion relation which has as input the low-energy experimental data for the cross-section of e\(^+\)e\(^-\) \(\rightarrow q\bar{q}\). Only better experimental data or better analysis of existing data will allow this error to be reduced substantially (see also the contribution by Beenakker et al.). The interference of t-channel \(\gamma\) and s-channel \(Z\) is a kind of 'pollution' of the LABH process. This interference is small: below 0.5\% for the second generation of the luminosity detectors located at lower angles. Unfortunately, it is not small enough to be treated in the Born approximation. The QED 'photonic' corrections to this contribution are important because it varies strongly with the total centre-of-mass energy. Furthermore, it is also affected by the up-down interferences (see contribution by Beenakker et al.). Light fermion pair (ee, \(\mu\mu\))-production and multiperipheral-type diagrams most probably contribute very little (< 0.1\%) but have to be examined quantitatively at the present experimental precision level of \(\approx 0.1\%\).

What is the significance of the precise measurement of the luminosity, and therefore of the absolute cross-section for the program of the precision tests of the standard electroweak model (SEM) at LEP? As is well known, the measurement of the invisible width of \(Z\) conveniently parametrized in the number of massless neutrinos \(N_\nu\) is affected directly by \(\Delta\mathcal{L}/\mathcal{L}\). The experimental deviation \(N_\nu - 3\) would signal very important non-standard physics (a new channel of \(Z\) decay or deviation of neutrino coupling constants from the SEM). It is also well known that \(N_\nu - 3\) and \(\sigma_{\text{tot}}(M_Z)\) have very little sensitivity to details of the SEM such as the masses of the Higgs boson and top quark. The error of the luminosity measurement also affects the measurement of the Z decay width into electrons \(\Gamma_e\). This quantity is one of the main sources (together with asymmetries) of our knowledge of the electroweak mixing angle.

In the sections that follow, the reader will find further details on the present status of the theoretical calculations of the LEP/SLC luminosity cross-section.

S. Jadach
Higher-Order Radiative Corrections to Bhabha Scattering at Low Angles: The YFS Monte Carlo Approach

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Abstract

In this contribution we present new numerical results for the QED second-order radiative corrections to the low-angle Bhabha cross-section from the new Monte Carlo event generator, BHLUMI4.0. Our main concern is the precision of these calculations. We discuss quantitatively both technical precision — numerical problems — and physical precision — higher orders. The results presented here will be essential in the future reduction of the overall theoretical uncertainty in the measurement of the luminosity at LEP below the present 0.25\% level.

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1 Basics and notation

Luminosity at LEP/SLC is measured with the help of the low-angle Bhabha (LABH) process, $e^+e^- \rightarrow e^+e^-$, in the angular range below 100 mrad. The luminosity determined in this way provides absolute normalization of the cross-section of all other processes in the $e^+e^-$ scattering. The LABH cross-section is therefore not of physical interest in itself. On the contrary, it is regarded as completely known from theory, from Quantum Electrodynamics (QED). However, although in principle the LABH cross-section is calculable in perturbative QED with arbitrary precision (except for a small hadronic correction), it is subject to theoretical uncertainties due to a truncation of the perturbative expansion and to a limitation of the calculational tools (computer programs). All LEP/SLC experiments use theoretical calculations for LABH, based on works published by some of the present authors three years ago [1]. This calculation has an overall theoretical/technical precision of 0.25% and is embodied in the form of the Monte Carlo event generator [2] BHLUMI version 2.0. This error was acceptable in 1991 but now, with an improvement in the experimental precision of a factor of two or more, it dominates the present overall luminosity error. It is therefore quite urgent to reduce the theoretical error of the QED calculation to a precision level of at least 0.1%.

The backbone of the 0.25% theoretical precision estimate [1] is due to missing second-order $O(\alpha^2 L^2) - 0.15\%$ — and $O(\alpha^2 L) - 0.9\%$ — contributions in the matrix element encoded in the Monte Carlo calculations. Here $L = \ln(|t|/m^2)$ is the so-called big log in the leading-logarithmic (LL) approximation where $t$ is the $t$-channel transfer — of order 1 GeV; see Fig. 1 for a pictorial definition of the LL approximation.

The first of the above contributions (0.15%) also includes the technical precision of the Monte Carlo programs due to programming bugs, rounding errors, quality of random numbers, etc. It is illustrated in Fig. 2, taken from Ref. [1] as a difference of three Monte Carlo subgenerators of BHLUMI 2.0. These are: (i) multiphoton $O(\alpha)_{\text{exp}}$ BHLUMI, (ii) $O(\alpha)$ OLDBIS (without exponentiation) and, (iii) $O(\alpha^3)_{\text{LL}}$, the leading-logarithmic (collinear photon emission) event subgenerator LUMLOG. The differences in the three Monte Carlo subgenerators provide a solid estimate of the technical precision. In addition, subgenerators (ii) and (iii) have separate estimates of their technical precision at the level below 0.05%, coming in the case of (ii) from an independent comparison [3] with a semianalytical calculation, and in the case of (iii) with another Monte Carlo [4]. The comparison in Fig. 2 therefore provides an estimate of the technical precision mainly for the multiphoton BHLUMI subgenerator, which did not have any other independent analytical or Monte Carlo cross-check$^1$.

As for the physical precision, which is mainly due to a truncation of the perturbative calculation [1], the dominant (beyond first-order) correction of $O(\alpha^2 L^2)$ was under good control because it was calculated using the $O(\alpha^3)_{\text{LL}}$ subgenerator LUMLOG [4]. The hybrid Monte Carlo calculation OLDBIS plus LUMLOG includes the entire $O(\alpha^2 L^2)$ correction for the integrated cross-section, but due to the zero-angle collinear emission of photons, LUMLOG is not very suitable for experimental analysis where various fine-grain inclusive/multiphoton distributions are checked in the process of reducing the systematic

---

$^1$At the time it seemed unthinkable to analytically integrate the total cross-section of the $O(\alpha)_{\text{exp}}$ BHLUMI.
Figure 1: QED perturbative leading and subleading corrections. Rows represent corrections in consecutive perturbative orders — the first row is the Born contribution. The first column represents the leading logarithmic (LL) approximation and the second the next-to-leading (NLL) approximation. Terms selected for (a) second- and (b) third-order pragmatic expansion are limited by an additional line.

Figure 2: We plot the difference [1] of $\sigma_{B2}$ of BHLUMI 2.0 and $\sigma_{O+L}$ from OLDBIS and LUMLOG. It represents the missing $O(\alpha^2 L^2)$ bremsstrahlung correction in the BHLUMI version 2.0 event generator [2] together with its technical precision. The difference in the cross-sections (divided by Born) is calculated for the symmetric and asymmetric calorimetric trigger shown in Fig. 3 as a function of the energy cut $z_{\text{min}}$. Dotted lines mark the 0.15% limit. Vacuum polarization, $Z$ and $s$ channel $\gamma$ are switched off.
A simple calorimetric trigger

Forward hemisphere

\[ \theta_1 \]

\[ \theta_1^{W_{\text{max}}} \]

\[ \theta_1^{W_{\text{min}}} \]

\[ E_1 \]

Backward hemisphere

\[ \theta_2 \]

\[ \theta_2^{N_{\text{max}}} \]

\[ \theta_2^{N_{\text{min}}} \]

Acceptance (trigger) condition

\[ E_1 E_2 > z_{\text{min}} E_{\text{beam}}^2 = (1 - x_{\text{max}}) E_{\text{beam}}^2 \]

Figure 3: Geometry and acceptance of the simple calorimetric luminosity detector similar to LCAL in the ALEPH experiment. Each side of the detector consists of six calorimetric blocks. A single block measures the total energy of electrons and photons. For accepted events it is required that at least one pair of back-to-back blocks (two shaded blocks in the plot) have enough energy to fulfill the condition \( E_1 E_2 > z_{\text{min}} E_{\text{beam}}^2 = (1 - x_{\text{max}}) E_{\text{beam}}^2 \). Shown is the asymmetric type of angular acceptance; thick lines limit wide/narrow \( \theta \)-range for forward/backward hemispheres.

The obvious development path of the above calculation scheme was the following:

a) To implement the \( \mathcal{O}(\alpha^2 L^2) \) missing part of the matrix element in the multiphoton exponentiated subgenerator of BHLUMI2.0.

b) To provide a new, independent analytical cross-check of the new matrix element.

c) To improve the estimate of the next dominant bremsstrahlung-type corrections — i.e. of \( \mathcal{O}(\alpha^2 L) \) and \( \mathcal{O}(\alpha^3 L^3) \) corrections.

d) To again estimate other higher-order corrections like light pairs, vacuum polarization, remnant of s-channel Z-exchange etc.
In Section 2 we discuss step (a) and in Section 3 step (b). Step (c) is mainly covered in Section 4, but some preliminaries will be given in the preceding section. Point (d), and to some extent also point (c), have already been elaborated [5] in the literature.

Before we get down to more detail, let us explain/define several useful concepts, approximations and terms typical for QED calculations of the LABH cross-section, which will be used or referred to in the course of this contribution.

**Up-down interference:** At angles below 100 mrad all ‘photonic’ corrections in which the additional photon line connects the upper electron line with the lower positron line — the so-called up-down interference — are strongly suppressed. This phenomenon, which we call ‘suppression of the up-down interference’, was conjectured and proved numerically, using an $O(\alpha)$ calculation [3]. In all presented calculations we exploit this approximation.

The Monte Carlo technique, used heavily in all presented work, is nothing more or less than the technique of the exact (up to statistical error) integration over the multiparticle phase space. Here we implicitly assume that the differential cross-section is always explicitly written as a product of the matrix element squared, multiplied by the phase-space integration element (this is not true for many QCD calculations).

The Yennie-Frautschi-Suura (YFS) exponentiation is a technique of summing exactly all infrared singularities to infinite order that provides us with exclusive multiphoton differential distributions. The real hard photons co-exist with the soft photons and there is no need to introduce any additional parameters/cuts to separate them. The multiphoton distributions, with virtual corrections as well, are derived from Feynman diagrams and are calculated/improved order by order. In other words, the $O(\alpha^n)$ calculation exponentiated in the YFS scheme, when truncated back to $O(\alpha^n)$, coincides exactly with the ordinary un-exponentiated $O(\alpha^n)$ calculation. This applies for exclusive multiphoton distributions, and consequently for all inclusive distributions and the total cross-section.

**Trigger** is our short-hand name for the set of kinematical cuts which define accepted events for the LABH total cross-section. Real experimental triggers of LEP/SLC detectors are calorimetric — all photons and electrons satisfying the angular acceptance condition $\theta_{\min} < \theta < \theta_{\max}$ are registered without any distinction being made among them, and the minimum total energy required is $x = E/E_{\text{beam}} > 1 - X_{\text{max}}$ in the forward and backward hemisphere simultaneously. In practice, $\theta_{\min,\max}$ in the forward and backward direction are quite often taken differently (asymmetric trigger). Also, in the real experiment, the association of photons and electrons into a single cluster with energy $E$ and average angle $\theta$ is a little more involved — See Ref. [6] for more details on the various types of experimental triggers. In the calculations we often use as an example the algorithm describing the example of the trigger very close to the new ALEPH SICAL detector. This trigger is defined quite precisely in Fig. 4. It is rather obvious that this kind of trigger cannot be treated analytically and Monte Carlo is in this case the only way to calculate the total cross-section.

## 2 New $O(\alpha^2)_{\text{prog}}^{\exp}$ BHLUMI4.0 Monte Carlo

In the following we characterize the new $O(\alpha^2)_{\text{prog}}^{\exp}$ exponentiated matrix element implemented in the new version of Monte Carlo BHLUMI4.0. For reasons of space, we cannot
Figure 4: Geometry and acceptance of the SICAL luminosity subdetector in the ALEPH experiment. Each side of the subdetector consists of \(16 \times 32\) pads. Single-pad measures the total energy of electrons and photons. A pad of maximum energy and its \(3 \times 7\) neighbourhood is called a cluster. Total energy registered in the cluster is denoted as \(E^{cl}_i\) and the average position (weighted with energy) is denoted \((\theta^{cl}_1, \phi^{cl}_1), i = 1, 2\). By definition \(\phi_1 = \phi_2\) for back-to-back particles. We depict asymmetric type of the angular acceptance; thick lines limit wide/narrow \(\theta\)-range for forward/backward hemispheres. Pads of the cluster which spill over the angular range (thick lines) are used to determine total energy and average position of the cluster (see the example of the backward hemisphere).

include here the full definition of the matrix element used in the program (it would need more than five pages). Nevertheless, we will attempt to characterize all its essential properties.

In \(\mathcal{O}(\alpha^2)\), in order to reach the 0.1% physical precision level, it is probably enough to add to the Monte Carlo matrix element, beyond the regular \(\mathcal{O}(\alpha)\), the dominant second-order contribution of \(\mathcal{O}(\alpha^2L^2)\) — if possible, with exponentiation. This type of calculation, which we denote as \(\mathcal{O}(\alpha^2\text{prog})\), is depicted in Figs. 1a and 5. In fact, in BHLUMI4.0 we include two examples of the \(\mathcal{O}(\alpha^2)\text{prog}\) matrix element, marked A and B, which differ by \(\mathcal{O}(\alpha^2L)\) and \(\mathcal{O}(\alpha^2)\) terms. This also is illustrated in Fig. 5. Why are we free to make two choices and why is it profitable? While \(\mathcal{O}(\alpha)\) distributions come directly from the Feynman diagrams (no freedom!) the additional \(\mathcal{O}(\alpha^2L^2)\) contributions we derive more simply by twice convoluting the Altarelli–Parisi kernel. The same kind of IL Ansatz was used successfully in the YFS2 and YFS3 Monte Carlo programs [7–9]. For the \(\mathcal{O}(\alpha^2L^2)\)
Figure 5: Perturbative content of the matrix element of BHLUMI version 4.0. The correct/complete contributions of $\mathcal{O}(\alpha^2)_{\text{prag}}$ are above the dotted line. The diagram right illustrates perturbative content of the difference of the two types A and B of the $\mathcal{O}(\alpha^2)_{\text{prag}}$ matrix elements.

contribution, the soft limit is improved by hand\(^2\). The well-known behaviour and the finite transverse momenta of photons are introduced in the distributions using the soft limit as a model. Obviously, the above procedure has some freedom in the construction of the matrix element, but how much? Let us first note that (even without exponentiation!) the above procedure creates some non-zero contributions of $\mathcal{O}(\alpha^2 L^1)$ and $\mathcal{O}(\alpha^2 L^0)$. The two types of $\mathcal{O}(\alpha^2)_{\text{prag}}$ matrix elements may have different such contributions. The important advantage of our $\mathcal{O}(\alpha^2 L^2)$ Ansatz is that it is simple, quick in the computer evaluation, and that its LL content — of primary interest — is explicit and therefore very easy to control. As we have already mentioned, the LL Ansatz for the $\mathcal{O}(\alpha^2 L^2)$ comes before the YFS exponentiation. Exponentiation introduces new non-zero contributions of $\mathcal{O}(\alpha^3)$ and higher orders. Among them, the $\mathcal{O}(\alpha^3 L^3)$ contribution will be numerically dominant. Since YFS exponentiation is well founded physically, these higher-order terms substantially improve the perturbative convergence of the calculation — as was proven explicitly for the $\mathcal{O}(\alpha^3 L^3)$ terms [10].

Note that option B for the matrix element, degraded to $\mathcal{O}(\alpha)_{\text{prag}}$ exponentiated, is identical to the matrix element in the published BHLUMI2.0 program. BHLUMI4.0 is also backward compatible with BHLUMI2.0 for the OLDBIS and LUMLOG subgenerators, and they are still included. In fact the LUMLOG generator with LL matrix element (exponentiated and unexponentiated) up to the third-order is extended a little, because emission of photons is now included not only in the initial but also in the final state. It is done as previously in the zero transverse-momentum approximation. For completeness we depict the perturbative content of the LUMLOG and OLDBIS event generators in Fig. 6.

\(^2\)In the LL approximation the correct soft limit is not reproduced in the general case.
Figure 6: Perturbative content of the matrix element of the hybrid Monte Carlo calculation OLDBIS plus LUMLOG.

3 Technical precision

Generally speaking, technical precision is obtained by doing two technically very different calculations and taking the difference. The most powerful method is to take the difference between the Monte Carlo and analytical calculation \([4]\). The serious disadvantage of this method is that it can be applied only for certain, rather simple kinds of trigger. In the case of BHLUMI4.0 even the existence of one such trigger for which the above method can be applied is of no small importance! The method by which two different Monte Carlo calculations are compared can be applied for a wider family of cuts. Its very serious disadvantage is that if one encounters — as is always the case — an intolerably big difference between the two Monte Carlo results, then it is very difficult, often impossible, to find the source of the difference (debug the corresponding Monte Carlo programs).

We are in the process of determining the technical precision of BHLUMI4.0 using an elaborate multistep method. The steps are:

1. Invent an ‘academic trigger’ for which analytical integration of the BHLUMI4.0 cross-section is feasible down to a precision of \(3 \times 10^{-4}\). This precision level requires a calculation of \(\mathcal{O}(\alpha^3)_{\text{prog}}\).

2. Perform an analytical calculation and debug the BHLUMI4.0 Monte Carlo program and the corresponding semianalytical calculation/program until \(|\sigma_{\text{MC}} - \sigma_{\text{analyt}}| < 3 \times 10^{-4}\) is obtained for the ‘academic trigger’.

3. Do the same in the (easier) cases of OLDBIS and LUMLOG.

4. Take the difference of BHLUMI4.0 minus (OLDBIS plus LUMLOG) and explore it analytically and numerically in every detail down to a \(3 \times 10^{-4}\) precision.

5. Do an ‘adiabatic transition’ from the academic trigger to the realistic trigger using a series of intermediate triggers and looking carefully at the evolution of the differ-
ence BHLUMI4.0 minus (OLDBIS plus LUMLOG). It should be understood to a precision comparable to $3 \times 10^{-4}$ — better than $5 \times 10^{-4}$, for example.

In the above scenario we take advantage of the extremely important fact that the technical precision for (OLDBIS plus LUMLOG) is practically zero for any kind of trigger!

In the following we will show results from step 1 in the method outlined above: (a) define a set of ‘academic’ cuts used for the semianalytical integration of the above matrix element over multiphoton phase space, (b) briefly characterize methods used in the analytical integration and the class of corrections kept in the analytical phase space integration (not the same as in the matrix element), and (c) show the numerical agreement of the Monte Carlo (including the new matrix element) with the semianalytical formula down to the 0.03% level (technical precision).

Ad (a): The most important criterion used to define our set of kinematical cuts for semianalytical integration of the $\mathcal{O}(\alpha^2)_{\text{prag}}$ new matrix element over the multiphoton phase space (the so-called ‘academic trigger’) is that this semianalytical integration is at present very feasible. We define the cuts of our ‘academic trigger’ as: $|t_{\text{min}}| < |t| < |t_{\text{max}}|$ and $V < V_{\text{max}}$, where $t$ is the four-momentum transfer squared and the variable $V$ represents some kind of measure of the total energy carried away by all emitted real photons. We require that $0 < V < 1$ represent the condition of completeness of the phase space and $V < \epsilon$ the condition that all photons are soft. The $V$-variable we actually define, in terms of the four momenta, as:

$$V = 1 - \frac{2(p_1 p_2) |t|}{[2(p_1 p_2) + 2(p_1 K_p)]^2} - \frac{2(q_1 q_2) |t|}{[2(q_1 q_2) + 2(q_1 K_q)]^2},$$

(1)

where $p_i = 1, 2$ are the four momenta of the incoming and outgoing electron, $q_i = 1, 2$ are the four momenta of the incoming and outgoing positron and, $K_p$ and $K_q$ are the total of the four momenta of all photons emitted from electron and positron lines.

Ad (b): With the above definition of the phase space window, it is rather straightforward to integrate the $\mathcal{O}(\alpha^2)_{\text{prag}}$ matrix element, keeping all terms within the $\mathcal{O}(\alpha^2)_{\text{prag}}$ approximation. This we found insufficient for the purpose of establishing a technical precision at the 0.03% level because some terms beyond $\mathcal{O}(\alpha^2)_{\text{prag}}$ — especially for partially incomplete results — are of that order. We have therefore decided to follow the integration of the $\mathcal{O}(\alpha^3)_{\text{prag}}$ approximation; see also Fig. 1b. It means that terms of $\mathcal{O}(\alpha^3 L)$, due to our LL Ansatz, and terms of $\mathcal{O}(\alpha^3 L^3)$, due to exponentiation, are integrated analytically over the phase space (with the academic trigger) exactly!

The resulting integrated cross section is not very complicated and reads as follows:

$$\sigma^{(2)}_H(t_{\text{min}}, t_{\text{max}}, V_{\text{max}}) = \int_{t_{\text{min}}}^{t_{\text{max}}} dt \int_0^{V_{\text{max}}} dV \, \rho^{(2)}_{\text{loc}}(t, V)$$

$$\rho^{(2)}_{\text{loc}}(t, V) = \eta \, F(2\gamma) \, e^{2\Delta_{\text{YFS}}(\gamma)} \, 2\gamma V^{2\gamma-1} \left\{ 1 + \gamma + \gamma^2/2 \right\}$$

$$+ \eta \, F(2\gamma) \, e^{2\Delta_{\text{YFS}}(\gamma)} \, V^{2\gamma} \left\{ \gamma(-2 + V) + \frac{\alpha}{\pi} \ln (1 - V)(-4 + 4V - 2V^{-1}) \right.$$

$$+ \gamma^2(-2) + \gamma^2 \ln (1 - V)(3 - 3V/2 - 2V^{-1})$$

$$+ \gamma^3(-7V/4) + \gamma^3 \ln (1 - V)[5/4 + V/2 - 2V^{-1}]$$

$$351$$
\begin{align}
+ \gamma^3 \ln (1 - V)^2[-5/8 + 5V/16 + (1/4)V^{-1}] + \gamma^3 \text{Li}_2(V)(2 - V) \\
+ \gamma \frac{\alpha}{\pi} [1/4 + 11V - (13/4)(2 - V)^{-1} + (1/2)(2 - V)^{-2} - 6(2 - V)^{-3} + 2(1 - V)^{1/2}] \\
+ \gamma \frac{\alpha}{\pi} \ln (1 - V)[39/4 - 19V/4 - 2V^{-1}] \\
- 2(2 - V)^{-1} + (2 - V)^{-2} - (1/2)(2 - V)^{-3} - (3/2)(1 - V)^{1/2}] \\
+ \gamma \frac{\alpha}{\pi} \ln (1 - V/2)[-9/2 + 3V/4 - 2V^{-1}] + 2(2 - V)^{-2} - 4(2 - V)^{-3}] \\
+ \gamma \frac{\alpha}{\pi} \ln (1 - V)^2[19/8 - 41V/16 + V^{-1}] + \gamma \frac{\alpha}{\pi} \ln (1 - V)\ln(2 - V)(-1/2 + V/4) \\
+ \gamma \frac{\alpha}{\pi} \ln (1 - V)\ln(V)(12 - 10V) + \gamma \frac{\alpha}{\pi} \ln (1 - V)\ln(V/2)(-6 + 5V) \\
+ \gamma \frac{\alpha}{\pi} \ln (1 - V/2)(3/2 - 11V/4) \\
+ \gamma \frac{\alpha}{\pi} \ln (1 - V/2)^2(3/4 - 5V/8) + \gamma \frac{\alpha}{\pi} \text{Li}_2(1/2)(-3/2 + 11V/4) \\
+ \gamma \frac{\alpha}{\pi} \text{Li}_2[(1 - V)/(2 - V)](1/2 - V/4) + \gamma \frac{\alpha}{\pi} \text{Li}_2[1/(2 - V)](1 - 5V/2) \\
+ \gamma \frac{\alpha}{\pi} \text{Li}_2[-V/2(1 - V)](6 - 5V) + \gamma \frac{\alpha}{\pi} \text{Li}_2[1 - (1 - V)^{-1/2}](6 - 5V) \\
- \xi \gamma \chi(V)/(1 - V) \biggl) \label{2}
\end{align}

where \( \gamma = 2(\alpha/\pi)(L - 1) \), \( b_0 = \chi(\xi) \), \( \chi(x) \equiv [1 + (1 - x)^2]/2 \), \( \xi = |t|/s \), \( F(x) \equiv \exp(-C\chi)/\Gamma(1 + x) \), and \( \Delta_{YFS}(\gamma) = \gamma/4 - (\alpha/\pi)(1/2 + \pi^2/6) \). Note that the \( \mathcal{O}(\alpha^2) \) part of the formula is very compact and that its LL content is identical to the non-singlet, second-order, structure function of the photon in the electron\(^5\). The terms of \( \mathcal{O}(\alpha^2L^3) \) and \( \mathcal{O}(\alpha^2L) \), which represent most of the formula, finally turn out to be numerically small — in fact below 0.04%.

Ad (c): In Fig. 7a we show a comparison of Eq. (2) with the Monte Carlo BHLUMI4.0. Although the integration over \(|t|\) and \(V\) is analytically feasible, we do it numerically with the help of the standard Gauss technique: the integrand is a very smooth function, suitable for this method (peaks are removed either by change of variables or subtraction). As we see, the Monte Carlo and semianalytical result differ by up to 0.03%. We conclude that for the above ‘academic trigger’ we have obtained a 0.03% technical precision — i.e. we have attained the goals of steps 1 and 2.

In the multistep procedure outlined above we have gone through steps 1 to 4 and have the first numerical results for step 5. In step 3 we obtained the simple semianalytical results \( \sigma_0^{(1)} \) and \( \sigma_L^{(3)} \), which agree with the corresponding Monte Carlo OLDBIS and LUMLOG for ‘academic trigger’ to better than 0.02%. The next numerical result, shown in Fig. 7b, is relevant to step 4 and demonstrates that the step is completed. As we see, the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) is under control down to 0.05% because the Monte Carlo and analytical results agree at this precision level. In the next section we further discuss the result of Fig. 7b.

\(^5\) This is due to the fact that the variable \( V \) in the LL has a very simple meaning.
The crucial question now is whether we can extend this result to triggers other than our ‘academic trigger’ — in particular whether we can port our result to realistic experimental triggers. This part — step 5 — is still under development and we cannot show all relevant partial results due to space restrictions.

4 Physical precision

The precision estimate in our previous work [1] was based to a large extent on the calculation of the difference BHLUMI2.0 minus (OLDBIS plus LUMLOG), which for BHLUMI2.0, represented the missing $O(\alpha^2 L^2)$ plus technical precision. With the new matrix element in BHLUMI4.0, which includes the complete $O(\alpha^2 L^2)$ contribution, we look immediately into the same three-generator difference. The result of such a comparison, with the scale on the vertical axis inflated by a factor of almost ten, is presented in Fig. 8. It is done for the same trigger type and angular range as in the earlier publication, calorimeter of the ELCAL/ALEPH type [1], and also for the new ALEPH/SICAL-type detector at a lower angular range.

At first sight, the new result in Fig. 8 looks completely compatible with the earlier published result of Fig. 2, with the difference again well within 0.15%. Of course, the interpretation of this difference is not the same now as previously. The multiphoton Monte Carlo BHLUMI4.0 includes $O(\alpha^2 L^2)$ corrections, hence the plotted difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) is potentially dominated by the $O(\alpha^2 L^2)$, $O(\alpha^3 L^3)$ and technical precision. (The estimate of the technical precision from Fig. 7 does not apply.
Figure 8: We plot the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) (a) for the ALEPH/ELCAL type calorimeter/trigger (angular range 60–120 mrad) as previously defined [1] and (b) for ALEPH/SICAL detector (25–60 mrad). Dotted lines mark the same 0.15\% limit as in Fig. 2.
Here automatically, due to the different type of trigger.) The $O(\alpha^2 L^2)$ is absent from the new results of Fig. 8. These new results are very encouraging but should be treated as preliminary — they will soon be subjected to a new round of tests.

In the following we present additional numerical results that will allow us to better understand the meaning of the results in Fig. 8. The immediate question to answer is: How big was the missing $O(\alpha^2 L^2)$ correction in the published version of BHLUMI2.0? For the answer, we need only look at the difference BHLUMI4.0 minus BHLUMI2.0. Since the published BHLUMI2.0 includes the matrix element of type A, we examine the difference $O(\alpha^2)_{\text{prag,A}} - O(\alpha^2)_{\text{prag,A}}$. It is plotted in Fig. 9a for the SICAL detector of ALEPH. We see that the result is small in comparison with the typical values/estimates 0.25%-0.5% quoted in previous papers [4, 1] for the $O(\alpha^2 L^2)$ corrections. We simply conclude that the choice of the matrix element in BHLUMI2.0 was very lucky! To see it more clearly, we show a similar quantity for the matrix element of type B in Fig. 9b. Here, the situation looks more 'normal'. The missing $O(\alpha^2 L^2)$ contribution for the experimentally relevant $X_{\text{max}} \simeq 0.5$ is $-0.25\%$. Of course, the next logical question is: Do the two $O(\alpha^2)_{\text{prag}}$ results of type A and B agree? Yes, they do. And, as we see in Fig. 9c, the corresponding difference is tiny indeed — just $\simeq 0.01\%$!

All the above cross-checks, together with the results from the previous section, give us a strong hint that the new $O(\alpha^2)_{\text{prag}}$ matrix element in BHLUMI4.0 is correctly implemented. Nevertheless, since the problem of high technical precision for a realistic trigger is still not solved, we say that the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) in the plots of Fig. 8 represent the missing $O(\alpha^3 L^3)$, $O(\alpha^2 L)$ and the technical precision. In fact the $O(\alpha^3 L^3)$ can be practically eliminated from this list. Using the semianalytical formula (2) we can calculate exactly the missing $O(\alpha^3 L^3)$. The $O(\alpha^3 L^3)$ missing terms are known [10] and can be easily included in Eq. (2)! The effect of such a modification on the cross-section is shown in Fig. 9c. It is negligible: below 0.02%. This result is not that surprising, because it was shown [10] that the YFS exponentiation sums up the LL part of higher orders very efficiently. (We suspect that it may not hold for the $O(\alpha^2)$ calculation without YFS exponentiation.) Note that the above result extends for any calorimetric experimental trigger because it is of a pure LL character. We conclude, therefore, that Fig. 8 contains only the missing $O(\alpha^2 L)$ and technical precision.

In view of the above discussion, can we already improve on the total precision of the luminosity cross-section? First of all, we previously estimated [1] that the $O(\alpha^2 L)$ contribution is generically of an order of 0.1%. In fact the value 0.09% was used and summed up with the 0.15% estimate of the missing $O(\alpha^2 L^2)$ from Fig. 2 to obtain a total bremsstrahlung error of 0.24%. Since we now know that the missing $O(\alpha^2 L^2)$ was absent in the (old) Fig. 2 by chance and in the (new) Fig. 8 by construction, these figures show $O(\alpha^2 L)$ and technical precision. The 0.15% spread of the curves in Fig. 8 is very compatible with the generic estimate of 0.1%. But since the generic estimate and the spread of the curves now represent the same thing, in order to avoid double counting, instead of their sum we should take the maximum of the two. This gives us roughly a 0.15% estimate of the total bremsstrahlung uncertainty in BHLUMI4.0 — i.e. the corresponding physical and technical precision. This is a net improvement over the published 0.24%. In order to improve on the above we need to have a better, separate, estimate of the technical precision for the experimental trigger below 0.05% and a better estimate (by
direct calculation of the $O(\alpha^2 L)$ missing contribution. Of course, inclusion of the error from the vacuum polarization, light fermion pairs, etc., will increase the error, but not by too much. This more detailed analysis will be presented elsewhere.

The central question of the physical precision of the BHLUMI4.0 problem is now obviously the following: How big is the missing second-order subleading $O(\alpha^2 L)$ correction? It would be best to calculate this object directly and there are attempts in this direction by the authors and others [11]. For the moment let us gather all indirect information on this subject. On the one hand, the earlier [1] estimate of 0.1% is in good agreement with the 0.15% variation in Fig. 8. On the other hand, there is some indication that the actual value may be smaller. This comes mainly from the analytical inspection of the difference BHLUMI4.0 minus (OLDBIS plus LUMLOG) for the academic trigger — see also Fig. 7b — and confirmed to some extent by Fig. 9c for the real experimental trigger.

**Table 1**

The canonical coefficients indicating the generic magnitude of various leading and subleading contributions up to third order. The big log $L$ is calculated for $\theta = 25$ mrad.

<table>
<thead>
<tr>
<th>Canonical coefficients</th>
<th>non-calorimetric case</th>
<th>calorimetric case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(\alpha L)$</td>
<td>$\frac{\alpha}{\pi} 4L$</td>
<td>$140 \times 10^{-3}$</td>
</tr>
<tr>
<td>$O(\alpha)$</td>
<td>$2\frac{1}{2} \frac{\alpha}{\pi}$</td>
<td>$2.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>$O(\alpha^2 L^2)$</td>
<td>$\frac{1}{2} (\frac{\alpha}{\pi} 4L)^2$</td>
<td>$10 \times 10^{-3}$</td>
</tr>
<tr>
<td>$O(\alpha^2 L)$</td>
<td>$\frac{\alpha}{\pi} (\frac{\alpha}{\pi} 4L)$</td>
<td>$0.35 \times 10^{-3}$</td>
</tr>
<tr>
<td>$O(\alpha^3 L^3)$</td>
<td>$\frac{1}{3!} (\frac{\alpha}{\pi} 4L)^3$</td>
<td>$0.45 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

The closer look into $O(\alpha^2 L)$ terms in the analytical formula for BHLUMI4.0 minus (OLDBIS plus LUMLOG) reveals that almost all such terms are numerically unimportant because the value of the coefficient $(\alpha/\pi)^2 4L = 0.035\%$ is small. All terms which do matter numerically have special reasons for this. They are bigger because they include enhancement factors which can be understood and traced back to certain peculiarities of the calculation. In the case of Fig. 7b all these enhancement factors reflect either a lack of exponentiation in OLDBIS or a zero-transverse-momentum approximation in LUMLOG. (They are typically logarithms of cut-off parameters and $\zeta_2 = \pi^2/6$ in the virtual corrections.) The fact that the result shown in Fig. 7b is bigger than the canonical 0.035% reflects these artefacts in the OLDBIS plus LUMLOG and not the overlooking of such contributions in BHLUMI4.0. The important lesson for future evaluation of $O(\alpha^2 L)$ contributions is that any such contribution above 0.035% has to be checked and explained.
Figure 9: Plots (a)–(c) show Monte Carlo results for the SICAL/ALEPH detector and (d) shows analytical result for the ‘academic’ trigger. All cross-sections are divided by the Born value and plotted as a function of the energy cut $X_{\text{max}}$ or $V_{\text{max}}$. In (a) we demonstrate the ‘missing second-order’ of the already-published BHLUMI2.0 as the difference between it and the new version, BHLUMI4.0. In (b) we plot the same quantity for matrix element type B. Plot (c) demonstrates the difference in results between A and B type $O(\alpha^2)_{\text{prag}}$ matrix elements. In (c) we show for the ‘academic’ trigger how the cross-section would change if the matrix element B were upgraded to $O(\alpha^3)_{\text{prag}}$. 
separately because it needs a special reason — the enhancement factor — to exist! We have done the above exercise a little more systematically and, looking into coefficients in the Eq. (2), we have obtained all typical ‘canonical’ coefficients for various leading and subleading contributions up to the third-order, these being included in Table 4. Of course, the coefficients are smaller for calorimetric detection of the final-state electrons and we tried also to estimate this effect. Note that the canonical coefficients in Table 4 agree very well with: for \( \mathcal{O}(\alpha^2 L^2) \), Fig. 9b; for \( \mathcal{O}(\alpha^2 L) \), Fig. 9c; and for \( \mathcal{O}(\alpha^3 L^3) \), Fig. 9d.

5 Summary and outlook

The status of our work on the QED corrections for the luminosity cross section can be summarized as follows:

- The \( \mathcal{O}(\alpha^2) \text{prog} \) exponentiated matrix element in the new version of BHLUMI 4.0 is implemented and we have a lot of high-precision (technical precision: \( 3 \times 10^{-4} \)) evidence that it was done correctly.

- A technical precision as high as \( 3 \times 10^{-4} \) has been established for BHLUMI 4.0 for the special ‘academic’ type of trigger (cut-offs).

- The OLDBIS/LUMLOG tandem is used to port the above high technical precision to realistic examples of triggers under development.

- There are indications that the main \( \mathcal{O}(\alpha^2 L) \) contribution to physical precision is below 0.1% — even 0.03% possibly — but we have to stick to a conservative estimate of 0.1%, which provides us with the new BHLUMI 4.0 total bremsstrahlung uncertainty of 0.15% for the generic experimental trigger. This is a considerable improvement over the previously published value of 0.24%.

In our future work we plan to extend the technical precision \( 3 \times 10^{-4} \) to truly realistic experimental triggers, implementing the \( \mathcal{O}(\alpha^2 L) \) part of the matrix element or providing a more solid numerical evaluation/estimate of its magnitude. Realization of the above will definitely allow total precision to be brought below 0.1%, which will match the best experimental errors.

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References


The Theoretical Precision in Small-Angle Bhabha Scattering at LEP: Comparisons Between Different Approaches

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Abstract

We present the initial results of a systematic comparison of the numerical results given by some of the available event generators or Monte Carlo programs for small-angle Bhabha scattering at LEP, selecting a common set of configuration parameters (channels, phase-space region and physical cuts). Their agreement provides good support for the theoretical precision claimed by the various approaches.

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1 Introduction

The general preface to this section, dedicated to low-angle Bhabha scattering, explains why the theoretical error in the calculation of the process is requested to become less than 0.05%.

Such a request increases enormously the complexity of the problem and makes it difficult to be sure that every single correction has been included at the requested precision, and that no bias due to the approach has been introduced.

Probably only the agreement of the results of the different approaches and independent calculations can provide the necessary confidence.

Therefore we have begun a systematic comparison, whose first step is reported here, that will provide a control on the partial results of presently available computer programs for the low-angle Bhabha scattering cross-section.

In this first step we compare the results for the total cross-sections, mostly limited to photonic corrections, with undressed final fermions (an unrealistic case in LEP experiments), in order to test the precision of the programs and the differences due to alternative theoretical approaches.

We do not repeat here the description of the programs, which can be found in the references provided.

As the total centre-of-mass energy is not relevant to this discussion, all numbers are obtained at the value 95.17 GeV. The final electron and positron scattering angles are between three and eight degrees, and no acollinearity or acoplanarity cut is imposed.

We present a comparison between the results of the event generators OLD BIS [1], LUMLOG [2] and BHLUMI [3], presently used by LEP collaborations, and the results obtained by means of our event generator, BHAGEN94 [4], suitably modified to match the conditions of the other programs.

The analysis goes through the following steps:

- Comparison of $\mathcal{O}(\alpha)$ terms with OLD BIS
- Comparison of $\mathcal{O}(\alpha^2 L_t^2)$ terms with LUMLOG
- Comparison of the complete and resummed $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2 L_t^2)$ terms with BHLUMI.

We also report the results of the 40THIEVES program, recently improved in hard photon treatment [5], and kindly provided to us by the authors.

2 $\mathcal{O}(\alpha)$ comparison

We have extracted from BHAGEN94 an event generator of $\mathcal{O}(\alpha)$ — i.e. including up to first-order correction in $\alpha$ — that we call BG94-FO, as much as possible keeping the structure of the program unchanged.

This is achieved by analytically expanding in $\alpha$ all the formulae and then keeping only the $\mathcal{O}(\alpha)$ terms; the separation between soft and hard radiation is reintroduced through the emitted energy fraction cut $\epsilon$. 
In Table 1 we present a cross-section comparison with the event generator OLDBIS, taken from the BHLUMI set, Version 2.01 of September 1991.

We allow variation on the cut parameter $\epsilon$, corresponding to $XK0$ in OLDBIS, and on the maximum of the emitted energy fraction $\Delta$, $\Delta = 1 - E_{\text{min}}/E_{\text{beam}}$ in BG94-FO and $\Delta = 1 - VMAXE$ in OLDBIS.

To avoid unnecessary complications in BG94-FO we select only the photon $t$-channel and switch off the vacuum polarization. And the same is done in OLDBIS by selecting the parameter $\text{KEYRAD} = 2$.

The final electron and positron scattering angles vary between $3^\circ$ and $8^\circ$, the total energy is always at $95.17$ GeV, and no acollinearity cut is imposed. In OLDBIS this condition is obtained by selecting the internal routine TRIGAS0, with $\text{KEYTRI}=0$.

In the first line of Table 1 are the results for the total cross-section of the program OLDBIS — the values are given in nanobarns and the number in parenthesis is the statistical variance on the last digits (always smaller than $0.01\%$).

We then report three results, corresponding to successive levels of improvement in our program BG94-FO. In the second line are the results, indicated by BG94-FO-OLD, from the mentioned $O(\alpha)$ expansion of the program BHAGEN94. The difference with the program OLDBIS is up to $0.87\%$ at $\epsilon = 10^{-4}$ and $\Delta = 0.7$, and decreases with the decreasing values of $\Delta$ and increasing values of $\epsilon$. To help the comparison, on the right of Table 1 are the ratios of the values with those of OLDBIS.

As OLDBIS is a well tested program, the results clearly indicate that the source of the disagreement — at this level of accuracy — is in the hard photon treatment of BHAGEN94. In fact, the reduction of the hard photon phase space, and therefore its contribution, can be obtained by lowering the emitted energy fraction $\Delta$, or by raising the cut $\epsilon$, separating the soft and hard treatment.

Of course, increasing the value of $\epsilon$ requires the corrections of order $\epsilon$ or more to be introduced in order to leave the physical result independent — but this is not our present concern. We simply look for the stability of the agreement between the programs for the same value of $\epsilon$.

In particular, the analysis of contributions identifies the main cause of the difference in the treatment of the hard emission in the final state. For this correction, in place of the value coming from the the analytic formula approximately integrated in the final photon energy, we substitute in our program the numerical value obtained by enlarging the generation to the final photon energy, so that the approximation is only in the angular treatment. The new results, reported in the third line and indicated by BG94-FO-NEW, show less of a difference with OLDBIS: up to $0.28\%$ at $\epsilon = 10^{-4}$ and $\Delta = 0.5$.

The next results in the fourth line, indicated by BG94-FO-EXACT, are obtained by adding to the soft-emission part of the cross-section a new result for the hard-emission part. The later was obtained with a recently finished separate event generator, which calculates the one photon radiative Bhabha process with complete matrix element and exact kinematics — which will be illustrated separately elsewhere. In this last case the agreement is excellent: well inside both the statistical variance, which for the comparison is less than $0.02\%$, and the tested technical precision of OLDBIS, also $0.02\%$.

From this analysis we can conclude that the soft-emission treatment gives results with the precision of $0.02\%$, while the hard-emission of BG94-FO-NEW is still larger by $0.3\%$. 363
In the last line of Table 1 we report the values of the $O(\alpha)$ version of the improved numerical program 40THIEVES [5], provided by the authors, and imposing the same conditions. These values are also in good agreement (about 0.03%) with OLDBIS and BG94-FO-EXACT.

| Table 1 |
|------------------|----|----|----|----|----|----|
| $\Delta$   | 0.3 | 0.5 | 0.7 | 0.3 | 0.5 | 0.7 |
| $\epsilon = 0.1$ |     |     |     |     |     |     |
| OLDBIS     | 32.1371(7) | 33.7681(7) | 34.3980(7) |     |     |     |
| BG94-FO-OLD | 32.2360(7) | 33.9507(6) | 34.6281(6) | 1.0031 | 1.0054 | 1.0067 |
| BG94-FO-NEW | 32.1747(5) | 33.8409(6) | 34.4484(5) | 1.0012 | 1.0022 | 1.0015 |
| BG94-FO-EXACT | 32.1339(4) | 33.7662(5) | 34.3978(6) | 0.9999 | 0.9999 | 1.0000 |
| $\epsilon = 0.002$ |     |     |     |     |     |     |
| OLDBIS     | 31.481(1) | 33.112(1) | 33.742(1) |     |     |     |
| BG94-FO-OLD | 31.637(2) | 33.354(2) | 34.0287(14) | 1.0050 | 1.0073 | 1.0085 |
| BG94-FO-NEW | 31.544(2) | 33.202(2) | 33.8172(14) | 1.0020 | 1.0027 | 1.0022 |
| BG94-FO-EXACT | 31.476(1) | 33.110(1) | 33.7401(14) | 0.9998 | 0.9999 | 0.9999 |
| $\epsilon = 0.0001$ |     |     |     |     |     |     |
| OLDBIS     | 31.467(2) | 33.098(2) | 33.728(2) |     |     |     |
| BG94-FO-OLD | 31.625(3) | 33.344(3) | 34.020(3) | 1.0050 | 1.0074 | 1.0087 |
| BG94-FO-NEW | 31.532(3) | 33.192(3) | 33.807(3) | 1.0021 | 1.0028 | 1.0023 |
| BG94-FO-EXACT | 31.465(2) | 33.098(2) | 33.728(3) | 0.9999 | 1.0000 | 1.0000 |
| 40THIEVES  | 31.456(3) | 33.087(3) | 33.719(3) | 0.9997 | 0.9997 | 0.9997 |

3 $O(\alpha^2 L_t^2)$ comparison

The event generator BHAGEN94 includes up to second-order leading logarithmic corrections, $O(\alpha^2 L_t^2)$, $O(\alpha^2 L^2)$ and $O(\alpha^2 L_t L)$, where $L = \ln((s/m^2)_{\epsilon})$ and $L_t = \ln(-t/m^2)$.

In the derivation of this correction, from Feynman graph considerations [4], we observed that the analytical value for the initial radiation state in the $s$-channel is the same as that which comes from the expansion of the resummed formula obtained in [6], using the YFS method, with a particular choice of scale.

We have also performed a numerical check with the event generator LUMLOG [2]. Here the resummed formula of [6] taken from the BHLUMI set, Version 2.00 of September
1991, is implemented, which has a tested technical precision of 0.02%.

The total cross-section values (in nanobarns) are calculated at a centre-of-mass energy of 95.17 GeV, and for the final electron and positron scattering angles, $3^\circ \leq \theta_\pm \leq 8^\circ$.

In the first line of Table 2 the results of LUMLOG are reported for various entries of the emitted energy fraction $\Delta$.

To meet the LUMLOG conditions the program BHAGEN94 has been modified in the following way:

- the same cuts and LUMLOG condition $1 - x_1 x_2 < \Delta$ are imposed, where $x_1$ and $x_2$ are the energy fraction carried by the final fermions
- the function, appearing in BHAGEN94, $\beta(s, \theta_\pm) = \beta_e(s) + \beta_{\text{int}}(\theta_\pm)$, where $\beta_e(s) = 2(\alpha/\pi)(L - 1)$ and $\beta_{\text{int}}(\theta) = 4(\alpha/\pi)\ln(tan \theta/2)$, is calculated in the new centre-of-mass rest frame, after initial emission; this choice is only made so that it conforms with LUMLOG. The choice of the scale does, of course, affect the physical result, but in the present technical tests we want to compare only the higher-order contributions.

In the second line, labelled BG94-LUMLOG, are listed the results obtained using the generation procedure of BHAGEN94 and the resummed function of LUMLOG: this is a check of the generation procedure. The statistical variance is less than 0.01% and the agreement is within the technical precision of LUMLOG, which is 0.02%, as can easily be verified in the ratios to LUMLOG reported on the right in Table 2.

In the third line, labelled BG94-SIMPLE, are the results of a version of BHAGEN94 simplified to match the LUMLOG conditions:

- only the photon $t$-channel contribution is selected
- the vacuum polarization is switched off
- final photon emission corrections of $O(\alpha)$ and $O(\alpha^2 L_i^3)$ are switched off
- non-leading corrections of $O(\alpha)$ are switched off.

The statistical variance is about 0.01% and, in this case as well, the agreement is of the order of the technical precision of LUMLOG. Thus it can be concluded that the left-over corrections, $O(\alpha^3 L_i^3)$ and higher, due to initial emission, contribute less than 0.03%. The $O(\alpha^2 L_i^3)$ for final emission cannot be tested with the presently available version of LUMLOG.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>LUMLOG</td>
<td>33.696(4)</td>
<td>34.4135(5)</td>
<td>34.5715(7)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BG94-LUMLOG</td>
<td>33.704(1)</td>
<td>34.422(1)</td>
<td>34.5794(7)</td>
<td>1.0002</td>
<td>1.0002</td>
<td>1.0002</td>
</tr>
<tr>
<td>BG94-SIMPLE</td>
<td>33.702(4)</td>
<td>34.419(4)</td>
<td>34.578(3)</td>
<td>1.0002</td>
<td>1.0002</td>
<td>1.0002</td>
</tr>
</tbody>
</table>

Table 2

Total cross-section values (nanobarns) and ratios to LUMLOG values.
4 Complete and resummed results comparison

Finally, we compare our BHAGEN94 event generator in its normal shape with the BHLUMI event generator, taken from the BHLUMI set, Version 2.01 of September 1991.

Again, to avoid unnecessary complications, we start by selecting in BHAGEN94 only the photon $t$-channel. We do the same in BHLUMI by making the value of the variables DELZ and DELS equal zero. The total energy is at 95.17 GeV and the electron and photon scattering angles are $3^\circ \leq \theta_{\pm} \leq 8^\circ$. The internal subroutine TRIGAS0 is selected for the trigger in BHLUMI.

In Table 3 total cross-section values (in nanobarns) are given for various entries of the emitted energy fraction $\Delta = 1 - VMAXE$.

In the first line are the BHLUMI results. In the second line, labelled with BG94-OLD, are the results from the old version of BHAGEN94: the difference is up to 0.8% for $\Delta = 0.7$, as can easily be seen in the lower part of Table 3, where the ratios of the results to those of BHLUMI are explicitly given to help the comparison.

In the third line, labelled BG94-NEW, are the results from the new version of BHAGEN94, after improvements for the final hard photon emission following the first-order comparison. The generation of the final photon energy has replaced the approximated analytic formula previously used.

Another small correction to the BHAGEN94 resummed program is due to the introduction of the leftover higher-order terms in the normalization factor of the generating procedure, which although known in closed form is calculated through its expanded form for convenience.

The new values are in better agreement with BHLUMI: the difference is less than 0.25% and the agreement is better for smaller values of $\Delta$.

The discrepancy of up to 0.25% can be attributed to the still approximated treatment of the final hard photon correction, as highlighted in the $O(\alpha)$ comparison. We expect from that experience that the inclusion in the resummed BHAGEN94 program of the exact hard emission, through the complete matrix element of the one-photon radiative Bhabha process with exact kinematics, will decrease the values of 0.2–0.3%, bringing the agreement to the same level as in the $O(\alpha)$ comparison. This line of work is presently being followed.

In the fourth line of Table 3 we report the results obtained in the same conditions by the authors of the improved version of 40THIEVES [5]: the agreement with BHLUMI is inside 0.1%, with a statistical variance of 0.04%.
Table 3
Total cross-section values (nanobarns) and ratios to BHLUMI values.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>BHLUMI</td>
<td>14.433(3)</td>
<td>20.588(4)</td>
<td>28.975(5)</td>
<td>33.302(6)</td>
<td>34.977(7)</td>
<td>35.711(9)</td>
</tr>
<tr>
<td>BG94-OLD</td>
<td>14.4645(7)</td>
<td>20.628(1)</td>
<td>29.074(4)</td>
<td>33.507(6)</td>
<td>35.244(7)</td>
<td>36.003(4)</td>
</tr>
<tr>
<td>BG94-NEW</td>
<td>14.4464(7)</td>
<td>20.602(1)</td>
<td>29.014(1)</td>
<td>33.377(2)</td>
<td>35.066(2)</td>
<td>35.763(1)</td>
</tr>
<tr>
<td>40THIEVES</td>
<td></td>
<td></td>
<td></td>
<td>33.306(6)</td>
<td>34.965(4)</td>
<td>35.681(6)</td>
</tr>
</tbody>
</table>

|       |       |       |       | 1.0022 | 1.0019 | 1.0034 |
|       |       |       |       |       | 1.0062 | 1.0076 |
|       |       |       |       |       |       | 1.0082 |
| BG94-OLD|       |       |       | 1.0009 | 1.0007 | 1.0013 |
|       |       |       |       |       | 1.0023 | 1.0025 |
|       |       |       |       |       |       | 1.0015 |
| 40THIEVES|       |       |       |       |       |       |

|       |       |       |       |       | 1.0001 | 0.9997 |
|       |       |       |       |       |       | 0.9992 |

5 Conclusions

We have performed a detailed comparison of the $O(\alpha)$ and $O(\alpha^2 L_1^2)$ photonic corrections. This has allowed us to identify the source of the initial discrepancy (about 0.8%) between BHAGEN94 and BHLUMI.

After improvements, we obtained an agreement better than 0.02% at $O(\alpha)$ with the program OLDBIS.

Of the $O(\alpha^2 L_1^2)$ terms in BHAGEN94, the ones which are also included in the presently available program LUMLOG agree at the 0.03% level.

The resummed BHAGEN94 program has a discrepancy of up to 0.25% with BHLUMI, which can be attributed to the still approximated treatment in BHAGEN94 of the final hard photon correction. Improvements in progress are expected to bring the agreement to the same level as in the $O(\alpha)$ comparison.

Other non-photonic corrections have to be tested at the same accuracy level.

Finally, the requested precision has to be obtained in the description of real processes (calorimetric dressing and real experimental cuts), for which the comparison of the distributions of the generated events is needed.

Acknowledgements

Useful conversations with S. Jadach and Z. Was in pinpointing the source of the initial discrepancy are gratefully acknowledged. We thank O. Nicrosini for providing us the results of 40THIEVES. One of us (HC) is grateful to the Bologna Section of the INFN and to the Department of Physics of Bologna University for their support and kind hospitality.

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References


Small Angle Bhabha Scattering for LEP *

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Abstract

We present the results of our calculations to a one-, two-, and three-loop approximation of the $e^+e^- \rightarrow e^+e^-$ Bhabha scattering cross-section at small angles. All terms contributing to the radiatively corrected cross-section, within an accuracy of $\delta \sigma/\sigma = 0.1\%$, are explicitly evaluated and presented in an analytic form. $O(\alpha)$ and $O(\alpha^2)$ contributions are kept up to next-to-leading logarithmic accuracy, and $O(\alpha^3)$ terms are taken into account to the leading logarithmic approximation. We define an experimentally measurable cross-section by integrating the calculated distributions over a given range of final-state energies and angles. The cross-sections for exclusive channels as well as for the totally integrated distributions are also given.

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1 Introduction

An accurate verification of the Standard Model is one of the primary aims of LEP [1]. While electroweak radiative corrections to the s-channel annihilation process and to large angle Bhabha scattering allow a direct extraction of the Standard Model parameters, small angle Bhabha cross-section affects, as an overall normalization condition, all observable cross-sections, and represents an equally unavoidable condition towards a precise determination of the Standard Model parameters. The small angle Bhabha scattering process is used to measure the luminosity of electron–positron colliders. At LEP an experimental accuracy on the luminosity of

$$\frac{\delta \sigma}{\sigma} < 0.001$$

will soon be reached [2]. However, to obtain the total accuracy, a systematic theoretical error must also be added. This precision calls for an equally accurate theoretical expression for the Bhabha scattering cross-section in order to extract the Standard Model parameters from the observed distributions. An accurate determination of the small angle Bhabha cross-section and of the luminosity directly affects the determination of absolute cross-sections such as, for example, the determination of the invisible width and of the number of massless neutrino species $N_{\nu}$ [3].

In recent years a considerable attention has been devoted to the Bhabha process [4, 5]. However, the accuracy reached, is still inadequate. According to these evaluations the theoretical estimates are still incomplete, moreover they are in disagreement with each other, by up to 0.5% — far from the required theoretical and experimental accuracy [2].

The process that will be considered in this work is that of Bhabha scattering when electrons and positrons are emitted at small angles with respect to the initial electron and positron directions. We have examined the radiative processes inclusively accompanying the main $e^+e^- \rightarrow e^+e^-$ reaction at high energies, when both the scattered electron and positron are tagged within the counter aperture.

We assume that the centre-of-mass energies are within the range of the LEP collider $2\sqrt{s} = 90–200$ GeV, and the scattering angles are within the range $\theta \sim 10–150$ mrad. We assume that the charged particle detectors have the following polar angle cuts:

$$\theta_1 < \theta_\pm = \theta_1 \theta_2 \equiv \theta < \theta_3 , \quad \theta_2 < \theta_+ = \theta_2 \theta_3 ^\prime < \theta_4 , \quad 0.01 \lesssim \theta_i \lesssim 0.1 \text{ rad} ,$$

where $\vec{p}_1, \vec{p}_1^\prime, (\vec{p}_2, \vec{p}_2^\prime)$ represent the momenta of the initial and of the scattered electron (positron) in the centre-of-mass frame.

In this paper we present the results of our calculations of the Bhabha scattering cross-section with an accuracy of $O(0.1\%)$. The squared matrix elements of the various exclusive processes inclusively contributing to the $e^+e^- \rightarrow e^+e^-$ reaction are integrated in order to define an experimentally measurable cross-section according to suitable restrictions on the angles and energies of the detected particles. The various contributions to the electron and positron distributions, needed for the required accuracy, are presented using analytical expressions.
In order to define the angular range of interest and the implications on the required accuracy, let us first briefly discuss, in a general way, the angle-dependent corrections to the cross-section.

We consider $e^+e^-$ scattering at angles as defined in Eq. (2). Within this region, if one expresses the cross-section by means of a series expansion in terms of angles, the main contribution to the cross-section $d\sigma/d\theta^2$ comes from the diagrams for the scattering amplitudes containing one exchanged photon in the $t$-channel. These diagrams, as it is well known, show a singularity of the type $\theta^{-4}$ for $\theta \to 0$, e.g.

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}.$$  

Let us now estimate the correction of order $\theta^2$ to this contribution. If

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}(1 + c_1 \theta^2), \quad (3)$$

then, after integration over $\theta^2$ in the angular range as in Eq. (2), one obtains:

$$\int_{\theta_{\min}}^{\theta_{\max}} \frac{d\sigma}{d\theta^2} d\theta^2 \sim \theta_{\min}^{-2} \left(1 + c_1 \theta_{\min}^2 \ln \frac{\theta_{\max}^2}{\theta_{\min}^2}\right). \quad (4)$$

We see that, for $\theta_{\min} = 50$ mrad and $\theta_{\max} = 150$ mrad (we have taken the case where the $\theta^2$ corrections are maximal), the relative contribution of the $\theta^2$ terms is about $5 \times 10^{-3} c_1$. Therefore, the terms of relative order $\theta^2$ must only be kept in the Born cross-section where the coefficient $c_1$ is not small. In higher orders of the perturbative expansion the coefficient $c_1$ contains at least one factor $\alpha/\pi$ and therefore these terms can safely be omitted. This implies that, within our accuracy, only radiative corrections from the scattering type diagrams contribute. Furthermore, one should take into account only diagrams with one photon exchanged in the $t$-channel, since, according to the generalized eikonal representation, the large logarithmic terms from the diagrams with the multi-photon exchange are cancelled.

Having, as a final goal for the experimental cross-section, the relative accuracy as in Eq. (1), and by taking into account that the minimal value of the squared momentum transfer $Q^2 = 2e^2(1 - \cos \theta)$ in the region (2) is of the order of 1 GeV$^2$, we may omit in the following also the terms appearing in the radiative corrections of the type $m^2/Q^2$ with $m$ equal to the electron ($m_e$), or the muon ($m_\mu$) mass.

The contents of this paper can be outlined as follows. In Section 2 we discuss the Born cross-section $d\sigma^B$, taking the $Z^0$ boson exchange into account, and compute the corrections to it caused by the virtual and real soft photon emission. We present the results, as discussed above, in the form of an expansion in terms of the scattering angle $\theta$. We then define an experimentally measurable cross-section, $\sigma_{\text{exp}}$, which is obtained by tagging the scattered electron and positron within a suitable range of polar angles and energies. We introduce the ratio $\Sigma = \sigma_{\text{exp}}/\sigma_0$ by normalizing $\sigma_{\text{exp}}$ with respect to the cross-section $\sigma_0 = 4\pi \alpha^2/e^2 \theta^2$. In Section 2, by using a simplified version of the differential cross-section for the small angle scattering, we discuss the contribution to $\sigma_{\text{exp}}$ from the
single bremsstrahlung process. In Section 3 we find all corrections of $O(\alpha^2)$ to $\sigma_{\text{exp}}$ caused by two virtual and real photon emissions. In Section 4 we consider $O(\alpha^2)$ corrections caused by $e^+e^-$ pair emission. In Section 5 we derive the leading logarithmic corrections to the $\alpha^3$ order by using the structure function method. In Section 6 we estimate the contributions of the neglected terms. Finally, in Section 7 we give the results obtained in terms of the ratio $\Sigma$ as functions of the experimental parameters.

A more detailed derivation of these results will be presented elsewhere [6].

### 2 Born and one-loop soft and virtual corrections

The Born cross-section for Bhabha scattering within the Standard Model is well known [7]:

$$
\frac{d\sigma^B}{d\Omega} = \frac{\alpha^2}{8s} \{ 4B_1 + (1 - c)^2B_2 + (1 + c)^2B_3 \},
$$

where

\begin{align*}
B_1 &= \left(\frac{s}{t}\right)^2 \left[ 1 + (g_v^2 - g_a^2)\xi \right]^2, \\
B_2 &= \left[ 1 + (g_v^2 - g_a^2)\chi \right]^2, \\
B_3 &= \left(\frac{s}{t} + (g_v + g_a)^2 \left(\frac{s}{t}\xi + \chi\right) \right)^2,
\end{align*}

\begin{align*}
\chi &= \frac{\Lambda s}{s - m_Z^2 + i M_Z \Gamma_Z}, \\
\xi &= \frac{\Lambda t}{t - M_Z^2},
\end{align*}

\begin{align*}
\Lambda &= \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha} = (\sin 2\theta_w)^{-2}, \\
g_a &= -\frac{1}{2}, \\
g_v &= -\frac{1}{2}(1 - 4 \sin^2 \theta_w), \\
s &= (p_1 + p_2)^2 = 4\epsilon^2, \\
t &= -Q^2 = (p_1 - p_1')^2 = -\frac{1}{2} s (1 - c), \\
c &= \cos \theta, \\
\theta &= \overrightarrow{p_1}\overrightarrow{p_1'}.
\end{align*}

Here $\theta_w$ is the Weinberg angle. In the small angle limit

$$
c = \cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \ldots.
$$

Expanding the result (5) we have

$$
\frac{d\sigma^B}{\theta d\theta} = \frac{8\pi \alpha^2}{\epsilon^2 \theta^4} \left( 1 - \frac{\theta^2}{2} + \frac{9}{40} \theta^4 + \delta_{\text{weak}} \right),
$$

where $\epsilon = \sqrt{s}/2$ is the electron or positron initial energy. The contribution from weak interactions $\delta_{\text{weak}}$, connected with diagrams with $Z^0$ boson exchange, is given by the expression:

$$
\delta_{\text{weak}} = 2\frac{g_v^2}{4} \xi - \frac{\theta^2}{4} (g_v^2 + g_a^2) R \epsilon \chi + \frac{\theta^4}{32} (g_v^4 + g_a^4 + 6 g_v^2 g_a^2) |\chi|^2.
$$
From Eq. (7) one can see that the contribution $c^w_1$ of the weak correction $\delta_{\text{weak}}$ into the coefficient $c_1$ in Eq. (3) is

$$c^w_1 \lesssim 2g_v^2 + \frac{(g_v^2 + g_a^2)}{4} \frac{M_Z^2}{\Gamma_Z} + \frac{\theta^2}{\Gamma_{\text{max}}} \frac{(g_v^4 + g_a^4 + 6g_v^2g_a^2)}{32} \frac{M_Z^2}{\Gamma_Z^2} \sim 1 \quad (8)$$

According to our previous discussion after Eq. (4) this means that the contribution connected with the $Z^0$ boson exchange diagrams does not exceed 0.3%. We shall therefore neglect the radiative corrections to weak contributions, since they could contribute at most with terms $\lesssim 10^{-4}$.

In the pure QED case one-loop radiative corrections to the Bhabha cross-section were calculated a long time ago [8]. Taking into account the contribution coming from the emission of soft photons with energy less than a given finite threshold $\Delta \epsilon$ as well, one obtains for $d\sigma^{[1]}_{\text{QED}}/dc$ in a one-loop approximation

$$d\sigma^{[1]}_{\text{QED}}/dc = d\sigma^{B}_{\text{QED}}(1 + \delta_{\text{virtual}} + \delta_{\text{soft}}) \quad (9)$$

where $d\sigma^{B}_{\text{QED}}$ is the Born cross-section in the pure QED case (i.e. it is equal to $d\sigma^{B}$ with $g_a = g_v = 0$) and

$$\delta_{\text{virtual}} + \delta_{\text{soft}} = 2 \frac{\alpha}{\pi} \left\{ 2 \left[ 1 - \ln \left( \frac{4e^2}{m^2} \right) + 2 \ln ctg \frac{\theta}{2} \right] \ln \frac{\epsilon}{\Delta \epsilon} + \int_{\cos^2 \frac{\theta}{2}}^{\sin^2 \frac{\theta}{2}} \frac{dx}{x} \ln(1 - x) \\
- \frac{23}{9} + \frac{11}{6} \ln \left( \frac{4e^2}{m^2} \right) \right\} + \frac{\alpha}{\pi} \frac{1}{(3 + c^2)^2} \left[ \frac{\pi^2}{3} (2c^4 - 3c^3 - 15c) \\
+ 2 (2c^4 - 3c^3 + 9c^2 + 3c + 21) \ln^2 \sin \frac{\theta}{2} \\
- 4 (c^4 + c^2 - 2c) \ln^2 \cos \frac{\theta}{2} - 4 (c^3 + 4c^2 + 5c + 6) \ln^2 tg \frac{\theta}{2} \\
+ \frac{2}{3} (11c^3 + 33c^2 + 21c + 111) \ln \sin \frac{\theta}{2} + 2 (c^3 - 3c^2 + 7c - 5) \ln \cos \frac{\theta}{2} \\
+ 2 (c^2 + 3c^2 + 3c + 9) \delta_t - 2 (c^3 + 3c)(1 - c) \delta_s \right\} .$$

The value $\delta_t (\delta_s)$ is defined by the contributions to the photon vacuum polarization function $\Pi(t)$ [$\Pi(s)$] as follows:

$$\Pi(t) = \frac{\alpha}{\pi} \left( \delta_t + \frac{1}{3} L - \frac{5}{9} \right) + \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 L \quad (10)$$

where

$$L = \ln \frac{Q^2}{m^2} \quad , \quad Q^2 = -t = 2e^2(1 - c) \quad (11)$$

and we have taken into account the leading logarithmic part of the two-loop corrections to the polarization operator. In the Standard Model $\delta_t$ contains contributions of muons, tau-leptons, W-bosons, and hadrons:

$$\delta_t = \delta_t^\ell + \delta_t^\tau + \delta_t^W + \delta_t^H \quad , \quad \delta_s = \delta_t (Q^2 \rightarrow -s) \quad (12)$$
and the first three contributions are theoretically calculable:

\[ \delta^\mu_t = \frac{1}{3} \ln \frac{Q^2}{m^2_\mu} - \frac{5}{9}, \]

\[ \delta^\tau_t = \frac{1}{2} v_\tau \left( 1 - \frac{1}{3} v_\tau^2 \right) \ln \frac{v_\tau + 1}{v_\tau - 1} + \frac{1}{3} v_\tau^2 - \frac{8}{9}, \quad v_\tau = \left( 1 + \frac{4m^2_\tau}{Q^2} \right)^{\frac{1}{2}}; \]

\[ \delta^w_t = \frac{1}{4} v_w \left( v_w^2 - 4 \right) \ln \frac{v_w + 1}{v_w - 1} - \frac{1}{2} v_w^2 + \frac{11}{6}, \quad v_w = \left( 1 + \frac{4m^2_w}{Q^2} \right)^{\frac{1}{2}}. \quad (13) \]

The contribution of the hadrons \( \delta^H_t \) can be expressed through the experimentally measurable cross-section of the e\(^+\)e\(^-\) annihilation [9].

In the limit of small scattering angles we can present Eq. (9) in the following form:

\[
\frac{d\sigma^{(1)}_{\text{QED}}}{dc} = \frac{d\sigma^{R}_{\text{QED}}}{dc} \left[ 1 - \Pi(t) \right]^{-2} (1 + \delta),
\]

\[
\delta = \frac{2}{\pi} \left[ 2(1 - L) \ln \frac{1}{\Delta} + \frac{3}{2} L - 2 \right] + \frac{\alpha}{\pi} \theta^2 \Delta \theta + \frac{\alpha}{\pi} \theta^2 \ln \Delta,
\]

\[
\Delta \theta = \frac{3}{16} l^2 + \frac{7}{12} l - \frac{19}{18} + \frac{1}{4} (\delta - \delta_e),
\]

\[
\Delta = \frac{\Delta \epsilon}{\epsilon}, \quad l = \ln \frac{Q^2}{s} \simeq \ln \frac{\theta^2}{4}. \quad (14)
\]

This representation gives us a possibility to verify explicitly that the terms of a relative order \( \theta^2 \) in the radiative corrections are small. Taking into account that the large contribution proportional to \( \ln \Delta \) disappears when we add the cross-section for the hard emission, one can verify once more that such terms can be neglected. Therefore in higher orders we will omit the annihilation diagrams as well as multiple-photon exchange diagrams in the scattering channel. The second simplification is justified by the generalized eikonal representation for the amplitudes at small scattering angles [10].

Let us introduce now the dimensionless quantity

\[
\Sigma = \frac{Q^2_1 \sigma_{\text{ee}}}{4\pi \alpha^2},
\]

with \( Q^2_1 = \epsilon^2 \theta^2 \) where \( \sigma_{\text{ee}} \) represents the experimentally observable cross-section:

\[
\Sigma = \frac{Q^2_1}{4\pi \alpha^2} \int dx_1 \int dx_2 \theta(x_1 x_2 - x_c) \int d^2 q_1 \theta^c_1 \int d^2 q_2 \theta^c_2 \frac{d\sigma^{e+e-\rightarrow e^+e^-}(|\vec{q}_1, x_2) \otimes (\vec{q}_1, x_1) + x}}{dx_1 d^2 q_1 dx_2 d^2 q_2} \quad (15)
\]

where \( x_{1,2} \) and \( \vec{q}_{1,2} \) are, respectively, the energy fractions and the transverse components of the electron and positron momenta in the final state, \( sx_c \) is the experimental cut-off on their squared invariant mass \( sx_1 x_2 \), and the functions \( \theta^c_1 \) which take into account the angular cuts in Eq. (2) are defined as:

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\[
\theta^*_1 = \theta \left( \frac{\vec{q}_1}{x_1 \epsilon} \right) \theta \left( \frac{\vec{q}_1}{x_1 \epsilon} - \theta_1 \right) \quad , \quad \theta^*_2 = \theta \left( \frac{\vec{q}_2}{x_2 \epsilon} - \frac{\vec{q}_2}{x_2 \epsilon} \right) \quad . \quad (16)
\]

We restrict ourselves further to the symmetrical case only:

\[
\theta_2 = \theta_1 \quad , \quad \theta_4 = \theta_3 \quad , \quad \rho = \frac{\theta_3}{\theta_1} > 1 \quad . \quad (17)
\]

Let us define \( \Sigma \) as a sum of exclusive contributions:

\[
\Sigma = \Sigma_0 + \Sigma^\gamma + \Sigma^{2\gamma} + \Sigma^{*+*-} + \Sigma^{3\gamma} + \Sigma^{*+*-*} \quad , \quad (18)
\]

where \( \Sigma_0 \) stands for a modified Born contribution, \( \Sigma^\gamma, \Sigma^{2\gamma}, \) and \( \Sigma^{3\gamma} \) stand for the contributions of one-, two-, and three-photon emissions (both real and virtual), \( \Sigma^{*+*-} \) and \( \Sigma^{*+*-*} \) represent the emission of virtual or real (soft and hard) pairs without and with the accompanying real or virtual photon.

By integrating Eq. (6) with the use of the full propagator for the \( t \)-channel photon, which takes into account the growth of the electric charge at small distances, we obtain:

\[
\Sigma_0 = \theta^2 i \int_{\theta^*_1}^{\theta^*_2} \frac{d\theta^2}{\theta^4} [1 - \Pi(t)]^{-2} + \Sigma_W + \Sigma_\theta \quad , \quad (19)
\]

where

\[
\Sigma_W = \theta^2_i \int_{\theta^*_1}^{\theta^*_2} \frac{d\theta^2}{\theta^4} \delta_{\text{weak}} \quad ,
\]

is the correction due to the weak interactions and the term

\[
\Sigma_\theta = \theta^2_i \int_{1}^{\rho^2} \frac{dz}{z} [1 - \Pi(-zQ_1^2)]^{-2} \left( -\frac{1}{2} + \frac{\theta^2_1}{40} \right) \quad ,
\]

comes from the expansion of the Born cross-section in Eq. (5) in powers of \( \theta^2 \).

The one-loop contribution \( \Sigma^\gamma \) comes from one-photon emission (real and virtual). By adding to Eq. (14) the cross-section for hard emission calculated using a simplified version of the differential cross-section for small angle scattering [11] we obtain:

\[
\Sigma^\gamma = \frac{\alpha}{\pi} \int_{1}^{\rho^2} \frac{dz}{z^2} [1 - \Pi(-zQ_1^2)]^{-2} \left\{ \int_{x_0}^{1} dx \left[ (L_z - 1)P(x) \right. \right. \\
\left. \left. [1 + \theta(x^2\rho^2 - z)] + \frac{1 + x^2}{1 - x} k(x, z) \right] - 1 \right\} \quad , \quad (20)
\]

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where

\[ P(x) = \left( \frac{1 + x^2}{1 - x} \right)_+ = \lim_{\Delta \to 0} \left\{ \frac{1 + x^2}{1 - x} \theta(1 - x - \Delta) + \left( \frac{3}{2} + 2 \ln \Delta \right) \delta(1 - x) \right\} , \quad (21) \]

is the non-singlet splitting kernel and

\[ k(x, z) = \frac{(1 - x)^2}{1 + x^2} [1 + \theta(x^2 \rho^2 - z)] + L_1 + \theta(x^2 \rho^2 - z) L_2 + \theta(z - x^2 \rho^2) L_3 . \quad (22) \]

Here \( L_z = \ln \frac{z Q_0^2}{m^2} \) and

\[ L_1 = \ln \left( \frac{z^2 (z - 1) (\rho^2 - z)}{(x - z) (x \rho^2 - z)} \right) , \quad L_2 = \ln \left( \frac{(z - x^2) (x^2 \rho^2 - z)}{x^2 (x - z) (x \rho^2 - z)} \right) , \]

\[ L_3 = \ln \left( \frac{(z - x^2) (x^2 \rho^2 - z)}{(x - z) (x \rho^2 - z)} \right) . \quad (23) \]

### 3 Two-photon emission

Let us now consider the corrections of the order of \( \alpha^2 \). They come from the two-photon emission as well as from the pair production, real and virtual. The virtual and soft real photon corrections can be obtained by using the results of Refs. [11]–[14].

The corresponding contributions to \( \Sigma \) are:

\[ \Sigma_{S+V}^{\gamma} = \Sigma_{VV} + \Sigma_{VS} + \Sigma_{SS} = \left( \frac{\alpha}{\pi} \right)^2 \rho^2 \int dz \, z^{-2} [1 - \Pi(-z Q_0^2)]^{-2} R_{S+V}^{\gamma} . \quad (24) \]

It is convenient to present the \( R_{S+V}^{\gamma} \) in the following way:

\[ R_{S+V}^{\gamma} = r_{S+V}^{(\gamma\gamma)} + r_{S+V(\gamma\gamma)} + r_{S+V}^{\gamma} \]

\[ r_{S+V}^{(\gamma\gamma)} = r_{S+V(\gamma\gamma)} = \left( L_z \left( 2 \ln^2 \Delta + 3 \ln \Delta + \frac{9}{8} \right) \right) 
+ \left( L_z \left( -4 \ln^2 \Delta - 7 \ln \Delta + 3 \xi_3 - \frac{3}{2} \xi_2 - \frac{45}{16} \right) \right) \]

\[ r_{S+V}^{\gamma} = 4 \left[ (L_z - 1) \ln \Delta + \frac{3}{4} L_z - 1 \right]^2 . \quad (25) \]

Here the quantity \( \Delta = \delta \epsilon / \epsilon \ll 1 \) is the maximal energy fraction carried by a soft photon.

The single hard photon radiation can be accompanied by real soft or virtual photons. It is useful to separate the cases of photons emitted by the same electron or positron or by both of them.
\[ d\sigma|_{H,S^+V} = d\sigma_H^{(S^+V)} + d\sigma_{H(S^+V)} + d\sigma_{H(S^+V)} + d\sigma_H^{(S^+V)} . \]  

In the case where one of the fermions emits the hard real photon and another interacts with a virtual or soft real photon, we find:

\[
\Sigma_H^{(S^+V)} + \Sigma_H^{(S^+V)} = 2\frac{\alpha}{\pi} \int_1^{1-\Delta} dx \frac{1 + x^2}{1 - x} \int_0^{\rho^2} dz z^{-2}[1 - \Pi(-zQ^2)]^{-2} \left\{ [1 + \theta(x^2\rho^2 - z)] \left( L_z - 1 \right) + k(x, z) \right\} \left( \frac{\alpha}{\pi} \right) \left( L_z - 1 \right) \ln \Delta + \frac{3}{4} L_z - 1 . \]  

A more complicated expression arises when both photons interact with the same fermion. In this case the cross-section can be expressed in terms of the Compton tensor with a ‘heavy photon’ [15]. We will consider below the case of the photon emission from the electron. An equal contribution arises from the positron activity. For the small angle detection of the final fermions we have [7]:

\[
\Sigma_H^{(S^+V)} = \Sigma_{H(S^+V)} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 \int_1^{\rho^2} \frac{dz}{z^2[1 - \Pi(-zQ^2)]^2} \int_{\alpha^2}^{1-\Delta} \frac{dx(1 + x^2)}{1 - x} L_z \left\{ 2 \ln \Delta - \ln x + \frac{3}{2} \right\} \left[ (L_z - 1)(1 + \theta) + k(x, z) \right] + \frac{1}{2} \ln^2 x + (1 + \theta)[-2 + \ln x - 2 \ln \Delta] \\
+ (1 - \theta) \left[ \frac{1}{2} L_z \ln x + 2 \ln \Delta \ln x - \ln x \ln(1 - x) \\
- \ln^2 x + \int_0^{1-\Delta} \frac{dt}{t} \ln (1 - t) \right. \left. - \frac{x(1 - x) + 4x \ln x}{2(1 + x^2)} \right\} - \frac{(1 - x)^2}{2(1 + x^2)} \right\} \]  

where \( \theta = \theta(x^2\rho^2 - z) \).

Let us consider now the contribution from the emission of two hard real photons. One can distinguish two cases: double photon bremsstrahlung, a) in opposite directions along electron and positron momenta, and b) in the same direction along electron or positron momenta.

The differential cross-section in the first case can be obtained by using the factorization of cross-sections in the impact parameter space [15]. It takes the following form [11, 7]:

\[
\Sigma_H^H = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dz z^{-2}[1 - \Pi(-zQ^2)]^{-2} \int_{\alpha^2}^{1-\Delta} dx_1 \int_{\alpha^2}^{1-\Delta} dx_2 \\
\frac{1 + x_1^2}{1 - x_1} \frac{1 + x_2^2}{1 - x_2} \Phi(x_1, z)\Phi(x_2, z) , \]  

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where

\[
\Phi(x, z) = (L_z - 1)[\theta(z - 1)\theta(\rho^2 - z) + \theta(z - x^2)\theta(\rho^2 x^2 - z)]
+ L_3[-\theta(x^2 - z) + \theta(z - x^2) + L_2 + \frac{(1 - x)^2}{1 + x^2}] \theta(z - x^2) \theta(x^2 \rho^2 - z)
+ \left[L_1 + \frac{(1 - x)^2}{1 + x^2}\right] \theta(z - 1) \theta(\rho^2 - z)
+ [\theta(1 - z) - \theta(z - \rho^2)] \ln \frac{(z - x)(\rho^2 - z)}{(x \rho^2 - z)(z - 1)}.
\]  

(30)

Let us now turn to the double hard photon emission in the same direction, and the hard $e^+e^-$ pair production. We distinguish the cases of the collinear and semi-collinear kinematics of the final particles. In the first case all particles produced move in the cones within the polar angles $\theta_0 << \theta_1$ centred along the charged particle momenta (final or initial). The region corresponding to the case when both photons are radiated outside these cones does not contain a large logarithmic contribution. In the semi-collinear region only one of the particles produced moves inside the cones, whilst the other moves outside them.

In the totally inclusive cross-section the dependence on the auxiliary parameter $\theta_0$ disappears, and the total contribution has the form:

\[
\Sigma^{HH} = \Sigma_{HH} = \frac{1}{4} \left(\frac{\alpha}{\pi}\right)^2 \frac{\rho^2}{x} \int dz \int d\frac{x}{z} \left[1 - \Pi(-z Q_1^2)^{-2}\right] \int dx \int_{-\Delta}^{1-\Delta} dx_1
\]

\[
\frac{I^{HH} L_z}{x_1(1 - x - x_1)(1 - x_1)^2}
\]

\[
I^{HH} = A \theta(x^2 \rho^2 - z) + B + C \theta[(1 - x_1)^2 \rho^2 - z].
\]  

(31)

Here

\[
A = \gamma \beta \left\{ \frac{L_z}{2} + \ln \frac{(\rho^2 x^2 - z)^2}{x^2 \rho^2 x(1 - x_1) - z^2} \right\}
\]

\[
+ [x^2 + (1 - x_1)^2] \ln \frac{(1 - x_1)^2(1 - x - x_1)}{x x_1} + \gamma A,
\]

\[
B = \gamma \beta \left\{ \frac{L_z}{2} + \ln \left[ \frac{x^2 (z - 1)(\rho^2 x^2 - z)(z - x^2)[z - (1 - x_1)^2][\rho^2 x (1 - x_1) - z]^2]}{(\rho^2 x^2 - z)[z - (1 - x_1)^2][\rho^2 (1 - x_1)^2 - z]^2[z - x (1 - x_1)]^2} \right] \right\}
\]

\[
+ [x^2 + (1 - x_1)^2] \ln \frac{(1 - x_1)^2 x_1}{x (1 - x - x_1)} + \delta_B
\]

\[
C = \gamma \beta \left\{ L_z + 2 \ln \left[ \frac{x \rho^2 (1 - x_1)^2 - z}{(1 - x_1)^2 \rho^2 x (1 - x_1) - z} \right] \right\}
\]

\[
- 2(1 - x_1) \beta - 2x (1 - x_1) \gamma,
\]  

(32)
\[ \gamma = 1 + (1 - z_1)^2, \quad \beta = z^2 + (1 - z_1)^2, \]
\[ \gamma_A = zx_1(1 - x - x_1) - x_1^2(1 - x - x_1)^2 - 2(1 - x_1)\beta, \]
\[ \delta_B = zx_1(1 - x - x_1) - x_1^2(1 - x - x_1)^2 - 2x(1 - x_1)\gamma. \]

One can verify [6] that the combinations
\[
\left( \frac{\alpha}{\pi} \right)^2 \int_1^\rho^2 \frac{dz}{z^2[1 - \Pi(-zQ_1^2)]^2} r_{\gamma'}^2 V + \Sigma_{S+V}^H + \Sigma_{HH}^H, \\
\left( \frac{\alpha}{\pi} \right)^2 \int_1^\rho^2 \frac{dz}{z^2[1 - \Pi(-zQ_1^2)]^2} r_{\gamma'}^2 V + \Sigma_{S+V}^H + \Sigma_{S+V}^H + \Sigma_{HH}^H, 
\]
do not depend on \( \Delta \) for \( \Delta \to 0. \)

The total expression \( \Sigma_{\gamma'} \) which describes the contribution to Eq. (18) from all (real and virtual) two-photon emissions is determined by the expressions in Eqs. (24), (25), (27), (28), (29), and (31).

Furthermore, it does not depend on the auxiliary parameter \( \Delta \) and can be written as follows:

\[
\Sigma_{\gamma'} = \Sigma_{\gamma'}^\gamma + 2 \Sigma_{S+V}^H + \Sigma_{V+S}^H + \Sigma_{V+S}^H + \Sigma_{S+V}^H + 2\Sigma_{HH}^H.
\]

The leading contributions \( \Sigma_{\gamma'} \), \( \Sigma_{\gamma} \) have the form:

\[
\Sigma_{\gamma'} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 \int_1^\rho^2 L_z^2 dz z^{-2}[1 - \Pi(-zQ_1^2)^2]^{-1} \int x e \left\{ \frac{1}{2} P^{(2)}(x) \left[ \theta(x^2 \rho^2 - z) + 1 \right] \\
+ \int \frac{dt}{t} P(t) P \left( \frac{x}{t} \right) \theta(t^2 \rho^2 - z) \right\},
\]

where

\[
P^{(2)}(x) = \int \frac{dt}{t} P(t) P \left( \frac{x}{t} \right) = \lim_{\Delta \to 0} \left\{ \left( 2\ln \Delta + \frac{3}{2} \right)^2 - 4\xi_2 \right\} \delta(1 - x)
+ 2 \left[ \frac{1 + x^2}{1 - x} \left( 2\ln(1 - x) - \ln x + \frac{3}{2} \right) \\
+ \frac{1}{2} (1 + x) \ln x - 1 + x \right] \theta(1 - x - \Delta),
\]

and

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\[
\Sigma_\gamma = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty L_z^2 \int \frac{dz}{z^2} \left[ 1 - \Pi(-Q^2 z) \right]^{-2} \frac{dx}{x_{e} x_{e'}} \int_0^1 dx_1 P(x_1)P(x_2) \left[ \theta(z - 1)\theta(\rho^2 - z) + \theta(z - \frac{x_1^2}{x_1} x_{e}'^2)\theta(\rho^2 - z) \right] . \tag{37}
\]

We see that the leading contributions to \(\Sigma^{2\gamma}\) can be expressed in terms of kernels for the evolution equation for structure functions. The function \(\Phi^{\gamma}\) in the expression (34) collects the next-to-leading contributions which cannot be obtained by the structure functions method [16, 13]. It has a form given explicitly in Ref. [6].

## 4 Pair production

In a similar way we consider also pair production. The corrections due to virtual \(e^+e^-\) pairs can be extracted from Ref. [12]. Using the expression for soft pair production cross-section [14] one obtains for the contribution of the virtual and soft real \(e^+e^-\) pairs to \(\Sigma\) the following result:

\[
\Sigma^{e^+e^-}_{S+V} = \left( \frac{\alpha}{\pi} \right)^2 \int \frac{d\rho^2}{\rho^2} L_z \int \frac{dz}{z^2} \left[ 1 - \Pi(-Q^2 z) \right]^{-2} R_{S+V}^{e^+e^-} ,
\]

\[
R_{S+V}^{e^+e^-} = L_z^2 \left( \frac{2}{3} \ln \Delta + \frac{1}{2} \right) + L_z \left( -\frac{17}{6} + \frac{4}{3} \ln^2 \Delta - \frac{20}{9} \ln \Delta - \frac{4}{3} \xi_2 \right) + O(1) . \tag{38}
\]

In this expression the quantity \(\Delta = \delta e/e << 1\) is the maximal energy fraction carried by a soft pair.

Here we have taken into account only \(e^+e^-\) pairs. An order of magnitude of the pair production correction is less than 0.5%. A rough estimate of the muon pair contribution gives less than 0.05% since \(\ln Q^2/m^2 \sim 3 \ln Q^2/m^2\). Contributions of pion and tau-lepton pairs give even smaller corrections. Therefore, within the 0.1% accuracy, we can omit any pair except \(e^+e^-\).

Let us consider now the hard pair production. In this case we can restrict ourselves only to the collinear region where the produced pair moves in the small cones within the polar angles \(\theta_0\) around the fermion momenta. Indeed, the non-leading semi-collinear region will give contributions of one order of magnitude smaller than the leading ones and therefore within the required accuracy one can neglect them [7].

The pair production contributions to \(\Sigma\) appear from two regions with the pair components moving respectively along initial and scattered electrons and analogously for positrons. This separation allows us to carry out the integration over the energy fraction of the pair components. The resulting contribution to \(\Sigma\) has, with both directions included, the form:
\[ \Sigma_{H^{+,-}}^{*} = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^{2} \int_{1}^{\rho^{2}} d \rho \int_{1}^{\rho^{2}} \frac{d z}{z} |1 - \Pi(-z Q_{s}^{2})|^{-2} \int_{x_{c}}^{1} dx \left[ R_{0}(x) \left[ \mathcal{L}^{2} \left[ 1 + \theta(x^{2} \rho^{2} - z) \right] + 4 \mathcal{L} \ln x \right] + 2 \theta(x \rho - 1) \theta(x^{2} \rho^{2} - z) \mathcal{L} f(x) + 2 f_{1}(x) \mathcal{L} \right] , \] (39)

where we put the auxiliary parameter \( \theta_{0}^{2} \) equal to \( \theta^{2} = z \theta_{1}^{2} \), and introduce the following notations:

\[ R_{0}(x) = \frac{2}{3} \frac{1 + x^{2}}{1 - x} + \frac{1}{3x}(4 + 7x + 4x^{2}) + 2(1 + x) \ln x , \] (40)

\[ f(x) = - \frac{131}{9} + \frac{136}{9} x - \frac{2}{3} x^{2} - \frac{4}{3x} - \frac{20}{9(1 - x)} + \frac{2}{3} \left( -4x^{2} - 5x + 1 + \frac{4}{x} \right) \ln (1 - x) + \frac{1}{3} \left( 8x^{2} + 5x - 7 - \frac{13}{1 - x} \right) \ln x - 2 \left( \frac{1}{1 - x} \right) \ln^{2} x \] (41)

\[ + \frac{4x^{2}}{1 - x} \left( \int_{0}^{x} dy \frac{\ln (1 - y)}{y} + 2(1 + x) \left[ \xi_{2} + \ln x \ln (1 - x) + \int_{x}^{\infty} dy \frac{\ln (1 - y)}{y} \right] \right) \cdot \] (42)

\[ f_{1}(x) = - \frac{116}{9} + \frac{151}{9} x + \frac{2}{3x} + \frac{4x^{2}}{3} - \frac{20}{9(1 - x)} + \frac{1}{3} \left[ 8x^{2} - 10x - 10 + \frac{5}{1 - x} \right] \ln x \] (40)

\[ + \frac{2}{3} \left[ -4x^{2} - 5x + 1 + \frac{4}{x} + \frac{4}{1 - x} \right] \ln (1 - x) - (1 + x) \ln^{2} x \] (42)

\[ + 2(1 + x) \left[ \xi_{2} + \int_{0}^{x} dy \frac{\ln (1 - y)}{y} + \ln x \ln (1 - x) \right] - \frac{4}{1 - x} \left( \int_{0}^{x} dy \frac{\ln (1 - y)}{y} \right) , \]

where \( f_{1}(x) = -xf(\frac{1}{x}) \).

The total contribution of virtual, soft, and hard pairs does not depend on the auxiliary parameter \( \Delta \) and can be written as follows:

\[ \Sigma_{S_{+V}}^{*} = \Sigma_{H^{+,-}}^{*} + \Sigma_{H^{+,-}}^{*} = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^{2} \mathcal{L}^{2} \int_{1}^{\rho^{2}} d \rho \int_{1}^{\rho^{2}} \frac{d z}{z} |1 - \Pi(-z Q_{s}^{2})|^{-2} \int_{x_{c}}^{1} dx \left[ R(x) \left[ \theta(x^{2} \rho^{2} - z) + 1 \right] + \left( \frac{\alpha}{\pi} \right)^{2} \mathcal{L} \phi_{s}^{+,-} \right] , \] (43)

where

\[ R(x) = 2 \left( 1 + x \right) \ln x + \frac{1}{3x} \left( 4 + 7x + 4x^{2} \right) + \frac{2}{3} P(x) . \] (44)

The explicit expression for \( \phi_{s}^{+,-} \) which collects the non-leading terms from Eqs. (38) and (39) is:

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\[
\phi^{+,-} = \frac{1}{2} \int_1^{\rho^2} dzz^{-2} \int_{x_s}^{1} dx \left[ \theta(x\rho - 1)\theta(x^2\rho^2 - z)\bar{f} + \bar{f}_1 - \frac{17}{6} - \frac{4}{3} \xi_2 \right], \tag{45}
\]

where
\[
\bar{f} = f + \frac{20}{9} \left[ \frac{1}{1 - x} - \left( \frac{1}{1 - x} \right)^+ \right] + \frac{8}{3} \left[ -\ln(1 - x) + \left( \frac{\ln(1 - x)}{1 - x} \right)^+ \right]. \tag{46}
\]

The same definition applies to \(\bar{f}_1\). The regularization corresponding to the \((\quad)^+\) prescription is as in Ref. [16].

5 \(O(\alpha^3)\) corrections

In order to evaluate the leading logarithmic contribution represented by terms of the type \((\alpha L)^3\) we use the iteration of the master equation obtained in Refs. [16, 13].

To simplify the analytical expressions we adopt here a realistic assumption about the smallness of the threshold for the detection of the hard subprocess energy. By neglecting the terms of the order of:
\[
x_c \left(\frac{\alpha}{\pi} L \right)^3 \sim 10^{-4}, \tag{47}
\]

one should consider only the emission by the initial particles. Three photon (virtual and real) contribution to \(\Sigma\) have the form:
\[
\Sigma^{3\gamma} = \frac{1}{4} \left(\frac{\alpha}{\pi} L \right)^3 \int_1^{\rho^2} dzz^{-2} \int_{x_s}^{1} dx_1 \int_{x_s}^{1} dx_2 \theta(x_1 x_2 - x_c) \left[ \frac{1}{6} \delta(1 - x_2) P^{(3)}(x_1) \theta(x_1 \rho - 1)\theta(x_1^2 \rho^2 - z) + \frac{1}{2} P^{(2)}(x_1) P(x_2)\theta_1 \theta_2 \right] \left[ 1 + O(x_c^3) \right], \tag{48}
\]

where \(P(x)\) and \(P^{(2)}(x)\) are given by Eqs. (21) and (36) correspondingly,
\[
\theta_1 \theta_2 = \theta \left( x - x_2 \right) \theta \left( \rho^2 \frac{x_2^2}{x_1^2} - z \right),
\]

\[
P^{(3)}(x) = \delta(1 - x) \Delta_t + \theta(1 - x - \Delta) \theta_t,
\]

\[
\Delta_t = 48 \left[ \frac{1}{2} \xi_3 - \frac{1}{2} \xi_2 \left( \ln \Delta + \frac{3}{2} \right) + \frac{1}{6} \left( \ln \Delta + \frac{3}{2} \right)^3 \right], \tag{49}
\]

\[
\theta_t = 48 \left\{ \frac{1}{2} \frac{1 + x^2}{1 - x} \left[ \frac{9}{32} - \frac{1}{2} \xi_2 + \frac{3}{4} \ln(1 - x) - \frac{3}{8} \ln x + \frac{1}{12} \ln^2(1 - x) \right.ight.
\]
\[
+ \frac{1}{12} \ln^2 x - \frac{1}{2} \ln x \ln(1 - x) \left. \right] + \frac{1}{8} (1 + x) \ln x \ln(1 - x) - \frac{1}{4} (1 - x) \ln(1 - x) \right.
\]
\[
+ \frac{1}{32} (5 - 3x) \ln x - \frac{1}{16} (1 - x) - \frac{1}{32} (1 + x) \ln^2 x - \frac{1}{8} (1 + x) \int_0^1 dy \frac{\ln(1 - y)}{y} \right\}. \]
The contribution to \( \Sigma \) of the pair production accompanied by the photon emission when both, pair and photons, can be real and virtual is given below (with respect to Ref. [16] we include also the non-singlet mechanism of the pair production):

\[
\Sigma_{e^+e^-\gamma} = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 \int dz \int dx_1 \int dx_2 \theta(x_1 x_2 - x_c) \left\{ \frac{1}{3} \left[ R^p(x_1) - \frac{1}{3} R^s(x_1) \right] \delta(1 - x_2) \theta(x^2_2 \rho^2 - z) + \frac{1}{2} P(x_2) R(x_1) \theta \right\} [1 + O(x_c^3)],
\]

where

\[
R(x) = R^s(x) + \frac{2}{3} P(x), \quad R^s(x) = \frac{1}{3} x \left( 4 + 7x + 4x^2 \right) + 2(1 + x) \ln x,
\]

\[
R^p(x) = R^s(x) \left[ \frac{3}{2} + 2 \ln(1 - x) \right] + (1 + x) \left[ -\ln^2 x - 4 \int_0^{1-x} \frac{\ln(1-y)}{y} dy \right] + \frac{1}{3} \left( -9 - 3x + 8x^2 \right) \ln x + \frac{2}{3} \left( -\frac{3}{x} + 8 + 8x + 3x^2 \right).
\]

The total expression for \( \Sigma \) in Eq. (18) is the sum of the contributions given in Eqs. (19), (20), (34), (43), (48) and (50). The quantity \( \Sigma \) is a function of the parameters \( x_c, \rho \), and \( Q_1^2 \).

### 6 Estimates of neglected terms

Let us now estimate the terms not taken into account here in accordance with the required accuracy:

a) Weak radiative corrections:

\[
\Sigma_{\text{weak}} \sim \frac{\alpha Q_1^2}{\pi M_Z^2} \lesssim 10^{-5}.
\]

b) Electromagnetic corrections to weak contributions, including interference terms:

\[
\Sigma_{\text{W-W}} \sim \delta_{\text{weak}} \left| \theta_1 \right| \frac{\alpha}{\pi} L \lesssim 10^{-4}.
\]

Here \( \delta_{\text{weak}} \) is given by Eq. (7).

c) Radiative corrections to the annihilation mechanism, including its interference with the scattering mechanism:

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\[ \Sigma_{\text{int}} \sim \theta_i^2 \frac{\alpha}{\pi} \lesssim 10^{-4} . \]  

Our explicit expressions for \( \Sigma \), without annihilation terms, coincide numerically with the results obtained at the same order in Ref. [17] by using exact matrix elements.

d) The interference between photon emissions by electron and positron:

\[ \Sigma_{\text{int}} \sim \theta_i^2 \frac{\alpha}{\pi} \lesssim 10^{-5} . \]  

This contribution is connected with terms violating the eikonal form [10] in the expression:

\[ A(s, t) = A_0(s, t)e^{i\phi(t)} + O \left( \frac{\alpha t}{\pi s} \right) . \]  

e) The interference terms between one- and two-photon mechanisms of pair production, including the effect final particles identity:

\[ \Sigma_{\text{int}}^{\text{pairs}} \sim \left( \frac{\alpha}{\pi} \right)^2 \lesssim 10^{-5} . \]  

f) The semi-collinear mechanism of pair production gives a contribution which contains a small factor \( \mathcal{L}^{-1} \) with respect to the collinear terms, and is numerically small: \( \Sigma_{\text{sc}}^{\text{pairs}} \lesssim 10^{-4} \). A more accurate estimate should be interesting.

g) The creation of heavy pairs \((\mu\mu, \pi\pi, \pi\pi, \ldots)\) is at least one order of magnitude smaller than the corresponding contribution due to the light particle production and is therefore not essential.

h) Higher-order corrections, including soft and collinear multi-photon contributions, can be safely neglected since they only give contributions of the type \((\alpha \mathcal{L}/\pi)^n\) for \( n \geq 4 \).

7 Results and their discussion

The total cross-section for the e\(^+\)e\(^-\) distribution is:

\[ \sigma = \frac{4\pi \alpha^2}{Q_1^2} \Sigma , \]  

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where $\Sigma$ is given by Eq. (18).

Let us define $\Sigma_0^0$ to be equal to $\Sigma_0|_{\Pi=0}$ [see Eq. (19)], which corresponds to the Born cross-section obtained by switching out the vacuum polarization contribution $\Pi(t)$ defined in Eq. (10). We obtain:

$$\sigma = \frac{4\pi\alpha^2}{Q^2} \Sigma_0^0 (1 + \delta_0 + \delta^\gamma + \delta^{2\gamma} + \delta^{\gamma+\gamma} + \delta^{2+\gamma} + \delta^{3+\gamma} + \delta^{4+\gamma})$$

where

$$\Sigma_0^0 = \Sigma_0|_{\Pi=0} = 1 - \rho^{-2} + \Sigma_W + \Sigma_\theta|_{\Pi=0} \simeq 1 - \rho^{-2},$$

and

$$\delta_0 = \frac{\Sigma_0 - \Sigma_0^0}{\Sigma_0^0}; \quad \delta^\gamma = \frac{\Sigma^\gamma}{\Sigma_0^0}; \quad \delta^{2\gamma} = \frac{\Sigma^{2\gamma}}{\Sigma_0^0}; \quad \cdots.$$  

In the Tables below we give the values of $\delta_\beta$ as well as their sum as functions of $x_c$ for the values of the parameters $\theta_1 = 1.61^\circ$, and $\theta_2 = 2.80^\circ$, defining the Narrow-Narrow ($NN$) case, and for $\theta_1 = 1.5^\circ$, and $\theta_2 = 3.15^\circ$, defining the Wide-Wide ($WW$) case as in Ref. [18].

Each of these contributions to $\Sigma$ has a sign which can be changed as a result of the interplay between real and virtual corrections. The cross-section corresponding to the Born diagrams for producing a real particle is always positive, whereas the sign of the radiative corrections depends on the order of perturbation theory for the virtual corrections: at odd orders it is negative, and at even orders it is positive. When the aperture of the counters is small the compensation between real and virtual corrections is not complete. In the limiting case of zero aperture only the virtual contributions remain giving a negative result. As a consequence we see that the radiative corrections for the $NN$ case are larger in absolute value than ones in the $WW$ case. These corrections depend on $\theta_1, x_c$, and $\rho$. When $x_c \to 1$ the corrections increase in absolute value.

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References


The values of $\delta^i$ in Eq. (58) for $\sqrt{s} = M_Z = 91.161$ GeV, $\sin^2 \Theta_W = 0.2283$, $\Gamma_Z = 2.4857$ GeV.

The 'NN counter' corresponds to $\rho = 1.74$ and the 'WW counter' to $\rho = 2.10$ as a function of $x_c$.

### Table 1

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<th>$x_c$</th>
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<th>$\delta^{\gamma\gamma}$</th>
<th>$\delta^{\gamma*\gamma}$</th>
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### Table 2

Values of $R_{NN}$ and $R_{WW}$ where $R$ represents the ratio of non-leading with respect to leading contributions and is defined as $R = \frac{(\frac{a}{\pi})^2 L \phi^{2\gamma} / \Sigma^{2\gamma}}{\sqrt{s} = M_Z = 91.161}$ GeV, $\sin^2 \Theta_W = 0.2283$, $\Gamma_Z = 2.4857$ GeV. The 'NN counter' corresponds to $\rho = 1.74$ and the 'WW counter' to $\rho = 2.10$ as above as a function of $x_c$.

<table>
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<th>$R_{WW}$</th>
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<tr>
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</table>

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Small Angle Bhabha Scattering at LEP:
a Semi-Monte Carlo Approach

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Abstract

A semi-Monte Carlo approach to the computation of the small angle Bhabha scattering cross section is presented. The formulation is based on a proper matching of exact $O(\alpha)$ results with all the higher-order corrections now under control. Some numerical predictions are shown and commented. An estimate of the theoretical accuracy of the approach is given.

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1 Introduction

In order to perform high-precision measurements at LEP, a very accurate knowledge of the small angle Bhabha scattering cross section is mandatory, since it is the basic tool to determine precisely the machine luminosity. The experimental precision of luminosity measurements is at present at the level of 0.1% and is foreseen to improve further in the near future. In order to exploit such an experimental precision, theoretical results of comparable accuracy are required. From this point of view, the comparison between different and independent approaches could become important in order, on the one hand, to assess the presently reached accuracy and, on the other, to eventually improve it in a significant way.

Within this framework, the aim of the present contribution is to describe the improvements carried out on a previous formulation of the Bhabha scattering cross section [1] and perform some comparisons with existing results. These improvements are especially designed to compute results in the small angle region, of interest for luminosity measurements. The main upgrading is the exact treatment, in the presence of realistic cuts, of the $O(\alpha)$ hard photon effects beyond the collinear approximation, so that the present formulation contains an exact $O(\alpha)$ result in the dominant $t$-channel $\gamma$-exchange contribution, plus $O(\alpha)$ leading-logarithmic corrections in all the other relevant channels, plus all the higher-order corrections now under control. Some numerical results are shown and commented. For a detailed comparison with the results of other codes the reader is referred to the contribution by M. Caffo, H. Czyż and E. Remiddi (see Ref. [2], in these proceedings).

2 Leading-logarithmic results

Computing QED corrections in a realistic set-up is a very involved problem, since the corrections, besides being large, critically depend on the experimental cuts such as energy or invariant mass thresholds, angular acceptance, acollinearity cut, and so on. As a first step, one can adopt the soft and/or collinear approximation which, by virtue of the dynamical features of photonic radiation, is already quite a good one. In this case, the structure function approach allows the leading-log corrected cross section in the laboratory frame to be written, taking into account all-orders photonic radiation, in the following form:

$$
\sigma_{LL}^{(\infty)}(\hat{s}) = \int_\Omega d\Omega \int_{A(\Omega)} dx_1 dx_2 D_1^{(\infty)}(x_1, Q^2)D_2^{(\infty)}(x_2, Q^2)J(x_1, x_2, \vartheta) \\
\times \frac{d\sigma}{d\Omega}(\hat{s}(x_1, x_2), \hat{t}(x_1, x_2, \vartheta))F^{(\infty)}(\xi_1, Q^2)F^{(\infty)}(\xi_2, Q^2). 
$$

The detailed description of Eq. (1) can be found in Refs. [3, 4].

The photonic content of the structure function $D^{(\infty)}(x, Q^2)$ can be directly read by its explicit expression [5, 6]:

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\[ D^{(\infty)}(x, Q^2) = \exp\left\{ \frac{1}{2} \beta(\frac{3}{4} - \gamma_E) \right\} \frac{1}{\Gamma(1 + \frac{1}{2} \beta)} - \frac{1}{4} \beta(1 + x) + \frac{1}{32} \beta^2 \left[ -4(1 + x) \ln(1 - x) + 3(1 + x) \ln x \right. \\
\left. - 4 \frac{\ln x}{1 - x} - 5 - x \right] , \]

where
\[ \beta = \beta(Q^2) = \frac{2\alpha}{\pi} \left[ \ln\left(\frac{Q^2}{m^2}\right) - 1 \right] . \]

The structure function is obtained by solving, according to well-established techniques, the Lipatov–Altarelli–Parisi evolution equation satisfied by \( D(x, Q^2) \) in terms of the one-loop expression for the splitting function \( \text{electron} \rightarrow \text{electron} + \text{photon} \). This allows one to resum the large mass logarithms from soft multiphoton emission, and to include hard collinear photon effects up to \( \mathcal{O}(\alpha^2) \).

The factors \( F^{(\infty)} \) describe final-state radiation. They are given by

\[ F^{(\infty)}(\xi, Q^2) = \int_0^1 dx \ D^{(\infty)}(x, Q^2) , \]

where \( \xi \)'s are defined as \( \xi_i = E_i^e / E_i(x_1, x_2, \vartheta), \ E_i^e \) and \( E_i(x_1, x_2, \vartheta) \) being the final fermions energy thresholds and energies, respectively.

The structure function Eq. (2) is a hybrid solution of the non-singlet evolution equation, obtained by first solving to all orders the evolution equation in the soft limit and then adding to the soft solution those finite order terms not included in the exponentiated result, obtained by means of an iterative solution truncated at \( \mathcal{O}(\alpha^2) \). When interested in a finite-order result, one has instead to use the proper iterative solution truncated at the desired order. For instance, the structure function

\[ D^{(\alpha)}(x, Q^2) = \delta(1 - x) + \frac{1}{4} \beta P(x) , \]

where \( P(x) \) is the splitting function

\[ P(x) = \frac{1 + x^2}{1 - x} - \delta(1 - x) \int_0^1 dz \ \frac{1 + z^2}{1 - z} , \]

is exactly the \( \mathcal{O}(\alpha) \) content of the distribution \( D^{(\infty)}(x, Q^2) \) of Eq. (2). This means that by substituting in Eq. (1) \( D^{(\infty)} \rightarrow D^{(\alpha)}, \ F^{(\infty)} \rightarrow F^{(\alpha)} \), and dropping all the spurious higher-order terms, one is left with exactly the up to \( \mathcal{O}(\alpha) \) content of the all-orders result of Eq. (1), which can be written as

\[ \sigma_{LL}^{(\alpha)}(s) = \int_R d\Omega \frac{d\sigma}{d\Omega} \left[ s(1, 1), t(1, 1, \vartheta) \right] \left[ 1 + \frac{1}{2} \beta \int_{E_0/E} ^1 dx \ P(x) \right] \\
+ \int_R d\Omega \int_{\Lambda(\alpha)} dx \ \frac{1}{4} \beta P(x) \left\{ J(x, 1, \vartheta) \frac{d\sigma}{d\Omega} [s(x, 1), t(x, 1, \vartheta)] \right. \\
\left. + J(1, x, \vartheta) \frac{d\sigma}{d\Omega} [s(1, x), t(1, x, \vartheta)] \right\} . \]
To match properly the leading-log results with the exact $O(\alpha)$ results, control of the $O(\alpha)$ content of the structure function is crucial, in order to avoid double counting. For convenience, it should be noted that the corrected cross sections $\sigma_{LL}^{(\infty)}(s)$ and $\sigma_{LL}^{(\alpha)}(s)$ are functionals of the kernel cross section $d\sigma/d\Omega$, which from now on will be referred to as $\Sigma^{(\infty)}[d\sigma/d\Omega]$ and $\Sigma^{(\alpha)}[d\sigma/d\Omega]$, respectively.

As it is well known, the scale entering the structure functions is not uniquely fixed by renormalization group techniques. In the small angle regime, a choice that allows the reabsorption of the leading angle-dependent terms to all orders is $Q^2 = -t$, which is understood from now on.

### 3 Hard photonic corrections

The complete squared matrix element of the (gauge invariant) $t$-channel diagrams associated with (hard) photon radiation from the electron line has been computed in the context of very small angle radiative Bhabha scattering [7]. Its full expression, including all $O(m^2)$, $O(m^4)$ and $O(m^6)$ terms, reads as follows:

$$|M|^2 = \frac{2(4\pi\alpha)^3}{t^2} \left\{ M_0 + m^2 M_2 + m^4 M_4 + m^6 M_6 \right\},$$

where the single terms $M_i$ are explicitly reported in Ref. [7].

The squared matrix element (8) coincides with the result obtained independently in Ref. [8]. The positron radiation squared matrix element can be obtained from Eq. (8) by means of obvious symmetry relations. Actually, in the kinematical region of interest for luminometry at LEP the only relevant terms in Eq. (8) are $M_0$ and $M_2$, the other ones being negligible. Also the up-down interference contributions, which in the small angle scattering region are known to be small, have been neglected.

The radiative Bhabha cross section has then been obtained by a numerical integration of the matrix elements, with the proper flux factors. In order to manage all possible experimental cuts a Monte Carlo integration technique has been adopted, together with proper variance-reducing tricks (namely the importance sampling) to take care of the collinear singularities present for photons nearly parallel to one of the charged lines and the singularity at small scattering angles. For future use, let us define the radiative Bhabha cross section as

$$\sigma_H^{\gamma}(k_0, \text{cuts}) = \int d[PS] \int d[PS] F \left( |M^-|^2 + |M^+|^2 \right) \Theta(\text{cuts}),$$

where $d[PS]$ is the phase-space volume element, $|M_{\pm}|^2$ are the squared amplitudes for the electronic and positronic radiation, $F$ is the proper flux-factor, $\Theta(\text{cuts})$ represents the rejection algorithm for implementing the experimental cuts, and $k_0$ is a minimum energy fraction of the radiated photon, defined as $k_0 = E_{\min}^\gamma / E$. 

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4 The Bhabha cross section in the luminometry region

The leading-logarithmic cross section of Eq. (1) can be attributed an accuracy of the order of 1%, which is already a good one with respect to the simplicity of the result, but in view of the present precision requirements it must be improved. A first step in such a direction can be to insert proper $K$-factors, taking into account non-logarithmic corrections going beyond the leading-log approximation. This solution, which again shares a great simplicity with the previous one, has already been explored in the literature [1, 9], and on the basis of the studies performed one can say that this improvement leads to an accuracy of order, say, 0.5%, the main drawback of the approach being the absence of truly hard photon effects beyond the collinear approximation. However, when aiming at a higher accuracy, hard photon effects must be included exactly, at least at $\mathcal{O}(\alpha)$.

Some additional comments are also in order here. The tree-level Bhabha cross section, including $\gamma$- and $Z^0$-exchange both in $s$- and $t$-channels, can be written in terms of ten contributions (see for instance [9]). In the luminometry region ($\theta < 10^\circ$) the contribution coming from the square of the $t$-channel $\gamma$-exchange diagram is by far the dominant one, all the others (in particular $\gamma_t-Z_s$ interference) adding up to a few per mil already at tree-level. This fact allows two main approximations, which can be adopted when aiming at an accuracy of the order of 0.1%. The first one involves computing the exact $\mathcal{O}(\alpha)$ hard photon corrections only to the $t$-channel $\gamma$-exchange, and treating all the other contributions at the leading-log level. Note that when aiming at an accuracy below 0.1% the exact $\mathcal{O}(\alpha)$ correction to $\gamma_t-Z_s$ interference becomes necessary [10]. The second one entails neglecting all non-QED corrections but vacuum polarization. Actually, a comparison between the Born + vacuum polarization and the improved Born approximation of the second paper of Ref. [1] shows that all non-QED corrections but vacuum polarization are below 0.01%. For the quark contribution, the vacuum polarization effect is included by using the commonly adopted parametrization of Ref. [11].

Now, by denoting $d\sigma_0/d\Omega$ as the full Bhabha tree-level cross section, $d\sigma_\gamma/d\Omega$ its $t$-channel $\gamma$-exchange contribution, and $\sigma^{S+V}_\gamma(k_0)$ the $t$-channel $\gamma$-exchange cross section including virtual corrections plus soft photons of energy up to $E_\gamma = k_0 E$ (see for instance [9]), the cross section in the luminometry region can be written as

$$\sigma_A(s) = \Sigma^{(\infty)}|d\sigma_0/d\Omega| - \Sigma^{(\alpha)}|d\sigma_\gamma/d\Omega| + \sigma^{S+V}_\gamma(k_0) + \sigma^H_\gamma(k_0, \text{cuts}),$$  \hspace{1cm} (10)

where $\Sigma^{(\infty)}|d\sigma_0/d\Omega|$ is the all-order leading-log corrected full Bhabha cross section of Eq. (1), $\Sigma^{(\alpha)}|d\sigma_\gamma/d\Omega|$ is the up to $\mathcal{O}(\alpha)$ leading-log corrected cross section, Eq. (7), limited to the $t$-channel $\gamma$-exchange contribution, and $\sigma^H_\gamma(k_0, \text{cuts})$ is the radiative Bhabha cross section of Eq. (9).

The difference $\Sigma^{(\infty)}|d\sigma_0/d\Omega| - \Sigma^{(\alpha)}|d\sigma_\gamma/d\Omega|$ contains the Born contribution of all the Bhabha channels but the $t$-channel $\gamma$-exchange one, plus their leading-log QED corrections resummed to all orders, plus the higher-order leading-log QED corrections to the $t$-channel $\gamma$-exchange contribution starting from $\mathcal{O}(\alpha^2)$; the exact up to $\mathcal{O}(\alpha)$ contribution for the $t$-channel $\gamma$-exchange term is then supplied by $\sigma^{S+V}_\gamma(k_0) + \sigma^H_\gamma(k_0, \text{cuts})$. Note that the procedure adopted to match the exact up to $\mathcal{O}(\alpha)$ results with leading-log corrected cross
sections is quite general and could be easily improved to take care of other exact \( \mathcal{O}(\alpha) \) corrections besides the \( t \)-channel \( \gamma \)-exchange one.

Equation (10) is in the additive form. This, on the one hand, has the advantage of keeping clear the perturbative content of the cross section but, on the other hand, does not respect the so-called classical limit, according to which the cross section must vanish in the absence of photonic radiation. The usual way out is by writing a cross section in a factorized form, which in the present case can be built starting from the very same ingredients of Eq. (10) as follows:

\[
\sigma_F(s) = (1 + C_{NL}^H) \Sigma^{(\infty)} [d\sigma_0/d\Omega],
\]

\[
C_{NL}^H = \left\{ \sigma^{S+V}_\gamma(k_0) + \sigma^{H}_\gamma(k_0, \text{cuts}) - \Sigma^{[\alpha]}[d\sigma_\gamma/d\Omega] \right\} / \int_R d\Omega \frac{d\sigma_0}{d\Omega}.
\]  

Equation (11) has the same \( \mathcal{O}(\alpha) \) content as Eq. (10). The expression \( \sigma^{S+V}_\gamma(k_0) + \sigma^{H}_\gamma(k_0, \text{cuts}) - \Sigma^{[\alpha]}[d\sigma_\gamma/d\Omega] \) contained in \( C_{NL}^H \) gives the \( \mathcal{O}(\alpha) \) non-logarithmic contributions, including hard non-collinear photon effects, and from this point of view Eq. (11) represents the true improvement with respect to the results of Ref. [1]. In order to give some quantitative feeling of the situation, some numerical results follow here. The set-up is defined as follows: \( E = 47.585 \text{ GeV}, 3^\circ < \vartheta_\pm < 8^\circ, E^\gamma_{\text{max}}/E = 0.5, \text{ with } M_Z = 91.17 \text{ GeV}, \Gamma_Z = 2.3107 \text{ GeV}, \text{ and } \sin^2 \vartheta_W = 0.2273. \) The quoted cross sections are obtained by considering \( t \)-channel \( \gamma \)-exchange only, and with the correction due to vacuum polarization included.

### Table 1

A sample output of the updated version of 40THIEVES. Cross sections in nb.

<table>
<thead>
<tr>
<th>( \Sigma^{[\alpha]} )</th>
<th>( \Sigma^{[\infty]} )</th>
<th>( \sigma^{[\alpha]} )</th>
<th>( \sigma_A )</th>
<th>( \sigma_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.167(1)</td>
<td>35.213(1)</td>
<td>34.899(3)</td>
<td>34.945(3)</td>
<td>34.965(3)</td>
</tr>
</tbody>
</table>

### Table 2

A sample comparison with BHAGEN94 and BHLUMI. Cross sections in nb.

<table>
<thead>
<tr>
<th>( \sigma_F )</th>
<th>( \sigma_{BG} )</th>
<th>( \sigma_{BH} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>34.965(3)</td>
<td>35.066(2)</td>
<td>34.977(7)</td>
</tr>
</tbody>
</table>

In Table 1 a sample of the output of the present updated version of 40THIEVES is shown. \( \Sigma^{[\alpha, \infty]} \) are the leading logarithmic results up to \( \mathcal{O}(\alpha) \), and all orders, respectively; \( \sigma^{[\alpha]} \) is the exact first-order cross section; \( \sigma_A, F \) are the complete results in the additive and factorized form, respectively. By comparing the pure leading-log result \( \Sigma^{[\infty]} \) with the full cross section (in the factorized form \( \sigma_F \)) it is worth noting that \( a \ posteriori \) the accuracy of the leading-log approximation is of the order of 1%. Moreover, one should
also notice the difference between $\sigma_F$ and $\sigma_A$ which, although equivalent at $O(\alpha)$, differ
in the present case by 0.06%, the difference increasing when tightening the photon energy
cut. For the sake of comparison, the old version of 40THIEVES provides in the same
situation $\sigma_{old} = 35.134(2)$ nb, which deviates from the last result for $\sigma_F$ by about 0.5%.
In Table 2 a comparison with the results of the codes BHAGEN94 [12] and BHLUMI [13]
($\sigma_{BG}$ and $\sigma_{BH}$, respectively) is shown.

Some comments are in order here. First of all, our exact $O(\alpha)$ cross sections agree
with the analogous results of other codes within a $3\sigma$ numerical error of 0.03%, as can
be seen in Ref. [2]; this assures that the $O(\alpha)$ results are fully under control. Secondly,
the uncontrolled terms start from the next to leading $O(\alpha^2)$ ones, which can be naively
estimated to be of the order of 0.01%. An investigation on the relevance of two-loop
contributions can be found in Ref. [14], in these proceedings. At last, by looking at the
full results of Table 2 it can be seen that the agreement with BHAGEN94 is within 0.3%
and with BHLUMI within 0.03%. Moreover, by looking at the more detailed comparisons
quoted in Ref. [2] one can see that the agreement of our results and those of BHAGEN94
is still within 0.3%, and the agreement with BHLUMI is generally below 0.1%. Since
the authors of Ref. [2] attribute the 0.3% discrepancy of BHAGEN94 to an approximate
treatment of hard photon effects, we estimate that the accuracy of our results is of the
order of 0.1% as far as QED corrections are concerned. Further comments on the results
can be found in Ref. [2], in these proceedings.

Collinear photons from final-state electrons, important in every calorimetric measure-
ment, are at last taken into account as follows. First of all, since their effect is essentially
to compensate the final-state large infrared logarithms in such a way that the whole final-
state contribution is almost non-logarithmic, in the leading logarithmic formulae $\Sigma^{(\infty)}$ and
$\Sigma^{(\alpha)}$ the final-state radiation factors are removed. The $O(\alpha)$ hard photon contribution
$\sigma_H(k_0, \text{cuts})$ is then modified, both in the factorized and additive formulae, according to

$$
\sigma_H(k_0, \text{cuts}) \rightarrow \sigma_H(k_0, \text{cuts}) + \frac{\alpha}{\pi} \left[ F_\text{coll}(X_1, \delta_c) + F_\text{coll}(X_2, \delta_c) \right] \int d\Omega \frac{d\sigma_\gamma}{d\Omega},
$$

$$
F^{(\alpha, \infty)} \rightarrow 1, \tag{12}
$$

where the expression for $F_\text{coll}(x, \delta_c)$ is given in Ref. [15], with $X_i = E_0^i / E$. Including
$F_\text{coll}$ is equivalent to computing $\sigma_H$ with a rejection algorithm that keeps events in which
a photon of arbitrary energy is collinear, within an angle $\delta_c$, with one of the final-state
electrons. The effect of $F_\text{coll}$ critically depends on the energy threshold; for instance, in the
present case it is of the order of 1%.

In conclusion, we have presented an updated version of 40THIEVES and shown some
comparisons with existing codes. The accuracy of the results in the luminometry region
can be currently estimated to be of the order of 0.1% as far as pure QED corrections are
concerned. The code, which is in the form of a stand-alone program, can be obtained,
upon request, from one of the authors.
Acknowledgements

The authors are grateful to Michele Caffo for useful discussions, and for having kindly provided the results of BHAGEN94 and BHLUMI quoted in Table 2.

References


[14] A. Arbuzov et al., contribution to this Report.

Z-boson-exchange Contributions to the Luminosity Measurements at LEP and c.m.s.-energy-dependent Theoretical Errors

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\textsuperscript{b} Institut de Fisica d'Altes Energies
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Abstract

The precision of the calculation of Z-boson-exchange contributions to the luminosity measurements at LEP is studied for both the first and second generation of LEP luminosity detectors. It is shown that the theoretical errors associated with these contributions are sufficiently small so that the high-precision measurements at LEP, based on the second generation of luminosity detectors, are not limited. The same is true for the c.m.s.-energy-dependent theoretical errors of the Z line-shape formulae.

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During recent years the experiments at LEP have made substantial progress with high-precision luminosity measurements [1]. The first generation of luminosity detectors finally reached a precision of 0.3–0.5% (see Table 1). In September 1992 the second generation of LEP luminosity detectors made their first appearance. At present the most precise detectors of this type are already approaching a precision of 0.05% (see Table 1).

### Table 1

Precision (in %) of the luminosity detectors at LEP for different periods of data taking [1]. Names and precision of the second generation of detectors are marked in bold. ALEPH SICAL has been operational since September 1992. The 1992 precision of LCAL was 0.37%. DELPHI STIC has been operational since the beginning of 1994.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ALEPH LCAL,SICAL</td>
<td>0.49</td>
<td>0.37</td>
<td>0.15</td>
<td>0.09</td>
</tr>
<tr>
<td>DELPHI SAT,STIC</td>
<td>0.8</td>
<td>0.5</td>
<td>0.38</td>
<td>0.28</td>
</tr>
<tr>
<td>L3 BGO,SLUM</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.16</td>
</tr>
<tr>
<td>OPAL FD,SiW</td>
<td>0.7</td>
<td>0.45</td>
<td>0.41</td>
<td>0.07</td>
</tr>
</tbody>
</table>

More details about the acceptances of the various detectors can be found in Ref. [1]. For the analysis in this paper we define a symmetric angular acceptance of 3.3–6.3° for the first generation of luminosity detectors and an asymmetric one for the second generation, consisting of a ‘narrow’ region of 1.61–2.8° in one hemisphere and a ‘wide’ region of 1.5–3.15° in the other. These angular ranges are motivated by the acceptances of the ALEPH LCAL and SICAL detectors, but can be considered as being typical for the luminosity detectors at LEP.

The most important contribution to the small-angle Bhabha scattering cross-section, i.e. the $t$-channel photon-exchange one, is described in detail in other contributions to this report [2]. A summary of the results on the interference between the $t$-channel photon exchange and the $s$-channel $Z$ exchange will be given here. This contribution vanishes at the $Z$ peak in the Born approximation. It is positive below the $Z$ peak and negative above. The maximum value of about 1% is reached roughly 1 GeV above and below the $Z$ peak for the first generation of LEP luminosity detectors [3]. The corresponding $O(\alpha)$ QED corrections can reach up to 50% of the lowest-order contribution above the $Z$ peak, and contribute about 0.2% at the $Z$ peak for the first generation of luminosity detectors. The relative contribution of these interference terms decreases with a decreasing Bhabha scattering angle, and is about four times smaller for the second generation of luminosity detectors.

The most important contributions to the lowest-order Bhabha cross-section for the angular region of the second generation of LEP detectors are shown in Table 2. We indicate the matrix elements appearing in different Born contributions by $G_c$, where $G$ stands for the exchanged gauge boson ($Z$ or $\gamma$) and $c$ for the channel ($s$ or $t$). The $t$-channel photon exchange dominates the total cross-section. The interference of the $s$-channel $Z$-boson exchange and the $t$-channel photon exchange varies strongly with energy and stays below 0.25%. The interference between $t$- and $s$-channel photon exchange is about 0.03%.

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The most important contributions to the lowest-order Bhabha cross-section in the typical angular region of the second generation of LEP luminosity detectors [3], calculated with BABAMC for a $Z$ width of 2.3098 GeV. The corresponding table for the angular region of the first generation of detectors can be found in Ref. [3]. Contributions that are not listed in this table are smaller than 0.001%.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>Born (nb)</th>
<th>$\gamma t - \gamma t$ (% Born)</th>
<th>$Z_s - \gamma t$ (% Born)</th>
<th>$\gamma s - \gamma t$ (% Born)</th>
<th>$Z_s - Z_s$ (% Born)</th>
</tr>
</thead>
<tbody>
<tr>
<td>89.161</td>
<td>111.29</td>
<td>+99.840</td>
<td>+0.192</td>
<td>-0.033</td>
<td>+0.000</td>
</tr>
<tr>
<td>89.361</td>
<td>110.80</td>
<td>+99.830</td>
<td>+0.202</td>
<td>-0.033</td>
<td>+0.000</td>
</tr>
<tr>
<td>89.661</td>
<td>110.07</td>
<td>+99.816</td>
<td>+0.216</td>
<td>-0.033</td>
<td>+0.000</td>
</tr>
<tr>
<td>90.036</td>
<td>109.17</td>
<td>+99.807</td>
<td>+0.225</td>
<td>-0.033</td>
<td>+0.001</td>
</tr>
<tr>
<td>90.411</td>
<td>108.25</td>
<td>+99.824</td>
<td>+0.207</td>
<td>-0.033</td>
<td>+0.001</td>
</tr>
<tr>
<td>90.786</td>
<td>107.28</td>
<td>+99.898</td>
<td>+0.134</td>
<td>-0.033</td>
<td>+0.001</td>
</tr>
<tr>
<td>91.161</td>
<td>106.25</td>
<td>100.031</td>
<td>+0.000</td>
<td>-0.033</td>
<td>+0.001</td>
</tr>
<tr>
<td>91.536</td>
<td>105.24</td>
<td>100.168</td>
<td>-0.137</td>
<td>-0.033</td>
<td>+0.001</td>
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<td>91.911</td>
<td>104.30</td>
<td>100.247</td>
<td>-0.215</td>
<td>-0.033</td>
<td>+0.001</td>
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<td>92.286</td>
<td>103.43</td>
<td>100.269</td>
<td>-0.237</td>
<td>-0.033</td>
<td>+0.001</td>
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<td>92.661</td>
<td>102.60</td>
<td>100.263</td>
<td>-0.231</td>
<td>-0.033</td>
<td>+0.000</td>
</tr>
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<td>92.961</td>
<td>101.96</td>
<td>100.250</td>
<td>-0.218</td>
<td>-0.033</td>
<td>+0.000</td>
</tr>
<tr>
<td>93.161</td>
<td>101.53</td>
<td>100.240</td>
<td>-0.208</td>
<td>-0.033</td>
<td>+0.000</td>
</tr>
</tbody>
</table>

A very big effort has been made to improve the precision of the luminosity calculation in order to match the experimental precision [3-10]. In the various computer programs the photon–$Z$ interference is introduced in the following way:

- in lowest order in the Monte Carlo program BHLUMI [4] — the basic Monte Carlo generator currently used by the LEP experiments;
- in $O(\alpha)$ in the Monte Carlo program BABAMC [5];
- in $O(\alpha) +$ leading-log (LL) $O(\alpha^2)$ in the semianalytical program ALIBABA [6];
- in leading-log in the semianalytical program 40THIEVES [7];
- in $O(\alpha) +$ part of $O(\alpha^2)$ in the Monte Carlo program BHAGEN [8].

In order to determine the technical precision of the $O(\alpha)$ calculation of the terms containing $Z$-boson exchange in BABAMC, the contributions from different terms in BABAMC were compared with the corresponding ones in ALIBABA for the acceptances of the first and second generation of LEP luminosity detectors [3]. These contributions include QED $O(\alpha)$ corrections, corrections to the $Ze$ vertex, and corrections associated with the vacuum polarization, the $Z$ self-energy, and the photon-$Z$ mixing self-energy.
In order to make this comparison meaningful ALIBABA was downgraded to a pure O(α) program, using BABAMC input parameters. The most striking consequence of this change was the use of the Z width value $\Gamma_Z = 2.3098\text{ GeV}$ in the comparison. This value results from the lowest-order calculation of $\Gamma_Z$ in BABAMC and hence does not contain effects originating from the Z self-energy. In the second stage of the comparison this was compensated for by changing $\Gamma_Z$ in both programs to 2.487 GeV, close to the actually measured Z width. No difference bigger than 0.03% between the values calculated by both programs was observed for the total Z-boson contribution or the individual terms. This was valid for both of the above-mentioned values of $\Gamma_Z$ and for the angular regions of both the first and second generation of LEP luminosity detectors [3]. Thus it was concluded that the technical precision of the O(α) calculation of the terms containing Z-boson exchange in BABAMC is 0.03%.

The contribution of terms containing Z-boson exchange calculated with the help of the ALIBABA program is shown in Figs. 1 and 2 both for the first and second generation of LEP luminosity detectors. The numerical values for the second generation of detectors are shown in Table 3, from which we can see that the higher-order corrections can be trivially reduced by a factor of two by changing $\Gamma_Z$ in BABAMC from 2.3098 GeV to 2.487 GeV. In order to avoid confusion we have indicated the lowest-order contribution corresponding to terms containing Z-boson exchange by $Z$-Born. This includes all $Z$-exchange contributions, rather than just the $Z\gamma_f$ interference terms, in order to facilitate the isolation of these terms in BABAMC.

The comparison of the calculation of the terms containing Z-boson exchange in BHAGEN with those in ALIBABA was made by the OPAL Collaboration [11]. The agreement between the results obtained with the BHAGEN program and the corresponding ALIBABA ones given in Ref. [3] was found to be better than 0.006% of the full Born cross-section.

The contribution of terms containing Z-boson exchange to the absolute (global) luminosity error is shown in Table 4. For the calculation of the errors it is assumed that the luminosity cross-section is calculated with the help of the Monte Carlo program BHLUMI, that the photon–Z interference contribution is removed from the BHLUMI cross-section and that it is calculated with the help of the BABAMC Monte Carlo. This is the current strategy of the LEP experiments with the exception of OPAL, which is calculating this contribution with BHAGEN. The error related to the photon–Z interference contribution is composed of three parts. The first one (entry 3) is the technical precision of the calculation of this contribution in BABAMC. The second one (entry 4) corresponds to the LL O(α²) terms calculated in ALIBABA but absent in BABAMC. It is assumed that the other missing terms, dominated by the sub-leading (SL) O(α²) terms, will not be bigger than the LL ones (entry 5).

The Z line-shape is described by three parameters: $\sigma_0$, $\Gamma_Z$, and $M_Z$. The sensitivity of these three parameters to shifts in the cross-sections in the different energy points measured during the 1993 three-point scan are presented in Table 5, together with the corresponding sensitivity of the number of neutrino generations $N_\nu$. The error of the absolute luminosity measurement influences the $\sigma_0$ measurement and $N_\nu$, but has no effect on the measurement of the $Z$ width and mass (see global shift in Table 5). On the other hand, the relative shift of one measured point against the other one(s) influences
the $\Gamma_Z$ and $M_Z$ measurements. The photon--$Z$ interference contribution to the luminosity measurement represents a typical example of a theoretical error contributing to such a shift. The LL $O(\alpha^2)$ contribution to this process presented in Table 3 and Fig. 2, as calculated with ALIBABA, depends on the c.m.s.-energy. Since this contribution is absent in BABAMC, experiments using this program do not correct for it.

The influence of this c.m.s.-energy-dependent relative theory error on the precision of the measurement of the $Z$ parameters at LEP has been studied in Ref. [12] for the three-point 1993 scan. The change in the various $Z$ parameters as a result of the absence of LL $O(\alpha^2)$ contributions in the photon--$Z$ interference contribution to the luminosity measurement is given in Table 6. Since non-calculated higher-order corrections also have to be taken into account, the total error will be taken to be twice the errors given in Table 6 in the same way as was done for the global error in Table 4.

The second contribution to the c.m.s.-energy-dependent theoretical errors comes from the line-shape fitting formulae. This too has been studied in Ref. [12]. The missing LL terms for the Yennie–Frautschi–Suura (YFS) [13] formulae have been estimated to be not bigger than the complete LL $O(\alpha^3)$ contribution, and the missing SL contribution to be not bigger than the SL $O(\alpha^2)$ contribution [14]. Both contributions are shown in Fig. 3a and their sum is shown in Fig. 3b. The YFS formulae are used in the fitting program MIZA [15]. The difference between the MIZA calculations and the ZFITTER [15] $O(\alpha^2)$

Table 3

Higher-order corrections to the $Z$-boson exchange contributions for the second generation of luminosity detectors, determined relative to the $O(\alpha)$ corrected results based on $\Gamma_Z = 2.3098$ GeV (diff.1) or $\Gamma_Z = 2.487$ GeV (diff.2) [3]. All cross-sections are taken from the corresponding ALIBABA versions and are presented as fractions (%) of the Born cross-section given in Table 2.

<table>
<thead>
<tr>
<th>$\sqrt{s}$</th>
<th>Z-Born</th>
<th>Z-Born + $O(\alpha)$</th>
<th>Z-Born + $O(\alpha)$ + h.o.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.3098</td>
<td>2.487</td>
<td>2.3098</td>
</tr>
<tr>
<td>89.161</td>
<td>+0.192</td>
<td>+0.185</td>
<td>+0.155</td>
</tr>
<tr>
<td>89.361</td>
<td>+0.202</td>
<td>+0.193</td>
<td>+0.161</td>
</tr>
<tr>
<td>89.661</td>
<td>+0.216</td>
<td>+0.204</td>
<td>+0.170</td>
</tr>
<tr>
<td>90.036</td>
<td>+0.226</td>
<td>+0.209</td>
<td>+0.176</td>
</tr>
<tr>
<td>90.411</td>
<td>+0.208</td>
<td>+0.187</td>
<td>+0.164</td>
</tr>
<tr>
<td>90.786</td>
<td>+0.135</td>
<td>+0.118</td>
<td>+0.124</td>
</tr>
<tr>
<td>91.161</td>
<td>+0.001</td>
<td>+0.001</td>
<td>+0.054</td>
</tr>
<tr>
<td>91.536</td>
<td>-0.136</td>
<td>-0.119</td>
<td>-0.026</td>
</tr>
<tr>
<td>91.911</td>
<td>-0.214</td>
<td>-0.193</td>
<td>-0.087</td>
</tr>
<tr>
<td>92.286</td>
<td>-0.236</td>
<td>-0.218</td>
<td>-0.122</td>
</tr>
<tr>
<td>92.661</td>
<td>-0.231</td>
<td>-0.218</td>
<td>-0.137</td>
</tr>
<tr>
<td>92.961</td>
<td>-0.218</td>
<td>-0.208</td>
<td>-0.140</td>
</tr>
<tr>
<td>93.161</td>
<td>-0.208</td>
<td>-0.200</td>
<td>-0.138</td>
</tr>
</tbody>
</table>
Summary of theoretical errors in the luminosity calculation for the two different angular regions studied in this paper [3]. In the calculation of the total theoretical error, errors (1) and (2) as well as (4) and (5) have been added linearly, the other ones quadratically.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>1st gener.</th>
<th>2nd gener.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $O(\alpha^2)$ LL BHLUMI</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>(2) $O(\alpha^2)$ SL BHLUMI</td>
<td>0.09</td>
<td>0.09</td>
</tr>
<tr>
<td>(3) $Z$ exchange $O(\alpha)$ BABAMC</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>(4) $Z$ exchange $O(\alpha^2)$ LL BABAMC</td>
<td>0.06</td>
<td>0.015</td>
</tr>
<tr>
<td>(5) $Z$ exchange $O(\alpha^2)$ SL BABAMC</td>
<td>0.06</td>
<td>0.015</td>
</tr>
<tr>
<td>(6) Vacuum polarization</td>
<td>0.08</td>
<td>0.05</td>
</tr>
<tr>
<td>Total theoretical error</td>
<td>0.28</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Error of different fit variables generated by variation of global and relative errors by 0.1% (taken from Ref. [12], but recomputed with different off-peak energies).

<table>
<thead>
<tr>
<th>Error</th>
<th>Change in $M_Z$(MeV)</th>
<th>$\Gamma_Z$(MeV)</th>
<th>$\sigma_0$(pb)</th>
<th>$N_L(\times10^{-3})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Global ($\uparrow\uparrow\uparrow$)</td>
<td>0.0</td>
<td>0.0</td>
<td>42</td>
<td>7.5</td>
</tr>
<tr>
<td>(b) Relative - peak shift($\cdot\uparrow\cdot$)</td>
<td>0.26</td>
<td>2.2</td>
<td>44</td>
<td>7.9</td>
</tr>
<tr>
<td>(c) Relative - wing shift($\downarrow\cdot\uparrow$)</td>
<td>0.8</td>
<td>0.0</td>
<td>2.9</td>
<td>0.7</td>
</tr>
<tr>
<td>(d) Relative - wing shift($\cdot\cdot\uparrow$)</td>
<td>0.9</td>
<td>1.1</td>
<td>1.5</td>
<td>0.4</td>
</tr>
<tr>
<td>(e) Relative - wing shift($\uparrow\cdot\cdot$)</td>
<td>0.7</td>
<td>1.1</td>
<td>4.3</td>
<td>0.9</td>
</tr>
</tbody>
</table>

formulæ is shown in Fig. 3c. This difference is larger than the error associated with the YFS approach shown in Fig. 3b. This apparent contradiction can be traced back to the fact that the YFS approach allows the resummation of soft photons and lepton pairs to all orders in a systematic way, substantially improving perturbative convergence [16], and accounting naturally for the interplay between hard and soft radiation. Therefore, for the treatment of the soft radiation phenomena, which constitutes the bulk of the initial-state corrections, the YFS approach is more precise than the one adopted in Ref. [17] and, therefore, the difference in Fig. 3c does not reflect the actual theoretical uncertainty but rather the bias introduced by using simpler approaches than the YFS one. Finally Fig. 3d displays the difference between the ZFITTER fitting formulæ and its full Standard Model
LL $O(\alpha^2)$ corrections to the $Z$-boson exchange contributions (in % Born) determined relative to the $O(\alpha)$ corrected results from BABAMC using $\Gamma_Z = 2.3098$ GeV or $\Gamma_Z = 2.487$ GeV. The right-hand side of the table shows the change of the fit parameters caused by these corrections in units of MeV for $M_Z$ and $\Gamma_Z$, in pb for $\sigma_0$, and in $10^{-3}$ for $N_\nu$. (Table taken from Ref. [12], but recomputed with different off-peak energies).

<table>
<thead>
<tr>
<th>BABAMC $\Gamma_Z$</th>
<th>Angular acceptance</th>
<th>c.m.s.-energy $M_Z-1.8$</th>
<th>$M_Z$</th>
<th>$M_Z+1.8$</th>
<th>Change of $M_Z$</th>
<th>$\Gamma_Z$</th>
<th>$\sigma_0$</th>
<th>$N_\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3098</td>
<td>second generation</td>
<td>+0.002</td>
<td>-0.018</td>
<td>-0.004</td>
<td>0.0</td>
<td>+0.37</td>
<td>-8.1</td>
<td>+1.5</td>
</tr>
<tr>
<td>2.4870</td>
<td>generation</td>
<td>+0.000</td>
<td>-0.014</td>
<td>-0.003</td>
<td>+0.1</td>
<td>+0.28</td>
<td>-6.3</td>
<td>+1.1</td>
</tr>
<tr>
<td>2.3098</td>
<td>first generation</td>
<td>-0.012</td>
<td>-0.078</td>
<td>-0.019</td>
<td>0.0</td>
<td>+1.6</td>
<td>-35.4</td>
<td>+6.4</td>
</tr>
<tr>
<td>2.4870</td>
<td>generation</td>
<td>+0.003</td>
<td>-0.058</td>
<td>-0.013</td>
<td>0.0</td>
<td>+1.2</td>
<td>-26.0</td>
<td>+4.7</td>
</tr>
</tbody>
</table>

prediction,\(^1\) representing the error caused by the use of simplified formulae in the fitting program.

Table 7 summarizes the c.m.s.-energy-dependent errors in the theoretical predictions originating from the sources discussed above, and the resulting shifts in the fit parameters. As can be seen in Figs. 3a–c the errors associated with ISR are roughly constant for energies at and below $M_Z$. As this constant part is already contained in the global ISR error, only the deviation from the constant value is used in the estimate of the c.m.s.-energy-dependent errors. In the second part of the table the individual errors associated with ISR and fitting formulae have been added.

In the third part of Table 7 the total c.m.s.-energy-dependent theoretical error is presented for the two extreme strategies adopted by the experiments. Case 1 represents an experiment based on measuring the luminosity cross-section in the typical angular region of the second generation of luminosity detectors, calculating the corresponding contribution originating from $Z$-boson exchange with the modified $\Gamma_Z$ BABAMC, and using the YFS line-shape formulae. Case 2 involves measuring the luminosity cross-section in the typical angular region of the first generation of luminosity detectors, calculating the corresponding contribution originating from $Z$-boson exchange with the default $\Gamma_Z$ BABAMC, and using the $O(\alpha^2)$ line-shape formulae. The total errors were calculated as twice the shifts from Table 6 (to take into account the non-calculated higher-order corrections) plus the errors given in the second part of Table 7. All numbers were added linearly (including the sign) since they either represent calculated corrections or correspond to errors that might be correlated. We see that in both cases the error in $M_Z$ is much smaller than the expected 1993 systematical error of 1.5 MeV [18] coming from the LEP energy

\(^1\)This difference was evaluated using ZFITTER 4.6 and is in good agreement with the numbers obtained with MIZA. By using the latest version of ZFITTER 4.9 the results are somewhat different: the difference at the $Z$ peak almost disappears but the energy slope of this difference (which is what matters for the energy-dependent errors) becomes two times larger. Nevertheless, since this is not the dominant uncertainty, this change does not affect sizably our conclusions.
Table 7

Estimated uncertainties (in per cent) in the prediction of the hadron cross-section originating from c.m.s.-energy-dependent errors and the resulting shifts in the fit parameters in the same units as in Table 6. In the second part of the table the errors from ISR and fitting formulae have been added. In the third part of the table the total errors of the fit parameters are presented for the different cases explained in the text. (Table taken from Ref. [12], but recomputed with different off-peak energies).

<table>
<thead>
<tr>
<th>Source</th>
<th>c.m.s.-energy</th>
<th>Change of</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_Z - 1.8$</td>
<td>$M_Z$</td>
</tr>
<tr>
<td>ISR (YFS)</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>ISR [O($\alpha^2$)]</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Fitting formulae</td>
<td>+0.01</td>
<td>+0.02</td>
</tr>
<tr>
<td>YFS + fitting formulae</td>
<td>-0.2</td>
<td>+0.1</td>
</tr>
<tr>
<td>O($\alpha^2$) + fitting formulae</td>
<td>+0.2</td>
<td>+0.5</td>
</tr>
<tr>
<td>Case 1</td>
<td>0.0</td>
<td>0.66</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.2</td>
<td>3.7</td>
</tr>
<tr>
<td>Case 3</td>
<td>0.2</td>
<td>1.3</td>
</tr>
</tbody>
</table>

measurements. In case 1 the relative theoretical errors will have a negligible influence on the total systematical error. In case 2, however, the error in $\Gamma_Z$ exceeds the expected systematical error of 1.5–2 MeV [18]. Also the errors in $\sigma_0$ and $N_\nu$ are approaching the present systematical errors of 120 pb and 0.023 [19], respectively, and will dominate them if the BHLUMI global error is reduced. In case 2 an improvement can be made by taking advantage of the known theoretical corrections. This can be done by calculating the contributions of $Z$-boson exchange to the luminosity measurement with the modified $\Gamma_Z$ BABAMC, by introducing a LL O($\alpha^2$) correction to these contributions from ALIBABA, and by adding the corrections to the O($\alpha^2$) line-shape formulae. The error will then be reduced to the values presented as case 3 in Table 7.

To summarize, the theoretical errors associated with the $Z$-boson-exchange contributions to the luminosity measurements at LEP are sufficiently small so that the high-precision measurements at LEP, using the second generation of luminosity detectors, are not limited. Also the c.m.s.-energy-dependent theoretical errors of the $Z$ line-shape formulae should not limit high-precision measurements of the $Z$ width and mass. More information on these topics can be obtained from Refs. [3, 12].

References

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conferences’, OPAL internal report OPAL TN–221.


and B304 (1988) 921 (E).

[18] LEP Energy Working Group, in preparation; the final error was not known when
writing this paper.

[19] The LEP Collaborations and The LEP Electroweak Working Group, preprint
Figure 1: Contribution of terms containing Z-boson exchange for the first and second generation of LEP luminosity detectors.
Figure 2: Higher-order corrections to the terms containing Z-boson exchange determined relative to the O(α) corrected results from BABAMC using Γ_{Z} = 2.3098 GeV or Γ_{Z} = 2.4870 GeV. Results are given for the two different angular regions studied in this paper. Insets show the contribution of the Z-boson-exchange contributions calculated in the Born approximation.
Figure 3: Different sources of uncertainties for the $e^+e^- \rightarrow$ hadrons line-shape formulae. (a) ISR corrections from LL $O(\alpha^3)$ and SL $O(\alpha^2)$ terms in the YFS approach. (b) Total estimate of missing higher orders in the YFS approach. (c) Difference between YFS and the complete $O(\alpha^2)$ calculation from Ref. [17]. (d) Differences between the ZFITTER fitting formulae and its full Standard-Model prediction.