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NEW LOOK ON THE THEORY OF INTEGRABLE
SYSTEMS

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Abstract

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General properties of canonical discrete transformations of integrable systems are discussed. A new group-theoretical interpretation for discrete transformation which allow one to describe the essential properties of integrable systems as a direct corollary of theory of representation of all of discrete transformations group is proposed.

Аннотация

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Обсуждаются общие свойства дискретных преобразований интегрируемых систем. Предложена новая теоретико-групповая интерпретация дискретных преобразований, позволяющая понять и описать наиболее существенные свойства интегрируемых систем как прямое следствие теории представлений группы дискретных преобразований.

1. Introduction

Recently it has become clear that the most important and characteristic properties of integrable systems may be understood and explained in a successful way on the base of discrete symmetries of such systems. Under discrete transformation (local in the simplest case) we will understand the substitution (mapping) of following kind [1]

$$\tilde{u} \equiv \phi(u) = \phi(u, u', u'', \dots), \quad (1.1)$$

where u is a vector function (unknown dynamical variables of the system under consideration), $u, u', u'', \dots$ are its derivatives of arbitrary (but limited) degree with respect to independent arguments of the problem. In the general case the relations of type (1.1) may be not local, this means that for obtaining $\tilde{u}$ as a function of u and its derivatives it will be necessary to solve some differential (or integral, or of some other kind) equation.

2. General Properties of Discrete Substitution

At first we want to enumerate the most important general properties of substitutions (1.1) as a result of observation of the sufficiently large number of integrable systems [2].

1. All equations of the given hierarchy possess the same discrete transformation.
2. Substitution (1.1) is invertible: this means that equations (1.1) may be resolved with respect to the "old" variables u which may be expressed as a functions of the "new" variables $\tilde{u}$ and their derivatives.
3. Substitution (1.1) is canonical [3],[4]. This fact can be expressed in two equivalent forms. There exists a single generating function from which by the rules of the theory of canonical transformations it is possible to obtain the explicit formulae of substitution (1.1). In other words this means that substitution (1.1) may be related to some Poisson structure [5] (and not single) which is invariant with respect to transformations described by substitution (1.1).
4. The conserved quantities of the theory are shifted on the divergence (with respect to space coordinates) under transformation (1.1).
5. Substitution (1.1) may be rewritten in the form of infinite chain of equations

$$\tilde{u}_{n+1} = \phi(u_n, u'_n, u''_n, \dots), \quad (2.1)$$

where u_n denotes the result of n -time application of transformation (1.1) to some initial function u_0 (possible solution of some integrable system). The general property of chains (2.1) consists in their integrability. This means that it is possible to obtain exact general solution of these chains under appropriate way of interrupting the chain (by using some "good" boundary conditions) on its both ends. For all integrable systems with rational spectral parameter (in the old inverse scattering method terminology) chains (2.1) coincide with equations of Toda lattice (Darboux transformation) or with some generalizations. For the situation in the case of elliptic spectral parameter see [6].

3. The Problems that May be Solved with the Help of Discrete Transformation

Now we enumerate the most important results that may be obtained with the help of discrete transformation (1.1).

1. It is possible to obtain a wide class of explicit solutions of integrable systems in the determinant (Hirota) form. These solutions depend on some number of arbitrary functions [2].

2. With an appropriate choice of these functions it is possible to extract those solutions which are either invariant with respect to some inner automorphism of dynamical system under consideration (partially, multisoliton solutions), or satisfy some other boundary conditions [7],[8].

3. The condition of invariance of Poisson structures with respect to transformation (1.1)

$$\phi'(u)J_n(u)\phi'^T(u) = J_n(\phi(u))$$

($\phi'(u)$ — is the Frechet derivative) allows one to obtain the explicit form of nonlocal Hamiltonian operators of an arbitrary order and after this to construct the whole hierarchy of integrable systems with given discrete substitution [3].

4. The Equation Determining the Discrete Substitution

As it was discussed above the knowledge of the discrete substitution allows one to give a solution for many problems of the theory of integrable systems. There remains only one "small" question how to choose the appropriate substitution out of an infinite number of possible ones?

We shall now formulate the equation whose solution is exactly the mapping (1.1) satisfying all necessary conditions to exploit it as discrete symmetry of some integrable system [9]. We will write this equation under the assumption of locality of substitution.

Let $\phi'(u)$ be a Frechet derivative corresponding to substitution (1.1)

$$\phi'(u) = \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial u'} D + \frac{\partial \phi}{\partial u''} D^2 + \dots$$

By $F(u)$ let us denote the vector column function components of which are some (may be nonlocal) functions of dynamical variables u and their derivatives $F_i(u) \equiv$

$F_i(u, u', u'', \dots u^{(n)})$. Then every solution of functional differential equation with shifted arguments

$$F(\phi(u)) = \phi'(u)F(u) \quad (4.1)$$

may be connected with equation of the evolution type

$$u_t = F(u),$$

which are invariant with respect to discrete transformation $\tilde{u} = \phi(u)$.

Equation (4.1) is a generalization of the well-known condition of integrability in the theory of differential equations. Indeed let us differentiate (1.1) with respect to some parameter on which "initial" function u depends on and denote $\dot{\phi}(u) \equiv F(\phi(u))$, $\dot{u} \equiv F(u)$ then (with all necessary words) we come to (4.1).

So if the mapping (substitution) is integrable (in the above sense) then it is possible to consider it as a discrete symmetry of some integrable system.

5. The Theory of Integrable Systems as a Corollary of Representation Theory of the Discrete Group of Integrable Mapping

We emphasize that the material of this subsection has to be considered not as a mathematically rigorous result but at the level of mathematical analogies. However, the following results are not in contradiction with the known facts from the theory of integrable systems. This allows one to hope that rigorous theory of representation of discrete group will be, in future, the foundation for complete theory of integrable systems.

Let us now compare equation (4.1) with definition of linear representation $T(g)$ of some group (for instance Lee group)

$$\Phi(gx) = T(g)\Phi(x), \quad (5.1)$$

where g is the group element, $T(g)$ is the group operator of representation, $\Phi(x)$ is the basis of the corresponding representation space.

The obvious correspondence takes place after comparison (5.1) with (4.1)

$$\Phi(x) \rightarrow F_n(u), \quad T(g) \rightarrow \phi'(u).$$

Using this correspondence let us interpret equation (4.1) at the group theoretical level. We have some discrete group of transformation the group element of which is acting exactly as a substitution $u \rightarrow \phi(u)$; $\phi'(u)$ (the Frechet derivative) is a linear representation of the group element; and at last $F_n(u)$ (the equations of hierarchy) is a basis in representation space. If this representation is irreducible (it is necessary to check this fact by independent methods) then all possible bases of this representation (solutions of equation (4.1) with different n) must be connected by an operator $W_{n,n'}$

$$F_n(u) = W_{n,n'} F_{n'}. \quad (5.2)$$

Certainly the same situation takes place in the theory of $(1+1)$ integrable systems. All equations of the same hierarchy are connected with "raising" operators constructed from the skew symmetrical (nonlocal) Hamiltonian operators $J_n = -J_n^T$

$$W_{n,n'} = J_n J_{n'}^{-1}. \quad (5.3)$$

Two equations (typically for the group representation theory) will be important for a further consideration

$$\phi'(u)W(u)\phi'(u)^{-1} = W(\phi(u)), \quad \phi'(u)J(u)\phi'(u)^T = J(\phi(u)), \quad (5.4)$$

where $\phi'(u)^T = \phi_u^T - D\phi_u^T + D^2\phi_u^T - \dots$, and $W(u), J(u)$ are unknown $s \times s$ matrix operators, the matrix elements of which are polynomial of some finite degree with respect to an operator of differentiation D (of its positive and negative degrees).

From (5.4) and (4.1) it follows immediately that if $F_n(u)$ is a solution of main equation (4.1) then $W^p(u)F_n(u)$ (p is an arbitrary natural number) will be another solution of the same equation.

The solution of second equation (5.4) under additional condition of its skew symmetry may be connected with a Poisson structure which is invariant with respect to the transformation of discrete symmetry. Skew symmetrical operators $J(u)$ are known as Hamiltonian ones. Two different solutions of second equation (5.4) (if it is possible to find them), say $J_1(u)$ and $J_2(u)$ in combination $J_1J_2^{-1}$ satisfy first equation (5.4). The operator $J_1J_2^{-1}J_1(u)$ is again the solution of the second equation of (5.4) and so on. This is the way how in the theory of integrable systems Hamiltonian operators commonly arise. It is necessary to find from independent assumption two different Poisson structures and after this to fulfil describing the above operation. In the problem of constructing Hamiltonian operators for integrable systems equations (5.4) were used in [3].

Only future investigations may give an answer to the question how deep the analogue between the theory of representation of discrete groups determined by (4.1) and the well known results of the same representation theory (say Lee groups) is.

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