 QUANTUM STABILITY OF ACCELERATED BLACK HOLES\textsuperscript{1} 

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ABSTRACT 

We study quantum aspects of the accelerated black holes in some detail. Explicitly shown is the fact that a uniform acceleration stabilizes certain charged black holes against the well-known thermal evaporation. Furthermore, a close inspection of the geometry reveals that this is possible only for near-extremal black holes and that most nonextremal varieties continue to evaporate with a modified spectrum under the acceleration. We also introduce a two-dimensional toy model where the energy-momentum flow is easily obtained for general accelerations, and find the behavior to be in accordance with the four-dimensional results. After a brief comparison to the classical system of a uniformly accelerated charge, we close by pointing out the importance of this result in the WKB expansion of the black hole pair-creation rate. 

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1 Introduction

The Hawking radiation from the stationary black hole [1], is by now a well-understood phenomenon within the semiclassical framework. Many quantum mechanical concepts, such as energy quanta and the occupation numbers, turned out to be coordinate-dependent ones, and this ambiguity leads to the particle creation in the presence of the event horizon. A black hole that has nonzero $T_{BH}$, emits thermal radiations and thereby loses its mass steadily. That is, unless it is eventually stabilized by a conserved local charge inside. The canonical example of the latter is given by the well-known Reissner-Nordstrom (RN) black holes [2], the minimal variant of which, namely the extremal case, has vanishing $T_{BH}$.

It turns out that there is another type of situation [3] when the Hawking’s thermal radiation vanishes, and that despite nonzero $T_{BH}$. To understand this, we need to recall a related phenomenon of the so-called acceleration heat bath [4][5]. Through a similar quantum effect as in black hole radiance, the usual Minkowskian vacuum feels like a heat bath to uniformly accelerated Rindler observers, with the acceleration temperature $T_A$ equal to the acceleration multiplied by $\hbar/2\pi$.

Now suppose a RN black hole of nonzero Hawking temperature $T_{BH}$ is undergoing a uniform acceleration.³ Furthermore, suppose that $T_{BH}$ equals $T_A$. Then, co-moving Rindler observers (or the laboratory observers) feels not only the thermal radiation from the black hole but also the acceleration heat bath of the same temperature. In effect, it is as if one put a small blackbody of temperature $T_{BH}$ inside a large thermal cavity of temperature $T_A = T_{BH}$. There will not be any net flow of energy, since everything is in thermal equilibrium, which means that the co-moving Rindler observers find the black hole stable against Hawking’s thermal evaporation.

Of course, these one-loop effects are notoriously observer-dependent, and must be analyzed carefully within the semiclassical framework. In particular, the acceleration heat bath is not a real entity to inertial observers (for the good reason that the acceleration heat bath is a particular manifestation of the usual Minkowski vacuum, as perceived by the Rindler observers only), and thus it is rather difficult to imagine why the black hole should not evaporate as seen by asymptotic inertial observers. But the point is that the Hawking radiation and the acceleration heat bath are of the same theoretical origin, and in a sense equally observer-dependent. The difference is only in the choice of the initial conditions. In Ref. [3], the author showed that in fact the asymptotic

³If the reader is uncomfortable with the uniform acceleration, he may as well imagine that the black hole is levitated by static electromagnetic field in a laboratory. More on the matter of the uniform acceleration can be found in section 6.
Inertial observers find neither the acceleration heat bath (which was expected) nor the Hawking’s thermal radiation (which was not): The black hole does not evaporate at all.

In this article, we wish to investigate the semiclassical physics of accelerated RN black holes in more detail, and consider the consequences. After presenting the corresponding Ernst geometry in Section 2, we begin with a toy model in Section 3. The toy model has all the relevant features of the Ernst spacetime, yet allows us to find the covariant one-loop energy-momentum expectation value in a straightforward manner. In section 4, we return to the four dimensional Ernst spacetime and rederive the trivial late-time Bogolubov transformation. In both cases, the conclusion is that the black hole in question is stabilized despite the nonzero $T_{BH}$. In Section 5, the matter of the evolution is addressed by asking when $T_A$ can be larger than $T_{BH}$. We find the most nonextremal black hole continues radiates while being accelerated and that the semiclassical evolution always stops with near-extremal black holes. In Section 6, after a comparison to the superficially similar (classical) system of a uniformly accelerated charged particle [8], we discuss the implication of our findings in the context of RN black hole pair-creation.

2 Uniformly Accelerated RN Black Holes

The geometry we will consider is that of the Ernst metric [6]. This represents two oppositely charged magnetic RN black holes uniformly accelerated away from each other, where the driving force is an external magnetic field that diminishes rapidly away from the axis of symmetry. Let us first write down the Ernst metric in a new coordinate system.

$$g = \frac{A^2}{(1 + rAx)^2} \left\{ -F(r) \, ds^2 + F(r)^{-1} \, dr^2 + r^2 \, G(x)^{-1} \, dx^2 + r^2 \, G(x) \, A^{-4} \, d\phi^2 \right\}. \quad (1)$$

Starting with the Ernst metric in Ref. [10], we performed a coordinate redefinition $r = -1/Ax$ and also rescaled the time coordinate by $A$. The new coordinate $r$ plays the role of the usual radial coordinate only near the black hole horizon, as is easily seen from the form of $F(r)$.

$$F(r) \equiv -A^2 r^2 G(-1/Ax) = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r} - A^2 r^2). \quad (2)$$

$A$ is a function of $r$ and $x$ in general:

$$A \equiv \left\{ 1 + \frac{Bx}{2} \sqrt{r_+ r_-} \right\}^2 + \frac{B^2 r^2}{4A^2(1 + rAx)^2} G(x), \quad (3)$$

where $B$ is approximately the magnetic field strength that drives the uniform acceleration.
It takes some investigation to discover that the Killing coordinate $s$ here is actually a Rindler time coordinate [10], so that the “static” observers are in fact uniformly accelerated and thus confined within the Rindler wedges $LR$ and $RR$ in figure 1: They are Rindler observers.

![Diagram](image)

**Figure 1:** A schematic diagram for a pair of uniformly accelerated charged black holes. The black holes are represented by two hyperbolic world lines in each Rindler wedges.

The same quartic polynomial $G$ appears in all components of the metric. Call the four roots of it, $\xi_1, \xi_2, \xi_3, \xi_4$ in the ascending order. Then, $r = \tilde{r}_+ = -1/\xi_3 A$ and $r = r_A \equiv -1/\xi_3 A$ are respectively the outer or event horizon of the black holes and the acceleration horizon. On the transverse part of the geometry with the coordinates $x$ and $\phi$, there could be conical singularities at $x = \xi_3$ and $x = \xi_4$, which are easily resolved by adjusting the period of $\phi$ and also putting a constraint between parameters of the metric:

$$G'(x)/\Lambda^2 \bigg|_{x=\xi_3} = G'(x)/\Lambda^2 \bigg|_{x=\xi_4}. \quad (4)$$

When the black holes are of sufficiently small size ($r_\pm A \ll 1$), this constraint is easily seen to reproduce the Newton’s second law [10].

Although the geometry is rather complicated, one can easily recover the familiar structures if the relative size of the black holes is small ($r_\pm A \ll 1$). For instance, as the acceleration decreases
A metric resembles that of a static RN black hole in an increasingly larger region of the spacetime. For this limit, simply ignore terms proportional to the acceleration or the external magnetic field, and introduce an angular coordinate \( \theta = \cos^{-1} x \): \[ g \simeq -\tilde{F}(r) ds^2 + \tilde{F}(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad \tilde{F}(r) \equiv (1 - \frac{r_+}{r})(1 - \frac{r_-}{r}), \quad (5) \]

In this limit, therefore, \((r_+ + r_-)/2\) is the mass of the RN black hole while \(\sqrt{r_+r_-}\) is the magnetic charge inside. On the other hand, far away from the black hole \((rA \sim 1)\), one may introduce a new set of spatial coordinates \(\eta\) and \(\rho\) by \[ A^2 \eta^2 \simeq \frac{1 - r^2 A^2}{(1 + rAx)^2}, \quad A^2 \rho^2 \simeq A^2 r^2 \frac{1 - x^2}{(1 + rAx)^2}. \quad (6) \]

In these coordinates, the geometry far away from the black hole can be represented by the following approximate metric, \[ g \simeq \tilde{\Lambda}^2 (-A^2 \eta^2 ds^2 + d\eta^2 + dp^2) + \tilde{\Lambda}^{-2} \rho^2 d\phi^2, \quad \tilde{\Lambda} \equiv 1 + \frac{B^2 \rho^2}{4}, \quad (7) \]

which is simply the magnetic Melvin universe written in a Rindler-type coordinate system. From the curvature of each approximate metric, it is easy to see that the transition from the black hole geometry \((5)\) to the Melvin geometry \((7)\) must occurs at \(r^2 \sim r_+/B^2\) and \(\eta \approx 1/A, \rho \ll 1/A\).

We expect \(\eta^{-1}\) to play the role of the absolute acceleration of the local Rindler observers, and this tells us that the small black hole and co-moving observers nearby must be experiencing an acceleration \(\approx A\). For freely falling inertial observers far away from the black holes, on the other hand, the Rindler coordinate above is not appropriate, and we need to introduce yet another set of coordinates as follows, \[ T \equiv \eta \sinh Ax, \quad Z \equiv \eta \cosh Ax, \quad (8) \]

which are precisely the analogue of the ordinary Minkowski coordinate.

We finally come to the most essential part, namely the causal structure. Unlike a stationary black hole, this geometry possesses an extra horizon outside the event horizon. This in effect divides asymptotic infinities into two different classes: There are asymptotic infinities corresponding to \(x = -1/Ar = \xi_3\), and those corresponding to \(x = -1/Ar > \xi_3\). To reach the first, one need not cross the acceleration horizon at \(r = -1/\xi_3 A\), hence, they belong to the Rindler wedges \(LR\) or \(RR\). The second class on the other hand should belong to \(F\) or \(P\).

Now the point is that as long as we are concerned with a radiative process, we may as well safely ignore the first class of the infinities. The reason is simple: Most spacetime trajectories of quanta,
being either time-like or null, will eventually cross acceleration horizon into the region $F$ which is inaccessible to the Rindler observers [8]. In order to remain within the Rindler wedge forever, the particle must either itself have magnetic charge or be directed exactly parallel to the axis of the uniform acceleration. It is then easy to draw the Penrose diagram of this truncated spacetime. See figure 2.

![Penrose diagram of the Ernst spacetime with the Rindler infinities at $x = \xi_3 = -1/Ar$ excised. The other asymptotic infinities are indicated by the bold straight lines.](image)

Figure 2: Penrose diagram of the Ernst spacetime with the Rindler infinities at $x = \xi_3 = -1/Ar$ excised. The other asymptotic infinities are indicated by the bold straight lines.

Note that this Ernst spacetime actually represents a wormhole. But this wormhole geometry is different from those found in eternal black holes [2] in that, here, the two mouths (or equivalently the two black holes) share the same asymptotic future $F$.

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Throughout the article, we shall assume that there exits no light magnetic particle.
3 Toy Model

The Ernst geometry introduced above is only axially symmetric, which makes a separation of variable impossible even for the simplest field equation. In Ref. [3], this difficulty is side-stepped by setting up the problem in such a way that only the approximate behaviors of the eigenmodes near each horizon matter. Although, this approach turned out to be very powerful, it is not suitable for more general situations where the acceleration heat bath may not be in equilibrium with the black hole radiance.

![Penrose diagram of the charged black hole in De Sitter universe.](image)

Figure 3: Penrose diagram of the charged black hole in De Sitter universe. There is no distinction between the null and the timelike infinities, for the ambient universe is inflating.

The relevant region inside the broken curve is redrawn in figure 4 below.

On the other hand, if we only want to understand the system qualitatively, there is an easy way out: consider a vastly simplified model that has rough characters of such accelerated RN black holes, namely a RN black hole in De Sitter universe.\textsuperscript{5} Written in the static coordinate, the metric is given by,

\[ \tilde{g} = -\mathcal{F}(R) \, dt^2 + \mathcal{F}(R)^{-1} \, dR^2 + R^2 \, d\Omega^2, \quad \mathcal{F}(R) = 1 - \frac{2M}{R} + \frac{Q^2}{R^2} - \frac{H^2 R^2}{3} \]

where \( Q \) is the electromagnetic charge inside the black hole. For sufficiently small cosmological constant \( 3H^2 \), the geometry at small \( R \) is just that of the ordinary RN black hole.

This geometry has three horizons, outermost of which, the so-called De Sitter horizon, can be thought of as the analogue of the acceleration horizon of the Ernst geometry. Relative coordinate

\textsuperscript{5}The Hawking effect in the De Sitter universe was first studied in Ref. [11].
transformations between inertial observers near respective horizons are similar to those of the Ernst geometry, provided that the surface gravities in one case are identified with those of the other. As was asserted in Ref. [3], it is precisely the Bogolubov transformation between such two sets of inertial observers in the Ernst geometry that are relevant for the asymptotic Hawking radiation. By considering this toy model, therefore, we should be able to capture a qualitative semiclassical character of the accelerated black holes.

The physical reason why we could expect a similar behavior from this model as in uniformly accelerated black holes, is quite simple: the inflation of the ambient De Sitter universe. Although we cannot say that the black hole undergoes an acceleration any more, the asymptotic observer perceives the black hole as being uniformly “accelerated” away because the ambient space itself is expanding exponentially. Semiclassically, the acceleration heat bath of the Ernst geometry would be replaced by the thermal behavior associated with the De Sitter horizon, but the net effect is again that certain non-inertial observers near the black hole find a thermal equilibrium whenever the respective surface gravities \( \kappa_{bh} \) and \( \kappa_{dS} \) coincides,

\[
\kappa_{bh} = -\frac{F(R_{bh})}{2}, \quad \kappa_{dS} = -\frac{F(R_{dS})}{2},
\]

where \( R_{bh} \) and \( R_{dS} \) are the radii of the black hole event horizon and the De Sitter horizon respectively. As is easily seen from the form of \( F \), \( \kappa_{dS} \) reduces to the Hubble constant \( H \) for the sufficiently small black hole \( (M, Q \to 0) \).

The spherical symmetry gives us another opportunity to simplify the problem: the dimensional reduction. In effect, we shall consider the problem in the S-wave sector. Quantizing an effectively two-dimensional conformal field \( \phi = \phi(t, R) \) that is coupled to the reduced two-dimensional metric \( \tilde{g}^{(2)} \):

\[
\int L = \int \sqrt{-\tilde{g}^{(2)}} (\nabla \phi)^2, \quad \tilde{g}^{(2)} = -F(R) dt^2 + \frac{1}{F(R)} dR^2,
\]

various quantities of interest may be obtained from the conformal anomaly alone [12], as we will do in this section.

Introducing a tortoise coordinate \( r_* \), between the event horizon and the De Sitter horizon,

\[
r_* = \int^R d\tilde{R} \frac{1}{F(\tilde{R})},
\]

and defining the retarded and the advanced null coordinates \( \tilde{u} = t - r_* \) and \( \tilde{v} = t + r_* \), the energy-momentum conservation translates to certain inhomogeneous differential equations for \( T_{\tilde{u}\tilde{u}} \) and \( T_{\tilde{v}\tilde{v}} \),
with the inhomogeneous term proportional to the conformal anomaly \( \sim \langle T_{\alpha\beta} \rangle \) or equivalently the two-curvature \( \sim \mathcal{F}^\mu \). Then, it is an elementary exercise to solve them for the outward energy flux [13]:

\[
\langle T_{\alpha\beta} \rangle \sim \mathcal{H} \left[ \partial_\alpha \rho - \partial_\beta \rho \partial_\beta \rho \right] + t_{\alpha\beta} (\bar{u}), \quad \rho \equiv \frac{1}{2} \log \mathcal{F}.
\] (13)

The unknown function \( t_{\alpha\beta} \) is to be determined by the choice of the initial condition. Alternatively, one may think of this flux as the S-wave sector contribution to the actual four-dimensional flux, except that possible backscattering effect is ignored. Of course, one need to insert an appropriate scale factor in order to recover four-dimensional S-wave flux density.

![Figure 4](image-url)

**Figure 4:** Radiation from the black hole toward De Sitter horizon \((\kappa_{\text{DS}} < \kappa_{\text{SH}})\) is depicted schematically. The vacuum chosen is of the Unruh type.

In any case, the only thing that remains to be done is choosing the initial condition that will fix \( t_{\alpha\beta} \). Before we do this, however, we first need to know what are the good coordinates near the horizons. As in Ref. [3], the approximate behavior of these Kruskal coordinates near the horizons are completely determined by the respective surface gravities. For the retarded time coordinates, specifically, we find,

\[
U_{\text{DS}} \simeq \frac{1}{\kappa_{\text{DS}}} e^{\kappa_{\text{DS}} \bar{u}}, \quad U_{\text{SH}} \simeq -\frac{1}{\kappa_{\text{SH}}} e^{-\kappa_{\text{SH}} \bar{u}}.
\] (14)
Note that $U_{1b}$ vanishes at the future event horizon ($\tilde{u} = \infty$).

In order for the future event horizon to remain smooth at one-loop, it is necessary that the local energy density (as measured by infalling inertial observers) remains finite at one-loop. But the latter is given by the quantity $(T_{\tilde{u}u})/(d\tilde{u}/dU_{1b})^2$ near the future event horizon, which means that this is possible only if $t_{\tilde{u}u}$ cancels the leading contribution from the derivatives of $\rho$, near the future event horizon $u \to \infty$. It turns out that this prescription is also sufficient whenever $\kappa_{1b} > 0$.

Once we demand such an Unruh type initial condition on an initial surface, say $\tilde{v} = 0$, the late-time behavior of the outward flux $(T_{\tilde{u}u})$ along $\tilde{v} = \infty$, is uniquely determined:

$$\langle T_{\tilde{u}u} \rangle \sim \frac{\hbar}{\pi} [\kappa_{1b}^2 - \kappa_{dS}^2] \quad \text{as} \quad \tilde{u} \to \infty.$$  

(15)

If the De Sitter horizon were absent ($H = 0$ so that $\kappa_{dS} = 0$), the hypersurface $\tilde{v} = \infty$ would be the asymptotic future. In that case, it is easy to see that Eq. (15) does reproduce the usual late-time thermal radiation from the RN black hole, since the radiative flux from a blackbody (in 1+1 dimension) of temperature $T$ scales like $T^2/\pi$. Recall that the Hawking temperature is given by the surface gravity $\kappa_{1b}$ multiplied by $\hbar/2\pi$. The universal nature of the Hawking radiation is already apparent here, in that the above behavior (15) does not depend on the details of the initial condition other than the regularity of the future event horizon. Neglected here are non-universal subleading terms that are exponentially small as $\tilde{u} \to \infty$.

In the presence of the De Sitter horizon ($\kappa_{dS} > 0$), however, there is one more step to be taken. First of all, $\tilde{u}$ is not a good coordinate at the future De Sitter horizon ($\tilde{v} = \infty$). Using the local inertial coordinate $U_{dS}$ instead, the energy flux that crosses the De Sitter horizon into the asymptotic future, is given by the following,

$$\text{Outward Energy Flux} \simeq \langle T_{U_{dS}U_{dS}} \rangle \sim \frac{\hbar}{\pi} [\kappa_{1b}^2 - \kappa_{dS}^2] \frac{1}{\kappa_{dS}^2 U_{dS}^2}.$$  

(16)

as $U_{dS} \to \infty$. At last, we see that the effect of the extra horizon on the Hawking radiation is two-fold.

The first is the red-shift factor $(\kappa_{dS} U_{dS})^{-2} \simeq e^{-2\kappa_{dS} \tilde{u}}$ that originates from the fact that the ambient universe is undergoing an inflationary expansion. The physical distance between the source (i.e., the black hole) and the inertial observer propagating along a fixed advanced time outside the cosmological horizon, grows exponentially with $\tilde{u}$, and so does the relative red-shift factor. The effective temperature is red-shifted likewise by a factor $(\kappa_{dS} U_{dS})^{-1}$. Is there an analogous effect in the case of uniformly accelerated black holes?
In the Ernst spacetime, consider the inertial observers on the axis of rotational symmetry and between the two accelerated RN black holes. If energy quanta from the black holes reach them, the energy must be Doppler-shifted due to the instantaneous relative motion. It is matter of a simple Lorentz transformation to see that the appropriate red-shift factor scales as \( e^{-\hat{\eta}} \) at large dimensionless proper time \( \hat{s} \simeq A \hat{s} \) of the black hole. Since the black hole is at the Rindler coordinate \( \eta \simeq 1/A \), we may rewrite this in terms of the retarded time \( (T - Z) \sim e^{-\hat{\eta}}/A \) (see section 2 for the notation). This tells us that the effective temperature would be red-shifted by the factor \( (A(T-Z))^{-1} \) to such inertial observers at the retarded time \( T - Z \). Identifying \( A \) with \( \kappa_{dS} \) and \( T - Z \) with \( U_{dS} \), this produces the same kind of the red-shift factor as above.

The second, much less intuitive, effect is to replace \( \kappa_{ib}^2 \) by \( \kappa_{ib}^2 - \kappa_{dS}^2 \). This has little to do with the instantaneous relative motions, as is easily seen from the fact that the same modification exists in the static coordinate \((\hat{u}, \hat{v})\). The physical interpretation of this is that the presence of the extra horizon completely altered the original thermal spectrum. In particular, when the two surface gravities coincides \( (\kappa_{ib} = \kappa_{dS}) \), the leading Doppler-shifted Hawking radiation is completely turned off. As was shown in Ref. [3] and will be reexamined in the next section, the same phenomenon occurs for certain accelerated black hole.

Also note that if \( \kappa_{dS} \) is larger than \( \kappa_{ib} \), this asymptotic flux has a negative sign, meaning the black hole is actually accreting energy rather than evaporating. There is nothing wrong with this since the ambient space comes with a uniform density of energy in the form of the cosmological constant. Furthermore, as we will see in Section 5, the typical classical geometries are such that the Hawking temperature is almost always larger than that of the extra horizon outside. Such accretion processes, if any, are always stopped before the black hole deviates far from the extremality.

### 4 Vanishing Hawking Radiation

Now back to the real model of the four-dimensional Ernst spacetime. In this section, for the sake of the completeness, we want to reiterate the main arguments of Ref. [3] but in a slightly more detailed fashion. Let us first define the surface gravities of the relevant horizons in Ernst spacetime:

\[
\kappa_{BH} \equiv \frac{F'(\hat{r}_+)}{2}, \quad \kappa_A \equiv -\frac{F'(\hat{r}_A)}{2},
\]  

(17)

which are related to the temperatures by \( T_{BH} = \hat{\eta}\kappa_{BH}/2\pi \) and \( T_A = \hat{\eta}\kappa_A/2\pi \). Note that when the size of the black holes are relatively small \( (r_+ A \ll 1) \), \( A \simeq \kappa_A \) can be regarded as the acceleration of the black hole.
Since we are mostly interested in the cases where the co-moving Rindler observers find a complete thermal equilibrium, we will require that the Hawking temperature be equal to the acceleration temperature. In terms of the surface gravity, therefore, we demand that

$$\kappa \equiv \kappa_{BH} = \kappa_A.$$  (18)

In some cases, most notably when $r_+ \gg r_-$ so that the black hole mass is much larger than its charge, $\kappa_{BH} > \kappa_A$ is always true and this constraint can never be met. When the non-extremal RN black holes in question are sufficiently close to the extremality, on the other hand, it is possible to achieve this fine-tuning [7]. In fact, this constraint is naturally imposed if the two black holes were pair-created via a wormhole-type instanton [9].

Having the Ernst metric instead of the toy model of the previous section gives rise to several difficulties, mostly due to the lack of the spherical symmetry and the complicated pattern of the Doppler effects. Fortunately, it turned out that none of these problems matter if we are interested in the null Hawking effect when $\kappa_{BH} = \kappa_A$, provided that we work with the eigenmodes in the Rindler coordinate and construct other inertial eigenmodes via the analyticity argument of Unruh [5].

For this purpose, it is most convenient to introduce a tortoise-like coordinate $z$ where $\tilde{r}_+ \leq r \leq r_A$,

$$z \equiv \int^r d\tilde{r} \frac{1}{F(\tilde{r})},$$  (19)

which logarithmically approaches $-\infty$ at the event horizon and $+\infty$ at the acceleration horizon. Consider the field equation of a free scalar $\Psi$ that may have a quadratic curvature coupling,

$$\nabla^2 \Psi = M^2 \Psi + \cdots.$$  (20)

After rescaling the eigenmodes $\Psi^{(w,m)}$, for each Rindler frequency $w > 0$ and the quantized angular momentum $m$,

$$\Psi^{(w,m)} = e^{\pi i w_+} \frac{(1 + r_A x)}{r} \left[ \Phi^{(w,m)}(r, x) e^{im\phi} \right],$$  (21)

we find the following equation that must be solved for the eigenmodes:

$$w^2 \Phi^{(w,m)} + \frac{\partial^2}{\partial z^2} \Phi^{(w,m)} = F(r(z)) \left\{ \frac{1}{r^2} \left[ - \frac{\partial}{\partial x} G(x) \frac{\partial}{\partial x} + \frac{m^2 A^2}{G(x)} \right] + U_{\text{eff}} \right\} \Phi^{(w,m)}.$$  (22)

$U_{\text{eff}}$ is a bounded function of coordinates $z$ and $x$, and in particular contains the scalar mass term and the possible curvature couplings. Note that the right-hand-side of Eq. (22) has the overall
factor \( F(r(z)) \) that vanishes exponentially \( \sim e^{-2\kappa|z|} \) as \( |z| \to \infty \). In the same limit, \( \Lambda \) is reduced to functions of \( x \) only, \( \Lambda \to \Lambda_4 \equiv \Lambda(r = r_4) \) or \( \Lambda \to \Lambda_{BH} \equiv \Lambda(r = \hat{r}_+) \).

For any mode with finite transverse physical momentum along \( x \) and \( \phi \) directions, the exponentially small factor \( F(z) \) causes an effective separation of variables occurs near the horizons. Then, introducing two null coordinates \( u = s - z \) and \( v = s + z \) in \( L \) and also in \( R \), we find the following general behavior, near each horizon, of the future-directed Rindler eigen-modes \( \Psi_{L}^{(u,m)} \) and \( \Psi_{R}^{(v,m)} \) that have respective supports in either \( L \) or \( R \) only,

\[
\Psi_{L}^{(u,m)} \sim e^{-iuu} C_{\lambda m}(x)e^{im\phi} \quad \text{or} \quad e^{-iuu} C_{\lambda m}(x)e^{im\phi} \quad \text{in } L, \quad \Psi_{L}^{(v,m)} = 0 \quad \text{in } R, \quad (23)
\]

\[
\Psi_{R}^{(v,m)} \sim e^{ivv} C_{\lambda m}(x)e^{im\phi} \quad \text{or} \quad e^{ivv} C_{\lambda m}(x)e^{im\phi} \quad \text{in } R, \quad \Psi_{R}^{(u,m)} = 0 \quad \text{in } L. \quad (24)
\]

The positive sign in (24) is because \( (u,v) \) grow toward past rather than future in the region \( R \). The same set of symbols \( C_{\lambda m} \) and \( \lambda \) are used to denote eigenfunctions and eigenvalues for two different eigenvalue problems at each horizon. The relevant operators are obtained from the one inside the square bracket in Eq. (22), by replacing \( \Lambda \) by \( \Lambda_4 \) or by \( \Lambda_{BH} \). In particular, due to the lack of the spherically symmetry, an eigenmode that has a definite \( \lambda \) near the event horizon will not have a definite \( \lambda' \) near the acceleration horizon. Similarly, possible backscattering will also mix the left-moving and the right-moving modes. But since none of these details matter, as we will find out shortly, we shall keep just one superscript \( w \) form now on.

Let us be reminded that, for each Rindler mode \( \Psi^{(u)} \) with positive \( u \), there exists a time-reversed negative mode \( \Psi^{(-u)} \) that propagates backward but otherwise of the same form: The complete Hilbert space is spanned by both positive and negative modes. But the point is, such labels as future-directed and past-directed are inherently observer-dependent. A purely future-directed mode in one coordinate system could be a mixture of both future-directed and past-directed modes as perceived by another coordinate system. Hence, the so-called Bogolubov transformation, which maps one basis to the other, is such that a vacuum with respect to one set of observers can actually be an excited state with respect to the other. And this is exactly the origin of both the Hawking radiation and the acceleration heat baths [1][5].

In this regard, it is important to realize that \((u,v)\) are not good coordinates near the horizons and must be traded off in favor of the Kruskal coordinates that play the role of advanced and retarded times for local inertial observers. The approximate coordinate transformation, near the respective horizons, are easily determined in terms of the surface gravity \( \kappa \). Calling the Kruskal
coordinates near the event horizon \((U_1,V_1)\), we find,

\[
\begin{align*}
\kappa u &\simeq -\ln(-U_1) \quad \text{in L}, & \kappa u &\simeq -\ln(+U_1) \quad \text{in R}, \\
\kappa v &\simeq +\ln(+V_1) \quad \text{in L}, & \kappa v &\simeq +\ln(-V_1) \quad \text{in R}.
\end{align*}
\]

(25) (26)

For the other Kruskal coordinates \((U_2,V_2)\) near the acceleration horizon, we simply replace \((U_1,V_1)\) by \((U_2,V_2)\) and reverse every single sign on the right-hand-side.

![Figure 5: Various null coordinates near the horizons. \(U_1 = 0\) or \(V_1 = 0\) at the event horizon, while \(U_2 = 0\) or \(V_2 = 0\) at the acceleration horizon. All Kruskal coordinates increase toward future. The Rindler-type null coordinates \((u,v)\), however, increase toward future only in L, and actually increase toward past in R.](image)

At last, we are ready to obtain the eigenmodes of positive frequencies with respect to inertial observers. For the purpose, we may use Unruh’s characterization of positive frequency [5]: through a simple analyticity argument, it is easy to see that a positive frequency mode must be analytic and bounded in the lower-half-plane of the complexified time coordinate. For instance, a positive frequency mode as detected by inertial observers near the event horizon must be analytic in \(U_1\) (with fixed \(V_1\)) and in \(V_1\) (with fixed \(U_1\)) throughout their lower-half-planes. For more detail, see Section II of Ref. [5].

Since the Rindler modes are defined in either L or R, they are defined only on the half-lines of Kruskal coordinates. To construct the eigenmodes that are appropriate for inertial observers, one
expresses $\Psi_L^{(u)}$ and $\Psi_R^{(u)}$ in terms of $(U_i, V_i)$ for $i = 1, 2$ using the coordinate transformations above in (25) and (26), and analytically continue the logarithms through lower-half-planes of each Kruskal coordinates. Then the resulting modes have positive frequencies with respect to inertial observers, in addition to having the supports on the entire spans of Kruskal coordinates. For inertial observers near the event horizon of the black holes, the positive frequency modes $\Psi_B^{(u)}$ are

$$\Psi_{BH}^{(u)} \simeq N_u[\Psi_L^{(u)} + e^{-\pi u/\kappa} \Psi_R^{(-u)}], \quad \Psi_{BH}^{(-u)} \simeq N_u[\Psi_R^{(u)} + e^{-\pi u/\kappa} \Psi_L^{(-u)}],$$

(27)

where $N_u \equiv 1/\sqrt{1 - e^{-2\pi u/\kappa}}$. Expressions for the negative modes $\Psi_B^{(-u)}$ can be found likewise.

For an ordinary nonaccelerated black holes ($\kappa = \kappa_{BH}$, $\kappa_A = 0$), this would be the end of the story: The null coordinates $(u, v)$ become the asymptotic retarded and advanced time coordinates, so that requiring the smooth future event horizon necessarily implies proliferation of particles (associated with $\Psi_L^{(u)}$'s or $\Psi_R^{(u)}$'s here.) at asymptotic infinities. The spectrum would follow the Bose-Einstein distribution $\sim e^{-\pi u/\kappa} N_u^2$ [1].

However, with the uniformly accelerated black hole, $(u, v)$ are no longer good asymptotic inertial coordinates. Rather, $(U_2, V_2)$ are. The geometry (1) becomes the background Melvin spacetime far away from the black holes, but written in terms of Rindler-like coordinate. This is particularly clear when the black holes are relatively small ($r_A \ll 1$), as we observed in section 2. The Minkowski-type coordinates $T$ and $Z$ in (8), are related to our Kruskal coordinates by $U_2 = T - Z$ and $V_2 = T + Z$, which makes it quite explicit that $U_2$ and $V_2$ are the retarded and the advanced time coordinates appropriate for asymptotic inertial observers.

Near the acceleration horizon, the situation is identical to the above, thanks to the identical surface gravity $\kappa_A = \kappa = \kappa_{BH}$, except that the relative positions of L and R are switched. Calling the asymptotic inertial modes $\Psi_A^{(u)}$'s, we find near the acceleration horizon:

$$\Psi_{AR}^{(u)} \simeq N_u[\Psi_R^{(u)} + e^{-\pi u/\kappa} \Psi_L^{(-u)}], \quad \Psi_{AL}^{(u)} \simeq N_u[\Psi_L^{(u)} + e^{-\pi u/\kappa} \Psi_R^{(-u)}].$$

(28)

When compared to (27), the particular thermal nature of this immediately tells us that the Rindler observers must find some sort of thermal equilibrium. On the other hand, the Bogolubov transformation relevant for the asymptotic observers is found by combining (27) and (28), but this does not lead to any mixing between the positive and negative modes:

$$\Psi_B^{(u)} \Rightarrow \Psi_A^{(u)}, \quad \Psi_B^{(-u)} \Rightarrow \Psi_A^{(-u)}.$$ 

(29)

Therefore, we find that no late-time Hawking radiation reaches the asymptotic inertial observers.
As noted earlier, the respective forms of the Rindler modes near each horizon must be taken with a grain of salt, since the nontrivial effects of the local geometry mixes up left-moving modes with the right-moving modes and also require rather complicated $x$-dependence near at least one of the horizons. Combining the two Bogolubov transformations, in general, we must include a unitary transformation $U$ that reflects the complicated effects of local geometry between the two horizons. But, since the Rindler time $s$ is a Killing coordinate, $U$ has to commute with the Rindler energy operator $i\partial_s$. Therefore, with the Bogolubov transformations above that depend only on the Rindler frequency $\omega$, this additional complication cannot alter our conclusion above.

Actually there is a more direct way to understand this result. Consider the direct transformations between the two sets of Kruskal coordinates when the two surface gravities match:

$$U_2 \sim -\frac{1}{U_1}, \quad V_2 \sim -\frac{1}{V_1}.$$  \hspace{1cm} (30)

This transformation is easily seen to preserve the lower-half-planes of the Kruskal time coordinates, being an $SL(2,R)$ generator, which in turn explains why the Bogolubov transformation thereof is trivial to the leading approximation. In fact, this transformation (30) is exactly what we would have found if we had been considering a freely falling extremal RN black hole that has zero Hawking temperature and thus no late-time Hawking radiation. 

There is still some subtlety that remains to be addressed. Note that our derivations above utilize the so-called late-time approximation, as in most derivations of black hole radiance. While this precludes any analogue of Hawking's result [1], we still may not be able to account for possible transient behaviors that might show up before the black hole and its environment settle down to a steady state.

Exactly how does such a nontrivial result show up beyond the trivial Bogolubov transformation above? The effect comes from at least two different sources. The obvious one is the fact that the above coordinate transformation is only approximate and is valid very close to the horizons. The less obvious factor, which one can easily overlook, is that the ranges of different Kruskal coordinates above overlap with each other completely only in the maximally extended spacetime. For more realistic cases without past event horizon, it is typically the case that only the half-lines of each Kruskal coordinates are relevant, and this leads to certain nontrivial result.

Fortunately, none of these complicated effects introduce an analogue of the Hawking radiations. Rather, they merely lead to a finite and small one-loop correction of the black hole geometry. In the toy model of Section 3, this kind of transient behavior could be found in the subleading
contributions which are suppressed by at least two powers of $\kappa_{ib} U_{ib} \approx -\epsilon^{-\alpha_b \tilde{a}}$ as $\tilde{a} \to \infty$, and which depends on the detailed history. In fact, such an effect that shifts the black hole mass by a finite and small factor $\sim \tilde{a}/r_0^2$, was previously observed for static extremal RN black hole [13] directly as well as for the case of pair-produced near-extremal RN black holes [7] indirectly. Once a steady state (with $\kappa_A = \kappa_{BH}$) is reached, in any case, the main result (29) tells us that no further radiation may escape into the asymptotic future beyond the acceleration horizon.

5 Evolution

Let us consider the Ernst spacetime (1) at arbitrary accelerations. The first fact we need to understand, is that the acceleration cannot really be arbitrary. The interpretation of this geometry as a pair of black holes presumes that there are at least three distinct roots of the quartic polynomial $G$, which is false for some values of $r_+ A$.

The necessary and sufficient condition for the accelerated black hole interpretation, turned out to be $r_+ A < 2/\sqrt{27}$ [10]. The physical reason is clear: As either the black hole size or the strength of the acceleration becomes too large, the event horizon and the acceleration horizon eventually have to merge with each other and then subsequently disappear. When they merge ($r_+ A = 2/\sqrt{27}$), we find $\xi_2 = \xi_3 = -\sqrt{3}$ and $\xi_4 = \sqrt{3}/2$. For the rest of the section, we shall work with the harmless assumption that the smallest root $\xi_1$ is given by $-1/r_- A$. Even if this is not the case, it does not alter our conclusion as long as $r_- A$ is a positive number. The latter is again required for the physical interpretation of the Ernst metric.

In general Ernst geometry, the ratio between the two temperatures can be conveniently written as follows,

$$\frac{T_A}{T_{BH}} = \frac{\kappa_A}{\kappa_{BH}} = \frac{(\xi_4 - \xi_3)(\xi_3 - \xi_1)}{(\xi_4 - \xi_2)(\xi_2 - \xi_1)}$$

(31)

As we decrease $r_+ A$ from $2/\sqrt{27}$, $\xi_4$ spans between $\sqrt{3}/2$ and 1, and $\xi_3$ between $-\sqrt{3}$ and 1, which means that $(\xi_4 - \xi_3)$ is always of order one. Now, there are two different cases when this ratio is one: either $\xi_3 = \xi_5$ or $(\xi_2 - \xi_1) = (\xi_1 - \xi_5) \sim 1$. The first is the unphysical case where the two horizons merge while the second, for $r_- A < \tilde{r}_+ A \ll 1$, can be translated into the following,

$$\frac{(r_+ - r_-)}{4\pi r_+^2} \approx \frac{A}{2\pi},$$

(32)

which is clearly the condition of equal temperatures.

When can $T_A$ be larger than $T_{BH}$? One must have either $(\xi_4 - \xi_3) > (\xi_4 - \xi_2)$ or $(\xi_4 - \xi_3) > (\xi_2 - \xi_1)$, but the first cannot be satisfied simply because $\xi_4 < \xi_3$ by definition. On the other hand,
the latter condition can be met only for those black holes sufficiently near the extremality. Why?
In order to be far from the extremality, \( A r_\pm = -1/\xi_1 \) must be much smaller than \( A \tilde{r}_+ = -1/\xi_2 \).
But this implies \( (\xi - \xi_1) \gg 1 \sim (\xi_2 - \xi) \), which together with \( \xi_1 < \xi_2 \) tells us that the ratio \( T_A/T_{BH} \) in Eq. (31) is necessarily smaller than the unity. In particular, for relatively small black holes \( (r_- A < \tilde{r}_+ A \ll 1) \), it is easy to see that \( T_A/T_{BH} \) monotonically decreases as a function of the black hole mass and equals to one when \( (r_+ - r_-)/r_+ \simeq 2r_+ A \ll 1 \). Note that the heat capacity of the black hole is positive if \( (r_+ - r_-)/r_+ \ll 1 \).

In short, \textit{black holes far from the extremality will always have its Hawking temperature larger than the acceleration temperature}. Actually, this property appears quite general and not very specific to the particular solution we are using. For instance, if we replace the Ernst metric by the C-metric \[14\] where the black holes are accelerated by a pair of string-like singularities instead, we find the exactly same phenomenon.\[6\] Therefore, black holes that are sufficiently far away from the extremality must continue to evaporate (with a somewhat modified spectrum) even during the acceleration. In particular, this suggests that the Schwarzschild black hole cannot be stabilized by a uniform acceleration.

This observation immediately precludes a possible disaster. Suppose that the black hole actually absorbs net energy from the acceleration heat bath when \( T_{BH} < T_A \). What happens if any RN black hole (of negative heat capacity) can undergo an arbitrarily large acceleration? The black hole would begin to accrete energy quanta from the acceleration heat bath. With a fixed external force that drives the acceleration, \( T_A \) may decrease due to the increasing mass, but so should \( T_{BH} \) at a similar rate due to the negative heat capacity. Then, the process might continue forever, leading to an arbitrarily massive black hole.

In contrast, near-extremal black holes with positive heat capacity may undergo a large acceleration \( (T_A > T_{BH}) \); but the subsequent accretion of mass must increase \( T_{BH} \) and at the same time decrease \( T_A \). After a while, the evolution will take the state to a thermal equilibrium at a new temperature that lies in between the two original temperatures.

The upshot is that \textit{the uniform acceleration does not introduce any runaway behavior} that involves a non-extremal RN black holes accreting unlimited amount of energy quanta from the acceleration heat bath. Instead, it simply shifts the ground state from the extremal black hole to a near-extremal variety whose Hawking temperature is equal to the acceleration temperature. Similar considerations may be applied to the toy model of section 3 with the same conclusion.

\[6\] The C-metric is obtained by setting \( \Lambda = 1 \) in the metric \( (1) \) and readjusting the periodicity of \( \phi \) coordinate.
6 Discussions

One might ask whether the eternal uniform acceleration of the black hole is really physical. Should we not consider only finite processes? In fact, such skepticisms surfaced time and again in regard to the superficially similar classical system of a uniformly accelerated charge. As early as 1910's, physicists debated whether a uniformly accelerated charge really emits Bremmstrahlung or not. The problem was that the uniformly acceleration does not involve any radiation backreaction on the charge itself, which seemed to suggest that there is no source for the radiation energy. In contrast, as soon as one considers only finite processes where the charge experiences a finite net momentum transfer, the integrated radiation energy matches exactly the total work done by the radiation backreaction. Maybe it is not “legitimate” to consider such infinite processes within the framework set by the classical electrodynamics, one may argue.

But the equivalence principle tells us that there is nothing esoteric about such uniform accelerations. If we let the charge levitate in a static electromagnetic field in a laboratory, this is equivalent, according to Einstein, to a uniform acceleration of strength equal to the gravitational field of the earth. Furthermore, even if we consider finite processes only, we still need to explain the differential form of the energy conservation not just integrated version thereof.

These observations tell us that the issue of the eternal uniform acceleration itself must be a red-herring, and that there should be a resolution within the conventional physics. In fact, the classical physics of a uniformly accelerated charged has been thoroughly understood by Boulware in early 80's [8]: The uniformly accelerated charge does emit Bremmstrahlung into the asymptotic future across the acceleration horizon, but the radiation energy does not originate from the charge. Rather the destructive interference between the Coulomb field and the radiation field reduces the field energy near the charge just the right amount so that the net radiation energy is entirely explainable without violating energy conservation. Whenever the uniform acceleration is disrupted, the radiation backreaction transfer some energy from the charge to the surrounding Coulomb field again, but during the uniform acceleration the total energy of the electromagnetic field remains constant [15].

Throughout this paper, we have been asking what happens quantum mechanically if we replace the charge by a uniformly accelerated charged black hole. Obviously, the classical radiations associated with electromagnetic (and gravitational) fields must be still observable to freely falling

\footnote{With appropriate regularization of the point-like charged particle, of course.}
observers, since classical observers at large spatial distances cannot distinguish a point-like source from a black hole. But what about the one-loop effects?

There is one particular case where the analogy with the above classical system appears almost complete, that is, when the laboratory observers find an equilibrium state even at one-loop level, on account of the fact that the Hawking temperature and the acceleration temperature are fine-tuned to be equal. If we had extended to the analogy all the way, we would have believed that the inertial observers detects the black hole radiance at arbitrary late-time while co-moving observers may still insist upon no evaporation. However, a careful one-loop analysis above revealed that this is not the case at all. All observers actually agree when it comes to the Hawking radiation: There is none whatsoever.

It would have been a very difficult situation, if different observers found different evolutions of the black hole. It would have meant among others that the mass of the black hole is observer-dependent and maybe so is the Bekenstein-Hawking entropy. While such behaviors are not entirely inconceivable, considering the complicated nature of the gravity, still it is gratifying to know that the simplest explanation available is also the correct one.

One of the more important implications of our findings above, concerns the natural vacuum state associated with the Ernst geometry. Since the Euclidean version of the geometry, when $\kappa_{BH} = \kappa_A$, is the wormhole-type instanton that pair-creates near-extremal RN black holes [9], the matter of the natural vacuum is of fundamental importance in the WKB estimate of the tunneling rates [10][7].

Recall that the natural vacuum of a “freely falling” Euclidean black hole is the so-called Hartle-Hawking state, which entails an asymptotic heat bath of $T_{BH}$. If one insists upon a zero-temperature state at large distances (in order to maintain finite total energy, for instance), such as the Boulware state, this leads to a severe divergence of the energy-momentum expectation at the event horizon. And this happens precisely because of the nontrivial Bogolubov transformation that relates the asymptotic inertial observers to their counterpart near the event horizon.

In contrast, our findings above imply that a vacuum state of the Ernst space may resemble the Hartle-Hawking vacuum near the black hole yet asymptotically behaves as an zero-temperature state of Boulware type. It is as if the acceleration heat baths that surround the pair-created black holes are of a naturally finite size. In a previous work [7] on the matter of RN black hole pair-creation, the author has asserted that the natural vacuum is characterized precisely by such
Why is this vacuum state especially important for the semiclassical estimate of the tunneling rate? Note that, within the semiclassical framework where the classical geometry is treated also as a dynamical object, a consistent WKB expansion must include the gravitational backreaction to the quantum energy-momentum. That is, in order to obtain a consistent \((n+1)\)-th order WKB estimate of the tunneling rate, we must first obtain the quantum-corrected instanton geometry that solves the Euclidean Einstein equation with the total energy-momentum to \(n\)-th order. Naively, up to the one-loop order, one may start with the zero-th order classical Euclidean instanton solution.

Now suppose that the natural Euclidean vacuum state involved an asymptotic heat bath, as one would expect for a stationary black hole. Although the one-loop energy density could be made arbitrarily small by considering large black holes with small Hawking temperatures, the total energy thereof would always diverge since the heat bath should entail a finite uniform energy density in the asymptotic region. Then, the one-loop corrected instanton would have a very different asymptotic geometry from the one we started with, and it becomes rather unclear whether and in what sense the quantum correction might be small enough to justify the WKB expansion to begin with. More specifically, while it is conceivable that there exists a physical prescription of performing a systematic expansion in this case, it is hardly obvious whether one will obtain the right answer from the naive approach based on the classical instanton solution, e.g., as in Ref. [16].

Fortunately, with the present modified vacuum, the heat bath is cut off essentially where the Euclidean black hole is truncated, that is, at \(r^0 \sim r_+/B^2\) in the notation of section 2. The total one-loop energy must be finite, and in the weak field limit or equivalently in the small acceleration limit, the dominant contribution is near the Euclidean black hole. Subsequently, the only gravitational backreaction to such one-loop energy-momentum is to change the local geometry near the black hole a little bit, and in particular manifest itself in a small one-loop correction to the black hole mass of order \(\sim \pi_4/r_+\) while the temperature is held fixed [7]. For small enough \(\pi_4/r_+^3\), therefore, we find that the systematic WKB approximation based on the classical instanton is sensible thanks to this unusual nature of the vacuum.

In summary, we studied the semiclassical evolution of accelerated Reissner-Nordstrom black hole, and found that the uniform acceleration actually shift the final ground state from the extremal to a near-extremal variety. A similar phenomenon is also found in the toy model of charged black holes in a dimensionally reduced De Sitter universe. Our findings also provide an important
consistency check for the WKB expansion of the RN black hole pair-creation tunneling rates.

References


