DISTRIBUTIONS OF FOURIER MODES OF COSMOLOGICAL DENSITY FIELDS

Zuhui Fan and J. M. Bardeen

Department of Physics, FM-15, University of Washington, Seattle, WA 98195

ABSTRACT

We discuss the probability distributions of Fourier modes of cosmological density fields using the central limit theorem as it applies to weighted integrals of random fields. It is shown that if the cosmological principle holds in a certain sense, i.e., the universe approaches homogeneity and isotropy sufficiently rapidly on very large scales, the one-point distribution of each Fourier mode of the density field is Gaussian whether or not the density field itself is Gaussian. Therefore, one-point distributions of the power spectrum obtained from observational data or from simulations are not a good test of whether the density field is Gaussian.

PACS number(s): 98.80_k, 98.80.BP, 98.60.Eg
I. INTRODUCTION

In the standard cold dark matter (CDM) model, primordial fluctuations are generated during inflationary stage, and the simple linear perturbation theory predicts Gaussian fluctuations [1]. On the other hand, some alternative inflationary theories [2] [3] and topological defect mechanisms [4] [5] involve nonlinear processes which generate non-Gaussian perturbations. Thus it is interesting to test the consistency of a theory with observations in this regard. Besides analyzing the spatial distribution of galaxies [6] [7] and the pattern of microwave background radiation anisotropies [8], some research groups considered the one-point distribution of the power spectrum of the density field as a test of the Gaussianity [9] [10] [11] [12] [13]. It is found that for the QDOT IRAS redshift survey, the distribution is almost exactly an exponential distribution which is consistent with the Gaussian distribution for both the real and imaginary parts of a Fourier mode [13]. Suginohara and Suto [14] did simulations up to very nonlinear stages, analyzed the one-point distribution for both the power spectrum and the phase of a Fourier mode, and also got results consistent with Gaussian distributions. On the other hand, we know that in highly nonlinear stages the one-point density distribution in real space is very non-Gaussian. This raises the question, namely, what can one really conclude from the statistics of the one-point distribution of the power spectrum?

The Fourier transform of the cosmological density field is basically a weighted
sum of the densities over the space. If the universe consists of a large number of dense clumps (e.g., galaxies, clusters), and those clumps are independent of each other, then by virtue of the central limit theorem, the real and imaginary components of each individual Fourier mode are Gaussian distributed although the density distribution itself could be highly non-Gaussian. Kaiser and Peacock [10] generated some test catalogs, and analyzed the distributions of Fourier modes. They found that even if only a few independent clumps existed in a test catalog, the one-point probabilities of individual Fourier modes would be well described by Gaussian distributions.

On the other hand, if there are some correlations on scales larger than the typical size of a clump (as most of the cosmological models predict), what is the one-point probability of an individual Fourier mode? In this paper, it is shown that if the random density field satisfies certain mixing conditions, the central limit theorem guarantees that the one-point distribution of an individual Fourier component is Gaussian whether or not the field itself is Gaussian.

The paper is organized as following: In section II, we introduce some concepts and discuss an application of the central limit theorem to weighted integrals of a random field. In section III, we consider the Fourier transform of a random field and see how it satisfies the conditions in section II. Section IV contains further discussion and conclusions. The appendix provides an approximate estimate of the mixing rate, which is introduced and used in the section II, in terms of the
two-point joint probability.
II. THE CENTRAL LIMIT THEOREM

In this section, we discuss a central limit theorem for weighted integrals of random scalar fields in terms of constraints on the field moments, correlations and the weighting functions. The theorem and its context is clearly discussed by Ivanov and Leonenko [16]*.

Let us first consider a random variable $\delta$. Without losing generality, we assume the average $< \delta > = 0$ and $\delta \in [f_l, f_u]$, where $f_l \in R^1$ and $f_u \in R^1$. Let $p(\delta)$ be the probability density function of $\delta$, and a set $A_\delta$ be $\{ \delta : f_l \leq a_1 \leq \delta \leq b_1 \leq f_u \}$. Then the probability that $\delta$ taking values on the set $A_\delta$ is

$$ P(A_\delta) = \int_{a_1}^{b_1} p(\delta) d\delta \quad ,$$

and the normalization condition is

$$ \int_{f_l}^{f_u} p(\delta) d\delta = 1 \quad .$$

Now consider a random scalar field $\delta(\vec{x})$ satisfying $< \delta(\vec{x}) > = 0$, $\vec{x} \in R^3$. We assume that the probability density function and the value range of $\delta(\vec{x})$ are the same for all $\vec{x} \in R^3$. Let $\Delta_1$ and $\Delta_2$ be two regions in $R^3$. The distance between $\Delta_1$ and $\Delta_2$ is defined as

$$ r = \min |\vec{x} - \vec{y}|, \quad \vec{x} \in \Delta_1, \quad \vec{y} \in \Delta_2 \quad .$$

In order to apply the central limit theorem, we need to introduce the concept of the mixing rate [17]. Let us denote positions on $\Delta_1$, $\Delta_2$ by $\{\vec{x}_i, i = 1, 2, \ldots\}$,

* See the reference 16, Chapter I.
\{\tilde{y}_j, j = 1, 2, \ldots\}, respectively. Let \(A\) be a set generated by the random variables \(\{\delta(\tilde{x}_i), i = 1, 2, \ldots\}\) and

\[
a_i \leq \delta(\tilde{x}_i) \leq b_i, \quad a_i \geq f_i, \quad b_i \leq f_u, \quad i = 1, 2, \ldots,
\]

(2.3)

and let \(B\) be generated by the random variables \(\{\delta(\tilde{y}_j), j = 1, 2, \ldots\}\) with the form

\[
c_j \leq \delta(\tilde{y}_j) \leq c_j, \quad c_i \geq f_i, \quad c_j \leq f_u, \quad j = 1, 2, \ldots.
\]

(2.4)

Then the probability \(P(A), P(B)\) and \(P(AB)\) are

\[
P(A) = \{ \prod_{i=1}^{\tilde{b}_i} d\delta(\tilde{x}_i) \} \ p[\delta(\tilde{x}_1), \delta(\tilde{x}_2), \ldots],
\]

(2.5)

\[
P(B) = \{ \prod_{j=1}^{\tilde{c}_j} d\delta(\tilde{y}_j) \} \ p[\delta(\tilde{y}_1), \delta(\tilde{y}_2), \ldots],
\]

(2.6)

and

\[
P(AB) = \{ \prod_{i,j=1,2,\ldots}^{\tilde{b}_i, \tilde{c}_j} d\delta(\tilde{x}_i) \int_{\tilde{a}_i}^{\tilde{b}_i} d\delta(\tilde{x}_i) \} \ p[\delta(\tilde{x}_1), \delta(\tilde{x}_2), \ldots, \delta(\tilde{y}_1), \delta(\tilde{y}_2), \ldots].
\]

(2.7)

respectively, where \(p[\delta(\tilde{x}_1), \delta(\tilde{x}_2), \ldots]\) is the joint probability density function of \(\{\delta(\tilde{x}_i), i = 1, 2, \ldots\}\), \(p[\delta(\tilde{y}_1), \delta(\tilde{y}_2), \ldots]\) is the joint probability density function of \(\{\delta(\tilde{y}_j), j = 1, 2, \ldots\}\), and \(p[\delta(\tilde{x}_1), \delta(\tilde{x}_2), \ldots, \delta(\tilde{y}_1), \delta(\tilde{y}_2), \ldots]\) is the joint probability density function of \(\{\delta(\tilde{x}_i), \delta(\tilde{y}_j), i = 1, 2, \ldots, j = 1, 2, \ldots\}\). The dependence of the random variables defined on \(\Delta_1\) and \(\Delta_2\) is characterized by the Rosenblatt dependence rate \([17]\) which is defined as

\[
\alpha(\Delta_1, \Delta_2) = \max \left| P(AB) - P(A)P(B) \right|,
\]

(2.8)
where the maximum is taken over all possible values of \( \{a_i, b_i \in [f_1, f_a], \ c_j, e_j \in [f_1, f_s], \ i = 1, 2, ..., \ j = 1, 2, ...\} \). The mixing rate \( \alpha(r_0) \) is defined as

\[
\alpha(r_0) = \max \alpha(\Delta_1, \Delta_3) ,
\]

where the maximum is taken over all pairs of \( \Delta_1 \) and \( \Delta_2 \) lying at a distance at least \( r_0 \). It is clear that in general the mixing rate \( \alpha(r_0) \) is not only related to the two-point but also to all higher order correlations.

With these definitions, we now discuss the central limit theorem, and introduce a column vector \( s_V(\tilde{\theta}) = [s_{V}^{(1)}(\tilde{\theta}), ..., s_{V}^{(r)}(\tilde{\theta})]^t \), where \( t \) stands for the transpose, with components defined by

\[
s_{V}^{(i)}(\tilde{\theta}) = \int_{V} g_{V}^{(i)}(\vec{x}, \tilde{\theta}) \delta(\vec{x}) d\vec{x} ,
\]

where \( g_{V}(\vec{x}, \tilde{\theta}) = [g_{V}^{(1)}(\vec{x}, \tilde{\theta}), ..., g_{V}^{(r)}(\vec{x}, \tilde{\theta})]^t \) is a column vector of functions, and \( \tilde{\theta} \in \Theta \). In the situation we are going to discuss in the next section, \( \Theta = R^3 \).

We assume that the following conditions are satisfied:

(i). As \( V \to \infty \), there exists a function \( \sigma^2(\tilde{\theta}) \), taking on values in a set of positive definite \( r \times r \) matrices such that

\[
\lim_{V \to \infty} [s_{V}(\tilde{\theta})s_{V}^t(\tilde{\theta}) - \sigma^2(\tilde{\theta})] = 0
\]

uniformly in \( \tilde{\theta} \in \Theta \).

(ii). For any \( i \in \{1, ..., r\} \), the function \( g_{V}^{(i)}(\vec{x}, \tilde{\theta}) \) satisfies the condition:

For \( V \to \infty \), there exists a constant \( k_{3}^{(i)} > 0 \) (possibly depending on \( i \)) such
that
\[ \max_{\delta \in \Theta, \bar{\epsilon} \in V} |g_{\bar{\epsilon}}^{(i)}(\bar{x}, \bar{\theta})| \leq k_3^{(i)}/\sqrt{V} \quad . \] (2.12)

(iii). There exists a constant \( \gamma > 0 \) such that
\[ < |\delta(\bar{x})|^2 + \gamma > \leq k_1 < \infty, \quad \bar{x} \in \mathbb{R}^3 \quad . \] (2.13)

(iv). The mixing rate \( \alpha(r_0) \) of the random field \( \delta(\bar{x}) \) satisfies
\[ \alpha(r_0) \leq k_2 r_0^{-3-\epsilon} \quad , \quad \epsilon \gamma > 6 \quad , \] (2.14)

where \( k_1 \) is a constant.

Then as \( V \to \infty \), the random vector \( s_\bar{\epsilon}^{(i)}(\bar{\theta}) \) has, uniformly in \( \Theta \), an asymptotically multidimensional normal distribution \( N_r(0, \sigma^2(\bar{\theta})) \) with zero mean vector and covariance matrix \( \sigma^2(\bar{\theta})^* \).

In the next section, we will discuss the Fourier transform of a cosmological density field using this central limit theorem.

* See the reference 16, p41, Theorem 1.7.5
III. FOURIER TRANSFORM OF A RANDOM FIELD

The Fourier transform of a random field $\delta(\vec{x})$ in a 3-dimensional space is defined as

$$\delta(\vec{k}) = \lim_{V \to \infty} \frac{1}{\sqrt{V}} \int_V \exp(i\vec{k} \cdot \vec{x})\delta(\vec{x})d^3\vec{x} \ .$$

Here we assume $\delta(\vec{x})$ is a homogeneous, isotropic and real random field, which satisfies $< \delta(\vec{x}) > = 0$.

The Fourier transform $\delta(\vec{k})$ can be written as a sum of the real part $A(\vec{k})$ and the imaginary part $iB(\vec{k})$, where $A(\vec{k})$ and $B(\vec{k})$ are defined as

$$A(\vec{k}) = \lim_{V \to \infty} A_v(\vec{k}) \ ,$$

and

$$B(\vec{k}) = \lim_{V \to \infty} B_v(\vec{k}) \ ,$$

respectively, where

$$A_v = \frac{1}{\sqrt{V}} \int_V \cos(\vec{k} \cdot \vec{x})\delta(\vec{x}) \ ,$$

and

$$B_v = \frac{1}{\sqrt{V}} \int_V \sin(\vec{k} \cdot \vec{x})\delta(\vec{x}) \ .$$

We consider the vector $s_v(\vec{k}) = [A_v(\vec{k}), B_v(\vec{k})]$. In terms of the notations of section II, we have $r = 2$ and

$$g_{v}^{(1)}(\vec{x}, \vec{k}) = \frac{1}{\sqrt{V}} \cos(\vec{k} \cdot \vec{x}) \ .$$
\[ g^{(2)}_{V}(\tilde{x}, \tilde{k}) = \frac{1}{\sqrt{V}} \sin(\tilde{k} \cdot \tilde{x}) \quad . \tag{3.5} \]

We discuss the conditions of the central limit theorem in the section II one by one.

(i). To evaluate the covariance matrix \( \sigma^2(\tilde{k}) \), three elements should be considered:

\[
\sigma^2_{11}(\tilde{k}) = \lim_{V \to \infty} < A_V(\tilde{k}) A_V(\tilde{k}) > \quad ,
\]

\[
\sigma^2_{22}(\tilde{k}) = \lim_{V \to \infty} < B_V(\tilde{k}) B_V(\tilde{k}) > \quad ,
\]

and

\[
\sigma^2_{12}(\tilde{k}) = \lim_{V \to \infty} < A_V(\tilde{k}) B_V(\tilde{k}) > \quad .
\]

First let us consider \( \sigma^2_{11} \).

\[
\sigma^2_{11}(\tilde{k}) = \lim_{V \to \infty} \int_V \int_V d^3\tilde{x}_1 d^3\tilde{x}_2 \frac{1}{\sqrt{V}} \cos(\tilde{k} \cdot \tilde{x}_1) \frac{1}{\sqrt{V}} \cos(\tilde{k} \cdot \tilde{x}_2) < \delta(\tilde{x}_1) \delta(\tilde{x}_2) > \quad . \tag{3.6}
\]

Because \( \delta(\tilde{x}) \) is homogeneous, isotropic and \( < \delta(\tilde{x}_1) >= 0, < \delta(\tilde{x}_2) >= 0 \), the correlation \( < \delta(\tilde{x}_1) \delta(\tilde{x}_2) >= \xi(|\tilde{x}_1 - \tilde{x}_2|) \) only depends on \( |\tilde{x}_1 - \tilde{x}_2| \). One can readily show that

\[
\sigma^2_{11}(\tilde{k}) = \frac{1}{2} \lim_{V \to \infty} \int_V d^3\tilde{x} \cos(\tilde{k} \cdot \tilde{x}) \xi(|\tilde{x}|) \quad .
\]

We know that the power spectrum averaged over the ensemble of realizations is related to the correlation function \( \xi(\tilde{x}) \) by

\[
< P(\tilde{k}) >= \lim_{V \to \infty} \int_V d^3\tilde{x} \exp(i\tilde{k} \cdot \tilde{x}) \xi(\tilde{x}) \quad .
\]
Since $\delta(\vec{x})$ is homogeneous, isotropic and real, it follows that

$$< P(\vec{k}) > = < P(-\vec{k}) > = < P(k) > \text{ and } \xi(\vec{x}) = \xi(|\vec{x}|).$$

Thus

$$\lim_{V \to \infty} \int_V \delta^3 \vec{r} \cos(\vec{k} \cdot \vec{r}) \xi(|\vec{r}|) = < P(k) > ,$$

and

$$\lim_{V \to \infty} \int_V \delta^3 \vec{r} \sin(\vec{k} \cdot \vec{r}) \xi(|\vec{r}|) = 0 .$$

Then

$$\sigma_{11}^2(\vec{k}) = \frac{1}{2} < P(k) > . \quad (3.7)$$

Similarly one can show that

$$\sigma_{22}^2(\vec{k}) = \frac{1}{2} < P(k) > , \quad (3.8)$$

and

$$\sigma_{12}^2(\vec{k}) = 0 . \quad (3.9)$$

For any reasonable cosmological density field, the average power spectrum $< P(k) >$ exists. Therefore the condition for the existence of the covariance matrix $\sigma^2(\vec{k})$ is satisfied.

(ii). From (3.4), (3.5), $|\cos(\vec{k} \cdot \vec{r})| \leq 1$, and $|\sin(\vec{k} \cdot \vec{r})| \leq 1$, we can always choose $k_3 = 1$ so that condition (ii) in section II is satisfied.

Condition (iii) of section II constrains the statistical properties of the field $\delta(\vec{x})$. The existence of moments higher than the second is not a problem for most models of cosmological density field.
Condition (iv) of section II concerns the behavior of the mixing rate $\alpha(r_0)$. In general this mixing rate is related to all order of correlations and thus is difficult to be calculated precisely. Under certain approximations, however, we can estimate the behavior of $\alpha(r_0)$. In the Appendix, we make such an approximation: Between two well separated regions $\Delta_1$ and $\Delta_2$, only two-point joint probabilities are considered, while within each region full correlations are taken into account. We argue in the Appendix that under plausible assumptions, the reduced two-point joint probability of $\delta_i$ at $\vec{x}_i$ and $\delta_j$ at $\vec{y}_j$ can be written as

$$\tilde{p}(\delta_i, \delta_j, |\vec{x}_i - \vec{y}_j|) = \sum_{n = n_0}^{\infty} A_n(\delta_i, \delta_j) F_n^i(|\vec{x}_i - \vec{y}_j|),$$

where $F_n^i(|\vec{x}_i - \vec{y}_j|) \ll 1$ when $|\vec{x}_i - \vec{y}_j|$ is very large. $A_n(\delta_i, \delta_j)$ is independent of $F_n^i(|\vec{x}_i - \vec{y}_j|)$ and $n_0$ is the power index such that $A_n$ vanishes for $n < n_0$. Here $n_0 \geq 1$.

Under these approximations

$$\alpha(r_0) \sim C'_0 \max_{|\vec{x}_i - \vec{y}_j| \geq r_0} |F_n^{ij}(|\vec{x}_i - \vec{y}_j|)|,$$

where $C'_0$ is a finite constant, and the maximum is taken over all pairs of $(\vec{x}_i, \vec{y}_j)$ with $|\vec{x}_i - \vec{y}_j| \geq r_0$. The condition (iv) of section II requires that

$$\alpha(r_0) \leq k_2 r_0^{-3-\epsilon}, \quad e\gamma > 6.$$

Then in order the inequality (3.12) to be satisfied, we need

$$\max_{|\vec{x}_i - \vec{y}_j| \geq r_0} |F_n^{ij}(|\vec{x}_i - \vec{y}_j|)| \leq C_{14} r_0^{-3-\epsilon},$$
where $C_{14}$ is a finite constant. As discussed in the Appendix, for a non-Gaussian field which is a local functional of a Gaussian field, $F_{ij}(|\vec{x}_i - \vec{y}_j|)$ is the two-point correlation function of the Gaussian field. The power index $n_0$ in (3.10) depends on the specific form of the non-Gaussian field. For example, for a field $\Phi = \phi^2 - <\phi^2>$, where $\phi$ is a Gaussian field and $<\phi^2>$ is the average of $\phi^2$, the power index $n_0$ is 2. If the two-point correlation function of a Gaussian field satisfies condition (3.13) (in this case $n_0 = 1$), all non-Gaussian fields which are local functionals of the Gaussian field satisfy the mixing rate condition.

Reduced higher order joint probabilities are related to higher order reduced correlations. Observations indicate that higher order of reduced correlations drop faster than the two-point correlation function [15] [18] [19]. Thus it is a reasonable approximation for a cosmological density field to ignore reduced higher order joint probabilities.

As a case where the mixing rate condition is violated, we consider a random field $\delta(\vec{x})$ which is a convolution of two Gaussian fields $\phi_1$ and $\phi_2$

$$\delta(\vec{x}) = \int d^3 \vec{x}_1 \phi_1(\vec{x}_1) \phi_2(\vec{x} - \vec{x}_1) . \quad (3.14)$$

Clearly, for each $\vec{x} \in \mathbb{R}^3$, $\delta(\vec{x})$ is related to the fields $\phi_1$ and $\phi_2$ in the whole space. Thus the probabilities of the field $\delta(\vec{x})$ is always related to the joint probabilities of all $\phi_1$ and all $\phi_2$ in the whole space. It is not surprising that some complicated correlations for the random field $\delta(\vec{x})$ exist and the mixing rate condition is not satisfied. In fact, we know that the Fourier transform of $\delta(\vec{x})$ is the product of
the Fourier transform of $\phi_1$ and $\phi_2$

$$\delta(\vec{k}) = \phi_1(\vec{k}) \phi_2(\vec{k}) \ ,$$

(3.15).

so $\delta(\vec{k})$ is not Gaussian distributed.

If a field $\delta(\vec{x})$ satisfies the conditions of the central limit theorem, then the the joint distribution of the real and imaginary parts of its Fourier transform, namely, $A(\vec{k})$ and $B(\vec{k})$, is simply

$$p(A, B) dAdB = \left( \frac{1}{\pi <P(\vec{k})>} \right) \exp \left[ -\left( \frac{A^2 + B^2}{<P(\vec{k})>} \right) \right] dAdB \ .$$

(3.16)

If we change the variables from $A(\vec{k})$ and $B(\vec{k})$ to $P(\vec{k})$ and $\theta_{\vec{k}}$, where $\theta_{\vec{k}}$ is the phase of the $\vec{k}$ mode, we get

$$p \left[ P(\vec{k}), \theta_{\vec{k}} \right] = \exp \left[ -\frac{P(\vec{k})}{<P(\vec{k})>} \right] dP d\theta_{\vec{k}} \ ,$$

(3.17)

that is, $P(\vec{k})$ is exponentially distributed, and $\theta_{\vec{k}}$ is uniformly distributed.
IV. DISCUSSION

In this paper we showed that as long as the mixing rate of the density field drops fast enough at large distances, the central limit theorem guarantees normal distributions for the real and imaginary parts of a Fourier mode of the field.

Reduced higher order joint probabilities are related to the reduced higher order correlations. Observational evidences show that on large scales, these correlations do drop faster than the two-point correlation function does [15] [18] [19]. Therefore for the mixing rate, it is reasonable to only consider the two-point joint probabilities between two well separated regions. Under such an approximation, we estimate the mixing rate. If the reduced two-point joint probability of $\delta_i$ at $\vec{x}_i$ and $\delta_j$ at $\vec{y}_j$ depends on a function $F_{ij}(|\vec{x}_i - \vec{y}_j|)$ in such a way

$$\tilde{p}(\delta_i, \delta_j, |\vec{x}_i - \vec{y}_j|) = \sum_{n=-\infty}^{\infty} A_n(\delta_i, \delta_j) F_{ij}^n(|\vec{x}_i - \vec{y}_j|),$$

where $|F_{ij}^n(|\vec{x}_i - \vec{y}_j|)| \ll 1$ when $|\vec{x}_i - \vec{y}_j|$ is very large, and $A_n(\delta_i, \delta_j)$ is independent of $F_{ij}^n(|\vec{x}_i - \vec{y}_j|)$, the mixing rate has the same behavior as

$$C'_0 \max_{|\vec{x}_i - \vec{y}_j| \geq r_0} |F_{ij}^n(|\vec{x}_i - \vec{y}_j|)|,$$

where $C'_0$ is a finite constant. Thus if

$$\max_{|\vec{x}_i - \vec{y}_j| \geq r_0} |F_{ij}^n(|\vec{x}_i - \vec{y}_j|)| \leq C_{15} r_0^{-3-\epsilon}$$

at large $r_0$, where $\epsilon > 0$, the mixing rate condition is satisfied. The condition on

$$\max_{|\vec{x}_i - \vec{y}_j| \geq r_0} |F_{ij}^n(|\vec{x}_i - \vec{y}_j|)|$$

is equivalent to requiring that

$$\max_{|\vec{x}_i - \vec{y}_j| \geq r_0} \tilde{p}(\delta_i, \delta_j, |\vec{x}_i - \vec{y}_j|) \leq C_{15} r_0^{-3-\epsilon}.$$
This is a sensible requirement, and is easily satisfied by a cosmological density field.

Statistical properties of the large-scale structure in the universe provide important clues for understanding the universe. Theoretically, to analyze statistical properties of a random field, one must generate many different realizations of the field and calculate the statistical quantities over these realizations. Practically, since there is only one universe, we have to use the “ergodic” theorem in the position space to deal with cosmological density field. Therefore it is important to have what is in some sense “a fair sample”. The basic requirement for “a fair sample” is that quantities within the sample should represent the quantities of the whole universe without being strongly biased by the correlations of the quantities within the sample. If the universe is statistically homogeneous and isotropic at very large scales, all correlations within “a fair sample” should have no significant effect on the statistical quantities and can be ignored. For such samples, the Fourier modes of the density field are Gaussian distributed whether or not the density field itself is Gaussian. Since in reality one can never be certain whether or not we really have a fair sample, we must be cautious to interpret the statistical results on the distributions of the Fourier modes. If non-Gaussian distributions of individual Fourier modes are found for a set of samples and this non-Gaussianity disappears when larger samples are taken, it is most likely that the original sample is not a fair sample, rather than that there is a genuine non-Gaussian behavior.
If the non-Gaussianity persists when we go to larger and larger samples, then one might begin to believe that there is some physics in it.

One class of non-Gaussian models are topological defect seeded models, e.g., cosmic strings [5], or textures [4]. In those models, the density perturbations are due to randomly distributed seeds, and each seed has a density profile. If the density profiles of all seeds are the same, and are denoted by $f(\vec{x} - \vec{x}_i)$, where $\vec{x}_i$ is the position of a seed, the density field can be written as

$$\delta(\vec{x}) = \sum_{i=1}^{N} f(\vec{x} - \vec{x}_i), \quad (4.1)$$

where the sum is over all the seeds. We can write (4.1) as

$$\delta(\vec{x}) = \int d\vec{x}' g(\vec{x}') f(\vec{x} - \vec{x}'), \quad (4.2)$$

where

$$g(\vec{x}') = \sum_{i=1}^{N} \delta^{(3)}(\vec{x}' - \vec{x}_i), \quad (4.3)$$

and $\delta^{(3)}(\vec{x}' - \vec{x}_i)$ is the three-dimensional Dirac $\delta$-function. Then the Fourier transform of $\delta(\vec{x})$ is

$$F_\delta(\vec{k}) = F_g(\vec{k}) F_f(\vec{k}), \quad (4.4)$$

where $F_g(\vec{k})$ is the Fourier transform of the function $g$, and $F_f(\vec{k})$ is the Fourier transform $f$. Since $f$ is not a random function, $F_f(\vec{k})$ is not random. The distribution of $F_\delta(\vec{k})$ depends on the statistics of $F_g(\vec{k})$. We have

$$F_g(\vec{k}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \exp(ik \cdot \vec{x}_i). \quad (4.5)$$
If the seeds are uncorrelated and $N \rightarrow \infty$, $F_\delta(\vec{k})$ is Gaussian distributed and so is $F_\delta(\vec{k})$ by the central limit theorem for the discrete summation[20]. Certainly, if within an observational region only one or two seeds exist, one may get some non-Gaussian signal in the distributions of the Fourier transform [20]. But in that case, one needs to go to larger samples in order to get statistically solid conclusions.

In principle, the appropriate way to analyze the distributions of the Fourier modes (power spectrum) is to generate a large number of realizations of the cosmological density field. For each realization, the Fourier transform is made, and the average power spectrum $< P(k) >$ for each $k$ is obtained by averaging over all realizations. Then the distribution of $P(k)/ < P(k) >$ for each $k$ is analyzed. Practically, since we only have one universe, and can do limited number of simulations, some kind of “ergodic” assumptions are always used when analyzing observational or simulation data. For example, in [12] and [13] the distributions of the power spectrum in $k$-space are analyzed, i.e., to cumulate the numbers of $k$ which have the same value of $P/ < P >$, where $< P >$ is the average power based on some physical considerations. Then the distributions of those numbers against $P/ < P >$ are analyzed. In these analyses, we should be very careful to use an appropriate $< P >$. For example, as pointed out in [13], because of the small scale clustering, the average $< P(k) >$ is larger than the purely random walk estimation for large $k$. If one uses $< P >$ estimated by ignoring the clustering effect,
one may get a power spectrum distribution which is different from the exponential distribution expected from Gaussian distributions of Fourier modes. This difference is caused by using an inappropriate average power spectrum, rather than the existence of some genuine non-Gaussianity.

By the above discussion, we see that the one-point distributions of Fourier modes are not good statistical tests of the Gaussianity of the density field. One possible way to distinguish Gaussian and non-Gaussian statistics is to go to high order correlations. For example, we can consider the two-point correlation of the power spectra $F(\vec{k}, \vec{k}^\prime) = \langle P(\vec{k})P(\vec{k}^\prime) \rangle$, which is related to the four-point correlation of Fourier components [13]. For a Gaussian random field $\phi(\vec{x})$,

$$F(\vec{k}, \vec{k}^\prime) = \langle P(\vec{k}) \rangle \langle P(\vec{k}^\prime) \rangle$$

$$+ \delta^{(3)}(\vec{k} + \vec{k}^\prime) \langle P(\vec{k}) \rangle \langle P(\vec{k}^\prime) \rangle$$

$$+ \delta^{(3)}(\vec{k} - \vec{k}^\prime) \langle P(\vec{k}) \rangle \langle P(\vec{k}^\prime) \rangle \quad \text{(4.1)}$$

If $\vec{k} + \vec{k}^\prime \neq 0$ and $\vec{k} - \vec{k}^\prime \neq 0$, then

$$F(\vec{k}, \vec{k}^\prime) = \langle P(\vec{k}) \rangle \langle P(\vec{k}^\prime) \rangle \quad \text{(4.2)}$$

\text{i.e., for a Gaussian field, the correlation of the power spectrum for different } k \text{-mode is simply the product of the power spectrum of each mode.}

For a non-Gaussian field, it is expected that there are some additional terms in the correlation of the power spectrum. To illustrate this point, as an example, we consider a field $\zeta(\vec{x})$ which is the square of a Gaussian random field $\phi(\vec{x})$.

$$\zeta(\vec{x}) = \phi^2(\vec{x}) \quad \text{(4.3)}$$
Detailed calculations show that if $\vec{k} \neq -\vec{k}'$ and $\vec{k} \neq \vec{k}'$, 

$$F(\vec{k}, \vec{k}') = \langle P(\vec{k}) \rangle \langle P(\vec{k}') \rangle$$

$$+ 8 \int \frac{d^3 \vec{k}_1}{(2\pi)^3} < |\phi(\vec{k}_1)|^4 |\phi(\vec{k} - \vec{k}_1)|^2 |\phi(\vec{k}_1 + \vec{k}')|^2 |\phi(\vec{k} - \vec{k}' - \vec{k}_1)|^2 >$$

$$+ 8 \int \frac{d^3 \vec{k}_1}{(2\pi)^3} < |\phi(\vec{k}_1)|^2 |\phi(\vec{k} - \vec{k}_1)|^2 |\phi(\vec{k}_1 - \vec{k}')|^2 |\phi(\vec{k} + \vec{k}' - \vec{k}_1)|^2 >$$

(4.4)

It can be seen that there are some extra terms added up to $\langle P(\vec{k}) \rangle \langle P(\vec{k}') \rangle$.

Generally these extra terms depend on the specific form of the non-Gaussian field.

Therefore at this level, it is possible to distinguish Gaussian and non-Gaussian fields.

\textbf{ACKNOWLEDGEMENTS}

We are very grateful for helpful discussions with P. Guttorp and N. Kaiser.

And we are very thankful to N. Kaiser for providing us their preprint. One of us

(JMB) acknowledges useful discussions with other participants at an Aspen Center for Physics Workshop. This work was supported in part by the Department of Energy under Grant No. DE-FG06-91ER40614.
REFERENCES


Appendix

Estimation of the Mixing Rate

In this appendix, we discuss the approximate behavior of the mixing rate.

Consider \( |P(AB) - P(A)P(B)| \) as discussed in section II. If the random variables in \( \Delta_1 \) are completely independent of the random variables in \( \Delta_2 \), then \( P(AB) = P(A)P(B) \), and \( |P(AB) - P(A)P(B)| = 0 \) as expected. If the two sets of random variables are weakly correlated, we could only consider the first order corrections to \( P(AB) \) in addition to \( P(A)P(B) \). We consider a statistically homogeneous and isotropic random field. For two random variables \( \delta_i \) at \( \bar{x}_i \) and \( \delta_j \) at \( \bar{y}_j \), their joint probability is \( p(\delta_i, \delta_j, |\bar{x}_i - \bar{y}_j|) \). Then

\[
\tilde{p}(\delta_i, \delta_j, |\bar{x}_i - \bar{y}_j|) = p(\delta_i, \delta_j, |\bar{x}_i - \bar{y}_j|) - p(\delta_i)p(\delta_j)
\]

(A.1)
is the reduced two-point joint probability which represents the contribution to \( p(\delta_i, \delta_j, |\bar{x}_i - \bar{y}_j|) \) due to the correlation between \( \delta_i \) and \( \delta_j \). The first order approximation of \( P(AB) - P(A)P(B) \) means that between two well separated regions, only reduced two-point joint probabilities are considered. Let us consider the contribution from the correlation between \( \delta_i \in \Delta_1 \) and \( \delta_j \in \Delta_2 \). After \( \delta_i \in \Delta_1 \) having been picked out, since \( \delta_i \) is correlated with the rest of the variables within \( \Delta_1 \), the joint probability for the rest of the random variables within \( \Delta_1 \) should be the conditional joint probability with \( \delta_i \) fixed, which is denoted by \( p(\delta(\bar{x}_1), ..., \delta(\bar{x}_{i-1}), \delta(\bar{x}_{i+1}), ...|\delta(\bar{x}_i)) \). Similarly, the joint probability for the rest of the variables within \( \Delta_2 \) should be \( p(\delta(\bar{y}_1), ..., \delta(\bar{y}_{j-1}), \delta(\bar{y}_{j+1}), ...|\delta(\bar{y}_j)) \), the condi-
tional probability with \( \delta_j \) fixed. Therefore the contribution to \( P(AB) - P(A)P(B) \) from the correlation between \( \delta_i \) and \( \delta_j \) has the form

\[
P_{ij} = \left\{ \int_{a_{i,k}}^{b_{i,k}} d\delta(x_i) \int_{a_{j,k}}^{b_{j,k}} d\delta(y_j) \right\} 
\]

\[
\Pi_{p_i=1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots} \int_{a_{i,k}}^{b_{i,k}} d\delta(x_{p_i}) \int_{a_{j,k}}^{b_{j,k}} d\delta(y_{p_j}) 
\]

\[
p(\delta(x_1), \ldots, \delta(x_{i-1}), \delta(x_{i+1}), \ldots | \delta(x_i)) \]

\[
\times p(\delta(y_1), \ldots, \delta(y_{j-1}), \delta(y_{j+1}), \ldots | \delta(y_j)) \}.
\]

The first order correction to \( P(AB) - P(A)P(B) \) should include contributions from all pairs of random variables between \( \Delta_1 \) and \( \Delta_2 \). When \( r_0 \) is sufficiently large, approximately we have

\[
P(AB) - P(A)P(B) \sim \sum_{i,j=1,2,\ldots} P_{ij},
\]

where \( P_{ij} \) has the form (A.2)

Since we are interested in the behavior of the mixing rate with large separation between the two regions, we would like to separate the correlations between \( \delta_i \) and \( \delta_j \), which depends on \( |x_i - y_j| \) for a statistically homogeneous and isotropic random field, from the statistics for each of the random variables. First, let us consider a non-Gaussian field \( \delta(x) \) which is a local functional of a Gaussian field \( \phi(x) \), i.e., \( \delta(x) = \delta[\phi(x)] \). The joint probability between \( \delta(x_i) \) and \( \delta(y_j) \) is related to the joint probability of \( \phi(x_i) \) and \( \phi(y_j) \). We assume that for a fixed value of \( \delta(x) \), there are \( m \) values of \( \phi(x) \), \( k = 1, \ldots, m \) corresponding to it, it then follows

\[
p[\delta(x_i), \delta(y_j)] = \sum_{k_i=1, k_j=1}^{m} p[\phi^{(k_i)}(x_i), \phi^{(k_j)}(y_j)]
\]

\[
\left\{ \frac{1}{|d\delta[\phi(x_i)]/d[\phi(x_i)]|_{k_i}} \right\} \left\{ \frac{1}{|d\delta[\phi(y_j)]/d[\phi(y_j)]|_{k_j}} \right\},
\]

(A.4)
where the factor \( \{1/|d\delta[\phi(x_i)]/d[\phi(x_i)]|_{k_i}\}\{1/|d\delta[\phi(y_j)]/d[\phi(y_j)]|_{k_j}\} \) comes from the Jacobi transformation taking value at \( \phi^{(k_i)}(x_i) \) and \( \phi^{(k_j)}(y_j) \). It is well known that the two-point joint probability \( p[\phi(x_i), \phi(y_j)] \) of a Gaussian field \( \phi \) with \(<\phi^2> = 1 \) can be expanded as

\[
p[\phi(x_i), \phi(y_j)] = \sum_{n=0}^{\infty} \frac{\xi_n(\|x_i - y_j\|)}{n!} \Phi^{(n+1)}[\phi(x_i)]\Phi^{(n+1)}[\phi(y_j)], \tag{A.5}
\]

where \( \xi_n(\|x_i - y_j\|) \) is the two-point correlation function of the Gaussian field \( \phi(x) \), and

\[
\Phi^{(n+1)}[\phi(x_i)] = \frac{1}{2\pi} \exp[-\phi^2(x_i)/2]H_n[\phi(x_i)], \tag{A.6}
\]

where \( H_n \) is the Chebyshev-Hermite polynomials *.

Then (A.4) can be written as

\[
p[\delta(x_i), \delta(y_j)] = \sum_{n=0}^{\infty} \xi_n(\|x_i - y_j\|)A_n[\delta(x_i), \delta(y_j)], \tag{A.7}
\]

where

\[
A_n[\delta(x_i), \delta(y_j)] = \frac{1}{n!} \sum_{k_i=1}^{m_i} \sum_{k_j=1}^{m_j} \left\{ \left( \frac{1}{|d\delta[\phi(x_i)]/d[\phi(x_i)]|_{k_i}} \right) \Phi^{(n+1)}[\phi^{(k_i)}(x_i)] + \left( \frac{1}{|d\delta[\phi(y_j)]/d[\phi(y_j)]|_{k_j}} \right) \Phi^{(n+1)}[\phi^{(k_j)}(y_j)] \right\}. \tag{A.8}
\]

The reduced two-point probability is

\[
\tilde{p}[\delta(x_i), \delta(y_j)] = \sum_{n=1}^{\infty} \xi_n(\|x_i - y_j\|)A_n[\delta(x_i), \delta(y_j)]. \tag{A.9}
\]

For a specific non-Gaussian field, the index \( n \) of the first nonvanishing \( A_n[\delta(x_i), \delta(y_j)] \) can be greater than 1. We can write (A.9) as

\[
\tilde{p}[\delta(x_i), \delta(y_j)] = \sum_{n=1}^{\infty} \xi_n(\|x_i - y_j\|)A_n[\delta(x_i), \delta(y_j)], \tag{A.10}
\]

* See the reference 16, Page 55
where \( n_0 \) is the index such that all \( A_n (n < n_0) \) vanish. The dependence of \( \tilde{p}[\delta(x), \delta(y)] \) on \( |x - y| \) is thus shown explicitly in (A.10).

Based on the above considerations, we make several assumptions:

(i). \( \tilde{p}(\delta_i, \delta_j, |x - y|) \) depends on a function \( F_{ij}(|x - y|) \) and can be written as

\[
\tilde{p}(\delta_i, \delta_j, |x - y|) = \sum_{n=n_0}^{\infty} A_n(\delta_i, \delta_j) F_n(|x - y|),
\]

(A.11)

where \( A_n(\delta_i, \delta_j) \) is independent of \( F_{ij}(|x - y|) \). When \( F_{ij}(|x - y|) = 0 \), (A.11) implies that \( \delta_i \) and \( \delta_j \) are independent of each other and \( \tilde{p}(\delta_i, \delta_j, |x - y|) = 0 \). For a non-Gaussian field which is a local functional of a Gaussian field \( \phi \),

\( F_{ij}(|x - y|) = \xi_\phi(|x - y|) \) as shown above. The random variables can have value ranges extending to infinity, so it is necessary to keep all the terms in the expansion (A.11).

(ii). When \( |x - y| \) is large, \( F_{ij}(|x - y|) \ll 1 \).

By (A.11), \( P_{ij} \) in (A.2) can be written as

\[
P_{ij} = \sum_{n=n_0}^{\infty} F_n(|x - y|) C_{n,ij}
\]

(A.12)

where

\[
C_{n,ij} = \left\{ \int_{x_1}^{y_1} d\delta(x_1) \int_{y_1}^{x_1} d\delta(y_1) A_n(\delta_1, \delta_j) \right. \\
\left. \prod_{p=1}^{j} \delta(x_{p-1}) \delta(x_p) \int_{x_{q-1}}^{y_{q-1}} d\delta(x_q) \int_{y_q}^{x_q} d\delta(y_q) \\
p(\delta(x_1), ..., \delta(x_{i-1}), \delta(x_{i+1}), ..., |\delta(x_i)) \\
\prod_{q=1}^{j} \delta(y_q), ..., \delta(y_{j-1}), \delta(y_{j+1}), ..., |\delta(y_j)) \right\},
\]

(A.13)
is a function of \((\bar{x}_1, \ldots, \bar{x}_i, \ldots, a_1, b_1, \ldots, a_i, b_i, \ldots)\) and \((\bar{y}_1, \ldots, \bar{y}_i, \ldots, c_1, e_1, \ldots, c_i, e_i, \ldots)\). Since \(C_{n_0}\) is finite, by the assumption (ii), we only consider \(F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)\) and ignore all the other higher order of \(F_{ij}(|\bar{x}_i - \bar{y}_j|)\). Then

\[
P(AB) - P(A)P(B) \sim \sum_{i,j} F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)C_{n_{n_{ij}}} \tag{A.14}
\]

For a specific random field, \(F_{ij}\) may have a certain value range. It is clear that \(F_{ij} = 0\) must be an allowable value.

We also have the following relation:

\[
|P(AB) - P(A)P(B)| \leq \sum_{i,j} |F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)C_{n_{n_{ij}}}| \leq \max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} |F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)| \sum_{i,j} |C_{n_{n_{ij}}}|, \tag{A.15}
\]

where the maximum is taken over all possible \(\bar{x}_i \in \Delta_1\) and \(\bar{y}_j \in \Delta_2\), and

\[
\max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} |P(AB) - P(A)P(B)| \sim \max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} |F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)| \sum_{i,j} |C_{n_{n_{ij}}}| \leq \max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} |F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)| \max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} \sum_{i,j} |C_{n_{n_{ij}}}|, \tag{A.16}
\]

where the maximum on the left hand is taken over all possible integration ranges.

On the right hand of (A.16), the first maximum is taken over all possible \(\bar{x}_i \in \Delta_1\) and \(\bar{y}_j \in \Delta_2\), and the second maximum is in the same sense as the maximum on the left hand.

We know that \(|P(AB) - P(A)P(B)| \leq 2\) regardless of both the specific values of \(F_{ij}\) (within the value range) and the specific integration ranges for the random variables. Then the sum \(\sum_{i,j} |C_{n_{n_{ij}}}|\) should converge. Concerning the approximate behavior when the separation between \(\Delta_1\) and \(\Delta_2\) is sufficiently large, we have

\[
\max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} |P(AB) - P(A)P(B)| \sim C_{n_0} \max_{\bar{x}_i \in \Delta_1, \bar{y}_j \in \Delta_2} |F_{ij}^{n_0}(|\bar{x}_i - \bar{y}_j|)|, \tag{A.17}
\]
where $C_0$ is a finite constant. By the definition of the mixing rate $\alpha(r_0)$ in (2.9), we have

$$\alpha(r_0) \sim C'_0 \max_{|x_i - y_j| \geq r_0} |F_{x_j}^n(|x_i - y_j|)|,$$

(A.18)

where $C'_0$ is a finite constant and the maximum is taken over all pairs of $(x_i, y_j)$ with $|x_i - y_j| \geq r_0$. 