Quantum Smearing of Spacetime Singularity

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Abstract

We propose a quantum mechanical generalization of affine length of a curve which gives a criterion whether a classical singularity is smeared out by quantum fluctuation of spacetime geometry. An explicit example is demonstrated in the $(2+1)$ -dimensional black hole model.

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Introduction

There is a general belief that quantum gravity effects will somehow resolve singularities, existence of which have been proven in classical general relativity.

Quantum cosmology is partly motivated by the intuition that singularity can be avoided by considering quantum effects of spacetime. The singularity is black hole singularity from quantum mechanical point of view because of cosmic censorship conjecture.² In the case of naked singularities, which appear in some models of gravitational collapse in which many people are skeptical about its true existence in the real world, violent quantum fluctuation of metrics will eventually wipe out the singularity. Or at least, one cannot say for sure that naked singularities exist, if quantum effects of spacetime at Planck scale are taken into account.

A spacetime singularity is a completely different thing from singularities which appear in classical electromagnetism or field theory in which they are normally expressible by distributions. In general relativity, singularity, because of the nonlinearity of gravitational force, is characterized by the incompleteness of geodesics. More precisely, if a causal geodesic curve is incomplete with a finite affine length, in a spacetime which has no further extension, we say that the spacetime is singular. The famous singularity theorem by Hawking and Penrose states that such a singularity always exists if (a) energy condition, (b) causality and (c) causality hold, and if (d) a trapped surface exists.

The main address in this paper is to propose a criterion for a singularity in quantum mechanically smeared away. We shall demonstrate some examples.

² Penrose and Hawking singularity theorems study the vacuum expectation value of the metric tensor to determine the global structure of spacetime. This approach seems to be limited in the regime of quantum gravity.

Figure 1: The Penrose diagram for the Schwarzschild black hole showing the singularity curve to the singularity

2 Generalized Affine Parameter

As briefly explained in the introduction, spacetime singularities are characterized by a geodesic of a finite affine length which ends nowhere in spacetime. Roughly, a free-falling observer will finish his history after a finite proper time. On the other hand, if the affine length is infinite the would-be singularity is infinitely far away so that it is not a singularity at all. In two senses, this definition of singularity is unsatisfactory. One thing, the geodesic incompleteness seems too restrictive to define the spacetime singularity. For example, a noninertial observer with a rocket of a finite amount of fuel is a perfectly physical observer, but we need to tell him whether singularity exists or not. Second, the definition of geodesic is a classical one so that it is rather difficult to generalize the definition of singularities on the basis of geodesics in quantum gravity. Very fortunately, a favorable generalization of spacetime singularity is available in classical general relativity which was first proposed by Penrose and Schmidt[2]. The time-like or null geodesic curve is replaced by a causal curve so that we can make some instructions to a space traveler with a finite amount of fuel. He also ingeniously defined the so-called generalized affine parameter, which roughly measures how long it takes to reach the singularity. If it is finite, the space traveler should worry his fate. The spacetime which is assumed to be non-extendible. We also assume the spacetime is equipped with tetrads and connections. The generalized affine parameter associated with a causal curve $\alpha : t \in [0, 1) \rightarrow M$ is

$$|\alpha| = \int_0^1 dt \sqrt{\sum_{a=1}^4 V^a(t) V^a(t)}. \quad (1)$$

a component of the tangent vector to the causal curve at α with respect to the parallelly transported tetrad from an initial tetrad at the initial point $\alpha(0)$. We can say that if the parameter is finite for a causal curve which goes to the singularity, the causal curve is *b-incomplete* (b for bundle). One of the definitions of the generalized affine parameter is the positive quantity inside the square root. The generalized affine parameter is the same as the ordinary affine parameter when the curve is a geodesic. The b-completeness implies the geodesic completeness. The generalized affine length $|\alpha|$ depends on the initial tetrad as well as the point. However, for new components of the tangent vector with respect to the parallelly transported tetrads from a fixed initial tetrad, which is obtained by a finite Lorentz transformation, it has been shown that there is some constant $C > 0$ such

$$C \sum_{a=1}^4 V^a V^a \leq \sum_{a=1}^4 V'^a V'^a \leq C^{-1} \sum_{a=1}^4 V^a V^a. \quad (2)$$

where C is finite or infinite does not depend on the choice of the point and the chosen frame there. To express the generalized affine length in a way to suit quantum gravity we use an expression for $V^a(t)$ in terms of the connection:

$$V^a(t) = V^\mu e_\mu^a [P \exp \int_0^t \omega]_b^a, \quad (3)$$

where e_μ^a are the tangent vector and the frame tetrad at the point α . ω is the spin connection and P denotes the path ordered

exponential. Here we propose a quantum mechanical generalization of the

$$|\alpha|_q = \int_0^1 dt \sqrt{\sum_{a=1}^4 \langle (V^\mu e_\mu^a [P \exp \int_0^t \omega]_b^a)^2 \rangle}, \quad (4)$$

Figure 2: A conceptual picture of an infinite generalized affine parameter near a singularity due to large quantum fluctuation

where $\langle \rangle$ means the expectation value with respect to a quantum state of the geometry. Here we stress that the curve is fixed in the coordinate system without referring to the metric, which is now a quantum variable. In this section we study a classical test particle in quantum geometry. We will return to this point in the final section. The idea behind this definition of b-(in)completeness is the following. Suppose we approach a point $\alpha(1)$ which is classically a singularity. So $|\alpha|$ is finite classically, but due to quantum mechanical fluctuation of the connection may become infinite. This implies the observer will never reach the singularity in a finite lifetime due to the violent fluctuation of the connection.

In the following sections, we will demonstrate how our definition of smearing of singularity works in a simple (2+1)-dimensional black hole

the (2+1)-dimensional model is for the illustration only. One can compute $|\alpha|_q$ in any model of quantum gravity.

BTZ Black Hole

We briefly review the (2+1)-dimensional black hole found by [1] in the simplest circular symmetric case. The model is given by a spacetime which contains the Einstein-Hilbert action and the cosmological constant, in (2+1)-dimensions the Riemann tensor can be expressed in terms of the Ricci tensor $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$. Hence the Einstein equation gives us

$$S = -\frac{1}{16\pi G} \int d^3x \sqrt{-g} [R - 2\Lambda], \quad (5)$$

where Λ is the negative cosmological constant and R is the scalar curvature. In (2+1)-dimensions the Riemann tensor can be expressed in terms of the Ricci tensor $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$. Hence the Einstein equation gives us

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \quad (6)$$

The spacetime is a constant curvature spacetime with a section being $\Lambda = -1/l^2$. The simplicity of the (2+1)-dimensional spacetime is that it contains no gravitational wave modes and no Newtonian gravity. One might wonder what is meant by quantization of black holes in (2+1)-dimensions. There are no gravitons to quantize and much more to the formation of a black hole in the physical (3+1)-dimensional spacetime. The point is, however, the existence of the global modes which carry mass and angular momentum of a black hole, which appear as discrete elements of the identification of spacetime by a discrete element of the covering space of spacetime geometry. In the present case the covering space is the anti-de Sitter spacetime and the isometry group is $SO(2, 2)$. To see this will be embedding the anti-de Sitter space in the (4+1)-dimensional flat space with $++--$ signature. Let the coordinates of the flat space be $X^A, A = 0, 1, 2, 3$. The anti-de Sitter space is defined by the condition:

$$(X^0)^2 + (X^1)^2 - (X^2)^2 - (X^3)^2 = -1/l^2. \quad (7)$$

We introduce the parameterization which is convenient to describe the inside of the black hole as

$$\begin{aligned} X^0 &= \sqrt{1 - \hat{r}^2/l^2} \sinh \hat{t}/l, & X^2 &= \sqrt{1 - \hat{r}^2/l^2} \cosh \hat{t}/l, \\ X^1 &= \hat{r}/l \cosh \hat{\theta}, & X^3 &= \hat{r}/l \sinh \hat{\theta}. \end{aligned}$$

Recall that we are actually interested in the inside of the black hole, where we are going to study the behavior of the generalized affine parameter $\tau = 0$. The induced metric becomes

$$\begin{aligned} ds^2 &= (dX^0)^2 + (dX^1)^2 - (dX^2)^2 - (dX^3)^2, \\ &= (1 - \hat{r}^2/l^2)d\hat{t}^2 - (1 - \hat{r}^2/l^2)^{-1}d\hat{r}^2 + \hat{r}^2d\hat{\theta}^2 \end{aligned}$$

From the construction it is easy to see that the full isometry group is $SO(2, 2)$. It may be more illuminating to rewrite the metric in terms of $\hat{\theta} = \sqrt{M}\theta, \hat{t} = \sqrt{M}t, \hat{r} = r/\sqrt{M}$. We obtain

$$ds^2 = (M - r^2/l^2)dt^2 - (M - r^2/l^2)^{-1}dr^2 + r^2d\theta^2$$

The global structure of the spacetime of this metric is shown in Fig. 2. The circular symmetric black hole is identified with $\theta = 0$ with $\theta = 2\pi$, with $\hat{\theta} = 2\pi\sqrt{M}$. Therefore the circular symmetric BTZ black hole spacetime can be represented as a quotient space of the anti-de Sitter spacetime by a certain amount of Lorentz boost, which is a discrete subgroup of the full isometry group $SO(2, 2)$. The mass of the black hole M is treated as a parameter of the identification, a Teichmüller parameter. The surface $r = 0$ turns out to be a space-like causal singularity as shown in Fig. 3, though the curvature is finite. So it is a rather mild singularity. In the next section we shall discuss a canonical quantization of the BTZ black hole system, in which the mass M will be treated as a dynamical variable.

4 A Quantization of the Circular Symmetric BTZ Black Hole

It is by now well-known that there are many ways to quantize (2+1)-dimensional gravity and probably this ambiguity will also go on in higher dimensions.

Penrose diagram for the circular symmetric BTZ black hole

dimensional quantum gravity. For our present purpose, it is now an example calculation in a specific scheme of quantization and it is not the issue addressed in this paper to decide which is the correct one. We start with the first order formalism in which the spin connection are treated as independent variables. Only they are related by the metricity condition. We adopt the solution to quantization[4] by making the ansatz for the triad and the spin connection forms,

$$\begin{cases} e^0 &= d\hat{r}/\sqrt{(1-\hat{r}^2/l^2)} \\ e^1 &= \alpha\sqrt{(1-\hat{r}^2/l^2)}dt \\ e^2 &= \beta\hat{r}d\phi \end{cases} \quad (11)$$

$$\begin{cases} \omega_1^0 &= -\alpha\hat{r}/l^2 dt \\ \omega_2^0 &= \sqrt{(1-\hat{r}^2/l^2)}\beta d\theta \\ \omega_2^1 &= 0 \end{cases} \quad (12)$$

the metric is given by

$$\begin{aligned} &= -(e^0)^2 + (e^1)^2 + (e^2)^2 \\ &= \alpha^2(1-\hat{r}^2/l^2)d\hat{t}^2 - (1-\hat{r}^2/l^2)^{-1}d\hat{r}^2 + \beta^2\hat{r}^2d\hat{\theta}^2, \end{aligned} \quad (13)$$

parameters α and β are related to the scales of the boost angles $\hat{t}$, constant, the triad and the spin connection satisfy the Einstein

We now regard the parameters as functions of $\hat{r}$ and carry out the radial quantization, because inside the horizon the radial coordinate plays a role of time. The curvature 2-form is calculated to be

$$\begin{aligned} R_1^0 &= -R^2 = -\frac{1}{l^2}e^0e^1 - \frac{\dot{\alpha}\hat{r}}{\alpha^2}e^0e^1 \\ R_2^0 &= R^1 = -\frac{1}{l^2}e^0e^2 + \frac{\dot{\beta}(1-\hat{r}^2/l^2)}{\beta\hat{r}}e^0e^2 \\ R_2^1 &= R^0 = -\frac{1}{l^2}e^1e^2, \end{aligned}$$

where the dot indicates the differentiation with respect to $\hat{r}$. In forms, the action can be written as

$$S = -\frac{2}{16\pi G} \int [-e_a R^a + \frac{1}{l^2}e^0e^1e^2],$$

so that the direct substitution and partial integration give

$$S = -\frac{1}{4G} \int \alpha\dot{\beta}drdt.$$

Here we have performed the integration over the angle ϕ . It is noted that $p_\beta = -\frac{1}{4G}\alpha$ is the canonical momentum conjugate to α . The Hamiltonian vanishes. Note that there are no contributions from the boundary terms. This can be confirmed from the fact that the equations derived from (16): $\dot{\alpha} = 0$ and $\dot{\beta} = 0$ reproduce the Einstein equations with the previous ansatz for the triad and the spin connection. Thus the quantization is straightforward;

$$[\alpha(t), \beta(t')] = 4Gi\hbar\delta(t-t').$$

At this stage one might be puzzled that the parameters $\alpha(t)$ and $\beta(t)$ appear in the "spatial" metric do not commute with each other because by making the ansatz for the triad and spin connection we have chosen a certain gauge, which makes a commutative Poisson bracket and a commutative Dirac bracket. This phenomenon has already been observed by Carlip in his analysis of the torus universe[4]. The parameter β is the scale of the boost angle between two spacetime points which we identified in the previous section, this scale β is directly related to the mass of the BTZ black hole. Actually, $\beta^2 = M$. The physical meaning of α is less clear. If we are allowed to continue our metric definition

tside and then Wick rotate the coordinate t , α would give
period of the Killing time t in the imaginary direction. So α is
to the Hawking temperature or the "opening angle" in the
ch of Carlip and Teitelboim [5].

um Length

e shall study the generalized affine parameter defined in §3
ory of the (2+1)-dimensional BTZ black hole analyzed in
tion. This model has a technical advantage that the met-
ne Minkowskii metric and connection becomes flat near the
ity. For comparison we start with a classical computation
d affine parameter for a causal curve which approaches the
arity.

al case

stration, consider a curve along which

$$e^1 = 0, \quad e^2 = ke^0 \quad (18)$$

constant. For the time-like curve, $|k| < 1$, while $|k| = 1$ for
We are interested in the case that the causal curve approaches
ingularity $r = 0$. It is therefore sufficient to consider a curve
that we can ignore ω_1^0 and use an approximation, $\omega_2^0 \approx \beta d\phi$.
ath ordered product $P \exp \int \omega$ is simply a Lorentz boost in
near $r = 0$. Namely,

$$P \exp \int_0^t \omega_b^a = V^\mu e_\mu^0(1, 0, k) \begin{pmatrix} \cosh \theta & 0 & \sinh \theta \\ 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \end{pmatrix} \quad (19)$$

(18), the absolute value of the coordinate θ gets large like
 β as r goes to zero as far as k is non-zero. Correspondingly
affine parameter (4) tends to

$$\begin{aligned} |\alpha| &= \int dt \sqrt{\sum_{a=1}^4 \langle (V^\mu e_\mu^b [P \exp \int \omega]_b^a)^2 \rangle} \\ &\approx \int_0^t dr (1 + |k|) \exp[|\phi|]/\sqrt{2} \\ &\approx \int_0^t dr (1 + |k|) r^{-|k|}/\sqrt{2}. \end{aligned}$$

It is now clear that for a time-like curve, $|k| < 1$ the above inte-
verges so that the path can reach the singularity $r = 0$ in a finite α .
On the other hand, for some null or space-like paths $|k| \geq 1$, the
affine parameter diverges; the singularity is infinitely far away. Ac-
cording to the classical definition of singularity, $r = 0$ is a singu-
lar point. We have at least one causal curve which is b-incomplete.

5.2 a gaussian wave packet

The previous classical analysis is suggestive in the sense that
space-like curves contribute more to the generalized affine par-
ameter than the time-like one. Intuitively, in quantum gravity, the metric
fluctuates between time-like and space-like. This fluctuation
that the curve fluctuates between time-like and space-like. This
will effectively make the generalized affine parameter longer. Let
us consider a gaussian wave packet spreading around a mass $M_0 = \beta_0^2$ with

$$\Psi(\beta) = \exp \left[-\frac{(\beta - \beta_0)^2}{2\Delta} \right].$$

The expectation value $\sum_{a=0}^2 \langle (V^\mu e_\mu^b [P \exp \int_0^t \omega]_b^a)^2 \rangle$, is read-
ily to be proportional to

$$\begin{aligned} &\int d\beta \exp[2\beta\theta] |\Psi(\beta)|^2 \\ &= \text{const.} \times \exp[2\beta_0\theta + \Delta\theta^2]. \end{aligned}$$

We obtain

$$|\alpha|_q \approx \int_0^t dr \exp \left[-|k| \log r + \frac{\Delta}{2\beta_0^2} (|k| \log r)^2 \right],$$

for the path in the previous subsection with β replaced by
 $-|k| \log r / \beta_0$. (The choice β_0 is arbitrary, because it can be ab-
sorbed into the definition of Δ). We chose this for the comparison to the previous classical case.
It is meaningless to define time-like or null curve in the theory
with a fluctuating metric, which is fluctuating. It is clear that the generalized aff-
ine parameter $|\alpha|_q$ is divergent however small the mass fluctuation Δ is. Roughly

tion brings about an infinitely long random walk in the boost original universal covering anti-de Sitter space. We also note does not depend whether the path is time-like ($|k| < 1$), null ce-like ($|k| > 1$), unless $k \neq 0$.

c eigenstate

tonian is zero in our system described in §4, the time evolu- tion function is trivial. Among many possibilities consider the e metric operator:

$$= \alpha^2(1 - \hat{r}^2/l^2)d\hat{t}^2 - (1 - \hat{r}^2/l^2)^{-1}d\hat{r}^2 + \beta^2\hat{r}^2d\hat{\theta}^2. \quad (24)$$

t are arbitrary functions of r . Here α and β are considered as tors, which satisfy the canonical commutation relation (17). ke the commutation relation make sense at equal t 's, we re- $/T$, where T is the range of the variable t . The "equal t " relation now reads

$$[\alpha, \beta] = 4G\hbar/T. \quad (25)$$

problem

$$\widehat{ds^2}\psi(\beta) = ds^2\psi(\beta) \quad (26)$$

ly the same as the harmonic oscillator problem in elementary anics. The ground state wave function is given by

$$\psi_0(\beta) = \exp\left[-\frac{T}{8G\hbar\sqrt{1 - r^2/l^2}}r\left|\frac{d\theta}{dt}\right|\beta^2\right] \quad (27)$$

ized affine parameter is computed as

$$|\alpha|_q \approx \text{const.} \int_0^0 dr \exp\left[\frac{4G\hbar}{T} \frac{\sqrt{1 - r^2/l^2}}{r} \left|\frac{dt}{d\theta}\right|\theta^2\right]. \quad (28)$$

ately recognize that the generalized affine parameter is diver- eric case $\frac{dt}{d\theta} \neq 0$ and $\theta \neq 0$ at $r = 0$. Otherwise, the curve is d t are arbitrary functions of r , which satisfy the genericness s should be so because the mass $M = \beta^2$ is fluctuating in

the metric eigenstate, conforming with the previous analysis for wave packet. We can argue that any quantum state prepared for server should be an eigenstate of the observer's proper time. If say what is the exact value of the black hole is when he plung hole, because the black hole mass and the proper time are not si diagonalizable.

6 Summary and Discussions

We have proposed a criterion for quantum fluctuation to smear singularities which appear in classical gravity. That is, the spa to be smeared out if the quantum mechanically defined gene length is infinite for a generic curve approaching the classical si explicit example is demonstrated in the (2+1)-dimensional circ ic black hole. In this model, the computation of the Wilson trivial near the black hole origin, where the spin connection be

At this stage, we do not claim that the singularities are away. Probably it depends on quantum states and also types in classical solutions as well as Lagrangian models. We can the cases that the spacetime can be extendible in quantum g it cannot classically in analogy to tunneling phenomena. This scope of the present work. Without doubt, the physically in is the curvature singularity which appears in the classical solu hole and of cosmology. However, in this case the path order non-trivial. In general evaluation of the expectation value of th object may require some non-perturbative machinery in qua possibly numerical method.

In the classical relativity, the spacetime is singular if at lea curve is b-incomplete. It seems to the present author that in qu "at least one causal curve" should be replaced by "a generic cu a single curve has a measure zero contribution to the path i we quantize the position of a particle, though in the present the particle trajectory in the coordinate space in analogy to th criterion of the phase of gauge theories. This belief is alread the concluding statement of this section. The weakness of proposed in this paper is that we have to refer to the class

of the generalized affine parameter near the singularity.
 al spacetime structure obtained by the classical analysis,
 ere is the singularity let alone its quantum smearing. At
 resent author has no idea how to proceed without classical

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