Abstract

We propose a quantum mechanical generalization of affine length.

Quantum Smearing of Spacetime Singularity

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Introduction

There is a general belief that quantum gravity effects will somehow influence singularities, existence of which have been proven in classical general relativity. Quantum cosmology is partly motivated by the intuition that quantum mechanics, which appear in some models of gravitational collapse in black hole evaporation, may be avoided by considering quantum effects of spacetime and singularities. Many people are skeptical about its true existence and whether it can exist in the real world. In contrast, quantum fluctuations of metrics will eventually wipe out singularities. Or at least, one cannot say for sure that naked naked singularities exist in the first place, if quantum effects of spacetime at Planck scale are taken into account.

The spacetime singularity is a completely different thing from naked singularities, which appear in classical electromagnetism or field theory in physics. They are not normally expressible by distributions. In general relativity as well, because of the nonlinearity of gravitational force. The singularity theorem of Hawking and Penrose is that such a singularity always exists if (a) energy condition is violated and (c) causality holds, and if (d) a trapped surface exists.

The singularity in this paper is to propose a criterion for a spacetime singularity to determine its presence. We shall demonstrate examples which study the vacuum expectation value of the metric tensor and the local structure of spacetime. The second approach seems to be limited to this regime.

Figure 1: The Penrose diagram for the Schwarzschild black hole

2 Generalized Affine Parameter

As briefly explained in the introduction, spacetime singularities can be characterized by a geodesic of a finite affine length which exists nowhere in spacetime. Roughly, a free-falling observer will finish its history after a finite proper time. On the other hand, if the affine length is infinite, the would-be singularity is infinitely far away so that it is just a point at all. In two senses, this definition of singularity is unsatisfactory. One thing, the geodesic incompleteness seems too restrictive to detect spacetime singularities. For example, a noninertial observer in a rocket of a finite amount of fuel is a perfectly physical observer. We need to tell him whether singularity exists or not. Second, we need a definition of singularity on the basis of geodesics in quantum spacetime. Very fortunately, a favorable generalization of spacetime singularity is available in classical general relativity which was first discovered by Schmidt[2]. The time-like or null geodesic curve is replaced by a causal curve so that we can make some instructions to a space-like curve.

He also ingeniously defined the so-called affine parameter, which roughly measures how long it takes to reach a singularity. If it is finite, the space traveler should worry his fate at the spacetime which is assumed to be non-extendible. We also consider this spacetime is equipped with tetrads and connections. The general parameter associated with a causal curve : $t : t \in [0, 1] \rightarrow \mathcal{M}$ is...
\[ |\alpha| = \int_0^1 dt \left( \sum_{a=1}^{4} V^a(t) \dot{V}^a(t) \right). \]  

(1)

A component of the tangent vector to the causal curve at a point with respect to the parallelly transported tetrad from an initial tetrad at the initial point \( \alpha(0) \). We can say that if the affine parameter is finite for a causal curve which goes to the singularity, the causal curve is \( b \)-incomplete (b for bundle). One of the definitions of the generalized affine parameter is the positive quantity inside the square root. The generalized affine parameter is obtained from the ordinary affine parameter when the curve is a geodesic. The b-completeness implies the geodesic completeness.

The generalized affine length \(|\alpha|\) depends on the initial tetrad as well as the point. However, for new components of the tangent vectors with respect to the parallelly transported tetrads from a point of the initial tetrad, which is obtained by a finite Lorentz transformation, it has been shown that there is some constant \( C > 0 \) such that

\[ C \sum_{a=1}^{4} V^a \dot{V}^a \leq \sum_{a=1}^{4} V^a \dot{V}^a \leq C^{-1} \sum_{a=1}^{4} V^a \dot{V}^a. \]  

(2)

Figure 2: A conceptual picture of an infinite generalized affine parameter, large quantum fluctuation near the singularity.

where \( <> \) means the expectation value with respect to a quantum state of the geometry. Here we stress that the curve is fixed in the classical limit without referring to the metric, which is now a quantum variable when we study a classical test particle in quantum geometry. We will return to this point in the final section. The idea behind this definition of \( b \)-(in)completeness is the following. Suppose we approach a point \( \alpha(1) \) is classically a singularity. So \(|\alpha|\) is finite classically, a quantum mechanical fluctuation of the connection may be very important, the classical spacetime singularity so that the length of the curve in bundle space becomes infinite. This implies the observer will never reach the singularity in a finite lifetime due to the violent fluctuation of the connection.

In the following sections, we will demonstrate how our definition of smearing of singularity works in a simple \((2+1)\)-dimensional black hole solution.
The (2+1)-dimensional model is for the illustration only. One compute $|a|^2$ in any model of quantum gravity.

**4T Black Hole**

We briefly review the (2+1)-dimensional black hole found by the simplest circular symmetric case. The model is given by a contains the Einstein-Hilbert action and the cosmological dimensions,

$$S = -\frac{1}{16\pi G} \int d^3x \sqrt{-g} [R - 2\Lambda].$$

(5)

negative cosmological constant and $R$ is the scalar curvature. In (2+1)-dimensions the Riemann tensor can be expressed as $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R$. Hence the Einstein equation gives us

$$R_{\mu\nu} = \Lambda (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}).$$

(6)

The spacetime is a constant curvature spacetime with a scalar being $\Lambda = -1/l^2$. The simplicity of the (2+1)-dimensional black hole is that it contains no gravitational wave modes and no Newtontone one might wonder what is meant by quantization of black dimensions. There are no gravitons to quantize and much less gravitational force to cause the gravitational collapse which is to the formation of a black hole in the physical (3+1)-dimensional space-time. Nonetheless, the existence of the global modes which mass and angular momentum of a black hole, which appear as the identification of spacetime by a discrete element of the group of spacetime geometry. In the present case the covering the anti-de-Sitter spacetime and the isometry group is $SO(2,2)$. To see this will be embedding the anti-de Sitter space in the flat space with $++-+$ signature. Let the coordinates of the flat space be $X^A, A = 0, 1, 2, 3$. The anti-de Sitter space is a condition:

$$(X^0)^2 + (X^1)^2 - (X^2)^2 - (X^3)^2 = -1/l^2.$$  

(7)

We introduce the parameterization which is convenient to calculate inside of the black hole as

$$X^0 = \sqrt{1 - \tau^2/l^2} \sinh \theta/l, \quad X^2 = \sqrt{1 - \tau^2/l^2} \cosh \theta/l,$$

$$X^1 = \tau/l \cosh \theta, \quad X^3 = \tau/l \sinh \theta.$$

Recall that we are actually interested in the inside of the black hole, we are going to study the behavior of the generalized affine parameter $\tau = 0$. The induced metric becomes

$$ds^2 = (d\tau^2 + (dX^1)^2 - (dX^2)^2 - (dX^3)^2)$$

$$= (1 - \tau^2/l^2) d\tau^2 - (1 - \tau^2/l^2)^{-1} d\tau^2 + d\theta^2$$

From the construction it is easy to see that the full isometry group $SO(2,2)$. It may be more illuminating to rewrite the metric in $\hat{\theta} = \sqrt{M} \theta, \hat{\tau} = \sqrt{M} \tau, \hat{r} = \tau/\sqrt{M}$. We obtain

$$ds^2 = (M - \tau^2/l^2) d\tau^2 - (M - \tau^2/l^2)^{-1} d\tau^2 + d\theta^2$$

The global structure of the spacetime of this metric is shown by a certain amount of Lorentz boost, which is a discrete sub full isometry group $SO(2,2)$. The mass of the black hole $M$ as a parameter of the identification, a Teichmüller parameter $r = 0$ turns out to be a space-like causal singularity as shown in diagram (Fig.3), though the curvature is finite. So it is a rather similar. In the next section we shall discuss a canonical quantization of BTZ black hole system, in which the mass $M$ will be treated as a variable.

**4 A Quantization of the Circular Symmetric BTZ Black Hole**

It is now well-known that there are many ways to quantize dimensional gravity and probably this ambiguity will also go...
Penrose diagram for the circular symmetric BTZ black hole in three-dimensional quantum gravity. For our present purpose, it is useful to perform a calculation in a specific scheme of quantization where the spin connection is treated as independent variables. Only the metric is related by the metricity condition. We adopt the scheme of quantization [4] by making the ansatz for the triad and the connection:

\[
\begin{align*}
e^0 &= \frac{dt}{\sqrt{1 - r^2/\ell^2}} \\
e^1 &= \alpha \frac{\dot{r}}{\sqrt{1 - r^2/\ell^2}} dt \\
e^2 &= \beta \dot{\theta} d\theta \\
\omega^0_1 &= -\alpha \dot{r}/\ell^2 dt \\
\omega^0_2 &= \sqrt{(1 - r^2/\ell^2)} \beta d\theta \\
\omega^1_3 &= 0
\end{align*}
\]  

(11)

The metric is given by

\[
\begin{align*}
\text{g}_{\mu\nu} &= -(e^0)^2 + (e^1)^2 + (e^2)^2 \\
&= \alpha^2(1 - r^2/\ell^2)d\tau^2 - (1 - r^2/\ell^2)^{-1}dr^2 + \beta^2 r^2 d\theta^2. 
\end{align*}
\]  

(13)

We now regard the parameters as functions of $\tau$ and carry out the radial quantization, because inside the horizon the radial coordinate plays a role of time. The curvature 2-form is calculated to be

\[
\begin{align*}
\Omega_1 &= -R^2 = -\frac{1}{\ell^2} e^0 e^1 \frac{dt}{\sqrt{1 - r^2/\ell^2}} \\
\Omega_2 &= R = -\frac{1}{\ell^2} e^0 e^2 + \frac{8(1 - r^2/\ell^2)^{3/2}}{\ell^4} e^0 e^2 \\
\Omega_3 &= R^0 = -\frac{1}{\ell^2} e^1 e^2,
\end{align*}
\]

where the dot indicates the differentiation with respect to $\tau$. In these forms, the action can be written as

\[
S = -\frac{2}{16\pi G} \int [-\epsilon e R^a + \frac{1}{\ell^2} e^0 e^1 e^2].
\]

so that the direct substitution and partial integration give

\[
S = -\frac{1}{4G} \int \alpha \beta d\tau dt.
\]

Here we have performed the integration over the angle $\theta$. It is clear that $\rho = -\frac{1}{4G} \alpha$ is the canonical momentum conjugate to $\alpha$ of the Hamiltonian. Note that there are no contributions from terms. This can be confirmed from the fact that the equation derived from (16): $\dot{\alpha} = 0$ and $\dot{\beta} = 0$ reproduce the Einstein equations of the previous ansatz for the triad and the spin connection. This quantization is straightforward;

\[
[\alpha(t), \beta(t')] = 4Gi\hbar \delta(t - t').
\]

At this stage one might be puzzled that the parameters $\alpha(t)$ and $\beta(t)$ appear in the "spatial" metric do not commute with each other because by making the ansatz for the triad and spin connection we have chosen a certain gauge, which makes a commutative Poisson bracket into a non-commutative Dirac bracket. This phenomenon has already been discussed by Carlip in his analysis of the torus universe [4]. The parameter $\beta$ is defined as a boost angle between two spacetime points which we identified in the previous section. The scaling $\beta$ is directly related to the black hole mass. Actually, $\beta^2 = M$. The physical meaning of the parameter $\alpha$ is less clear. If we are allowed to continue our metric definition.
outside and then Wick rotate the coordinate $t$, $\alpha$ would give period of the Killing time $t$ in the imaginary direction. So $\alpha$ is to the Hawking temperature or the "opening angle" in the work of Carlip and Teitelboim [5].

**Luminous Length**

We shall study the generalized affine parameter defined in §3 of the (2+1)-dimensional BTZ black hole analyzed in §5. This model has a technical advantage that the metric $M_{\text{minkw}}$ and connection becomes flat near the horizon. For comparison we start with a classical computation of affine parameter for a causal curve which approaches the horizon.

**Rational Case**

Consider a curve along which

$$ e^1 = 0, \quad e^2 = ke^0 $$

constant. For the time-like curve, $|k| < 1$, while $|k| = 1$ for space-like curves and $|k| > 1$ for null curves. We are interested in the case that the causal curve approaches the singularity $r = 0$. It is therefore sufficient to consider a curve that we can ignore $\omega_1^0$ and use an approximation, $\omega_0^0 \approx 3d\phi$. The order in $r$ is simply a Lorentz boost in spatial direction $r = 0$. Namely,

$$ P \exp \int_0^\phi \omega^0 \wedge e^0_{\mu} \left( \begin{array}{ccc} \cosh \theta & 0 & \sinh \theta \\ 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \end{array} \right) \right) $$

(19)

the absolute value of the coordinate $\theta$ gets large like $\beta$ as $r$ goes to zero as far as $k$ is non-zero. Correspondingly, affine parameter (4) tends to

$$ |\alpha| \approx \int_0^\phi \sqrt{\sum_{\mu=1}^3 (V^\mu e^\mu _\phi P \exp \int \omega^0)^2} >.$$

$$ \approx \int_0^\phi dr (1 + |k|) \exp[|\phi|/\sqrt{2} \approx \int_0^\phi dr (1 + |k|) r^{-1/2}/\sqrt{2}.$$

It is now clear that for a light-like curve, $|k| < 1$ the above integral converges so that the path can reach the singularity $r = 0$ in a finite $\phi$. On the other hand, for some null or space-like paths $|k| > 1$, the affine parameter diverges; the singularity is infinitely far away. According to the classical definition of singularity, $r = 0$ is a singularity, we have at least one causal curve which is b-incomplete.

5.2 a Gaussian Wave Packet

The previous classical analysis is suggestive in the sense that null curves contribute more to the generalized affine parameter than the time-like one. Intuitively, in quantum gravity, the metric is not the curve fluctuates between time-like and space-like. This will effectively make the generalized affine parameter longer. Let a gaussian wave packet spreading around a mass $M_0 = \beta_0^2$ with

$$ \Psi(\beta) = \exp \left\{ -\frac{(\beta - \beta_0^2)^2}{2\Delta} \right\}.$$

The expectation value $\sum_{\mu=0}^2 (V^\mu e^\mu _\phi P \exp \int_0^\phi \omega^0)^2$, is readily to be proportional to

$$ \int d\beta \exp \{2\beta \theta \} |\psi(\beta)|^2 = \text{const.} \times \exp \{2\beta_0 \theta + \Delta \theta^2\}.$$

We obtain

$$ |\alpha| \approx \int_0^\phi dr \exp \left\{ -|k| \log r + \frac{\Delta}{2\beta_0^2} (|k| \log r)^2 \right\},$$

for the path in the previous subsection with $\beta$ replaced by $-|k| \log r / \beta_0$. (The choice $\beta_0$ is arbitrary, because it can be absorbed by $A$). We chose this for the comparison to the previous classical case. It is meaningless to define time-like or null curve in the geometric, which is fluctuating. It is clear that the generalized affine parameter $|\alpha|_a$ is divergent however small the mass fluctuation $\Delta$ is. Rough...
The condition brings about an infinitely long random walk in the boost or original universal covering anti-de Sitter space. We also note that does not depend whether the path is time-like (\( |k| < 1 \)), null-like (\( |k| > 1 \)), unless \( k \neq 0 \).

**C. eigenstate**

Hilbertian is zero in our system described in §4, the time evolution function is trivial. Among many possibilities consider the metric operator:

\[
\rho^2 = \alpha^2 (1 - \beta^2 / l^2) dt^2 - (1 - \beta^2 / l^2)^{-1} d\tau^2 + \beta^2 r^2 d\theta^2
\]  \( \text{(24)} \)

\( \alpha, \beta \) are arbitrary functions of \( t \). Here \( \alpha \) and \( \beta \) are considered as constants, which satisfy the canonical commutation relation (17). The commutation relation make sense at equal \( t \)'s, we rely on it. We define \( \mathcal{A} \) to be the range of the variable \( t \). The "equal \( t \)" relation now reads

\[
[\alpha, \beta] = 4G\hbar / \mathcal{A}.
\]  \( \text{(25)} \)

The problem

\[
\tilde{d}^2 \psi(\beta) = d^2 \psi(\beta)
\]  \( \text{(26)} \)

is the same as the harmonic oscillator problem in elementary statistics. The ground state wave function is given by

\[
\psi_0(\beta) = \exp \left[ - \frac{T}{8G\hbar \sqrt{1 - \beta^2 / l^2}} \int \frac{d\theta}{r} |\beta|^2 \right]
\]  \( \text{(27)} \)

The generalized affine parameter is computed as

\[
\alpha_0 = \text{const.} \int^0_0 dr \exp \left[ - \frac{4G\hbar}{T} \sqrt{1 - r^2 / l^2} \int \frac{d\theta}{r} |\beta|^2 \right].
\]  \( \text{(28)} \)

Unfortunately, recognize that the generalized affine parameter is divergent case \( \frac{d\theta}{dr} \neq 0 \) and \( \theta \neq 0 \) at \( r = 0 \). Otherwise, the curve is determined \( \alpha \) and \( t \) are arbitrary functions of \( r \), which satisfy the genericness of \( \beta \) to be so because the mass \( M = \beta^2 \) is fluctuating in

the metric eigenstate, conforming with the previous analysis for wave packet. We can argue that any quantum state prepared for the observer should be an eigenstate of the observer's proper time. If we say what is the exact value of the black hole mass when it plunges into the hole, because the black hole mass and the proper time are not simultaneous.

**6 Summary and Discussions**

We have proposed a criterion for quantum fluctuation to smear out singularities which appear in classical gravity. That is, the space-time is extendible if the quantum mechanically defined geodesic length is infinite for a generic curve approaching the classical singularity. An explicit example is demonstrated in the (2+1)-dimensional circular black hole. In this model, the computation of the Wilson line is trivial near the black hole origin, where the spin connection becomes trivial.

At this stage, we do not claim that the singularities are smeared out. Probably it depends on quantum states and also types of solutions in classical solutions as well as Lagrangian models. We can say the cases that the spacetime can be extendible in quantum gravity, it cannot classically in analogy to tunneling phenomena. This scope of the present work. Without doubt, the physically interesting is the curvature singularity which appears in the classical solution of black hole and of cosmology. However, in this case the path order is non-trivial. In general evaluation of the expectation value of the Wilson line may require some non-perturbative machinery in quantum gravity.

In the classical relativity, the spacetime is singular if at least one curve is b-incomplete. It seems to the present author that in quantum gravity, "at least one causal curve" should be replaced by "a generic curve" to a single curve has a measure zero contribution to the path integral. We quantize the position of a particle, though in the present case the particle trajectory in the coordinate space in analogy to the criterion of the phase of gauge theories. This belief is already in the concluding statement of this section. The weakness of the proposed in this paper is that we have to refer to the class...
aor of the generalized affine parameter near the singularity.

The actual spacetime structure obtained by the classical analysis, where is the singularity let alone its quantum smearing. At the present author has no idea how to proceed without classical

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