Construction of Free Energy of Calabi-Yau manifold embedded in $CP^{N-1}$ via Torus Actions

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Abstract

We calculate correlation functions of topological sigma model (A-model) on C.Y.hypersurfaces in $CP^{N-1}$ using torus action method. We also obtain path-integral representation of free energy of the theory coupled to gravity.

1 Introduction

Recently vast development occurs in the field of topological sigma model, especially from the mirror symmetry of the models having Calabi-Yau manifolds as target spaces [7]. In [3], Nagura and the author treated the topological sigma model on the Calabi-Yau hypersurfaces $M_N$ in $CP^{N-1}$. We derived $N-2$ point correlation function by the analysis of the solution of period integral equation which emerges from the deformation of the complex structure of $M_N$, the mirror counterpart of $M_N$(B-model). And we showed that translation of the calculation of B-model into the A-model by mirror map actually gives the correlation function of topological sigma model on $M_N$, i.e., the number of holomorphic maps satisfying the conditions imposed by external operator insertion. But the explicit evaluation was limited only for the degree 1 case. So there remains a problem of the calculation of the correlation functions from the A-model point of view for higher degree cases. Of course, search for the deeper understanding of mirror symmetry is very important, but we limit our interest to the above problem in this paper.

As we saw in [3], correlation functions of the A-model have geometrically very clear meaning. And their direct calculation relies on the explicit construction of moduli spaces for holomorphic maps from $CP^1$ to target space (in our case $M_N$). For degree 1, this can be done by taking the zero locus of a section of $Sym^N(U^*)$ of
Gr(2,N) (U is the universal bundle of Grassmannian). S.Katz constructed degree 2 moduli space as the zero locus of a section of $Sym^{N}(U^*)/Sym^{N-2}(U^*) \otimes \mathcal{O}_{P}(1)$ on $Gr(3,N)$. ($\mathcal{O}_{P}(1)$ is the taughntological sheaf on $P(Sym^{2}(U))$). These construction seems to become more complicated as the degree rises. But in [1], Kontsevich constructed the compact moduli space from $C P^{1}$ to complex manifold $V$ by introducing stable maps. Roughly speaking, these are maps from branched $C P^{1}$ (stable curves) to $V$. In the case where $V$ is $C P^{N}$ or hypersurfaces of $C P^{N}$, he also did some calculations of correlation functions (Gromov-Witten invariants) using this construction and by means of fixed point theorem under the torus action flow of $C P^{N}$. We find that his treatment of quintic hypersurface in $C P^{4}$ is very much like our elementary approach for $M_{N}$. So we thought we can calculate Gromov-Witten invariants for $M_{N}$ by this method.

We have to notice one difference. In [3], we treated the matter theory, but in Kontsevich’s formulation correlation functions are those for the theory coupled with gravity because of introduction of stable maps. After all, we found this method also works for $M_{N}$. Using the fact that 3-point functions are identical for both theories and that fusion rule holds in the matter theory (See Greene, Morrison, Plesser [4]), we reconstruct $N - 2$ point functions for matter theory and derived some identity.

This method has a by-product. It has the structure of the sum of tree graph amplitudes, so by using Feynmann rules, we can construct path-integral representation of the generating function of correlation functions (Free Energy). We will represent it at the end of this paper.

In section 2, we introduce the topological sigma model (A-model) and construct correlation functions as integrals of forms on moduli spaces. In section 3, we review the torus action method and in section 4, we do some explicit calculation of amplitudes and see these results are compatible with those of Greene, Morrison, Plesser [4], and S.Katz [9]. In section 5 we construct path-integral representation of free energy for $M_{N}$ (coupled with gravity).

This paper’s treatment is rather non-rigolous in comparison with the original one of Kontsevich, but we think it is more accessible and may arise some insight for generalization.

2 Correlation Function as an Integral of Forms on Moduli Spaces

2.1 Topological Sigma Model (A-Model)

2.1.1 lagrangian and weak coupling limit

Topological sigma model can be obtained by twisting $N = 2$ Supersymmetric Sigma Model on M. A-model corresponds to A-twist, which turns $\psi_{+}^{i}$ and $\psi_{-}^{i}$ in $N = 2$ Sigma Model into $\chi^{i}$, and $\psi_{+}^{i}$, $\psi_{-}^{i}$ into $\psi_{z}^{i}$, $\psi_{\bar{z}}^{i}$. In other words, A-twist means...
subtraction of half of $U(1)$ charge from world sheet spin quantum number. Then we obtain the following Lagrangian for the A-model,

$$L = 2t \int \sum d^2z \left( \frac{1}{2} g_{ij} \partial_z \phi^i \partial_z \phi^j + i \psi_i \bar{\partial} \chi^i + i \psi_i \bar{\partial} \chi^i - R_{ij} \psi_j \bar{\psi}_i \chi^j \bar{\chi}^i \right) \tag{2.1}$$

(2.1) is invariant under the BRST – transformation,

$$\delta \phi^i = i \alpha \chi^i$$
$$\delta \bar{\phi}^i = i \alpha \bar{\chi}^i$$
$$\delta \chi^i = \delta \bar{\chi}^i = 0$$
$$\delta \psi_i^2 = -\alpha \partial_z \phi^i - i \alpha \bar{\chi}^i \bar{\psi}_i \bar{\psi}_i$$
$$\delta \bar{\psi}_i = -\alpha \partial_z \phi^i - i \alpha \chi^i \psi_i$$ \tag{2.2}

This invariance allows us to consider only BRST-invariant observables. As in section 2, we define BRST operator $Q$ by $\delta V = -i \alpha \{Q, V\}$ for any field $V$. Of course, $Q^2 = 0$. We can rewrite the lagrangian (2.1) modulo the $\psi$ equation of motion as,

$$L = it \int \sum d^2z \{Q, V\} + t \int \sum \Phi^*(e) \tag{2.3}$$

where

$$V = g_{ij} (\psi^i \bar{\partial} \phi^i + \partial_z \phi^i \bar{\psi}_i) \tag{2.4}$$

and

$$\int \sum \Phi^*(e) = \int \sum (\partial_z \phi^i \bar{\partial} \phi^i g_{ij} - \partial_z \phi^i \bar{\partial} \phi^i g_{ij}) \tag{2.5}$$

(2.5) is the integral of the pullback of the Kähler form $e$ of $M$, and it depends only on the intersection number between $\Phi_4 (\Sigma)$ and $PD(e)$ ($PD(e)$ denotes the Poincare Dual of $e$), which equals the degree of $\Phi$. By an appropriate normalization of $g_{ij},$ we have

$$\int \sum \Phi^*(e) = n \tag{2.6}$$

where $n$ is the degree of $\Phi$. Next, we consider the correlation function of BRST-invariant observables $\{O_i\}$, i.e.

$$\langle \prod_{i=1}^k O_i \rangle = \int \sum D\phi D\psi D\chi e^{-L} \prod_{i=1}^k O_i \tag{2.7}$$

We have seen $\int \sum \Phi^*(e) = n$ and we decompose the space of maps $\phi$ into different topological sectors $\{B_n\}$ in each of which $deg(\Phi)$ is a fixed integer.

We can rewrite (2.7) as follows.

$$\langle \prod_{i=1}^k O_i \rangle = \int \sum D\phi D\psi D\chi e^{-L} \prod_{i=1}^k O_i = \sum_{n=0}^{\infty} e^{-nt} \int \sum D\phi D\psi D\chi e^{-it} \int \sum d^2z \{Q, V\} \prod_{i=1}^k O_i \tag{2.8}$$
And we set
\[ \langle \prod_{i=1}^{k} O_i \rangle_n = \int_{B_n} D\phi D\psi D\chi e^{-it \int_{\Sigma} d^2z \{ Q, V \}} \prod_{i=1}^{k} O_i \]  
(2.9)

We can easily see that \( \int_{\Sigma} d^2z \{ Q, V \} = \{ Q, \int_{\Sigma} d^2z V \} \), i.e. lagrangian is BRST exact. It follows from this and \( \{ Q, O_i \} = 0 \) that \( \langle \prod_{i=1}^{k} O_i \rangle_n \) doesn’t depend on the coupling constant \( t \) and we can take weak coupling limit \( t \to \infty \) in evaluating the path integral.

In this limit, the saddle point approximation of the path integral becomes exact. Saddle points of the lagrangian are given by
\[ \partial_z \phi^i = 0, \quad \partial_z \phi^7 = 0 \]  
(2.10)

(2.10) shows that the path-integral is reduced to an integral over the moduli space of holomorphic maps from \( \Sigma \) to \( M \) of degree \( n \). We will focus our attention to the case where \( \Sigma \) has genus \( 0 \). We denote this space as \( \mathcal{M}_{0, n}^M \).

### 2.1.2 The Ghost Number anomaly and BRST observables

In the previous subsection, we have seen that the path integral (2.7) is reduced to an integral over \( \mathcal{M}_{0, n}^M \) weighted by one loop determinants of the non zero modes. But in general, there are fermion zero modes which are given as the solution of
\[ D_z \chi^i = D_z \chi^7 = 0 \quad \text{and} \quad D_z \psi_2^i = D_z \psi_2^7 = 0. \]

Let \( a_n \) (resp. \( b_n \)) be the number of \( \chi \) (resp. \( \psi \)) zero modes. If \( M \) is a Calabi-Yau manifold, we can see from Riemann-Roch Theorem,
\[ w_n = a_n - b_n = \dim(M) \]  
(2.11)

The existence of fermion zero mode is understood as Ghost number anomaly, because lagrangian (2.1) classically conserves the ghost number. In path integration, these zero modes appear only in the integration measure except in \( \prod_{i=1}^{k} O_i \), and the correlation function \( \langle \prod_{i=1}^{k} O_i \rangle_n \) vanishes unless the sum of the ghost number of \( O_i \) is equal to \( w_n \).

\[ D_z \chi^i = 0 \ (\text{resp.} \ D_z \chi^7 = 0) \]  
can be considered as the linearization of the equation \( \partial_z \phi^i = 0 \ (\text{resp.} \ \partial_z \phi^7 = 0) \) and we can regard \( \chi \) zero mode as \( T \mathcal{M}_{0, n}^M \). \( w_n \) is usually called “virtual dimension” of \( \mathcal{M}_{0, n}^M \). In generic case \( b_n = 0 \) and \( \dim(\mathcal{M}_{0, n}^M) = w_n \) holds. Then we have \( \dim(\mathcal{M}_{0, n}^M) = a_n \). BRST cohomology classes of the A-model are constructed from the de Rham cohomology classes \( H^*(M) \) of the manifold \( M \). Let \( W = \omega_{1, \ldots, n} (\phi) d\phi^1 \wedge \cdots \wedge d\phi^i \) be an \( n \) form on \( M \). Then we define a corresponding local operator of the A-Model,
\[ O_W(P) = \omega_{1, \ldots, n} \chi^1 \wedge \cdots \chi^n(P) \]  
(2.12)

From (2.2) we have
\[ \{ Q, O_W \} = -O_{\delta W} \]  
(2.13)

which shows that if \( W \in H^*(M) \), \( O_W(P) \) is BRST-closed.
2.1.3 Evaluation of the Path Integral

Now we discuss how we can evaluate $\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle$. We take $\mathcal{O}$ to be $\mathcal{O}_{W_i}$ which is induced from $W_i \in H^*(M)$. By adding appropriate exact forms we can make $W_i$ into the differential form which has delta function support on $PD(W_i)$. Then $\mathcal{O}_{W_i}(P_i)$ is non zero only if

$$\phi(P_i) \in PD(W_i)$$

(2.14)

Then integration over $\mathcal{M}^M_{0,n}$ is restricted to $\mathcal{M}^M_{0,n}$, which consists of $\phi \in \mathcal{M}^M_{0,n}$ satisfying (2.14). In evaluating $\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle$, (2.14) imposes $\sum_{i=1}^{k} \dim(W_i)$ conditions, so $\dim(\mathcal{M}^M_{0,n}) = \dim(\mathcal{M}^M_{0,n}) - \sum_{i=1}^{k} \dim(W_i) = w_n + b_n - \sum_{i=1}^{k} \dim(W_i)$. But from the fact that ghost number of $\mathcal{O}_{W_i}$ equals $\dim(W_i)$ (contribution from $\chi$) and anomaly cancellation condition, we have $\dim(\mathcal{M}^M_{0,n}) = b_n$. In generic case where $b_n = 0$, $\mathcal{M}^M_{0,n}$ turns into finite set of points. Then we perform an one loop integral over each of these points. The result is a ratio of boson and fermion determinants, which cancel each other. Then contributions to $\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle$ in the generic case equals the number of instantons which satisfies (2.14), i.e.

$$\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle_{\text{generic}} = i\mathcal{M}^M_{0,n}$$

(2.15)

When $\dim(\mathcal{M}^M_{0,n}) = b_n \geq 1$, there are $b_n$ zero modes which we can regard as the fiber of the vector bundle $\nu$ on $M^M_{0,n}$. In this case, contributions to $\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle$ are known as the integration of Euler class $\chi(\nu)$ on $M^M_{0,n}$. If we consider $\nu$ as a $0$-dimensional vector bundle on a point in the generic case, we can apply the same logic there. We denote each component of $\mathcal{M}^M_{0,n}$ as $M^M_{0,n,m}$ and obtain

$$\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle_n = \sum_{m} \int_{M^M_{0,n,m}} \chi(\nu)$$

(2.16)

Hence from (2.8)

$$\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle = \sum_{n=0}^{\infty} \sum_{m=1}^{m_n} e^{-nt} \int_{M^M_{0,n,m}} \chi(\nu)$$

(2.17)

In algebraic geometry, generic instantons of degree $n$ corresponds to irreducible maps of degree $n$, and instantons of degree $n$ with non-zero zero mode to reducible maps which are $j$-th multiple cover of irreducible maps of degree $n/j$ ($j|n$). Let $M^M_{0,n,j,m}$ be the $m$-th connected component of moduli spaces which are $j$-th multiple cover of $n/j$-th irreducible instantons, and $\nu_{j,m}$ be vector bundle of $\psi$ zero modes on $M^M_{0,n,j,m}$. Then we have from (2.17),

$$\langle \prod_{i=1}^{k} \mathcal{O}_{W_i} \rangle = \sum_{n=0}^{\infty} \sum_{j|n} \sum_{m=1}^{m_n} e^{-nt} \int_{M^M_{0,n,j,m}} \chi(\nu_{j,m})$$

(2.18)
2.2 Reduction to an integral of Forms on Moduli Spaces

In this paper, we treat the topological sigma model (A-Model) of Calabi-Yau manifold embedded in $CP^{N-1}$. This manifold is realized as the zero-locus of section of $-K$ ($K$ is canonical line bundle of $CP^{N-1}$). Since $-K$ is equivalent to $NH$ as a line bundle ($H$ is hyperplane bundle of $CP^{N-1}$), we can take homogeneous polynomial of degree $N$ as the defining equation of $M_N$. For example,

$$M_N := \{(X_1, X_2, \cdots, X_N) / C^x \in CP^{N-1} | X_1^N + X_2^N + \cdots + X_N^N = 0\}$$ (2.19)

Observables of this model can be constructed from elements of $w \in H^w(M_N)$ which we denote as $O_w$, and in the following discussion we consider the observables which are included from the subring of $H^w(M_N, C)$ generated by Kähler form $\epsilon$ of $M_N$ (we denote it as $H_{CP^{N-1}}(M_N, C)$). One of the reason why we take this subring is that we can obtain it directly from $H^w(CP^{N-1}, C)$ and Poincare dual of its elements are analytic submanifold of $M_N$. More explicitly, elements of $H_{CP^{N-1}}(M_N, C)$ are given as $\epsilon^k (k = 1, 2, \cdots, N - 2)$ and Poincare dual of $\epsilon^k$ is the intersection of the zero locus of the section of $HP(CP^{N-1}, O(k \cdot H))$ and $M_N$. So in the following discussion we treat the observables $^\ast$

$$O_1, O_{c_1}, O_{c_2}, \cdots, O_{c_{N-2}}$$ (2.20)

Then the fact that Lagrangian of the topological sigma model is BRST-exact allows us to take the strong coupling limit and correlation functions of this model reduces to the integral of closed forms corresponding to the BRST closed observables on moduli spaces of holomorphic maps $f$ from Riemann surface $\Sigma_g$ to target space $M_N$ (we focus our attention to the case of $g = 0, i.e. CP^1$). When the target space is a hypersurface of simple projective space $CP^{N-1}$, we can classify moduli spaces by the degree $n = \{f(CP^1) \cap PD(\epsilon)\}$ and we denote the moduli space of degree $n$ as $M_{0,n}$. Dimension of $M_{0,n}^{M_N}$ which counts the number of $\chi$-zero modes is evaluated by the Riemann-Roch Theorem as follows.

$$\dim(M_{0,n}^{M_N}) := \dim(H^0(CP^1, f^*(T'M_N)))$$

$$= \dim(M_N) + \deg f \cdot c_1(KM_N) + \dim(H^1(CP^1, f^*(T'M_N)))$$

$$= \dim(M_N) + \dim(H^1(CP^1, f^*(T'M_N)))$$

$$= N - 2 + \dim(H^1(CP^1, f^*(T'M_N)))$$ (2.21)

where we used the Calabi-Yau condition $c_1(KM_N) = 0$. (2.21) tells us that the dimension of moduli space is independent of degree $n$.

First, we consider the generic case where $\dim(H^1(CP^1, f^*(T'M_N))) = 0$. From the above argument, we can heuristically represent correlation functions,

$$\langle O_{\nu_1}(z_1) O_{\nu_2}(z_2) \cdots O_{\nu_k}(z_k) \rangle_{n, \text{generic}}$$

*When coupled to gravity $O_1$ correspond to puncture operator $P$ but in the small phase, space $P$ insertion is suppressed except for constant map sector because of puncture equation and as we know from the later discussion of topological selection rule, ghost number of inserted operator must be less than $N - 3$. So it suffices to consider only $N - 4$ elements $O_1, O_{c_2}, \cdots, O_{c_{N-4}}$.}
\[
= \int_{\mathcal{M}_{0,n}^{M_N}} \alpha(O_{e_i^1}) \wedge \alpha(O_{e_i^2}) \wedge \cdots \wedge \alpha(O_{e_i^k})
\]

(2.22)

where \( \alpha(O_{e_i}) \) is the closed form on \( \mathcal{M}_{0,n}^{M_N} \) induced from \( O_{e_i} \). Since the form degree of \( \alpha(O_{e_i}) \) equals the ghost number of \( O_{e_i} (= \dim(e^i) = j) \), correlation functions are nonzero only if the following conditions are satisfied.

\[
\text{dim}(\mathcal{M}_{0,n}^{M_N}) = \sum_{i=1}^{k} j_i \\
\iff N - 2 = \sum_{i=1}^{k} j_i \quad (2.23)
\]

If we take \( e^i \) as the forms which has the delta function support on \( PD(e^i) \), then from \( (\ ) \), \( \alpha(O_{e_i}) \) can be interpreted as the constraint condition on \( f \),

\[
\alpha(O_{e_i}) \leftrightarrow f(z_i) \in PD(e^i) \quad (2.24)
\]

(2.24) imposes \( (\dim(e^i) - 1) + 1 = j \) independent conditions on \( \mathcal{M}_{0,n}^{M_N} \) \( ((\dim(e^i) - 1) \) corresponds to the degree of freedom which makes \( f(CP^1) \cap PD(e^i) \neq \emptyset \) and 1 to the one which sends \( f(z_i) \) into \( PD(e^i) \). And from (2.23), what remains is the discrete point set of holomorphic maps \( f \) which satisfy (2.24) for all \( i \). Then we have

\[
\langle O_{e_i^1}(z_1)O_{e_i^2}(z_2)\cdots O_{e_i^k}(z_k) \rangle_{n,\text{generic}} = \{ f : CP^1 \xrightarrow{hol} M_N \lvert f(z_i) \in PD(e^i) \}
\]

(2.25)

Now, let us consider non-generic case. In this case, \( \dim(H^1(CP^1, f^*(T'M_N))) > 0 \) and moduli space have additional \( \dim(H^1(CP^1, f^*(T'M_N))) \) degrees of freedom.

We can see these degrees of freedom correspond to multiple cover maps by the following argument. A multiple cover map \( f \) can be decomposed into the form \( f = \tilde{f} \circ \varphi \) where \( \tilde{f} \) is irreducible map from \( CP^1 \) to \( M_N \) and \( \varphi \) represents the map from \( CP^1 \) to \( CP^1 \) of degree \( d \geq 2 \). Then let us count \( \dim(\mathcal{M}_{0,n}^{M_N}) \) by taking the (holomorphic) variation of \( \tilde{f} \circ \varphi \).

\[
\delta (f \circ \varphi) = \delta \tilde{f} \circ \varphi + \tilde{f} \circ \delta \varphi \quad (2.26)
\]

\( \delta \tilde{f} \) corresponds to the generic degrees of freedom, i.e,

\[
\text{dim}(\delta \tilde{f} \circ \varphi) = N - 2 \quad (2.27)
\]

\( \tilde{f} \circ \delta \varphi \) counts the deformation of the multiple cover map which can be realized using the section \( \varphi^* \in H^0(CP^1, \varphi^*(T'C_P^1)) \) as \( \varphi^* \partial_z \tilde{f} \). We can count these by using Riemann-Roch,

\[
\text{dim}(\tilde{f} \circ \delta \varphi) = \dim(H^0(CP^1, \varphi^*(T'C_P^1))) - 3 = 1 + d_{\varphi}(\varphi) \cdot e_1(T'C_P^1) - 3 = 2d - 2 \quad (2.28)
\]

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In (2.28) we subtract the double counted \( SL(2,C) \) which comes from the indetermination of the decomposition of \( f \), i.e,
\[
    f = \tilde{f} \circ \varphi \\
    = \tilde{f} \circ u \circ u^{-1} \circ \varphi \quad u \in SL(2,C)
\]  
(2.29)

After all we find additional \( 2d - 2 \chi \) zero modes. But we can also construct \( 2d - 2 \psi^i \) which comes from \( H^1(CP^1, f^*(T'M_N)) \). By the Kodaira-Serre duality, the following equation holds.
\[
    \dim(H^1(CP^1, f^*(T'M_N))) = \dim(H^0(CP^1, K \otimes f^*(T'M_N))) \\
    = \dim(H^0(CP^1, K \otimes f^*(\tilde{T}'M_N)))
\]  
(2.30)

and
\[
    g^{ij} \psi_{\tilde{z}j} \in H^0(CP^1, K \otimes f^*(\tilde{T}'M_N))
\]  
(2.31)

Lagrangian (2.1) has the \( R_{ij\psi \chi} \xi^i \chi^j \psi \chi \) term, so we can “kill” additional \( 2d - 2 \chi \) and \( \psi \) zero modes by expanding \( \exp(R_{ij\psi \chi} \xi^i \chi^j \psi \chi) \). So we don’t have to add extra operator insertions. Then, by integrating \( \psi \) zero-modes first, we have the Euler class \( \chi(\nu) \) where \( \nu \simeq H^1(CP^1, \varphi^*(T'M_N)) \). This leads to
\[
    \langle \mathcal{O}_{\xi^1}(z_1) \mathcal{O}_{\xi^2}(z_2) \cdots \mathcal{O}_{\xi^k}(z_k) \rangle_n = \int_{\mathcal{M}_0^N} \chi(\nu) \wedge \alpha(\mathcal{O}_{\xi^1}(z_1)) \wedge \cdots \alpha(\mathcal{O}_{\xi^k}(z_k))
\]  
(2.32)

We can refine (2.32) by using the argument which leads to (2.24) and define the evaluation map,
\[
    \varphi_i : \mathcal{M}^N_{0,n} \to M_N : f \mapsto f(z_i)
\]  
(2.33)

We have
\[
    \langle \mathcal{O}_{\xi^1}(z_1) \mathcal{O}_{\xi^2}(z_2) \cdots \mathcal{O}_{\xi^k}(z_k) \rangle_n = \int_{\mathcal{M}_0^N} \chi(\nu) \wedge \varphi_i^{*}(\xi^1) \wedge \cdots \varphi_i^{*}(\xi^k)
\]  
(2.34)

We can relate the non-generic part the correlation functions to ones of lower degree, because in such case \( f \) decomposes into \( f = \tilde{f} \circ \varphi \) where \( \deg(\varphi) = d \) and \( \deg(f) = n/d < N \). But good results are given only in the case of \( k = 3 \), which was derived by Greene, Aspinwall, Morrison and Plesser [4] [15]. Of course, if we use the fusion rule that holds in the matter theory, we can reduce the correlation functions into the product of three point functions and formally distinguish the non-generic part from the generic ones. But geometrical meaning is still not clear.

Then we slightly change our point of view. Since \( M_N \) is a hypersurface in \( CP^{N-1} \), we can see \( \mathcal{M}^N_{0,n} \) as a submanifold of \( \mathcal{M}^{CP^{N-1}}_{0,n} \) which consists of maps satisfying the following condition.
\[
    f : CP^1 \to CP^{N-1} \\
    f(z) \in M_N \quad \text{for all} \quad z \in CP^1
\]  
(2.35)
If we can realize the condition (2.35) as the closed forms $c_n(M_N)$ on $M_{0,n}^{CP^{N-1}}$, we have an alternate representation for the correlation functions as follows,

$$\langle \mathcal{O}_{c_{j1}}(z_1)\mathcal{O}_{c_{j2}}(z_2)\cdots \mathcal{O}_{c_{jk}}(z_k) \rangle_{n,alt,grav.}$$

$$= \int_{M_{0,n}^{CP^{N-1}}} c_n(M_N) \wedge \tilde{\phi}_1^*(c_{j1}) \wedge \cdots \wedge \tilde{\phi}_k^*(c_{jk})$$

$$\tilde{\phi}_i : M_{0,n}^{M_N} \to CP^{N-1} : f \mapsto f(z_i)$$

(2.36)

In (2.36), we can drop off the Euler class $\chi(\nu)$. This is because

$$\dim(H^1(CP^1, \tilde{\phi}^*(T'CP^{N-1}))) = \dim(H^0(CP^1, K \otimes (\tilde{\phi}^*(T'M_N)))) = 0$$

(2.37)

Dimension of moduli space does not jump in this case. Then naturally arises the question about the relation between (2.34) and (2.36). However we want to proceed further with the formula (2.36).

Then we want to use the torus action method invented by Kontsevich, and we couple gravity to the topological sigma model. Roughly speaking, we add to the moduli space “puncture” degrees of freedom. So for k-point correlation function, dimension of moduli space (we denote it as $M_{0,n,k}^{CP^{N-1}}$) increases by $k - 3$. $-3$ corresponds to deiving by automorphism of $CP^1$, i.e. $SL(2,C)$ which is induced by c-ghost zero-modes. And topological selection rule (2.23) is changed into

$$N - 2 + k - 3 = \sum_{i=1}^{k} j_i$$

$$\iff N - 5 = \sum_{i=1}^{k} (j_i - 1)$$

(2.38)

$M_{0,n,k}^{CP^{N-1}}$ can be roughly represented as follows,

$$M_{0,n,k}^{CP^{N-1}} \simeq \{(z_1, z_2, \cdots, z_k), f\} / SL(2, C) \quad f \in M_{0,n}^{CP^{N-1}}$$

(2.39)

where $u \in SL(2, C)$ acts

$$u \circ \{(z_1, z_2, \cdots, z_k), f\} = \{(u(z_1), \cdots, u(z_k)), (u^{-1})^* \circ f\}$$

(2.40)

This action of $SL(2, C)$ is compatible with the “evaluation map”

$$\phi_i : M_{0,n,k}^{CP^{N-1}} \to CP^{N-1}$$

$$\{(z_1, z_2, \cdots, z_k), f\} \mapsto f(z_i)$$

(2.41)

because $(u^{-1})^* f(u(z_i)) = f(z_i)$.

Then the integral representation of amplitudes (2.36) turns into

$$\langle \mathcal{O}_{c_{j1}}(z_1)\mathcal{O}_{c_{j2}}(z_2)\cdots \mathcal{O}_{c_{jk}}(z_k) \rangle_{n,alt,grav.}$$

$$= \int_{M_{0,n,k}^{CP^{N-1}}} c_n(M_N) \wedge \phi_1^*(c_{j1}^i(H)) \wedge \cdots \wedge \phi_k^*(c_{jk}^i(H))$$

(2.42)
where we used the fact that $e$ corresponds to the first chern class of hyperplane bundle $H$. Then we have to find the realization of $c_n(M_N)$. We can roughly do it as follows. First consider the coordinate representation of $\mathcal{M}_{0,n}^{CP^{N-1}}$,

$$
f : (s, t) \mapsto \left( \sum_{i=0}^{n} a_i^i s^{-i} t^i, \cdots, \sum_{i=0}^{n} a_N^i s^{-i} t^i \right) \quad (2.43)
$$

where $(a_i^j)$'s are the coordinates of $\mathcal{M}_{0,n}^{CP^{N-1}}$. Then the condition imposed by $c_n(M_N)$ is equal to

$$
f(CP^1) \in M_N \quad \text{for all}(s, t)
$$

$$
\iff \left( \sum_{i=0}^{n} a_i^i s^{-i} t^i \right)^N + \cdots + \left( \sum_{i=0}^{n} a_N^i s^{-i} t^i \right)^N = 0 \quad \text{for all}(s, t)
$$

$$
f^m(a_j^j) = 0 \quad (m = 0, 1, \cdots, Nn) \quad (2.44)
$$

where $f^m(a_j^j)$'s are the coefficient polynomial of $s^m t^{Nn-m}$ of the l.h.s of the second line of (2.44). This imposes $Nn + 1$ condition on $\mathcal{M}_{0,n}^{CP^{N-1}}$. We can describe this condition mathematically in terms of gravitational moduli space $\mathcal{M}_{0,n}^{CP^{N-1}}$. Let $\pi_j$ be a forgetfull map $\pi_j : \mathcal{M}_{0,n,j}^{CP^{N-1}} \to \mathcal{M}_{0,n,j-1}^{CP^{N-1}}$. Then for $j = 1$, the fiber of $\pi_1$ is $CP^1$. And consider the sheaf $\phi_1^*(NH)$ on $\mathcal{M}_{0,n}^{CP^{N-1}}$ where $NH$ corresponds to defining polynomial of $M_N$ and $H^0(\mathcal{M}_{0,n,1}^{CP^{N-1}}, \phi_1^*(NH))$ to the second line of (2.44) modulo $SL(2, C)$ equivalence. Then direct limit sheaf $R_1^0(\phi_1^*(NH))$ (we denote it as $E_{Nn+1}$). It locally equals $H^0(CP^1, f^*(O(NH))$ has rank $(Nn + 1)$. We can translate the operation in going from the second line of (2.44) to the third one into the evaluation of the zero locus of the section of $E_{Nn+1}$. Considering the map,

$$
\tilde{\pi}_k := \pi_1 \circ \pi_2 \circ \cdots \circ \pi_k : \mathcal{M}_{0,n,k}^{CP^{N-1}} \mapsto \mathcal{M}_{0,n,0}^{CP^{N-1}} \quad (2.45)
$$

We have

$$
c(M_N) = c_T(\tilde{\pi}_k^*(E_{Nn+1})) \quad (2.46)
$$

Finally the representation (2.42) turns into

$$
\langle \mathcal{O}_{c_1}(z_1) \mathcal{O}_{c_2}(z_2) \cdots \mathcal{O}_{c_l}(z_k) \rangle_{n, \text{alt,gr}, \text{av.}}
$$

$$
= \int_{\mathcal{M}_{0,n,k}^{CP^{N-1}}} c_T(\tilde{\pi}_k^*(E_{Nn+1})) \wedge \phi_1^*(c_1^1(H)) \wedge \cdots \wedge \phi_k^*(c_l^1(H)) \quad (2.47)
$$

Then we can calculate the correlation functions using the torus action method.

3 Review of the Torus Action Method

3.1 Introduction of the Torus Action and the Bott Residue Formula

Torus action method is the strategy to use the Bott residue formula [2] which reduces the integral of Chern classes of vector bundle on $X$ to the one on $X_f$ of the fixed point set of the torus action flow on $X$ to the case where $X$ is $\mathcal{M}_{0,n,k}^{CP^{N-1}}$. 

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First, let us introduce the torus action flow on $\mathbb{C}P^{N-1}$,

\[
T_t : \mathbb{C}P^{N-1} \to \mathbb{C}P^{N-1}
\]

\[
(X_1, X_2, \cdots, X_N) \mapsto (e^{\lambda_1 t}X_1, e^{\lambda_2 t}X_2, \cdots, e^{\lambda_N t}X_N)
\]

\[\quad (t \in C^\times) \quad (3.48)\]

where $\lambda_i \in C$ is the character of the flow. Then (3.48) induce the flow on $\mathcal{M}_{0,n,k}^{ CP_{N-1}}$ from the compatibility with the evaluation map.

\[
\phi_i(T_t((z_1, z_2, \cdots, z_k, f)/ \sim)) := T_t \circ \phi_i((z_1, z_2, \cdots, z_k, f)/ \sim)
\]

\[= T_t \circ f(z_i) \quad (3.49)\]

Next, we introduce the Bott residue formula. For simplicity, we use $X$ for $\mathcal{M}_{0,n,k}^{ CP_{N-1}}$. Let $\mathcal{E}_i, \cdots, \mathcal{E}_m$ be a holomorphic vector bundle on $X$, and $X_f$ be the fixed point set of $X$ under the flow (3.49). We can decompose $X_f$ as the sum of the connected components $X_\gamma$.

\[X_f = \bigcup \gamma X_\gamma \quad (3.50)\]

Then consider $\mathcal{E}_i|_{X_\gamma}$ and the normal bundle $\mathcal{N}_\gamma \simeq T'X|_{X_\gamma}/T'X_\gamma$ and decompose them into the eigen vector bundle under the torus action $T_t$, i.e.,

\[
\mathcal{E}_i|_{X_\gamma} \cong \bigoplus_{j=1}^{m_{\gamma_i}} \mathcal{E}_{i,j}^{\gamma_{i,j}}(\lambda_i)
\]

\[
\mathcal{N}_\gamma \cong \bigoplus_{j=1}^{m_{\gamma_j}} \mathcal{N}_{j,j}^{\gamma_{j,j}}(\lambda_j) \quad (3.51)\]

where

\[
T_t(\mathcal{E}_{i,j}^{\gamma_{i,j}}(\lambda_i)) = e^{t \lambda_i}(\lambda_i) T_t \mathcal{E}_{i,j}^{\gamma_{i,j}}(\lambda_i)
\]

\[
T_t(\mathcal{N}_{j,j}^{\gamma_{j,j}}(\lambda_j)) = e^{t \lambda_j}(\lambda_j) T_t \mathcal{N}_{j,j}^{\gamma_{j,j}}(\lambda_j) \quad (3.52)\]

and we set

\[
r_k(\mathcal{E}_{i,j}^{\gamma_{i,j}}(\lambda_i)) = r_\mathcal{E}(i,j)
\]

\[
r_k(\mathcal{N}_{j,j}^{\gamma_{j,j}}(\lambda_j)) = r_\mathcal{N}(j) \quad (3.53)\]

We can represent the total Chern class of $\mathcal{E}_{i,j}^{\gamma_{i,j}}(\lambda_i)$ and $\mathcal{N}_{j,j}^{\gamma_{j,j}}(\lambda_j)$ as the product of first Chern class of formal line bundles as follows.

\[
c(\mathcal{E}_{i,j}^{\gamma_{i,j}}(\lambda_i)|_{X_\gamma}) = \prod_{j=1}^{m_{\gamma_j}} \prod_{k=1}^{r_\mathcal{E}(i,j)} (1 + t \cdot e_{i,j,k}^{\gamma_{i,j}}(\lambda_i))
\]

\[
c(\mathcal{N}_{j,j}^{\gamma_{j,j}}(\lambda_j)) = \prod_{j=1}^{m_{\mathcal{N}}(j)} \prod_{k=1}^{r_\mathcal{N}(j)} (1 + t \cdot n_{j,j,k}^{\gamma_{j,j}}(\lambda_j)) \quad (3.54)\]
Top Chern classes are given as the coefficient form of $t^k$ of highest degree.

With these preparations, we introduce the Bott residue formula.

$$
\int_X \prod_i c_T^{a_i}(\xi_i) = 
\sum_\gamma \int_{X_{\gamma}} \prod_i \prod_{j=1}^{m_i} \prod_{k=1}^{n_i} \left( e_{i,j,k}^{\gamma,j}(\lambda_i) + f_{i,j}(\lambda_i) \right) a_i
$$

(3.55)

### 3.2 Construction of Fixed Point Set

Fixed points of $CP^{N-1}$ under $T_t$ are given by the projective equivalence

$$
p_i := (0, 0, \cdots, 0, \overbrace{1}^i, 0, \cdots, 0)
$$

(3.56)

Then, we can find the fundamental maps $l_{i,j}^d$ from $CP^1 \mapsto CP^{N-1}$ which remain fixed under $T_t$ as the degree $d$ maps which connect $p_i$ and $p_j$.

$$
l_{i,j}^d : (s, t) \mapsto (0, 0, \overbrace{1}^i, 0, \cdots, 0, s^d, 0, \cdots, 0, t^d, 0, \cdots, 0)
$$

(3.57)

Of course $l_{i,j}^d$ is kept fixed under $SL(2, C)$ equivalence. But now that we have coupled gravity with the theory, we have to consider the boundary components of moduli space of $CP^1$, i.e., stable curves. Stable curve $C$ with $k$-punctures is constructed with the set of $CP^1$'s $\{C_\alpha\}$ with punctures assigned on them and additional punctures of double singularity which connect two components of $C_\alpha$'s. Then we can translate the condition into the condition that the genus of stable curve is zero into imposing its arithmetic genus to be zero. In geometrical language, if we represent $C_\alpha$ as a line and define a figure with lines which intersect at singular punctures, this is equivalent to the non-existence of closed loops in it. This addition makes us to introduce stable maps which map stable curves to $CP^{N-1}$.

With these considerations, we can label the connected components of the fixed point set $M_{0,n,k,f}^{CP^{N-1}}$ with a tree graph $\Gamma$ with the following structure. We denote them by $M_{0,n,k}^{D^{CP^{N-1}}}(\Gamma)$. The rules of correspondences are,

1) The vertices $v \in Vert(\Gamma)$ correspond to the connected component $C_v$ of $f^{-1}(p_1, \cdots, p_N)$. $C_v$ can be a sum of connected irreducible components of $C$ or be a point.

2) The edges $e \in Edge(\Gamma)$ correspond to the irreducible component $C_\alpha$ mapped to $l_{i,j}^d$.

Then we have to add the additional structures to $\Gamma$. 

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1) We label each \( v \in Vert(\Gamma) \) by \( f_v \in \{1, 2, \ldots, N\} \) which is defined by \( p_f = f(C_v) \).

2) The \( k \)-punctures are distributed among the vertices \( v \in Vert(\Gamma) \). We represent this distribution by \( S_v \in \{1, 2, \ldots, k\} \).

3) We attach degree \( d_\alpha \) to each \( \alpha \in Edge(\Gamma) \) defined by the degree of \( l_{i,j}^\alpha \).

We have to set punctures on the vertices \( Vert(\Gamma) \) because if we put punctures on \( C^a \), they move with the flow \( T_t \), which contradicts with the assumption of fixed point sets. Then we can construct \( \mathcal{M}_{0,n,k}^{CP^{N-1}}(\Gamma) \) under conditions that emerges from the above three structures,

1) If \( \alpha \in Edge(\Gamma) \) connects \( v, u \in Vert(\Gamma) \), \( f_u \neq f_v \).

2) \( \{1, 2, \ldots, k\} = \bigsqcup_{v \in Vert(\Gamma)} S_v \).

3) \( \sum_{\alpha \in Edge(\Gamma)} d_\alpha = n \)

Then we have

\[
\mathcal{M}_{0,n,k}^{CP^{N-1}}(\Gamma) \cong \prod_{v \in Vert(\Gamma)} (\mathcal{M}_{0,S_v}(\Gamma))/(Aut(\Gamma)) \tag{3.58}
\]

where \( \mathcal{M}_{0,S_v} \) is the moduli space of complex structure of \( CP^1 \) with \( S_v \) punctures. It represents the gravitational degree of freedom of \( C_v \). According to Kontsevich, division by \( Aut(\Gamma) \) reflects the orbispace structure of \( \mathcal{M}_{0,n,k}^{CP^{N-1}} \). It may reflect the multiplicity of the degeneration of stable maps.

### 3.3 Determination of the contribution from Normal and Vector bundles

#### 3.3.1 Contributions from \( \mathcal{N}_{\mathcal{M}[\Gamma]}^{ab} \)

With these preparations, we determine the contribution from \( \mathcal{M}_{0,n,k}^{CP^{N-1}}(\Gamma) \) (in the following discussion we abbreviate the notation as \( \mathcal{M}(\Gamma) \)) to (3.55).

First, we calculate the contribution from \( \mathcal{N}_{\mathcal{M}[\Gamma]} \). Following Kontsevich, we will use the expression of vector bundles as the K-group \([ \cdot ]\), which translates sum and quotient operations into addition and subtraction. Then we have

\[
[N_{\mathcal{M}[\Gamma]}] = [T^\prime \mathcal{M}[\mathcal{M}[\Gamma]]] - [T^\prime \mathcal{M}(\Gamma)] \tag{3.59}
\]

If we set

\[
C = \bigcup_{\alpha} C_\alpha
\]

(where

\[
\bigcup_{\alpha}
\]

means a sum with double-singularity gluing operation), \([T^\prime \mathcal{M}[\mathcal{M}[\Gamma]]]\) consists of the following degrees of freedom,
1) Moving \( f(C) \) in \( C P^{N-1} \).

2) Resolution of singularities of \( C \), i.e., from \( xy = 0 \) to \( xy = \epsilon \).

3) Moving puncture degrees of freedom.

And we have

\[
[T'M][\mathcal{M}(\Gamma)] = \left[ H^0(C, f^*(T'CP^{N-1})) \right] + \sum_{\alpha \neq \beta, \alpha, \beta \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \neq \beta, \alpha \in E_{	ext{Edg}}(\Gamma)} [T'_\alpha C] + \sum_{\alpha \notin E_{\text{Edg}}(\Gamma)} [T'_\alpha C] - \sum_{\alpha} [H^0(C^\alpha, T'C^\alpha)] \tag{3.60}
\]

The last term of (3.60) corresponds to division by \( SL(2, C) \) of each component \( C^\alpha \).

From (3.58) \( \mathcal{M}(\Gamma) \) has continuous degrees of freedom which come only from \( C^\alpha \) mapped to a point, we have

\[
[T'M(\Gamma)] = \sum_{\alpha \neq \beta, \alpha, \beta \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \neq \beta, \alpha \in E_{	ext{Edg}}(\Gamma)} [T'_\alpha C] + \sum_{\alpha \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C] - \sum_{\alpha} [H^0(C^\alpha, T'C^\alpha)] \tag{3.61}
\]

where we used the fact that all the punctures lie in the component mapped to a point.

From (3.60) and (3.61), we have

\[
[N_M(\Gamma)] = \left[ H^0(C, f^*(T'CP^{N-1})) \right] + [N^\alpha_M(\Gamma)] \tag{3.62}
\]

where

\[
[N^\alpha_M(\Gamma)] := \sum_{\alpha \neq \beta, \alpha, \beta \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \neq \beta, \alpha \in E_{	ext{Edg}}(\Gamma), \beta \in E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C] - \sum_{\alpha} [H^0(C^\alpha, T'C^\alpha)] \tag{3.63}
\]

(3.64)

\[
[N^\alpha_M(\Gamma)] := \sum_{\alpha \neq \beta, \alpha, \beta \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \neq \beta, \alpha \in E_{	ext{Edg}}(\Gamma), \beta \in E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C] - \sum_{\alpha} [H^0(C^\alpha, T'C^\alpha)] \tag{3.64}
\]

(3.65)

\[
[N^\alpha_M(\Gamma)] := \sum_{\alpha \neq \beta, \alpha, \beta \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \neq \beta, \alpha \in E_{	ext{Edg}}(\Gamma), \beta \in E_{	ext{Edg}}(\Gamma)} [T'_\alpha C \otimes T'_\beta C] + \sum_{\alpha \notin E_{	ext{Edg}}(\Gamma)} [T'_\alpha C] - \sum_{\alpha} [H^0(C^\alpha, T'C^\alpha)] \tag{3.65}
\]
Then we determine the contribution from the first term of (3.62), and (3.64), (3.65), and (3.65).

First, consider the contribution from (3.64). Since \( \alpha, \beta \in \text{Edge}(\Gamma) \), \( T_z C_\alpha \) and \( T_z C_\beta \)'s are trivial as the line bundle on \( \mathcal{M}(\Gamma) \). Let \( C_\alpha \) and \( C_\beta \) be mapped to \( l_{i,j}^{\alpha} \) and \( l_{i,j}^{\beta} \).

\[
C_\alpha : (z_1, z_2) \mapsto (0, \cdots, 0, z_1^{d_\alpha}, 0, \cdots, 0, z_2^{d_\alpha}, 0, \cdots, 0) \quad (3.66)
\]

\[
C_\beta : (w_1, w_2) \mapsto (0, \cdots, 0, w_1^{d_\beta}, 0, \cdots, 0, w_2^{d_\beta}, 0, \cdots, 0) \quad (3.67)
\]

Local coordinate around \( z \in C_\alpha \cap C_\beta \) on \( C_\alpha \) and \( C_\beta \) are \( \frac{z_2}{z_1} \) and \( \frac{w_2}{w_1} \), and we have

\[
T_z^t C_\alpha \otimes T_z^t C_\beta \cong \frac{d}{d\left(\frac{z_2}{z_1}\right)} \otimes \frac{d}{d\left(\frac{w_2}{w_1}\right)} \quad (3.69)
\]

Definition of torus action (3.49) leads us to

\[
z_1 \mapsto z_1 e^{\frac{\lambda_1}{d_\alpha}} \\
z_2 \mapsto z_2 e^{\frac{\lambda_1}{d_\alpha}}
\]

\[
w_1 \mapsto w_1 e^{\frac{\lambda_1}{d_\beta}} \\
w_2 \mapsto w_2 e^{\frac{\lambda_1}{d_\beta}} \quad (3.70)
\]

and

\[
T_z^t C_\alpha \otimes T_z^t C_\beta \mapsto e^{\left(\frac{\lambda_1 - \lambda_\alpha}{d_\alpha} + \frac{\lambda_1 - \lambda_\beta}{d_\beta}\right)t} T_z^t C_\alpha \otimes T_z^t C_\beta \quad (3.71)
\]

The result is,

\[
\text{(Contribution from (3.64) to (3.55))} = \prod_{\substack{C_\alpha \cap C_\beta \neq \# \\ \alpha \neq \beta \in \text{Edge}(\Gamma)}} \frac{1}{\frac{\lambda_\alpha - \lambda_\beta}{d_\alpha} + \frac{\lambda_\alpha - \lambda_\beta}{d_\beta}} \quad (3.72)
\]

Again following Kontsevich, we introduce the notation “Flag” \( F = (v, \alpha) \) which represents edge \( \alpha \) with a direction specified by the source vertex \( v \). We define

\[
w_F := \frac{\lambda_{f_1} - \lambda_{f_2}}{d_\alpha} \quad (3.73)
\]

Then the r.h.s of (3.72) can be rewritten as follows.

\[
\prod_{v \in V \cap \text{rs}(\Gamma), \quad v \neq \alpha(v) \neq 2, \quad s = 0} \frac{1}{w_{F_1(v)} + w_{F_2(v)}} \quad (3.74)
\]

where \( \text{val}(v) \) represents the valency of \( v \) and \( F_1(v) \) and \( F_2(v) \) are the flags whose sources are \( v \). Note that in this case \( f^{-1}(v) \) is a point.
Next we consider the contributions from (3.65). Again from (3.61), \( T_{z} C_{a} \) is trivial as the line bundle on \( \mathcal{M}(\Gamma) \) but has an eigenvalue \( w_{F} \) as in the derivation of (3.71). On the other hand, \( T_{z}^{T} C_{a} \) has trivial torus action (because \( C_{a} \) is mapped to a point) but non-trivial line bundle on \( \mathcal{M}(\Gamma) \). And if \((\text{punctures on } C_{v}) \geq 3, \mathcal{M}_{0,S_v} \) is well-defined and we have

\[
(\text{Contribution from (3.65) to (3.55))} = \prod_{v \in \text{Vert}(\Gamma)} \left( \int_{\mathcal{M}_{0,S_v}(C_{v})+1}^{\mathcal{M}_{0,S_v}(C_{v})} \frac{1}{w_{F} + c_{1}(T_{z}^{T} C_{v})} \right)
(\text{val}(v) + \frac{1}{2} S_{v} \geq 3)
\]

where \( z_{F} \) represents the gluing point of \( C_{v} \) and \( F \). We can evaluate the r.h.s of (3.75) by expanding in terms of \( \frac{1}{w_{F}} \) and using the fact that \( c_{1}(T_{z}^{T} C_{v}) = -c_{1}(T_{z}^{T} C_{v}) \). Expansion coefficients are intersection numbers of Mumford-Morita class on the \( CP^{1} \)-moduli space, which is identified as the correlation function of gravitational descendants by Witten [11]. Continuing the calculation, we have

\[
(\text{r.h.s of (3.75))} = \prod_{v \in \text{Vert}(\Gamma)} \left( \sum_{d_{1} \cdots d_{s}, v_{a}(\gamma) \geq 0} \prod_{\mathcal{F}_{a}(\gamma)} w_{F}^{-d_{1}} \cdots w_{F}^{-d_{s}} \frac{\sigma_{d_{1}} \cdots \sigma_{d_{s}}}{\prod_{a} \mathcal{P}_{a}^{d_{a}} \prod_{b} \mathcal{P}_{b}^{d_{b}}} \right)\]
\[
(\text{3.76})
\]

\[
\langle \sigma_{d_{1}} \cdots \sigma_{d_{s}} \mathcal{P} \cdots \mathcal{P} \rangle \text{ is calculated in [10],}
\]

\[
\langle \sigma_{d_{1}} \cdots \sigma_{d_{s}} \mathcal{P} \cdots \mathcal{P} \rangle = \frac{(\text{val}(v) + \frac{1}{2} S_{v} - 3)!}{d_{1}! \cdots d_{\text{val}(v)}!} \]
\[
(\text{3.77})
\]

Combining (3.75), (3.76) and (3.77), we have

\[
(\text{Contribution from (3.65) to (3.55))} = \prod_{v \in \text{Vert}(\Gamma)} \prod_{\mathcal{F}_{a}(\gamma)} w_{F}^{-1} \left( \sum_{\mathcal{F}_{a}(\gamma)} w_{F}^{-1} \right)^{\text{val}(v) + \frac{1}{2} S_{v} - 3}
(\text{val}(v) + \frac{1}{2} S_{v} \geq 3)
\]
\[
(\text{3.78})
\]

Then we consider (3.65). Contributions of the first terms are, as before

\[
\prod_{\mathcal{C}_{a} \cap \mathcal{C}_{b} \neq \emptyset} \frac{1}{w_{F_{i}(\alpha)}}
\]
\[
(\text{3.79})
\]

where \( F_{i}(\alpha) \)'s are two flags having \( \alpha \) as their edges.

The second terms which represent the automorphism group degrees of freedom of edge components can be expressed by the tangent bundles on the inverse images of two vertices of the edges and scaling transformation degree of freedom fixing the
punctures (We denote it as [0]). In terms of the K-group, we have

\[- \sum_{\alpha \in Edg(\Gamma)} [H^0(C_\alpha, T'C_\alpha)]\]

\[= - \sum_{\alpha \in Edg(\Gamma)} ([T'_{z_1(\alpha)} C_\alpha] + [0] + [T'_{z_2(\alpha)} C_\alpha]) \]  

\[(3.80)\]

And contributions to (3.55) are

\[\prod_{\alpha \in Edg(\Gamma)} w_{F_1}(\alpha) \cdot w_{F_2}(\alpha) \cdot C([0])\]  

\[(3.81)\]

where \(C([0])\) represents the factor from \([0]\). Multiplying (3.79) and (3.81), what remains except for \(C([0])\) is the products of \(w_F\)'s whose edges have only one double singularity. In other words, the corresponding \(F = (v, \alpha)\) has \(val(v) = 1\) and \(f^{-1}(v)\) is a point. We have

\[(\text{Contributions from (3.65)}) = \prod_{s \in V_{\text{crf}(\Gamma)}} \prod_{s \in (s_{\alpha})} \prod_{s \in (s_{\alpha})} w_F \prod_{\alpha \in Edg(\Gamma)} C([0]) \]

\[(3.82)\]

After all, from (3.74),(3.78) and (3.82), we put all the factors from \([X^d_{\alpha}]\) into the form,

\[\prod_{v \in V_{\text{crf}(\Gamma)}} \prod_{l \in f_{\alpha}^{\text{flag}}} \prod_{P \in (s_{\alpha})} w_F^{-1} \left( \sum_{l \in f_{\alpha}^{\text{flag}}} w_F^{-1} \right)^{val(v) + 1} \prod_{\alpha \in Edg(\Gamma)} C([0]) \]

\[(3.83)\]

3.3.2 Determination of the contributions from \([H^0(C, f^*(T'C P^{N-1}))]\)

Since

\[f(C) = \bigcup_{\alpha \in Edg(\Gamma)} f(C_\alpha)\]

we can construct \([H^0(C, f^*(T'C P^{N-1}))]\) by gluing

\[\bigoplus_{\alpha \in Edg(\Gamma)} [H^0(C_\alpha, f^*(T'C P^{N-1}))]\]

at \(p_f\). This process can be described using exact sequences,

\[\bigoplus_{\alpha \in Edg(\Gamma)} H^0(C_\alpha, f^*(T'C P^{N-1})) \rightarrow \bigoplus_{v \in V_{\text{crf}(\Gamma)}} C^{val(v)-1} \otimes T'_{p_f} C P^{N-1} \rightarrow 0 \]

\[(3.84)\]

This contribution is then given as the contribution from the second term divided by the one from from the third term. As the independent basis of \(H^0(C_\alpha, f^*(T'C P^{N-1}))\) describing the deformation of \(f(C_\alpha)\) in \(C P^{N-1}\) where \(C_\alpha\) is

\[C_\alpha : (z_1, z_2) \mapsto (0, \cdots, 0, z^d_{1, \alpha}, 0, \cdots, 0, z^d_{2, \alpha}, 0, \cdots, 0)\]

\[(3.85)\]
we have

\[ (0, \ldots, 0, z_1^{d_0} + \epsilon z_1^m z_2^{d_a-m}, 0, \ldots, 0, z_2^{d_0}, 0, \ldots, 0) \]  

(3.86)

\[ (0, \ldots, 0, z_1^{d_0}, 0, \ldots, 0, z_2^{d_0} + \epsilon z_1^m z_2^{d_a-m}, 0, \ldots, 0) \]  

(3.87)

\[ (0, \ldots, 0, \epsilon z_1^m z_2^{d_a-m}, 0, \ldots, 0, z_1^{d_0}, 0, \ldots, 0, z_2^{d_0}, 0, \ldots, 0) \]  

(3.88)

We can write these basis in more sophisticated form,

\[ a) \quad (\frac{z_1}{z_2})^m X_i \frac{\partial}{\partial X_i} \quad (\text{for } -d_a \leq m \leq d_a) \]  

(3.89)

\[ b) \quad (\frac{z_1}{z_2})^m X_j \frac{\partial}{\partial X_k} \quad (0 \leq m \leq d_a) \quad k \neq i, j \]  

(3.90)

This expression directly leads us to

\[
\text{(Contribution from (3.90))} = \frac{1}{m \cdot w_F} \quad (m \neq 0) 
\]

(3.91)

\[
\text{(Contribution from (3.90))} = \frac{1}{m \cdot w_F + \lambda_j - \lambda_k} 
\]

(3.92)

And from \( T^i_{p_j} C P^{N-1} = \bigoplus j \neq p_j \frac{\partial}{\partial \left( \frac{X_j}{f_r} \right)} \),

\[
\text{(Contributions from } C^{v_{a\mid v}}(\cdot)^{-1} \otimes T^i_{p_j} C P^{N-1}) = \prod_{v \in Vert(\Gamma)} (\lambda_{f_r} - \lambda_j)^{v_{a\mid v}(\cdot)^{-1}} \]

(3.94)

Combining (3.92), (3.93), (3.93) and (3.94), we have

\[
\prod_{\text{flags}} \prod_{F=(v, \alpha)} \left( w_F \right)^{-d_0} \prod_{\alpha \in Edge(\Gamma)} \prod_{k \neq f_r, f_s, m=0} \prod_{\lambda_{f_r}} \left( \frac{m \lambda_{f_r} + (d_a - m) \lambda_{f_s}}{d_a} \right)^{-1} \prod_{\alpha \in Edge(\Gamma)} (d_a !)^{-2} (C([0]))^{-1} \prod_{v \in Vert(\Gamma)} (\lambda_{f_r} - \lambda_j)^{v_{a\mid v}(\cdot)^{-1}} (3.95)
\]

3.3.3 Factors from Vector Bundles \( \mathcal{E} \),

First, we calculate the factors from \( \mathcal{E}_{Nn+1} \). As we have mentioned in section 2, this fiber locally corresponds to \([H^0(C, f^*(\mathcal{O}(N \cdot H)))])]. We can construct it as in (3.83), by the exact sequences,

\[
\bigoplus_{\alpha \in Edge(\Gamma)} H^0(C_\alpha, f^*(\mathcal{O}(NH))) \rightarrow \bigoplus_{v \in Vert(\Gamma)} C^{v_{a\mid v}(\cdot)^{-1}} \otimes \mathcal{O}_{p_j}(NH) \rightarrow 0
\]

(3.96)
Then since the basis of $H^0(C_o, f^*(O(N \cdot H)))$ are given as
\[
\{ z_1^{Np_{1a}}, z_2^{Np_{2a}-1} z_2, \ldots, z_2^{Np_{da}} \},
\]  
and the section of $O_{\mathcal{P}f_1}(N H)$ is $X^{N}_{f_1}$, we have
\[
(\text{Contributions from } c_T(E_{Nn+1})) = \prod_{\alpha \in \text{Edges}(\Gamma)} \prod_{a=0}^{N_{\alpha}d_{\alpha}} (\frac{a \lambda_{f_{1a}} + (N - a) \lambda_{f_{2a}}}{d_{\alpha}})^{N_{\alpha}d_{\alpha}} \prod_{v \in \text{Vertices}(\Gamma)} (N \lambda_{f_v})^{1 - w_{a}(v)}
\]  
Next, we determine the factor from $\phi_{\mathcal{P}f_1}^i(c^{ji}_1(H))$. From the argument of §3.2, puncture $i$ lies on the vertex $v(i)$ of $\Gamma$, and $\phi_{\mathcal{P}f_1}^i(c^{ji}_1(H))$ reduces to $O_{\mathcal{P}f_1}^i(j_i H)$. This leads to
\[
(\text{Contributions from } \phi_{\mathcal{P}f_1}^i(c^{ji}_1(H))) = \lambda_{f_{1(i)}^{ji}}
\]

\subsection*{3.3.4 Local Appendix}

We have to divide the above factors by $\frac{1}{d_{\alpha}} Aut(\Gamma)$ coming from (3.58) and in practice, we have to multiply a factor $\frac{1}{d_{\alpha}}$ for each edge $\alpha$. We cannot justify the reason for this factor at this stage.

\section{Some Explicit Calculation of Amplitudes}

From the argument of section 2, we constructed representation of amplitudes as an integral of forms on $\mathcal{M}^\mathbb{C}_{g,n,k}^{N-1}$, and the strategy of section 3 enables us to compute them as a sum of amplitudes corresponding to tree graphs. For some examples, we calculate $\langle O_{\mathcal{P}N-4} \rangle_1, \langle O_{\mathcal{P}N-4} \rangle_2$ and $\langle O_{\mathcal{P}N-4} \rangle_3$. First, we write out tree graphs that contribute to the amplitudes up to degree 3. (See Fig.1.) In Fig.1, we abbreviate the external insertion of “punctures”. So in calculation, we have to add all the cases of external operator insertions of $\langle O_{\mathcal{P}N-4} \rangle$ to vertices. Note that the two character numbers (for example “\(\mathbb{I}\)” in Fig.1) of neighboring vertices never coincide with each other. Then direct application of the argument of section 3 leads to the following formula.

\[
\langle O_{\mathcal{P}N-4} \rangle_1 = \frac{1}{2} \sum_{i \neq j} (\lambda_i - \lambda_k)^{-1} (\lambda_j - \lambda_k)^{-1} \prod_{a=0}^{N} (a \lambda_i + (N - a) \lambda_j) \frac{\lambda_i^{N-4} - \lambda_j^{N-4}}{w_{F_1}}
\]  
\[
(\text{from (a)})
\]
\[
\langle O_{\mathcal{P}N-4} \rangle_2 = \frac{1}{2} \sum_{i \neq j} (\frac{1}{\lambda_j} \frac{\lambda_i^{N-4} w_{F_1} + \lambda_j^{N-4} w_{F_1}}{w_{F_2} + w_{F_3}} + \lambda_j^{N-4}) \frac{1}{w_{F_1} w_{F_2} w_{F_3} w_{F_4}} \prod_{n \neq j} (\lambda_j - \lambda_n)
\]
\[ \prod_{m_1 \neq i, j} (\lambda_i - \lambda_{m_1})^{-1}(\lambda_j - \lambda_{m_1})^{-1} \prod_{m_2 \neq j, k} (\lambda_j - \lambda_{m_2})^{-1}(\lambda_k - \lambda_{m_2})^{-1} \]
\[ \prod_{a_1=0}^{N} (a_1 \lambda_i + (N - a_1) \lambda_j) \prod_{a_2=0}^{N} (a_2 \lambda_j + (N - a_2) \lambda_k) \]
(from (b))
\[ + \frac{1}{4} \sum_{i \neq j} ((\lambda_i^{N-4} - \lambda_j^{N-4})w_{F_3} - \frac{1}{w_{F_2} w_{F_2}} \frac{1}{w_{F_1} w_{F_2}} \prod_{k \neq i, j} (\lambda_i - \lambda_k)^{-1}(\lambda_j - \lambda_k)^{-1}(\frac{\lambda_i + \lambda_j}{2} - \lambda_k)^{-1} \prod_{a=0}^{2N} (a \lambda_j + (2N-a) \lambda_k)) \]
(from (c))
\[ \langle O_{1, N-4} \rangle_3 = \frac{1}{2} \sum_{j \neq i, k \neq l} \left( (\lambda_i^{N-4} - \lambda_j^{N-4}) \frac{1}{w_{F_2} w_{F_3}} \frac{1}{w_{F_3} w_{F_4}} + \frac{1}{w_{F_3} w_{F_4}} \frac{1}{w_{F_4} w_{F_5}} \prod_{n_1 \neq j} (\lambda_j - \lambda_{n_1}) \prod_{n_2 \neq k} (\lambda_k - \lambda_{n_2}) \prod_{m_1 \neq i, j} (\lambda_i - \lambda_{m_1})^{-1}(\lambda_j - \lambda_{m_1})^{-1} \prod_{m_2 \neq j, k} (\lambda_j - \lambda_{m_2})^{-1}(\lambda_k - \lambda_{m_2})^{-1} \prod_{m_3 \neq k, l} (\lambda_k - \lambda_{m_3})^{-1}(\lambda_l - \lambda_{m_3})^{-1} \prod_{a_1=0}^{N} (a_1 \lambda_i + (N - a_1) \lambda_j) \prod_{a_2=0}^{N} (a_2 \lambda_j + (N - a_2) \lambda_k) \prod_{a_3=0}^{N} (a_3 \lambda_k + (N - a_3) \lambda_l)) \right) \]
(from (d))
\[ + \frac{1}{2} \sum_{i \neq j} \left( \frac{1}{4} (\lambda_i^{N-4} - \lambda_j^{N-4}) \frac{1}{w_{F_2} w_{F_3}} \frac{1}{w_{F_3} w_{F_4}} + \frac{1}{w_{F_3} w_{F_4}} \frac{1}{w_{F_4} w_{F_5}} \prod_{n_1 \neq j} (\lambda_j - \lambda_{n_1}) \prod_{n_2 \neq k} (\lambda_k - \lambda_{n_2}) \prod_{m_1 \neq i, j} (\lambda_i - \lambda_{m_1})^{-1}(\lambda_j - \lambda_{m_1})^{-1} \prod_{m_2 \neq j, k} (\lambda_j - \lambda_{m_2})^{-1}(\lambda_k - \lambda_{m_2})^{-1} \prod_{m_3 \neq k, l} (\lambda_k - \lambda_{m_3})^{-1}(\lambda_l - \lambda_{m_3})^{-1} \prod_{a_1=0}^{N} (a_1 \lambda_i + (N - a_1) \lambda_j) \prod_{a_2=0}^{N} (a_2 \lambda_j + (N - a_2) \lambda_k) \prod_{a_3=0}^{N} (a_3 \lambda_k + (N - a_3) \lambda_l)) \right) \]
\[
\frac{1}{N \lambda_j} \frac{1}{w_{F_1}^2 w_{F_2}^2 w_{F_3} w_{F_4}} \prod_{m \neq j} \left( \lambda_j - \lambda_m \right) \\
\prod_{m \neq i, j} \left( \lambda_i - \lambda_{m_1} \right)^{-1} \left( \lambda_j - \lambda_{m_1} \right)^{-1} \left( \frac{\lambda_i + \lambda_j}{2} - \lambda_{m_1} \right)^{-1} \\
\prod_{m_k \neq i, j} \left( \lambda_j - \lambda_{m_2} \right)^{-1} \left( \lambda_i - \lambda_{m_2} \right)^{-1} \\
\frac{2N}{\prod_{a_i=0}^N (\lambda_i a + (3N - a) \lambda_i)} \\
\quad \text{(from (e))} \\
+ \frac{1}{6} \sum_{i \neq j} \frac{1}{36 \prod_{i \neq j} w_{F_2} (\lambda_i^{N-4} - \lambda_j^{N-4})^2} \left( \frac{2 \lambda_i + \lambda_j}{3} - \lambda_m \right)^{-1} \left( \frac{\lambda_i + 2 \lambda_j}{3} - \lambda_m \right)^{-1} \left( \lambda_j - \lambda_m \right)^{-1} \\
\prod_{a_i=0}^N (\lambda_i a + (3N - a) \lambda_i) \\
\quad \text{(from (f))} \\
+ \frac{1}{6} \sum_{i \neq j} \frac{\lambda_i^{N-4}}{w_{F_1} w_{F_3} w_{F_5} w_{F_6}} w_{F_2} w_{F_4} \\
+ \frac{\lambda_i^{N-4}}{w_{F_2} w_{F_4} w_{F_5} w_{F_6}} w_{F_2} w_{F_4} \\
+ \frac{\lambda_i^{N-4}}{w_{F_1} w_{F_3} w_{F_5} w_{F_6}} \left( \frac{1}{w_{F_1}^2} + \frac{1}{w_{F_3}^2} + \frac{1}{w_{F_5}^2} \right) \\
\frac{1}{w_{F_1}^2 w_{F_2} w_{F_3} w_{F_4} w_{F_5} w_{F_6}} \prod_{m \neq j, i} \left( \lambda_i - \lambda_{m_1} \right)^{-1} \left( \lambda_j - \lambda_{m_1} \right)^{-1} \\
\prod_{m_k \neq i, l} \left( \lambda_l - \lambda_{m_2} \right)^{-1} \left( \lambda_i - \lambda_{m_2} \right)^{-1} \\
\prod_{m_k \neq i, k} \left( \lambda_i - \lambda_{m_3} \right)^{-1} \left( \lambda_k - \lambda_{m_3} \right)^{-1} \\
\prod_{a_i=0}^N (a_1 \lambda_i + (N - a_1) \lambda_j) \\
\prod_{a_2=0}^N (a_2 \lambda_i + (N - a_2) \lambda_k) \\
\prod_{a_3=0}^N (a_3 \lambda_i + (N - a_3) \lambda_l) 
\]
\[ \prod_{n \neq i} (\lambda_i - \lambda_n)^2 \]  
(from (g))  

(4.100)

These results are generically independent of the values \( \lambda_i \), so we set \( \lambda_i \) equals to 3. Similarly we calculate the amplitudes \( \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle_1, \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle_2, (\alpha + \beta = N - 3) \). The results are collected in Table 1.\textsuperscript{~}~1 Table 3.

Note that \( \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle_n = \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle_n \cdot n \). This implies that the Kähler equation of Gromov-Witten invariants holds for the amplitudes defined by (2.47). Assuming this relation for all amplitudes, the results of Table 1~ Table 3 coincides with the ones calculated from mirror symmetry [4].\textsuperscript{~}1 We can calculate the amplitudes for matter theory for example \( \langle \mathcal{O}_{\alpha} \cdots \mathcal{O}_{\gamma} \rangle \) by a rather cunning way. Fusion rules hold in the matter theory, so we can reduce the amplitudes into the products of three-point functions.

Consider the “matter” expansion

\[ \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \mathcal{O}_{\gamma} \rangle = N + \sum_{k=1}^{\infty} \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \mathcal{O}_{\gamma} \rangle_k e^{-kt} \]  

(4.101)

where \( t \) is the deformation parameter coupled to the Kähler form. By using fusion rules, and flat metric \( \eta_{\alpha \beta} = N \cdot \delta_{\alpha + \beta, N - 2} \),

\[
\langle \mathcal{O}_{\alpha} \cdots \mathcal{O}_{\gamma} \rangle \overset{N-2\text{times}}{=} N^{5-N} \prod_{i=1}^{N-4} \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \mathcal{O}_{\gamma} \mathcal{O}_{\delta} \mathcal{O}_{\epsilon} \rangle

= N \cdot \sum_{k=1}^{\infty} \langle \mathcal{O}_{\alpha} \cdots \mathcal{O}_{\gamma} \rangle_k e^{-kt} \]  

(4.102)

Then for example, \( \langle \mathcal{O}_{\alpha} \cdots \mathcal{O}_{\gamma} \rangle_1 \) can be calculated as

\[
\langle \mathcal{O}_{\alpha} \cdots \mathcal{O}_{\gamma} \rangle_1 = -\frac{1}{2} \sum_{i \neq j \neq i, j} \prod_{a \neq 0}^{N} (\lambda_i - \lambda_k)^{-1}(\lambda_j - \lambda_k)^{-1} \prod_{a=0}^{N-1} (a\lambda_i + (N - a)\lambda_j)

((N - 4)(\lambda_j^{N-2} - \lambda_i^{N-2}) - (N - 2)(\lambda_j^{N-3} - \lambda_i^{N-3}))

\frac{1}{(\lambda_i - \lambda_j)^3}

\]

(4.103)

\[
= N^{N+1} - (N - 2) \cdot N \cdot N! \left( \frac{1}{N-1} + \cdots + \frac{1}{N-1} \right)

- 2N \cdot N! \]  

(4.104)

\textsuperscript{1}Note that for three-point function, amplitudes of the matter theory and the ones of theory coupled with gravity coincide.
If we set $\lambda_i = i$, we can derive (4.104) from (4.104) by a rather clumsy but elementary calculation. So theoretically we can see the coincidence of the calculation from A-model and B-model to the arbitrary degree $n$.

5 Construction of Free Energy

In section 4, we see that we can calculate the amplitudes $\langle \cdot \rangle_{n, \text{grav.alt.}}$ for topological sigma model on $M_N$ coupled to gravity by torus action method. As we have seen in section 3 and section 4, this method has a structure of summing over tree graphs, so we can construct a representation of Path-Integral form of the generating function of all amplitudes, i.e., free energy.

First, let us write out explicitly the contribution from $\mathcal{M}(\Gamma)$ to the amplitude $\langle \mathcal{O}_{e_1} \cdots \mathcal{O}_{e_k} \rangle_{n, \text{alt,grav.}}$

\[
\text{(Contribution from $\mathcal{M}(\Gamma)$ to $\langle \mathcal{O}_{e_1} \cdots \mathcal{O}_{e_k} \rangle_{n, \text{alt,grav.}}$)}
\]

\[
= \frac{1}{\text{Aut}(\Gamma)} \prod_{v \in \text{Vert}(\Gamma)} \lambda_{f_s(v)}^{j_s(v)} \prod_{v \in \text{Vert}(\Gamma)} w_F^{-1} \left( \sum_{\text{flags } \mathcal{F} = (v, \alpha)} w_F^{-1} \right)^{\text{val}(v) + 1} S_{-3} \prod_{v \in \text{Vert}(\Gamma)} (\lambda_{f_s} - \lambda_{f_j})^{\text{val}(v)} \prod_{a \in \text{Edges}(\Gamma)} \frac{1}{d_a}
\]

\[
\prod_{\text{flags } \mathcal{F} = (v, \alpha)} (w_F)^{-d_a} \prod_{a \in \text{Edges}(\Gamma) \setminus (e_1, e_2)} \prod_{s \neq f_s, f_j} \frac{d_a}{(m \lambda_{f_s} + (d_a - m) \lambda_{f_j} - \lambda_k)^{-1}}
\]

\[
= \prod_{\text{flags } \mathcal{F} = (v, \alpha)} (d_a!)^{-2} \prod_{v \in \text{Vert}(\Gamma) \setminus \{\text{flags } \mathcal{F} = (v, \alpha)\}} \prod_{j \neq f_s} (\lambda_{f_s} - \lambda_{f_j})^{\text{val}(v)} \prod_{a = 0}^{N\lambda_{f_s}} \prod_{v \in \text{Vert}(\Gamma)} \frac{a \lambda_{f_s} + (N - a) \lambda_{f_s}}{d_a} \prod_{a = 0}^{N\lambda_{f_s}} (N \lambda_{f_s})^{1 - \text{val}(v)}
\]

Then we classify the factors into two groups. One is the factors from edges, and the other from vertices. The factors from the edges are

(i) $\prod_{\text{flags } \mathcal{F} = (v, \alpha)} w_F^{-1}$

(ii) $\prod_{v \in \text{Vert}(\Gamma) \setminus \{\text{flags } \mathcal{F} = (v, \alpha)\}} (\lambda_{f_s} - \lambda_{f_j})^{\text{val}(v)}$

(iii) $\prod_{\text{flags } \mathcal{F} = (v, \alpha)} (w_F)^{-d_a} \prod_{a \in \text{Edges}(\Gamma) \setminus (e_1, e_2)} \prod_{s \neq f_s, f_j} \frac{d_a}{(m \lambda_{f_s} + (d_a - m) \lambda_{f_j} - \lambda_k)^{-1}} \prod_{a \in \text{Edges}(\Gamma)} (d_a!)^{-2}$
(iv) \[
\prod_{\alpha \in \text{Edg}(\Gamma)} \prod_{(v_{1}, v_{2}) \in \text{vertices } \alpha \text{ of } \Gamma} \left( \frac{a \lambda_{f_{\alpha}} + (N - a) \lambda_{f_{\alpha}}}{d_{\alpha}} \right) \prod_{v \in V_{\text{eff}}(\Gamma)} (N \lambda_{f_{v}})^{-\text{val}(v)}
\]

(v) \[
\prod_{E \in \text{Edg}(\Gamma)} \frac{1}{d_{\alpha}}
\]

And the factors we can push into the contribution from vertices are,

(i) \[
\prod_{v \in V_{\text{eff}}(\Gamma)} \lambda_{f_{v}}^{i} \]

(ii) \[
\prod_{v \in V_{\text{eff}}(\Gamma)} \left( \sum_{\text{flags } F = (v, \alpha)} \prod_{v_{1} \neq v_{2}} w_{F}^{-\text{val}(v)} + 1, S_{i} - 3 \right)
\]

(iii) \[
\prod_{v \in V_{\text{eff}}(\Gamma)} \prod_{j \neq f_{v}} (\lambda_{f_{v}} - \lambda_{j})^{-1}
\]

(iv) \[
\prod_{v \in V_{\text{eff}}(\Gamma)} (N \lambda_{f_{v}})
\]

(5.106)

Then we introduce the field variables \( \phi_{i j, d} \), propagator \( g_{i j, d t j', d'} \), vertex \( C_{i j, d} \phi_{i j, d} \phi_{i j', d'} \phi_{i j, d} \) and external field source parameters \( t_{1}, \cdots, t_{N-4} \).

In this formulation, field variables correspond to the edges with characters \( i \) and \( j \) and degree \( d \), \( g_{i j, d t j', d'} \) remains nonzero only if \( i = j' \), \( j = j' \), \( d = d' \), and the nonzero value of propagator is given as the reciprocal of the product of (5.106) (i)\(~(v)\). Then we have

\[
g_{i j, d} := g_{i j, d t j', d'} = -d^{3}(\lambda_{i} - \lambda_{j})^{2} \prod_{k=1}^{N} \prod_{a=1}^{d-1} (a \lambda_{i} + (d - a) \lambda_{j} - d \lambda_{k}) \prod_{a=1}^{N} (a \lambda_{i} + (N d - a) \lambda_{j})
\]

(5.108)

Vertex \( C_{i j, d} \phi_{i j, d} \phi_{i j', d'} \phi_{i j, d} \) are constructed with pairing the factor \( \lambda_{i}^{j} \) to \( t_{k} \) as follows.

\[
\sum_{k=1}^{\infty} \sum_{i=1}^{N} \frac{1}{\prod_{j \neq i} N \lambda_{j}} \sum_{j_{1}, \cdots, j_{k} \neq i} \sum_{j_{1}, \cdots, j_{k} \neq j_{1}, \cdots, j_{k}} (v_{i j_{1}, d_{1}} + \cdots + v_{i j, d_{k}})^{k+1} \phi_{i j_{1}, d_{1}} \cdots \phi_{i j, d_{k}} t_{j_{1}} \cdots t_{j_{k}} (\lambda_{j_{1}}^{j_{1}} + \cdots + \lambda_{j_{k}}^{j_{k}})
\]

\[
= \sum_{i=1}^{N} \frac{1}{\prod_{j \neq i} N \lambda_{j}} \sum_{k=1}^{\infty} \sum_{j_{1}, \cdots, j_{k} \neq i} \sum_{j_{1}, \cdots, j_{k}, \cdots \neq j_{1}, \cdots, j_{k}} (v_{i j_{1}, d_{1}} + \cdots + v_{i j, d_{k}})^{k} \phi_{i j_{1}, d_{1}} \cdots \phi_{i j, d_{k}}
\]

\[
\exp((t_{1} \lambda_{i} + \cdots + t_{N-4} \lambda_{N-4})^{N} (v_{i j_{1}, d_{1}} + \cdots + v_{i j, d_{k}}))
\]

\[
\frac{d}{\lambda_{i} - \lambda_{j}}
\]

(5.109)

where \( 1/k! \) is the factor that produces \( 1/4! \text{Ad}(\Gamma) \) and \( 1/1! \) is the combinatorial factor in the insertion of the external operator. With these preparation, we have the path-integral representation of the free energy.

\[F_{M_{N}}(t_{1}, \cdots, t_{N-4})\]
\[
V := \sum_{n_1, \ldots, n_{N-4} \geq 0} \langle O_{n_1} \cdots O_{n_{N-4}} \rangle \frac{t_{n_1} \cdots t_{n_{N-4}}}{n_1! \cdots n_{N-4}!} \\
= \text{Res}_{z} \text{Res}_{\varphi} \left( \frac{1}{z} \log \det (g^{-1}) \right) \frac{1}{\hbar} \int d\varphi_{ij,d} \\
\left( -\frac{1}{2} \sum_{i,j,d} \partial^2 (z \lambda_i - z \lambda_j)^2 \prod_{k=1}^{N} \prod_{l=1}^{N-4} (a z \lambda_i + (d - a) z \lambda_j) \phi_{ij,d} \phi_{ji,d} \right) \\
+ \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{N z \lambda_i}{z \lambda_i - z \lambda_j} \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{v_{ij,d_1}}{z} + \cdots + \frac{v_{ij,d_k}}{z} \right) \phi_{ij,d_1} \cdots \phi_{ij,d_k} \\
\exp \left( (t_1 z + \cdots + t_{N-4} z^N - 4)(\frac{v_{ij,d_1}}{z} + \cdots + \frac{v_{ij,d_k}}{z}) \right) \right)
\]

where we introduce \( h \) and dummy variable \( z \) to pick up the portion that comes from tree graphs and satisfies the topological selection rule (2.38).

6 Conclusion

I believe that results of this paper are reasonably clear, so I just point out what remains to show, or consider in the future. First, the relation between the representations of \( \langle \ast \rangle_{n, \text{grav}} \) and \( \langle \ast \rangle_{n, \text{grav, alt}} \). We think that the situation corresponds to the case described by Witten [16], i.e., the zero locus of the section of \( \mathcal{E}_{Nn+1} \) and external forms are not always points but submanifolds of \( \mathcal{M}_{0,n,k}^{CP^{N-1}} \). In such cases, Euler classes on these submanifolds arise and \( \langle \ast \rangle_{n, \text{grav, alt}} \) reduces to \( \langle \ast \rangle_{n, \text{grav}} \). Second, the difference between the moduli space of matter theory and the one coupled to gravity. For the matter theory we asserted in [3] that moduli spaces can be constructed by subtracting boundary points from simple projective space. This statement is indirectly supported by the work of Morrison,Plesser [13] by use of gauged linear sigma model. But stable map approach seems to go in the opposite way, i.e., it adds boundary points to compactify the moduli space. So the problem of construction of moduli spaces of matter theory still remains.

For generalization to the worldsheet of higher genera, we can think of two naive additional approaches. One is the addition of loop amplitudes. The other is the introduction of gravitational correlation functions for higher genus in the calculation of (3.76). We tried the calculation of genus 1 amplitudes with these factors but cannot get good results. So, further consideration is needed.

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Figure 1: Tree Graphs up to Degree 3
Table 1: $\langle O_{e^{-4}} \rangle_{alt,grav}$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\langle O_{e^{-4}} \rangle_1$</th>
<th>$\langle O_{e^{-4}} \rangle_2$</th>
<th>$\langle O_{e^{-4}} \rangle_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2875</td>
<td>$\frac{18708175}{4}$</td>
<td>$\frac{8564970000}{9}$</td>
</tr>
<tr>
<td>6</td>
<td>60480</td>
<td>440899200</td>
<td>6255156284160</td>
</tr>
<tr>
<td>7</td>
<td>1009792</td>
<td>122240038536</td>
<td>$\frac{27473860157}{100330728}$</td>
</tr>
<tr>
<td>8</td>
<td>15984640</td>
<td>33397163702784</td>
<td>$\frac{1386812286427888746400}{9}$</td>
</tr>
<tr>
<td>9</td>
<td>253490796</td>
<td>9757818404032059</td>
<td>897560654227562367535680</td>
</tr>
<tr>
<td>10</td>
<td>4120776000</td>
<td>3151991359959750000</td>
<td>629888601165742651840000000</td>
</tr>
</tbody>
</table>

Table 2: $\langle O_{e^0} O_{e^{-4}} \rangle_{alt,grav}$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\langle O_{e^0} O_{e^{-4}} \rangle_1$</th>
<th>$\langle O_{e^0} O_{e^{-4}} \rangle_2$</th>
<th>$\langle O_{e^0} O_{e^{-4}} \rangle_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_1 = 2875$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_1 = 60480$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_1 = 1009792$</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_2 = 1707797$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_1 = 15984640$</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_2 = 37502976$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_1 = 253490796$</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_2 = 763954092$</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_3 = 1069047153$</td>
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<tr>
<td>10</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_2 = 4120776000$</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_2 = 15274952000$</td>
<td>$\langle O_{e^0} O_{e^{-4}} \rangle_2 = 27768048000$</td>
</tr>
</tbody>
</table>
Table 3: $\langle O_c O_{ch} \rangle_{2, alt, grav}$

| N=5 | $\langle O_c O_{ch} \rangle_2 = \frac{48568.5}{2}$ |
| N=6 | $\langle O_c O_{ch} \rangle_2 = 881798400$ |
| N=7 | $\langle O_c O_{ch} \rangle_2 = 244480077072$ |
| N=8 | $\langle O_c O_{ch} \rangle_2 = \frac{1021577190083}{2}$ |
| N=9 | $\langle O_c O_{ch} \rangle_2 = 66794327405568$ |
| N=10 | $\langle O_c O_{ch} \rangle_2 = 224340722909184$ |

$\langle O_c O_{ch} \rangle_2 = 19515636808064118$
$\langle O_c O_{ch} \rangle_2 = 93777295510651590$
$\langle O_c O_{ch} \rangle_2 = \frac{312074853388012521}{2}$

$\langle O_c O_{ch} \rangle_2 = 6309398271991949500000$
$\langle O_c O_{ch} \rangle_2 = 40342298393756700000$
$\langle O_c O_{ch} \rangle_2 = 100290980414189400000$