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Explicit and approximate solutions
of a class of evolution problems
in Hilbert space

by

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Abstract

Explicit closed-form solutions for a class of first order evolution problems in a Hilbert space are presented. On the basis of these explicit representations we construct approximations adapted to the regularity of the initial data.

Keywords: Evolution problems, Cayley transform, Laguerre polynomials, explicit representation

1. Introduction

The aim of this paper is to give an algorithmically constructive explicit representation of the solution of the evolution problem

$$\frac{du}{dt} + \varphi(t, A)u = 0, \quad u(0) = u_0, \quad (1)$$

where $u(t) : \mathbb{R}_+ \rightarrow H$ is a function with values in a Hilbert space H , $A = A^* > 0$ is a selfadjoint positive definite operator with the domain $D(A)$ dense in H . We assume that the function $\varphi(t, z)$ of the complex variable z satisfies the following conditions (B):

i)

$$\varphi(t, z) = \varphi_0(t, z) + \sum_{n=1}^m \varphi_n(t) z^n, \quad (2)$$

where

$$|\varphi_0(t, z)| \leq \phi(t) \quad \forall t \in [0, \infty), \quad \forall z \in Z = \left\{ z \mid \arg z \in \left[-\frac{\pi}{2m}, \frac{\pi}{2m} \right] \right\} \quad (3)$$

and the function $\phi(t)$ is bounded together with t and integrable;

ii) $\varphi_n(t)$ are bounded together with t and

$$\int_0^t \varphi_n(s) ds \geq 0 \quad \forall t \in [0, \infty), \quad n = \overline{1, m}; \quad (4)$$

iii)

$$\varphi(t, A) > 0 \quad \forall t \in [0, \infty). \quad (5)$$

For $\varphi(t, A) \equiv A$ this problem was studied in [1,2,4], where the following representation was found and justified:

$$u(t) = e^{-\gamma t} \sum_{k=0}^{\infty} (-1)^k L_k^{(0)}(2\gamma t) (y_{\gamma, k} + y_{\gamma, k+1}). \quad (6)$$

Here γ is an arbitrary positive number, $L_k^{(0)}(s)$ are the Laguerre polynomials and the sequence $\{y_{\gamma, k}\}_{k=0}^{\infty}$ is defined by

$$y_{\gamma, k+1} = T_{\gamma} y_{\gamma, k}, \quad k \geq 0, \quad y_{\gamma, 0} = u_0, \quad (7)$$

where $T_{\gamma} = (\gamma I - A)(\gamma I + A)$ is the Cayley transform. The recursion (7) is equivalent to the sequence of the following operator equations with the same operator but with various right-hand sides:

$$(\gamma I + A) y_{\gamma, k+1} = (\gamma I - A) y_{\gamma, k}, \quad k \geq 0, \quad y_{\gamma, 0} = u_0. \quad (8)$$

As an approximate solution there was considered the truncated sum

$$u^N(t) = e^{-\gamma t} \sum_{k=0}^N (-1)^k L_k^{(0)}(2\gamma t) (y_{\gamma, k} + y_{\gamma, k+1}). \quad (9)$$

We would like to point out the following important features of (6) and (9): 1) exclusively contractive operators T_{γ} are used; 2) the non-stationary problem is

decomposed into a sequence of stationary problems (8) (the variable t is eliminated and one can consider (6), (9) as a discretization-in-time of the non-stationary problem); 3) the accuracy of the approximation (9) automatically depends on the smoothness of the initial data, namely the following estimates hold true:

$$\begin{aligned} \sup_{t \in [0, \infty)} \|u(t) - u^N(t)\| &\leq cN^{-\sigma} \|A^\sigma x_0\| \\ \sup_{t \in [\varepsilon, \omega] \subset (0, \infty)} \|u(t) - u^N(t)\| &\leq cN^{-\sigma - \frac{1}{4}} \|A^\sigma x_0\| \quad \forall \varepsilon, \omega, \end{aligned} \quad (10)$$

provided that $x_0 \in D(A^\sigma)$. Here and in what follows the letter c stands for various constants independent of u and N .

In the present paper we extend these results onto the class of problems (1). Here, we use a new technique of analysis different from that of [1,2,4].

2. Representation of the solution

At first we consider an auxiliary problem

$$\frac{du}{dt} + \varphi \left(t, \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{1/m} \right) u(t) = 0, \quad u(0) = 1, \quad (11)$$

where $\gamma > 0$ is an arbitrary positive number and $|\zeta| < 1$. We can find the solution of (11) in the form

$$u(t) \equiv u(t, \zeta) = \exp \left(- \int_0^t \varphi \left(s, \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{1/m} \right) ds \right) \quad (12)$$

or as the series

$$u(t, \zeta) = \sum_{k=0}^{\infty} v_k(t) \zeta^k, \quad (13)$$

where by the Cauchy formula

$$\begin{aligned} v_k(t) &= \frac{1}{2\pi i} \int_{|\zeta|=\varrho_k} \exp \left\{ - \int_0^t \varphi_0 \left(s, \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{1/m} \right) ds \right. \\ &\quad \left. - \sum_{n=1}^m \int_0^t \varphi_n(s) ds \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{n/m} \right\} \frac{d\zeta}{\zeta^{k+1}} \quad \forall \varrho_k \in (0, 1). \end{aligned} \quad (14)$$

In order to estimate $v_k(t)$ we substitute

$$\begin{aligned} \varrho_k &= 1 - \frac{1}{k}, \\ \zeta &= \varrho_k e^{i\theta}, \quad \theta \in [0, 2\pi), \quad \gamma \frac{1+\zeta}{1-\zeta} = a + ib = r e^{i\varphi}, \end{aligned} \quad (15)$$

where

$$a = \gamma \frac{1 - \varrho^2}{1 - 2\varrho \cos \theta + \varrho^2}, \quad b = \frac{2\varrho \gamma \sin \theta}{1 - 2\varrho \cos \theta + \varrho^2}, \quad (16)$$

in (14). Obviously, we have $a > 0$, $b \in (-\infty, +\infty)$, hence $\varphi \in [-\pi/2, \pi/2]$.

Lemma 1. Let the conditions (B) are satisfied. Then there exist functions $M(t)$, $M_1(t)$ bounded together with t such that the estimates

$$|v_k(t)| \leq M(t) \quad \forall t \in [0, \infty), \quad (17)$$

$$|v'_k(t)| \leq k M_1(t) \quad \forall t \in [0, \infty) \quad (18)$$

hold uniformly in k .

Proof. By (14), (15), (3) and (4) we obtain for $\varrho = 1 - 1/k$

$$|v_k(t)| \leq \frac{c}{2\pi} \exp \left\{ \int_0^t \phi(s) ds \right\} \int_0^{2\pi} \exp \left\{ - \sum_{n=1}^m \int_0^t \varphi_n(s) ds r^{n/m} \cos \frac{\varphi n}{m} \right\} d\theta.$$

Since $\varphi n/m \in [-\pi/2, \pi/2]$ for $n = \overline{1, m}$, we have $\cos(\varphi n/m) \geq 0$. Hence,

$$|v_k(t)| \leq c \exp \left\{ \int_0^t \phi(s) ds \right\} \equiv M(t). \quad (19)$$

Next, we consider the first derivative

$$\begin{aligned} v'_k(t) &= \frac{1}{2\pi i} \int_{|\zeta|=\varrho_k} \exp \left\{ - \int_0^t \varphi_0 \left(s, \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{1/m} \right) ds \right. \\ &\quad \left. - \sum_{n=1}^m \int_0^t \varphi_n(s) ds \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{n/m} \right\} \\ &\quad \cdot \left\{ -\varphi_0 \left(t, \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{1/m} \right) - \sum_{n=1}^m \varphi_n(t) \left(\gamma \frac{1+\zeta}{1-\zeta} \right)^{n/m} \right\} \frac{d\zeta}{\zeta^{k+1}}. \end{aligned}$$

Proceeding as done above, we get

$$|v'_k(t)| \leq \frac{1}{2\pi} M(t) \int_0^{2\pi} \left\{ \phi(t) + \sum_{n=1}^m |\varphi_n(t)| r^{n/m}(\varrho_k) \right\} d\theta. \quad (20)$$

Simple calculations show that

$$\begin{aligned} r^2(\varrho, x) &= \gamma^2 \frac{1 - 2\varrho^2 + \varrho^4 + 4\varrho^2(1-x^2)}{(1-2\varrho x + \varrho^2)^2} = \gamma^2 \frac{(1+\varrho^2)^2 - 4\varrho^2 x^2}{(1-2\varrho x + \varrho^2)^2}, \\ \frac{\partial r^2}{\partial x} &= \frac{4\varrho(1+\varrho^2)}{(1-2\varrho x + \varrho^2)^2} > 0, \end{aligned}$$

where $x = \cos \theta \in [-1, 1]$ what yields that

$$\begin{aligned} r_{\max}^2(\varrho) &= \gamma^2 \frac{(1-\varrho^2)^2}{(1-\varrho)^4} = \gamma^2 \left(\frac{1+\varrho}{1-\varrho} \right)^2, \\ r_{\max}(\varrho_k) &= \gamma \frac{1+\varrho_k}{1-\varrho_k} = \gamma(2k-1). \end{aligned} \quad (21)$$

It follows from (20), (21) that

$$|v'_k(t)| \leq M_1(t) k$$

and the proof is complete.

By analogy with (13) we look for the solution of (1) in the following form

$$\mu(t) = \sum_{k=0}^{\infty} (-1)^k v_k(t) T_{\gamma}^k u_0, \quad (22)$$

where

$$-T_{\gamma} = (\gamma I - A^m)(\gamma I + A^m)^{-1} \quad (23)$$

is the Cayley transform of the operator A^m .

Remark 1. If $m = 1$, $\varphi_0 \equiv 0$, $\varphi_1 \equiv 1$ we get

$$v_k(t) = e^{-\gamma t} [L_k^{(0)}(2\gamma t) - L_{k-1}^{(0)}(2\gamma t)] \quad (24)$$

(see also [1,2,4]).

Together with (22) we consider the series

$$v(t) = \sum_{k=0}^{\infty} (-1)^k \dot{v}_k(t) T_{\gamma}^k u_0, \quad (25)$$

which one obtains by formal differentiation of (22). First of all we have to investigate the question about convergence of (22) and (25) in H .

Lemma 2. Let the functions $\varphi_n(t)$, $n = \overline{0, m}$ be bounded on any bounded interval $[0, T]$, the operator A be densely defined, $A = A^* > 0$ and $u_0 \in D(A^{m\sigma})$ for $\sigma > 0$. Then

- i) the series (22) converges in H uniformly in $t \in [0, T]$ and $u(t)$ is continuous on $[0, T]$ provided that $\sigma > 1$;
- ii) the series (25) converges in H uniformly in $t \in [0, T]$, $v(t)$ is continuous on $[0, T]$ and $v(t) = \dot{u}(t)$ provided that $\sigma > 2$;
- iii) $u(t) \in D(A^m) \forall t \in [0, T]$ provided that $\sigma > 1$.

Proof. Let E_{λ} be the spectral family corresponding to A^m and λ_0 be the minimal eigenvalue of A^m . Then we have

$$y_{\gamma, k} \equiv T_{\gamma}^k u_0 = \int_{\lambda_0}^{\infty} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^k \lambda^{-\sigma} dE_{\lambda} A^{\sigma m} u_0.$$

If k is sufficiently large, then the function

$$\psi(\lambda) = \frac{1}{\lambda^{2\sigma}} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^{2k}$$

takes its maximum

$$\psi_{\max} \equiv \psi(\lambda^*) = \frac{\sigma^{2\sigma}}{\gamma^{2\sigma} (k + \sqrt{k^2 + \sigma^2})^{2\sigma}} \left(1 - \frac{2\sigma}{\sigma + k + \sqrt{k^2 + \sigma^2}} \right)^{2k} \quad (26)$$

at the point $\lambda^* = \frac{\gamma}{\sigma} (k + \sqrt{k^2 + \sigma^2})$. It is evident from (26) that

$$\psi(\lambda) \leq \frac{c(\sigma)}{k^{2\sigma}},$$

where the constant $c(\sigma)$ is independent of k . Therefore, we have

$$\|y_{\gamma, k}\|^2 \leq \frac{c(\sigma)}{k^{2\sigma}} \|u_0^{\sigma}\|, \quad x_0^{\sigma} = A^{\sigma m} u_0 \quad (27)$$

and it results from Lemma 1 that the series (22) and (25) are majorized by the number series

$$c \sum_{k=1}^{\infty} k^{-\sigma} \quad \text{and} \quad c \sum_{k=1}^{\infty} k^{-\sigma+1},$$

respectively. This implies that the series (22) converges to a continuous function $u(t)$ if $\sigma > 1$ and the series (25) converges to a continuous function $v(t) = \dot{u}(t)$ provided that $\sigma > 2$. If $u_0 \in D(A^{\sigma m})$, $\sigma > 1$ then for the truncated sum $u^N(t)$ of the series (22) we have

$$\begin{aligned} \varphi(t, A) u^N(t) &= \varphi(t, A) \sum_{k=0}^N (-1)^k v_k(t) T_{\gamma}^k u_0 \\ &= \sum_{k=0}^N (-1)^k v_k(t) T_{\gamma}^k \varphi(t, A) u_0. \end{aligned} \quad (28)$$

Let us consider the series

$$u_{\varphi}(t) = \sum_{k=0}^{\infty} (-1)^k v_k(t) T_{\gamma}^k \varphi(t, A) u_0. \quad (29)$$

We have

$$\|T_{\gamma}^k \varphi(t, A) u_0\|^2 = \int_{\lambda_0}^{\infty} \frac{1}{\lambda^{2\sigma}} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^{2k} \varphi^2(t, \lambda^{1/m}) d \|E_{\lambda} u_0^{\sigma}\|^2.$$

It follows from (B) and from the assumptions of Lemma 2 that

$$\varphi(t, \lambda^{1/m}) \leq \phi(t) \lambda \quad \forall t \in [0, T], \forall \lambda \in [\lambda_0, \infty),$$

where the function $\phi(t)$ is bounded on bounded intervals. Proceeding as done before, we deduce analogously to (27)

$$\|T_{\gamma}^k \varphi(t, A) x_0\|^2 \leq \frac{c}{k^{2(\sigma-1)}} \|u_0^{\sigma}\|^2.$$

Hence, the series (29) is majorized by the number series

$$c \sum_{k=1}^{\infty} k^{-\sigma+1}$$

which converges for $\sigma > 2$ and so the series (29) converges in H uniformly in $t \in [0, T]$ to a continuous function $u_{\varphi}(t)$. Since $\lim_{N \rightarrow \infty} u^N(t) = u(t)$, the closedness of the operator $\varphi(t, A)$ implies that $u(t) \in D(\varphi(t, A)) = D(A^m)$ provided that $\sigma > 2$. The proof is complete.

Next, we prove that the function $u(t)$ given by (22) is the solution of (1) on a bounded interval $[0, T]$ in the following sense: 1) $u(t)$ is continuous on $[0, T]$ and continuously differentiable on $(0, T]$; 2) $u(t) \in D(\varphi(t, A)) \quad \forall t \in (0, T]$ and $u(t)$ satisfies (1).

Theorem 1. Let A be a densely defined, selfadjoint, positive operator, the function $\varphi(t, z)$ satisfies the conditions (B) and $x_0 \in D(A^{\sigma m})$ for $\sigma > 2$. Then the function $u(t)$ defined by (22) is the solution of (1).

Proof. It follows from Lemma 2 that $u(t) \in D(\varphi(t, A))$ and

$$\varphi(t, A) u(t) = u_{\varphi}(t) = \sum_{k=0}^{\infty} (-1)^k v_k(t) \int_{\lambda_0}^{\infty} \frac{1}{\lambda^{\sigma}} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^k \varphi(t, \lambda^{1/m}) dE_{\lambda} u_0^{\sigma}. \quad (30)$$

On the other hand, taking into account (12), (13) we can write down

$$\begin{aligned}\varphi(t, A) u(t) &= \int_{\lambda_0}^{\infty} \sum_{k=0}^{\infty} (-1)^k v_k(t) \frac{1}{\lambda^\sigma} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^k \varphi(t, \lambda^{1/m}) dE_\lambda u_0^\sigma \\ &= \int_{\lambda_0}^{\infty} \lambda^{-\sigma} \varphi(t, \lambda^{1/m}) \exp \left(- \int_0^t \varphi(s, \lambda^{1/m}) ds \right) dE_\lambda u_0^\sigma. \quad (31)\end{aligned}$$

The integral and the sum can be interchanged because a) the function

$$f(t, \lambda, N) = \sum_{k=0}^N (-1)^k v_k(t) \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^k$$

$\forall N \in \mathbb{N}$ is integrable on $[\lambda_0, \Lambda] \forall \Lambda > \lambda_0$ and $f(t, \lambda, N)$ converges to the function

$$w \left(t, \frac{\lambda - \gamma}{\lambda + \gamma} \right) = \exp \left(- \int_0^t \varphi(s, \lambda^{1/m}) ds \right) \quad (32)$$

uniformly in λ : b) the integral

$$I(N) = \int_{\lambda_0}^{\infty} f(\lambda, t, N) dE_\lambda$$

converges uniformly in $N \in \mathbb{N}$ due to the relations

$$|f(t, \lambda, N)| \leq M(t) \frac{\lambda + \gamma}{2\gamma}, \quad \int_{\lambda_0}^{\infty} (\lambda + \gamma) dE_\lambda = A + \lambda I.$$

The function (32) and the derivative

$$\frac{d}{dt} w \left(t, \frac{\gamma - \lambda}{\gamma + \lambda} \right) = \varphi(t, \lambda^{1/m}) \exp \left(- \int_0^t \varphi(s, \lambda^{1/m}) ds \right)$$

are continuous with respect to $\lambda \in [\lambda_0, \infty)$ and $t \in [0, T]$. The integral

$$I_1(t) = \int_{\lambda_0}^{\infty} \frac{d}{dt} w \left(t, \frac{\gamma - \lambda}{\gamma + \lambda} \right) dE_\lambda$$

converges uniformly in $t \in [0, T]$ due to the inequality

$$\frac{d}{dt} w \left(t, \frac{\gamma - \lambda}{\gamma + \lambda} \right) \leq \varphi(t, \lambda^{1/m}) \leq M_2(t)$$

with a function $M_2(t)$ bounded on bounded intervals. Therefore, we deduce from (22):

$$\frac{d}{dt} u(t) = \frac{d}{dt} \int_{\lambda_0}^{\infty} \sum_{k=0}^{\infty} (-1)^k v_k(t) \lambda^{-\sigma} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^k dE_\lambda u_0^\sigma$$

$$\begin{aligned}
&= \frac{d}{dt} \int_{\lambda_0}^{\infty} \lambda^{-\sigma} \exp \left(- \int_0^t \varphi(s, \lambda^{1/m}) ds \right) dE_{\lambda} u_0^{\sigma} \\
&= \int_{\lambda_0}^{\infty} \lambda^{-\sigma} \frac{d}{dt} \exp \left(- \int_0^t \varphi(s, \lambda^{1/m}) ds \right) dE_{\lambda} u_0^{\sigma} \\
&= - \int_{\lambda_0}^{\infty} \lambda^{-\sigma} \varphi(t, \lambda^{1/m}) \exp \left(- \int_0^t \varphi(s, \lambda^{1/m}) ds \right) dE_{\lambda} u_0^{\sigma}.
\end{aligned} \tag{33}$$

Now, it follows from (31). (33) that

$$\frac{d}{dt} u(t) + \varphi(t, A) u(t) = 0.$$

To complete the proof it suffices to observe that (12), (13) and (22) also imply $u(0) = u_0$.

Remark 2. The series (22) for $\sigma > 1$ can be understood as a generalized solution.

3. Approximate solution

We consider the truncated sum

$$u^N(t) = \sum_{k=0}^N (-1)^k v_k(t) T_{\gamma}^k u_0 \tag{34}$$

as an approximate solution. The next result concerns the accuracy of this solution.

Theorem 2. Under the assumptions of Theorem 1 with $\sigma > 1$ the estimate

$$\|u^N(t) - u(t)\| \leq c N^{-\sigma+1} \|u_0^{\sigma}\| \tag{35}$$

holds uniformly in $t \in [0, T]$.

Proof. Lemma 1 and (27) imply

$$\|u^N(t) - u(t)\| \leq c \|u_0^{\sigma}\| \sum_{k=N+1}^{\infty} k^{-\sigma} \leq c N^{-\sigma+1} \|u_0^{\sigma}\|,$$

which is what had to be proved.

Remark 3. In the case $m = 1$, $\varphi_0 \equiv 0$, $\varphi_1 \equiv 1$ we have two representations, namely (9) and (22) with $v_k(t)$ defined by (24). For the truncated sum (9) we have the estimates (9) and (10) whereas for the truncated sum (34) the estimate (35), which is worse than (9) and (10). This example shows that the rate of convergence (accuracy) can be improved by changing the summation procedure.

4. Examples

The problem

$$\begin{aligned}
\frac{\partial u}{\partial t} - \varphi_1(t) \frac{\partial^2 u}{\partial x^2} + \varphi_2(t) \frac{\partial^4 u}{\partial x^4} &= 0, \\
u(0, x) &= u_0(x), \quad x \in (0, 1), \\
u(t, 0) &= u(t, 1) = 0, \quad t \in [0, T], \\
\frac{\partial^2 u}{\partial x^2}(t, 0) &= \frac{\partial^2 u}{\partial x^2}(t, 1) = 0, \quad t \in [0, T]
\end{aligned} \tag{36}$$

belongs to the class (1), where $u_0(x)$, $\varphi_1(x)$, $\varphi_2(t)$ are given functions, the functions $\varphi_i(t)$, $i = 1, 2$ are bounded and integrable on $[0, T]$,

$$\int_0^t \varphi_i(s) ds \geq 0 \quad \forall t \in [0, T],$$

$m = 2$, $\varphi_0(t, z) \equiv 0$, $\varphi(t, z) = \sum_{n=1}^2 \varphi_n(t) z^n$, $D(A) = \{u \in W_2^2(0, 1) \mid u(0) = u(1) = 0\}$, $Au = -\partial^2 u / \partial x^2$. The auxiliary function (12) can be presented in the form

$$w(t, \zeta) = \exp \left(\int_0^t \varphi_1(s) ds \cdot \sqrt{\gamma \frac{1+\zeta}{1-\zeta}} \right) \exp \left(- \int_0^t \varphi_2(s) ds \cdot \gamma \frac{1+\zeta}{1-\zeta} \right). \quad (37)$$

Comparing the multipliers in (37) with the generating functions for the Laguerre polynomials $L_n^{(\alpha)}(t)$ [3]

$$(1-z)^{-\alpha-1} \exp \frac{tz}{z-1} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(t) z^n, \quad |z| < 1 \quad (38)$$

and for the Meixner polynomials $P_n(t)$ [6]

$$\begin{aligned} f(z) \exp(t\tilde{u}(z)) &= \sum_{n=0}^{\infty} \frac{P_n(t)}{n!} z^n, \\ f(0) &= 1, \quad \tilde{u}(0) = 0, \quad \tilde{u}'(0) = 1 \end{aligned} \quad (39)$$

with the functions $f(z) \equiv 1$, $\tilde{u}(z) = \sqrt{(1+z)/(1-z)} - 1$, we get the expansion (13) with the coefficients

$$\begin{aligned} v_k(t) &= \exp \left(\sqrt{\gamma} \int_0^t \varphi_1(s) ds - \gamma \int_0^t \varphi_2(s) ds \right) \sum_{j=0}^k \frac{1}{j!} P_j \left(\sqrt{\gamma} \int_0^t \varphi_1(s) ds \right) \\ &\quad \cdot \left[L_{k-j}^{(0)} \left(2\gamma \int_0^t \varphi_2(s) ds \right) - L_{k-j-1}^{(0)} \left(2\gamma \int_0^t \varphi_2(s) ds \right) \right]. \end{aligned}$$

The first ten Meixner polynomials are

$$\begin{aligned} P_0(t) &= 1, \quad P_1(t) = t, \quad P_2(t) = \frac{t(t+1)}{2}, \quad P_3(t) = \frac{t(3+3t+t^2)}{6}, \\ P_4(t) &= \frac{t(9+15t+6t^2+t^3)}{24}, \quad P_5(t) = \frac{t(45+75t+45t^2+10t^3+t^4)}{120}, \\ P_6(t) &= \frac{t(225+495t+330t^2+105t^3+15t^4+t^5)}{720}, \\ P_7(t) &= \frac{t(1575+3465t+2835t^2+1050t^3+210t^4+21t^5+t^6)}{5040}, \\ P_8(t) &= \frac{t(11025+29295t+25830t^2+11445t^3+2730t^4+378t^5+28t^6+t^7)}{40320}, \\ P_9(t) &= \frac{t(99225+263655t+265545t^2+130410t^3+36855t^4+6174t^5)}{362880} \\ &\quad + \frac{t(630t^6+36t^7+t^8)}{362880}, \\ P_{10}(t) &= \frac{t(893025+2735775t+2920050t^2+1611225t^3+511245t^4+101115t^5)}{3628800} \\ &\quad + \frac{t(12600t^6+990t^7+45t^8+t^9)}{3628800}. \end{aligned}$$

The solution of (36) is

$$u(t, x) = \sum_{k=0}^{\infty} (-1)^k v_k(t) y_{\gamma, k}(x), \quad (40)$$

where

$$\begin{aligned} y_{\gamma, k} &= T_{\gamma} y_{\gamma, k-1}, \quad k \geq 1, \quad y_{\gamma, 0} = u_0(x), \\ T_{\gamma} &= (\gamma I - A^2)(\gamma I + A^2)^{-1}. \end{aligned} \quad (41)$$

In other words, (41) is the sequence of the following boundary value problems

$$\begin{aligned} \frac{d^4 y_{\gamma, k}(x)}{dx^4} + \gamma y_{\gamma, k}(x) &= \frac{d^4 y_{\gamma, k-1}(x)}{dx^4} - \gamma y_{\gamma, k-1}(x), \\ y_{\gamma, k}(0) &= y_{\gamma, k}(1) = 0, \\ \frac{d^2 y_{\gamma, k}(0)}{dx^2} &= \frac{d^2 y_{\gamma, k}(1)}{dx^2} = 0. \end{aligned} \quad (42)$$

Let $G(x, \xi)$ be Green's function for the operator $A^2 + \gamma I$, which can be found explicitly using fundamental solutions

$$\begin{aligned} u_1(x) &= \cosh x \cdot \cos(\kappa x), \quad u_2(x) = \cosh(\kappa x) \sin(\kappa x), \\ u_3(x) &= \sinh(\kappa x) \cos(\kappa x), \quad u_4(x) = \sinh(\kappa x) \sin(\kappa x) \end{aligned}$$

of the homogeneous equation $u^{(4)}(x) + \gamma u(x) = 0$, $\gamma = 4\kappa^4$ (see [5]). Then one can find the sequence $\{y_{\gamma, k}\}$ numerically using (42) or by

$$\begin{aligned} y_{\gamma, k}(x) &= \int_0^1 G(x, \xi) \left[y_{\gamma, k-1}^{(4)}(\xi) - \gamma y_{\gamma, k-1}(\xi) \right] d\xi, \quad k = 0, 1, \dots \\ y_{\gamma, 0}(x) &= u_0(x) \end{aligned}$$

with support of computer algebra tools.

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