ENTE PER LE NUOVE TECNOLOGIE,
L'ENERGIA E L'AMBIENTE

Dipartimento Innovazione

SUPERSYMMETRIC PHASE-SPACE OPERATORS

GIUSEPPE DATTOLI, AMALIA TORRE

Centro Ricerche di Frascati, Roma

Submitted to *Nuovo Cimento A*

RT/INN/94/50

Manuscript received in final form on March 1994

Printed on March 1995

This report has been prepared, and distributed by: *Servizio Edizioni Scientifiche* - ENEA, Centro Ricerche Frascati, C.P. 65 - 00044 Frascati, Rome, Italy.

Published by ENEA, Direzione Relazioni Esterne, Viale Regina Margherita 125, Rome, Italy

SUMMARY

We consider a supersymmetric extension of the phase-space creation-annihilation operators and of the relevant formalism. We show that supersymmetric phase-space partner potentials can be introduced and the concepts of supersymmetric quantum mechanics can be almost naturally extended to the phase-space case.

RIASSUNTO

Si discute una estensione supersimmetrica degli operatori di creazione e distruzione definiti nello spazio delle fasi e del relativo formalismo. Si dimostra come si possano introdurre coppie di potenziali supersimmetrici e che i concetti della meccanica quantistica supersimmetrica possono essere estesi, in maniera piuttosto naturale, alla formulazione della medesima nello spazio delle fasi.

1. Introduction

The phase-space formalism¹ has been proved to be a powerful tool to deal with a variety of problems in classical and quantum optics.²

Within such a context, a class of Hermite polynomials with many variables and many indices and the associated orthogonal functions may play a role of crucial importance.³

The theory of these functions has been discussed in Ref. 4, where a formalism based on generalized forms of creation-annihilation operators has been developed. In this paper we go a step further than the results of Ref. 4, introducing the concepts of supersymmetric phase-space creation-annihilation operators and of supersymmetric phase-space mechanics. Using a point of view analogous to the one leading to the formulation of supersymmetric quantum mechanics,⁵⁻⁷ we will show that the concepts and formalisms associated with the phase-space creation-annihilation operators of Ref. 4 are not restricted to quadratic potentials only.

In this introductory section we review the formalism of Ref. 4, while Secs. 2 and 5 are devoted to its supersymmetric generalization.

We consider, for simplicity, a two-dimensional phase-space, whose vectors are specified by

$$(1) \quad \underline{z} = \begin{pmatrix} x \\ p \end{pmatrix}.$$

Along with (1), we introduce the operator

$$(2) \quad \underline{\hat{c}}_z = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial p} \end{pmatrix}.$$

According to Ref. 4 we introduce the phase-space (ps) creation-annihilation operators

$$(3) \quad \hat{\underline{A}}^+ = \frac{1}{2} \hat{M} \underline{z} - \underline{\hat{c}}_z,$$

$$\hat{\underline{A}}^- = \hat{M}^{-1} \underline{\hat{c}}_z + \frac{1}{2} \underline{z},$$

where $\hat{M}$ is the following 2×2 matrix:

$$(4) \quad \hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, c > 0$$

$$\Delta = ac - b^2 > 0.$$

The operators (3) are two-component operators and the associated "number" operator is

$$(5) \quad \left(\hat{\underline{A}}^+ \right)^T \hat{\underline{A}}^- = -\underline{\hat{c}}_z^T \hat{M}^{-1} \underline{\hat{c}}_z + \frac{1}{4} \underline{z}^T \hat{M} \underline{z}$$

$$+ \frac{1}{2} \underline{z}^T \cdot \underline{\hat{c}}_z - \frac{1}{2} \underline{\hat{c}}_z^T \cdot \underline{z},$$

where the superscript " T " denotes transpose. Noting that

$$(6) \quad \underline{\hat{c}}_z^T \underline{z} = 2 + \underline{z}^T \cdot \underline{\hat{c}}_z,$$

we find

$$(7) \quad \left(\hat{\underline{A}}^+ \right)^T \hat{\underline{A}}^- = -\underline{\hat{c}}_z^T \hat{M}^{-1} \underline{\hat{c}}_z + \frac{1}{4} \underline{z}^T \hat{M} \underline{z} - 1.$$

We can immediately make the proper correspondences. The operator $-\underline{\partial}_z^T \hat{M}^{-1} \underline{\partial}_z$ replaces, in this formulation, the second derivative operator, $\frac{1}{4} \underline{z}^T \hat{M} \underline{z}$ is a positive-definite quadratic form, 1 is associated with the vacuum field fluctuation of the ps problem under study (see Ref. 4).

The commutation properties of the $\hat{A}$ operators are, furthermore, stated as follows:

$$(8) \quad \left[\left(\hat{A}^- \right), \left(\hat{A}^+ \right) \right] = \left(\hat{A}^- \right)^T \left(\hat{A}^+ \right) - \left(\hat{A}^+ \right)^T \left(\hat{A}^- \right) = \underline{z}.$$

The "states" on which the above operators act are the ps harmonic oscillator states denoted by $|m, n\rangle$ and such that

$$(9) \quad \hat{A}^+ |m, n\rangle = \begin{pmatrix} \sqrt{m+1} |m+1, n\rangle & 0 \\ 0 & \sqrt{n+1} |m, n+1\rangle \end{pmatrix}$$

$$\hat{A}^- |m, n\rangle = \begin{pmatrix} \sqrt{m} |m-1, n\rangle & 0 \\ 0 & \sqrt{n} |m, n-1\rangle \end{pmatrix}.$$

In the (x, p) variables we find⁴

$$(10a) \quad \langle x, p | m, n \rangle = \mathcal{H}_{m,n}(x, p) = \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}} \frac{1}{\sqrt{m!n!}} H_{m,n}(x, p) e^{-\frac{1}{4} \underline{z}^T \hat{M} \underline{z}},$$

where $H_{m,n}(x, p)$ is the two-index, two-variable Hermite polynomial defined by

$$(10b) \quad H_{m,n}(x, p) = (-1)^{m+n} e^{\frac{1}{2} \underline{z}^T \hat{M} \underline{z}} \frac{\partial^{m+n}}{\partial x^m \partial p^n} e^{-\frac{1}{2} \underline{z}^T \hat{M} \underline{z}}.$$

The ground state is, therefore, specified by

$$(11) \quad \langle x, p | 0, 0 \rangle = \mathcal{H}_{0,0}(x, p).$$

It is also clear that

$$(12) \quad \left(\underline{\hat{A}}^+ \right)^T \underline{\hat{A}} \mathcal{H}_{m,n}(x, p) = (m + n) \mathcal{H}_{m,n}(x, p).$$

Before closing this section, it is worth stressing further that the $\underline{\hat{A}}$ operators are two-component operators and that the single components write

$$(13a) \quad \underline{\hat{A}}^+ = \begin{pmatrix} \hat{a}_{+,m} \\ \hat{a}_{+,n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(ax + bp) - \frac{\partial}{\partial x} \\ \frac{1}{2}(bx + cp) - \frac{\partial}{\partial p} \end{pmatrix}$$

$$\underline{\hat{A}}^- = \begin{pmatrix} \hat{a}_{-,m} \\ \hat{a}_{-,n} \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta} \left(c \frac{\partial}{\partial x} - b \frac{\partial}{\partial p} \right) + \frac{1}{2}x \\ \frac{1}{\Delta} \left(a \frac{\partial}{\partial p} - b \frac{\partial}{\partial x} \right) + \frac{1}{2}p \end{pmatrix}$$

and it can also be checked that

$$(13b) \quad [\hat{a}_{-,m}, \hat{a}_{+,n}] = \delta_{n,m}$$

$$[\hat{a}_{\pm,m}, \hat{a}_{\pm,n}] = 0 \quad \text{also for } n = m$$

We have so far given a very quick view of the properties of ps creation-annihilation operators; the results of this section are, however, sufficient for the forthcoming considerations.

2. Supersymmetric Phase-Space Operators

In this section we show that we can go beyond the previously introduced operators, discussing their supersymmetric (SUSY) version.

We have noted that the quadratic form $\frac{1}{2}\underline{z}^T \hat{M} \underline{z}$ replaces the ordinary quadratic potential and that an analogous expression involving the $\underline{\partial}_z$ operator replaces the second-order derivative of the differential equation defining the ordinary harmonic oscillator eigenfunction. We have, furthermore, noted that ps harmonic oscillator eigenfunctions exist and that their "ground-state" is specified by $\mathcal{H}_{0,0}(x,p)$.

We assume now that a potential $V_-(x,p)$ with ground state $\psi_{0,0}(x,p)$ exists. We furthermore assume that the ground-state energy can be adjusted so that $E_{0,0}^- = 0$. The ps Schrödinger equation for this state can be written as⁵⁻⁷

$$(14a) \quad H_-(x,p) \psi_{0,0}(x,p) = \left(-\underline{\partial}_z^T \hat{M}^{-1} \underline{\partial}_z + V_-(x,p) \right) \psi_{0,0}(x,p) = 0.$$

As a consequence $H_-(x,p)$ can be explicitly written as

$$(14b) \quad H_-(x,p) = -\underline{\partial}_z^T \hat{M}^{-1} \underline{\partial}_z + \frac{1}{\psi_{0,0}(x,p)} \left[\underline{\partial}_z^T \hat{M}^{-1} \underline{\partial}_z \psi_{0,0}(x,p) \right].$$

In analogy to what is done in supersymmetric quantum mechanics (SUSY QM) and to Eq. (3), we define the operators

$$(15) \quad \underline{\hat{S}}^+ = -\underline{\hat{\mathcal{C}}}_z - \frac{1}{\psi_{0,0}}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0})$$

$$\underline{\hat{S}}^- = \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z - \frac{\hat{M}^{-1}}{\psi_{0,0}}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0}),$$

where $\underline{\hat{\mathcal{C}}}_z \psi_{0,0} = \begin{pmatrix} \hat{\mathcal{C}}_x \psi_{0,0} \\ \hat{\mathcal{C}}_p \psi_{0,0} \end{pmatrix}$ and it can be easily seen that

$$(16) \quad \left(\underline{\hat{S}}^+ \right)^T \underline{\hat{S}}^- = \hat{H}_-;$$

In fact,

$$(17a) \quad \left(\underline{\hat{S}}^+ \right)^T \underline{\hat{S}}^- = -\underline{\hat{\mathcal{C}}}_z^T \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z + \underline{\hat{\mathcal{C}}}_z^T \left[\frac{1}{\psi_{0,0}} \hat{M}^{-1}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0}) \right] \\ - \frac{1}{\psi_{0,0}}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0})^T \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z + \frac{1}{(\psi_{0,0})^2}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0})^T \hat{M}^{-1}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0})$$

Noting that

$$(17b) \quad \underline{\hat{\mathcal{C}}}_z^T \left[\frac{1}{\psi_{0,0}} \hat{M}^{-1}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0}) \right] = -\frac{1}{(\psi_{0,0})^2}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0})^T \hat{M}^{-1}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0}) \\ + \frac{1}{\psi_{0,0}}(\underline{\hat{\mathcal{C}}}_z^T \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z \psi_{0,0}) + \frac{1}{\psi_{0,0}}(\underline{\hat{\mathcal{C}}}_z \psi_{0,0})^T \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z,$$

Eq. (15) immediately follows. On the other hand, we also get

$$(18a) \quad \left(\underline{\hat{S}}^+ \right)^T \underline{\hat{S}}^- = \hat{H}_+ = -\underline{\hat{\mathcal{C}}}_z^T \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z + V_+(x, p),$$

where

$$(18b) \quad V_+(x, p) = -V_-(x, p) + \frac{2}{(\psi_{0,0})^2} (\underline{\hat{\mathcal{C}}}_z \psi_{0,0})^T \hat{M}^{-1} (\underline{\hat{\mathcal{C}}}_z \psi_{0,0}).$$

The potentials $V_{\pm}(x, p)$ are the ps SUSY partner potentials. The commutator of the operators $\underline{\hat{S}}^{\pm}$ is also shown to be

$$(19) \quad \left[\underline{\hat{S}}^+, \underline{\hat{S}}^- \right] = -2 \underline{\hat{\mathcal{C}}}_z^T \left[\frac{1}{\psi_{0,0}} \hat{M}^{-1} (\underline{\hat{\mathcal{C}}}_z \psi_{0,0}) \right].$$

In analogy to what is done in SUSY QM we use as superpotential the quantity

$$(20) \quad \underline{W}(x, p) = -\frac{1}{\psi_{0,0}} \hat{M}^{-1} (\underline{\hat{\mathcal{C}}}_z \psi_{0,0})$$

and define the $\underline{\hat{S}}^{\pm}$ operators as

$$(21) \quad \underline{\hat{S}}^+ = -\underline{\hat{\mathcal{C}}}_z + \hat{M} \underline{W}(x, p)$$

$$\underline{\hat{S}}^- = \hat{M}^{-1} \underline{\hat{\mathcal{C}}}_z + \underline{W}(x, p)$$

SP SUSY partner potentials are

$$(22) \quad V_{\pm}(x, p) = \underline{W}^T \hat{M}^{-1} \underline{W} \pm \underline{\hat{\mathcal{C}}}_z^T \underline{W}.$$

The results we have so far obtained show that the concepts of supersymmetry, in the sense of Refs. 5-7, can be quite naturally extended to

the phase-space and thus complete the analysis developed in Refs. 4. We will conclude this section showing that $V_+(x, p)$ and $V_-(x, p)$ have the same energy spectrum. To prove this statement it is sufficient to recall that $\hat{\underline{S}}^\pm$ are two-component operators and (see Eqs. (14))

$$(23) \quad \hat{H}_- = \hat{S}_{+,m} \hat{S}_{-,m} + \hat{S}_{+,n} \hat{S}_{-,n}$$

and similarly for $\hat{H}_+$. The eigenfunctions $\psi_{n,m}^{(-)}$ of $\hat{H}_-$ have eigenvalues $E_{m,n}^{(-)} = E_m^{(-)} + E_n^{(-)}$. Therefore, we get

$$(24a) \quad \begin{aligned} \hat{H}_+ \left(\hat{S}_{-,m} \hat{S}_{-,n} \psi_{m,n}^{(-)} \right) &= \left(\hat{S}_{-,m} \hat{S}_{-,m} + \hat{S}_{-,n} \hat{S}_{+,n} \right) \left(\hat{S}_{-,m} \hat{S}_{-,n} \psi_{m,n}^{(-)} \right) \\ &= \left(E_m^{(-)} + E_n^{(-)} \right) \left(\hat{S}_{-,m} \hat{S}_{-,n} \psi_{m,n}^{(-)} \right) \end{aligned}$$

and also

$$(24b) \quad \hat{H}_- \left(\hat{S}_{+,m} \hat{S}_{-,n} \psi_{n,m}^{(+)} \right) = \left(E_m^{(+)} + E_n^{(+)} \right) \left(\hat{S}_{+,m} \hat{S}_{-,n} \psi_{n,m}^{(+)} \right).$$

It is, therefore, clear that

$$(25) \quad E_{n,m}^{(+)} = E_{n+1,m+1}^{(-)}$$

$$\psi_{n,m}^{(+)} = \frac{1}{\left(E_{n+1,m+1}^{(-)} \right)} \left(\hat{S}_{-,m} \hat{S}_{-,n} \right) \psi_{m+1,n+1}^{(-)}$$

The action of $\hat{S}_{-,m} \hat{S}_{-,n}$ is that of converting an eigenfunction of $\hat{H}_-(x, p)$ into one of $\hat{H}_+(x, p)$ and similarly for $\hat{S}_{+,m} \hat{S}_{+,n}$.

3. Concluding Remarks

The operators $\underline{\hat{A}}$ even though playing the role of creation annihilation operators are not Hermitian conjugate to each other. The operators⁴

$$(26) \quad \underline{\hat{A}}^+ = \frac{1}{2} \underline{z} - \hat{M}^{-1} \underline{\hat{c}}_z$$

$$\underline{\hat{A}}^- = \frac{1}{2} \hat{M} \underline{z} + \underline{\hat{c}}_z$$

are the adjoint to $\underline{\hat{A}}^-$ and $\underline{\hat{A}}^+$, respectively, and provide a set of creation annihilation operators for the functions⁴

$$(27) \quad \mathcal{Y}_{m,n}(x,p) = \frac{\sqrt{\Delta}}{\sqrt{2\pi}} \frac{1}{\sqrt{m!n!}} G_{m,n}(x,p) \exp \left\{ -\frac{1}{4} \underline{z}^T \hat{M} \underline{z} \right\}$$

$$G_{m,n}(x,p) = (-1)^{m+n} e^{\frac{1}{2} \underline{z}^T \hat{M} \underline{z}} \frac{\partial^{m+n}}{\partial \xi^m \partial \eta^n} e^{-\frac{1}{2} \underline{z}^T \hat{M} \underline{z}}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \hat{M} \underline{z}.$$

The functions $\mathcal{Y}_{m,n}(x,p)$ are orthogonal to $\mathcal{H}_{m,n}(x,p)$ in the sense that

$$(28) \quad \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp \mathcal{Y}_{m,n}(x,p) \mathcal{H}_{r,s}(x,p) = \delta_{m,r} \delta_{n,s}.$$

The "vacua" $G_{0,0}(x,p)$ and $\mathcal{H}_{0,0}(x,p)$ coincide; in addition

$$(29) \quad \left(\underline{\tilde{A}}^+ \right)^T \underline{\tilde{A}}^- \mathcal{Y}_{m,n}(x,p) = (m+n) \mathcal{Y}_{m,n}$$

and it is easily checked that $(\tilde{\underline{A}}^+)^T \tilde{\underline{A}}^-$ is provided by the same relation given by Eq. (7). The above considerations suggest that it is possible to define the analogous of (15) in the form

$$(30) \quad \hat{\underline{S}}^+ = -\hat{M}^{-1} \underline{\mathcal{C}}_z - \hat{M}^{-1} \left(\frac{1}{\psi_{0,0}} \underline{\mathcal{C}}_z \psi_{0,0} \right)$$

$$\hat{\underline{S}}^- = \underline{\mathcal{C}}_z - \frac{1}{\psi_{0,0}} \underline{\mathcal{C}}_z \psi_{0,0}.$$

The above operators are Hermitian conjugate to (15) and provide the same Hamiltonians $\hat{H}_\pm$. It is, therefore, clear that, in analogy to the case of the harmonic oscillator eigenfunctions, we can introduce a class of functions

$\tilde{\psi}_{m,n}^{(-)}$ and $\tilde{\psi}_{m,n}^{(+)}$ such that

$$(31) \quad \hat{H}_+ \left(\hat{\underline{S}}_{-,m} \hat{\underline{S}}_{-,n} \tilde{\psi}_{n,m}^{(-)} \right) = \left(E_m^{(-)} + E_n^{(-)} \right) \left(\hat{\underline{S}}_{-,m} \hat{\underline{S}}_{-,n} \tilde{\psi}_{n,m}^{(-)} \right)$$

$$\hat{H}_- \left(\hat{\underline{S}}_{+,m} \hat{\underline{S}}_{+,n} \tilde{\psi}_{n,m}^{(+)} \right) = \left(E_m^{(+)} + E_n^{(+)} \right) \left(\hat{\underline{S}}_{+,m} \hat{\underline{S}}_{+,n} \tilde{\psi}_{n,m}^{(+)} \right),$$

thus drawing the same conclusions as before, namely, the operators $\hat{\underline{S}}_{-,m} \hat{\underline{S}}_{-,n}$ destroy, or better, convert an eigenfunction $\tilde{\psi}_{m,n}^{(-)}$ of $\hat{H}_-$ into an eigenfunction $\tilde{\psi}_{m,n}^{(+)}$ of $\hat{H}_+$.

In this paper we have just shown that the ideas of SUSY QM can be extended to the ps formulations. We have seen that SUSY partner potentials can be introduced, too, and that the formalism easily generalizes that of ps creation-annihilation operators.⁴ In a forthcoming paper we will discuss the

relevance of the above results to recent formulation of ps quantum mechanics.⁸

REFERENCES

- ¹ Y.S. Kim and M.S. Noz, *Phase-Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).
- ² G. Dattoli and A. Torre, J. Math. Phys. **35**, 4451 (1994).
- ³ G. Dattoli, S. Lorenzutta, G. Maino, and A. Torre, J. Math Phys., to be published April (1995).
- ⁴ G. Dattoli and A. Torre, submitted for publication.
- ⁵ L.Gandenshtein, JETP Lett. **38**, 356 (1983).
- ⁶ K. Sukumar, J. Phys. A **18**, 2917 (1985).
- ⁷ R. Dutt, Avinash Khan, and U.P. Sukhatma, Am. J. Phys. **56**, 163 (1988) and references therein.
- ⁸ Go. Torres-Vega, J.H. Frederick, Phys. Rev. Lett. **67**, 2601 (1991).

Edito dall'Enea
Direzione Relazioni Esterne
V. le Regina Margherita, 125 - 00198 Roma
Finito di stampare nel mese di marzo 1995
presso il Tecnografico