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GENERAL SOLUTION
OF TWO-DIMENSIONAL MATRIX TODA
CHAIN EQUATIONS WITH FIXED ENDS

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Abstract

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It is shown that a two-dimensional matrix Toda chain is the group of discrete symmetry of the two-dimensional matrix of the non linear Shrodinger equation (matrix generalization of the Davey-Stewartson system). A general solution of this chain with definite boundary conditions is represented in an explicit form.

Аннотация

Лезнов А.Н., Юзбашьян Е.А. Общее решение двумерной матричной цепочки Toda с закрепленными концами: Препринт ИФВЭ 94-132. – Протвино, 1994. – 4 с., библиогр.: 6.

Показано, что двумерная матричная цепочка Toda является группой дискретной симметрии матричного уравнения Davey-Stewartson. Общее решение матричной цепочки Toda с фиксированными концами найдено в явном виде.

1. Introduction

Under a matrix Davey-Stewartson equation we understand the following system of two equations for two unknown $s \times s$ - matrix functions u, v :

$$u_t + au_{xx} + bu_{yy} - 2au \int dy(uv)_x - 2b \int dx(uv)_y u = 0, \quad (1.1)$$

$$-v_t + av_{xx} + bv_{yy} - 2a \int dy(uv)_x v - 2bv \int dx(uv)_y = 0,$$

where a, b are the arbitrary numerical parameters; x, y are the coordinates of two dimensional space. In a partial case $s = 1$, when the order of multipliers is not essential, (1.1) is the usual Davey-Stewartson equation in its original form [1].

2. Discrete Substitution

By direct but not very simple computations one can get convinced that system (1.1) is invariant with respect to the following change of unknown functions

$$\tilde{u} = \frac{1}{v} \quad \tilde{v} = [vu - (v_x v^{-1})_y]v \equiv v[uv - (v^{-1}v_y)_x]. \quad (2.1)$$

Substitution (2.1) is discrete transformation [2] with respect to which all equations of matrix Davey-Stewartson hierarchy are invariant. In the case of one-dimensional space this substitution was mentioned in [3].

Substitution (2.1) is invertible and "old" functions u, v may be represented in terms of the new ones as:

$$v = \frac{1}{\tilde{u}} \quad u = [\tilde{u}\tilde{v} - (\tilde{u}_y \tilde{u}^{-1})_x]\tilde{u} \equiv \tilde{u}[\tilde{v}\tilde{u} - (u^{-1}\tilde{u}_x)_y]. \quad (2.2)$$

Substitution (2.1) may be rewritten in the form of the infinite chain of equations

$$((v_n)_x v_n^{-1})_y = v_n v_{n-1}^{-1} - v_{n+1} v_n^{-1}, \quad (u_{n+1} = v_n^{-1}), \quad (2.3)$$

where under (v_{n-1}, u_{n-1}) one should understand the result of n -times applications of substitution (2.1) to some given matrix-functions (v_0, u_0) .

Generally chain (2.3) is infinite in both directions, but it may be interrupted under appropriate boundary conditions. The case when $v_{-1}^{-1} = v_{N+1} = 0$ we shall call as the matrix Toda chain with fixed ends.

In the scalar case $s = 1$ a general solution of the Toda chain with fixed ends was found in [4] for all series of simisimple algebras except of E_7, E_8 . In [5] this result was reproduced in terms of invariant root technique applicable to all semisimple series.

The goal of the present paper is to obtain the general solution of the matrix Toda chain with fixed ends in an explicit form.

3. General solution

At first let us observe that from (2.1) it follows

$$v_{n+1}u_{n+1} - v_nu_n = -((v_n)_x v_n^{-1})_y.$$

Keeping in mind that $u_0 = v_{-1}^{-1} = 0$, we obtain immediately

$$v_{n+1} = -[\sum_{t=0}^n ((v_t)_x v_t^{-1})_y] v_n = -v_n [\sum_{t=0}^n (v_t^{-1} (v_t)_y)_x]. \quad (3.1)$$

The single equation for determining one unknown function v_0 ($v_{N+1} = 0$) takes the form

$$\sum_{t=0}^N (v_t)_x v_t^{-1} = A_N(x) \quad \sum_{t=0}^N v_t^{-1} (v_t)_y = B_N(y), \quad (3.2)$$

where $A_N(x), B_N(y)$ are arbitrary $s \times s$ matrix functions of the corresponding arguments.

We shall use the following notations

$$X^r = (v_r)_x v_r^{-1} \quad S^n = \sum_{t=0}^n X^t$$

and the corresponding expressions with respect to y coordinate.

In these notations (3.1) may be rewritten as

$$v_{n+1} = -(S^n)_y v_n = (S^n)_y (S^{n-1})_y v_{n-1} = \dots \quad (3.3)$$

From (3.3) it follows that the recurrent relations for the determination of X^n, S^n are

$$X^n = (S_{xy}^{n-1} + S_y^{n-1} X^{n-1}) (S_y^{n-1})^{-1}, \quad S^n = (\sum_{t=0}^{n-1} [S_x^t + S^t X^t])_y (S_y^{n-1})^{-1}. \quad (3.4)$$

Let us consider at first solutions of (3.2) for the lower values of $N = 0, 1, 2, \dots$ which allow us to obtain the solution for the general case of an arbitrary N by induction.

3.1. $N=0$

In sum (3.2) we have only one term and for the v_0 we obtain obvious solution

$$v_0 = \phi_0(x) \bar{\phi}_0(y),$$

where $\phi_0, \bar{\phi}_0$ are arbitrary $s \times s$ matrix functions of their arguments.

3.2. N=1

Equation (3.2) with the help of (3.4) may be rewritten as ($S^0 \equiv X^0$)

$$(X_x^0 + X^0 X^0)_y = A_1(x)X_y^0, \quad X_x^0 + X^0 X^0 = A_1(x)X^0 + A_0(x).$$

Keeping in mind the definition of X^0 (Y^0) we obtain

$$v_{xx}^0 = A_1(x)v_x^0 + A_0(x)v^0, \quad v_{yy}^0 = v_y^0 B_1(y) + v^0 B_0(y). \quad (3.5)$$

System (3.5) has obviously the following general solution

$$v^0 = \phi_0(x)\bar{\phi}_0(y) + \phi_1(x)\bar{\phi}_1(y),$$

where as in the previous example $\phi_p(x)\bar{\phi}_p(y)$ are arbitrary matrix functions of their arguments.

3.3. N=2

We reproduce the consistent steps of computations in this case without detailed comments

$$S^2 = A_2(x), \quad S_x^0 + S^0 X^0 + S_x^1 + S^1 X^1 = A_2(x)S^1 + A_1(x).$$

From the last equality with (3.4) we have

$$X_{xx}^0 + 2X_x^0 X^0 + X^0 X_x^0 + (X^0)^3 = A_2(X_x^0 + (X^0)^2) + A_1 X^0 + A_0.$$

At last keeping in mind the definition of X^0 we finally obtain two equations

$$v_{xxx}^0 = A_2 v_{xx}^0 + A_1 v_x^0 + A_0 v^0, \quad v_{yyy}^0 = v_{yy}^0 B_2 + v_y^0 B_1 + v^0 B_0$$

with the general solution

$$v^0 = \sum_{t=0}^2 \phi_t(x)\bar{\phi}_t(y).$$

3.4. The case of arbitrary N

Let us define by induction the values S_p^n

$$S_p^n = \sum_{q=0}^n [(S_{p-1}^q)_x + S_{p-1}^q X^q] \quad (3.6)$$

with the boundary condition $S_0^q = 1$. Comparing (3.4) with (3.6) we notice that S^n from (3.4) coincides with S_1^n from (3.6). Keeping in mind (3.4) and definition (3.6) we immediately obtain the recurrent relation

$$S_p^n = (S_{p+1}^{n-1})_y [(S_1^{n-1})_y]^{-1}. \quad (3.7)$$

Using (3.7) it is possible to represent all unknown matrix-valued functions of the chain v_n only in terms of v_0 function. The system of two equations which determines v_0 (3.2) takes the form

$$v_{\underbrace{x \dots x}_{N+1}}^0 = A_N(x)v_{\underbrace{x \dots x}_N}^0 + \dots + A_0(x)v^0,$$

(3.8)

$$\underbrace{v^0_{y \dots y}}_{N+1} = \underbrace{v^0_{y \dots y}}_N B_n(y) + \dots + v^0 B_0(y),$$

from which we obtain

$$v^0 = \sum_{t=0}^N \phi_t(x) \bar{\phi}_t(y). \quad (3.9)$$

4. Conclusion

The main result of the recent paper is contained in unknown before (3.9) general solution of matrix Toda chain (2.2) with fixed ends. In scalar case ($s = 1$) this solution, as is well known, is closely connected with the theory of representations of semisimple algebras [6]. For us the connection of the proposed solution (3.9) with theory of the group representation (if it takes place) is not clear and we hope to come back to this interesting question in further publications.

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