NONLINEARITY AND WHITE NOISE

Margarida Faria
Ludwig Streit

CCM, Universidade da Madeira, P9000 Funchal
and
BIBoS, Universitat Bielefeld, D 33501 BIELEFELD

UMa-Mat 16/95

SCAN-9505160

CERN LIBRARIES, GENEVA

Sc9522

Nonlinearity and White Noise

Margarida de Faria
CCM, Universidade da Madeira
P 9000 Funchal

Ludwig Streit
BiBoS, Universität Bielefeld
D 33615 Bielefeld

and
CCM, Universidade da Madeira
P 9000 Funchal

Abstract

We introduce smooth and generalized nonlinear functionals of Gaussian white noise through series expansions and apply this technique to selfintersection local times of Brownian motion and to Burgers' equation.

1 Introduction

Gaussian white noise is a natural candidate whenever one tries to model random events occurring independently at different points in time (and/or space). Of course this strict independence is an idealization, but it is one of those which physics invokes frequently to facilitate a simple and transparent analysis, an analysis that in many instances would become much more cumbersome for models with "colored noise". It is thus not surprising that white noise appears as the velocity of Brownian motion modeled by the Wiener process

$$\omega(t) = \frac{d}{dt}B(t).$$

With its independence and invariance properties Gaussian white noise ω is ideally suited as a universal "coordinate system" for stochastic and infinite dimensional analysis.

A certain amount of care is needed if we want to profit from the advantages of white noise modeling in a nonlinear setting; a well-known example is stochastic integration where e.g. the distinction between the integrals in the sense of Ito

and of Stratonovich essentially results from carefully defining them as nonlinear functionals of white noise [12].

In the following section we shall introduce nonlinear functionals of white noise, and - in analogy to the usual test functions and distributions - we shall introduce spaces of smooth and of generalized functionals of white noise. Examples will be discussed in two further sections. We shall first study white noise expansions for intersection local times of Brownian motion, and then solutions of Burgers' equation with multiplicative and with additive noise.

2 Nonlinear Functionals of White Noise

For nonlinear functionals of white noise ω we shall make the ansatz

$$\varphi(\omega) = \sum_{n=0}^{\infty} \int d^n t \mathbf{F}_n(t_1, \dots, t_n) : \omega(t_1) \dots \omega(t_n) : . \quad (1)$$

Here

$$: \omega^{\otimes n}(t) : \equiv : \omega(t_1) \dots \omega(t_n) :$$

is the well-known "normal" or "Wick" or "Hermite product". It amounts to an orthogonalization

$$E(: \omega^{\otimes n}(s) :: \omega^{\otimes m}(t) :) = \delta_{m,n} n! \prod_{i=1}^n \delta(s_i - t_i)$$

and can e.g. be defined recursively

$$: \omega(t_1) \dots \omega(t_n) : \equiv : \omega(t_1) \dots \omega(t_{n-1}) : \omega(t_n) - \sum_{i=1}^{n-1} \delta(t_n - t_i) : \prod_{k \neq i}^{n-1} \omega(t_k) : .$$

In this fashion all the terms in the "chaos expansion" (1) are orthogonal to each other w.r. to the white noise measure μ , and we find immediately the L^2 norm of φ :

$$\|\varphi\|_{L^2(d\mu)}^2 = E(\varphi^* \varphi) = \sum_{n=0}^{\infty} n! \int d^n t |\mathbf{F}_n(t_1, \dots, t_n)|^2 .$$

On the rhs we recognize a Fock space norm:

$$\varphi \in L^2(d\mu) \sim \{F_n\} \in H_{Fock}$$

is the Gelfand-Ito-Segal isomorphism between square integrable white noise functionals and Boson Fock space vectors.

Smooth and generalized functionals are constructed in analogy to the usual test functions and distributions; we recall that test functions in Schwartz space are characterized by a sequence of norms

$$f \in S(R^n) \text{ iff } |f|_p < \infty \text{ for all } p.$$

Likewise we introduce for white noise functionals the norms

$$\|\varphi\|_{p,q}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |\mathbf{F}_n|_p^2 \quad (2)$$

for $p, q > 0$. Putting $\beta = 0$ for the moment we define the space (S) of smooth functionals by

$$(S) = \{\varphi : \|\varphi\|_{p,q} < \infty \text{ for all } p, q\}$$

and the space $(S)^*$ of generalized functionals ("Hida distributions" [12]) as its dual:

$$(S) \subset L^2(d\mu) \subset (S)^*.$$

Similarly, for $\beta = 1$, we have the Kondratiev spaces [17] $(S)^{\pm 1}$, with

$$(S)^1 \subset (S) \subset L^2(d\mu) \subset (S)^* \subset (S)^{-1}.$$

How does one identify such generalized functionals $\Phi \in (S)^*$? In many physics applications Φ will be given in terms of a "source functional", such as

$$T\Phi(f) = E\left(\Phi(\omega) e^{i \int \omega(t)f(t)dt}\right)$$

or

$$S\Phi(f) = E\left(\Phi(\omega) : e^{\int \omega(t)f(t)dt} : \right)$$

Fortunately, any of these expressions provides a complete characterization of Φ [12] [16].

Theorem: A functional $G(f)$, $f \in S(R)$, is the T-transform of a unique generalized white noise functional $\Phi \in (S)^*$ iff for all $f_i \in S(R)$, $G(zf_1 + f_2)$ is analytic in the whole complex plane and of second order exponential growth

$$|G(zf)| < ae^{b|zf|_p^2}.$$

Remarks: (1) The same is true for the S-transform. (2) Generalized functionals in the Kondratiev space are characterized by local analyticity and local boundedness of the source functionals. (3) Hence, in either case, linear combinations, but also products of source functionals are again admissible source functionals. This induces an algebraic structure on the space of generalized functionals, via

$$S(\Phi)S(\Psi) = S(\Phi \diamond \Psi);$$

as it turns out this product is simply the Wick product:

$$: \omega^{\otimes n}(s) : \diamond : \omega^{\otimes m}(t) :=: \omega^{\otimes n}(s) \omega^{\otimes m}(t) :$$

(4) For the Kondratiev space of generalized white noise functionals even more can be said. By Remark (2), analytic functions g of source functionals are again admissible, and this induces an analytic "Wick calculus" on distribution space [17]

$$g(S(\Phi)) = S(g^\diamond(\Phi))$$

with

$$g^\diamond(\Phi) \equiv \sum a_n \Phi^{\otimes n} \text{ for } g(z) = \sum a_n z^n. \quad (3)$$

3 Self Intersection Local Time of Brownian Motion

We consider d -dimensional Brownian motion $\mathbf{B}(t)$, i.e., d independent copies $B_i(t)$ of scalar Brownian motion. The expression

$$L(T) \equiv \int_0^T d^2 t \delta(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \quad (4)$$

is designed to "light up" whenever the Brownian motion path returns at time t_2 to a point where it already has passed at time t_1 . Physically this intersection local time has applications e.g. in quantum field theory [29] [30], and in Brownian models for polymers where an interaction potential such as L is intended to suppress double points of the polymer chain [32]. Mathematical efforts to make good sense of the heuristic ansatz as in (4) go back at least forty years [6]-[11], [15], [18]-[22], [24]-[34]: - what does it mean to have a Brownian path inside a Dirac δ -function, in what sense does the double integral exist, etc.? - , and not all the answers have been given yet. Clearly one is dealing with a nonlinear functional of Brownian motion, and it is not hard to see that these objects become more and more singular as the dimension d increases. Alternatively one might try to substitute white noise for Brownian motion as the fundamental variable, and it was somewhat surprising to see that some new and relevant facts emerge when one tries a systematic chaos expansion for L . As a first step we describe the δ -function by its S -transform

$$S(\delta(\mathbf{B}(t_2) - \mathbf{B}(t_1)))(\mathbf{f}) = \left(\frac{1}{\sqrt{2\pi|t_2 - t_1|}} \right)^d e^{-\frac{1}{2|t_2 - t_1|} \left| \int_{t_1}^{t_2} \mathbf{f}(u) du \right|^2} \quad (5)$$

One sees right away that this expression has the required analyticity and growth properties to assure that δ is indeed a generalized functional of vector valued white noise in the space $(S)^*$.

The kernel functions F_n in the chaos expansion for the local times L are then calculated from those of δ by the double integration over t_i ,

$$\begin{aligned} \mathbf{F}_{2n}(u_1, \dots, u_{2n}) &= (-)^n \left((n + \frac{d}{2} - 1)(n + \frac{d}{2} - 2)(2\pi)^{d/2} 2^n \vec{n}! \right)^{-1} \cdot \\ &\cdot \Theta(u) \Theta(1 - v) \cdot \\ &\cdot (1 - v^{-n-\frac{d}{2}+2} - (1 - u)^{-n-\frac{d}{2}+2} + (v - u)^{-n-\frac{d}{2}+2}) \end{aligned} \quad (6)$$

with $v \equiv \max_{1 \leq k \leq 2n} u_k$ and $u \equiv \min_{1 \leq k \leq 2n} u_k$, all other F_n vanish.

Note that one obtains all the kernel functions in closed form, depending only on the two variables u and v , and with the dominant singularity given by the term $(v - u)^{-n-\frac{d}{2}+2}$ which becomes singular when all the times $u_1, \dots, u_{2n}$ coincide, and increasingly so as the dimension d increases.

Some immediate consequences for $d=3$ are:

1. For $n > 0$, the F_n are integrable (F_0 is divergent).
2. The centered local time is a Hida distribution: $L_c = L - E(L) \in (S)^*$.
3. The higher moments of L_c are divergent.
4. This can be corrected by a multiplicative renormalization : using a Gaussian δ -sequence, $\delta_\epsilon \rightarrow \delta$, one can explicitly study the logarithmic renormalization (see also [34])

$$L_{ren,\epsilon} \equiv \frac{L_{c,\epsilon}}{|\log \epsilon|^{1/2}}$$

and its finite (Brownian) limit as $\epsilon \rightarrow 0$.

4 Burgers Equation with Noise

Evidently, nonlinear evolution equations under the influence of white noise will naturally give rise to nonlinear functionals of white noise. Burgers' equation is a case in point, its stochastic perturbation have been studied by various authors [1], [3], [5], [13], [14], see also the contribution of de Lillo at this conference.

In its simplest, one-dimensional, form the Burgers equation reads

$$u_t + u \cdot u_x - u_{xx} = F \quad (7)$$

where F is the stochastic force term.

We remark that for singular forces such as white noise the interpretation of equation (7) is not self-evident: this applies in particular to the non-linear term

which may not be well-defined as a pointwise product. In [1], [13], [14] it is interpreted as a Wick product:

$$u_t + u \diamond u_x - u_{xx} = F. \quad (8)$$

Denoting the S-transform by $\tilde{u}$ for short, equation (8) becomes

$$\tilde{u}_t + \tilde{u}\tilde{u}_x - \tilde{u}_{xx} = \tilde{F} \quad (9)$$

with pointwise multiplication in the non-linear term. The advantage of considering this transformed equation is that singular forces F may become smooth under the S-transform so that the well-known methods for resolving a forced Burgers equation apply.

Among the examples that can be handled in this way we mention for illustration the case of space time dependent white noise ω

$$F(x, t) = \omega(x, t)$$

for which (9) becomes simply

$$\tilde{u}_t(x, t) + \tilde{u}(x, t)\tilde{u}_x(x, t) - \tilde{u}_{xx}(x, t) = f(x, t)$$

with a smooth force $f \in S(R^2)$. One resolves this equation by the usual Cole-Hopf method and for suitable initial conditions the resulting $\tilde{u} = \tilde{u}(x, t; f)$ has the analyticity properties required to make it the S-transform of a Kondratiev distribution u which solves equation (8), see [13], [14] where the technical details of the method have been developed.

Recently de Lillo [5] has studied Burgers' equation with a multiplicative time-dependent noise φ , of the form

$$u_t + u \cdot u_x - u_{xx} = \varphi(t)u_x. \quad (10)$$

If $u = u(x, t)$ is such that it solves equation (10) for $\varphi \equiv 0$, then

$$u = u(x + \int_0^t \varphi(s)ds, t)$$

is easily seen - at least formally - to solve (10) with noise φ . The Wick ordered equation, on the other hand, reads

$$v_t + v \diamond v_x - v_{xx} = \varphi(t) \diamond v_x$$

and is solved by a functional v with S-transform

$$\tilde{v} = u(x + \int_0^t \tilde{\varphi}(s)ds, t).$$

For white noise, $\varphi = \omega$, in particular, $u = u(x + B(t), t)$, while $\tilde{v}(x, t; f) = u(x + \int_0^t f(s)ds, t)$. For more singular noise such as e.g. $\varphi(t) = \frac{d}{dt}\omega(t)$, the solution u of (10) becomes ill-defined. The Wick ordered Burgers equation for this example is solved by $\tilde{v}(x, t; f) = u(x + f(t), t)$, i.e., the solution v has the chaos expansion

$$v(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial x^n} u(x, t) : \omega^n(t) : .$$

A similar comparison of Burgers and Wick-Burgers equations is possible for additive time dependent white noise where the pointwise multiplication case can be found in [23], a rigorous treatment of the multiplication by noise on the rhs of (10) will be given in [2].

5. ACKNOWLEDGEMENTS

Financial support from STRIDE (Portugal) is gratefully acknowledged.

References

- [1] Benth, F., Streit, L.: *The Burgers equation forced by a generalized stochastic process in a non-Gaussian setting*. In preparation.
- [2] Benth, F., Deck, Th., Potthoff, J., Streit, L.: *Stochastic Burgers and related equations*. In preparation.
- [3] Bertini, L., Cancrini, N., Jona-Lasinio, G.: *Burgers Equation forced by conservative or nonconservative noise*. In "Stochastic analysis and applications in physics", A.I. Cardoso et al., eds., Kluwer 1994.
- [4] De Faria, M., Hida, T., Streit, L., Watanabe, H.: *Intersection Local Times as Generalized White Noise Functionals*. BiBoS preprint no. 642/7/1994.
- [5] De Lillo, S.: The Burgers equation under multiplicative noise. *Phys. Lett. A* **188** (1994) 305-308.
- [6] Dvoretzky, A., Erdős, P., Kakutani, S. Double points of paths of Brownian motion in the plane. *Bull. Res. Council Israel Sect. F* **3** (1954), 364-371.
- [7] Dynkin. E. B.: Polynomials of the occupation field and related random fields. *J. Funct. Anal.* **58** (1984), 20-52.
- [8] Dynkin. E. B.: Self-intersection gauge for random walks and for Brownian motion. *Ann. Probab.* **16** (1988), 1-57.
- [9] Dynkin. E. B.: Regularized self-intersection local times of planar Brownian motion. *Ann. Probab.* **16** (1988), 58-73.

- [10] Geman, D., Horowitz, J., Rosen, J.: A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.* **12** (1984), 86-107.
- [11] He, S.W., Wang, J.-G., Yang, W.-Q., Yao, R.-Q.: *Local Times of Self-Intersection for Multidimensional Brownian Motion*. Shanghai preprint.
- [12] Hida, T., Kuo, H.-H., Potthoff, J. and Streit, L.: *White Noise - An Infinite Dimensional Calculus*. Kluwer-Academic, 1993.
- [13] Holden, H., Lindstrøm, T., Øksendal, b., Ubøe, J.: The Burgers equation with a noisy force. *Comm. PDE*, 19 (1994), 119-141.
- [14] Holden, H., Lindstrøm, T., Øksendal, b., Ubøe, J.: *The stochastic Wick type Burgers equation*. Preprint, August 1994.
- [15] Inkeller, P., Perez-Abreu, V., Vives, J.: *Chaos expansions of double intersection local times of Brownian motion in R^d and renormalization*. Preprint 1993.
- [16] Kondratiev, Yu. G., Leukert, P., Potthoff, J., Streit, L., Westerkamp, W., *Generalized Functionals in Gaussian spaces - the Characterization Theorem Revisited*. Mannheim University Preprint, 175/1994.
- [17] Kondratiev, Yu., Leukert, P., Streit, L.: Wick calculus in Gaussian analysis. BiBoS preprint 1994, to appear in *Acta Appl. Math.*
- [18] Le Gall, J. F.: *Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan*. Sémin. de Prob. XIX, 1983/84. LNM 1123 (1985), 314-331. Springer, Berlin.
- [19] Le Gall, J. F.: Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.* 14 (1986), 1219-1244.
- [20] Lévy, P.: Le mouvement brownien plan. *Amer. J. Math.* **62** (1940), 440-487.
- [21] Nualart, D., Vives, J.: Smoothness of Brownian local times and related functionals. *Potential Analysis* **1** (1992), 257-263.
- [22] Nualart, D., Vives, J.: Chaos expansion and local times. *Publications Mathématiques* **36**(2), 827-836.
- [23] Orłowski, A., Sobczyk, K.: Solitons and shock waves under random external noise. *Rep. Math. Phys.* **27** (1989) 59-71.
- [24] Penrose, M. D.: On the existence of self-intersections for quasi-every Brownian path in space. *Ann. Probab.* **17** (1989), 482-502.
- [25] Perez-Abreu, V.: *Chaos Expansions: A Review*, CIMAT preprint, 1993.

- [26] Rosen, J.: A local time approach to the self-intersections of Brownian paths in space. *Comm. Math. Phys.* **88** (1983), 327-338.
- [27] Rosen, J.: Tanaka's formula and renormalisation for intersections of planar Brownian motion. *Ann. Probab.* **14** (1986), 1425-1251.
- [28] Rosen, J.: *A renormalized local time for multiple intersections of planar Brownian motion*. Sémin. de Prob. XX, 1984/85. LNM 1204 (1986), 515-531. Springer. Berlin.
- [29] Symanzik, K.: *Euclidean quantum field theory*. In "Local Quantum Theory" (R. Jost, ed.). Academic, New York (1969).
- [30] Varadhan, S. R. S.: *Appendix to "Euclidean quantum field theory"* by K. Symanzik. In "Local Quantum Theory" (R. Jost. ed.). Academic. New York (1969)
- [31] Watanabe, H.: The local time of self-intersections of Brownian motions as generalized Brownian functionals. *Lett. Math. Phys.* **23**, 1-9, 1991.
- [32] Westwater, J.: On Edward's model for long polymer chains. *Comm. Math. Phys.* **72** (1980), 131-174.
- [33] Wolpert, R.: Wiener path intersection and local time. *J. Funct. Anal.* **30** (1978), 329-340.
- [34] Yor, M.: *Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement brownien dans R^3* . Séminaire de Probabilité, 1983/84. LNM 1123 (1985), 350-365. Springer, Berlin.