A General Form of the Constraints in the Path Integral Formula

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Abstract

A form of the constraints, specifying a $D$-dimensional manifold embedded in $D + 1$ dimensional Euclidean space, is discussed in the path integral formula given by a time discretization. Although the mid-point prescription is privileged in the sphere $S^D$ case, it is more involved in generic cases. An interpretation on the validity of the formula is put in terms of the operator formalism. Operators from this path integral formula are also discussed.

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1. Introduction

Dynamical system, constrained on a $D$-dimensional manifold, $M^D$, which is now supposed to be given by the equation,

$$f(x) = 0,$$

(1.1)

where $x \equiv (x^1, \ldots, x^{D+1})$ is the $D+1$-dimensional Cartesian coordinate, can be described classically as follows: $f(x)$ is assumed to obey

$$(\nabla_x f(x))^2 \neq 0; \quad \forall x \in M^D,$$

(1.2)

where we have written $\nabla_x$ for the usual $\nabla$ vector. The equation of motion in a flat $D + 1$-dimensional space,

$$\ddot{x}^a = -\frac{\partial V(x)}{\partial x^a} \equiv -\partial_a V(x),$$

(1.3)

with $V(x)$ being a potential, is modified to

$$\Pi_{ab}(\nabla_x f)\dot{x}^b = -\Pi_{ab}(\nabla_x f)\partial_b V(x),$$

(1.4)

in $M^D$, where $\Pi_{ab}(X)$ is a projection operator,

$$\Pi_{ab}(X) \equiv \delta_{ab} - \frac{X^a X^b}{X^2},$$

(1.5)

onto the plane perpendicular to the vector $X$: $X^a \Pi_{ab}(X) = \Pi_{ab}(X)X^b = 0$. Here and hereafter repeated indices imply summation. The significance of (1.4) is easily grasped; since the motion is restricted on $M^D$ so that any deviation to the direction $\nabla_x f$ must be suppressed.

It is well-known that the Lagrangian,

$$L = \frac{\dot{x}^2}{2} - V(x) - \lambda f(x),$$

(1.6)

with $\lambda$ being the multiplier, leads to the equations (1.4) and (1.1). Also the canonical formalism can be developed under the guidance of Dirac[1]: regard (1.1) as the (primary) constraint

$$\phi_1(x) \equiv f(x) \quad (= 0; \quad \forall x \in M^D),$$

(1.7)
and consider the consistency condition: a Hamiltonian,

\[ H = H(p, x) + \lambda f(x) = \frac{p^2}{2} + V(x), \tag{1.8} \]

gives

\[ \dot{\phi}_1 = \{\phi_1, H\} = p \cdot \nabla_x f(x), \tag{1.9} \]

thus to find

\[ \phi_2(x) \equiv p \cdot \nabla_x f(x) \quad (= 0; \quad \forall x \in M^D). \tag{1.10} \]

(Here \( \{A, B\} \) designates the Poisson bracket.) They belong to the second class:

\[ \{\phi_1(x), \phi_2(x)\} = (\nabla_x f(x))^2 \neq 0, \tag{1.11} \]

on account of (1.2), which enables us to obtain the Dirac bracket,

\[ \{A, B\}_D \equiv \{A, B\} + \frac{1}{(\nabla_x f)^2} \left( \{A, \phi_1(x)\}\{\phi_2(x), B\} - (A \leftrightarrow B) \right). \tag{1.12} \]

Therefore we find

\[ \begin{align*}
\{x^a, x^b\}_D &= 0, \\
\{x^a, p_b\}_D &= \Pi_{ab}(\nabla_x f) = \delta_{ab} - \frac{\partial_a f \partial_b f}{(\nabla_x f)^2}, \\
\{p_a, p_b\}_D &= p_c \left( \partial_a \Pi_{cb} - \partial_b \Pi_{ca} \right) = p_c \frac{\partial_a \partial_c f \partial_b f - \partial_b \partial_c f \partial_a f}{(\nabla_x f)^2},
\end{align*} \tag{1.13} \]

those which correctly reproduce the equation (1.4).

As for quantum mechanics, a recipe of path integral quantization had been given by Faddeev [2] and later by Senjanovic [3](FS); the FS-formula reads formally

\[ \langle \phi | e^{-iTH} | \psi \rangle = \int D\mu \phi^* f(x) \exp \left[ i \int_{-T/2}^{T/2} dt \{p \cdot \dot{x} - H(p, x)\} \right] \psi(x), \tag{1.14} \]

with

\[ D\mu \equiv Dp Dx |\det \{\phi_1, \phi_2\}|^{1/2}\delta(\phi_1)\delta(\phi_2), \tag{1.15} \]

and \( x_f \equiv x(T/2), x_i \equiv x(-T/2) \). Here \( Dp \) and \( Dx \) are functional measures which must be specified somehow. The issue is then how to define the above functional measure properly.
to confirm the well-defined form of (1.14): the most well-known and primitive approach is to discretize the time, obtaining

$$D_p \mapsto \prod_j dp(j), \quad D_x \mapsto \prod_j dx(j). \quad (1.16)$$

In this approach it was stressed by the present author [4] that the mid-point prescription is privileged in the case of $D$-dimensional sphere $S^D$ given as

$$x^2 = \rho^2. \quad (1.17)$$

We try to generalize the case in this paper.

In section 2, we review the $S^D$ case. With this in mind, a generic case $f(x) = 0$ is discussed in section 3. The next section 4 deals with operators obtained from the path integral formula, then the final section 5 is devoted to discussion.

## 2. The case of $D$-dimensional sphere

The $D$-dimensional sphere $S^D$ is given, in view of (1.17), by

$$f(x) \equiv \frac{1}{2} \left( x^2 - \rho^2 \right) (= \phi_1). \quad (2.1)$$

The secondary constraint (1.10) is read as

$$\phi_2 \equiv p \cdot \nabla x f(x) = p \cdot x. \quad (2.2)$$

The FS-formula (1.14) and (1.15) in a discretized form is found as

$$\langle \phi | e^{-iTH} | \psi \rangle \equiv \lim_{N \to \infty} \prod_{j=0}^{N} \int d^{D+1} x(j) \delta(\phi_1(x(j)))$$

$$\times \prod_{j=1}^{N} \left( \frac{d^{D+1} p(j)}{(2\pi)^D} \delta(\phi_2(j)) \right) \left| \det \{ \phi_1(x(j)), \phi_2(j) \} \right|^{1/2}$$

$$\times \phi^*(x(N)) \exp \left[ i \sum_{j=1}^{N} \left\{ p(j) \cdot \Delta x(j) - \Delta t H(p(j), \pi(j)) \right\} \right] \psi(x(0)), \quad (2.3)$$

with

$$\Delta t \equiv \frac{T}{N}, \quad (2.4)$$
\[ \Delta x(j) \equiv x(j) - x(j-1), \quad (2.5) \]

and

\[ \overline{x}(j) \equiv \frac{x(j) + x(j-1)}{2}. \quad (2.6) \]

Here we have employed the mid-point prescription (2.6) to the argument of Hamiltonian, which can be interpreted as a consequence of the Weyl ordering [5][6]. The issue is to fix the form of \( \phi_2(j) \): the correct form has been found also as the mid-point type [4]:

\[ \phi_2(j) = p(j) \cdot \overline{x}(j). \quad (2.7) \]

The way to (2.7) can be convinced by the following discussion.

Consider \( T = 0 \) case: put \( N = 1 \) in (2.3) to obtain

\[ \langle \phi | \psi \rangle = \int d^{D+1}x \, d^{D+1}x' \, \delta \left( \frac{x^2 - \rho^2}{2} \right) \delta \left( \frac{x'^2 - \rho^2}{2} \right) \]
\[ \times \int \frac{d^{D+1}p}{(2\pi)^D} \delta \left( p \cdot x^{(\alpha)} \right) \left| x \cdot x^{(\alpha)} \right| \phi^*(x) e^{ip \cdot (x-x')} \psi(x'), \quad (2.8) \]

where we have written \( x, x' \), and \( p \) for \( x(1), x(0) \), and \( p(1) \) respectively and set the form of (2.2) as

\[ \phi_2(j = 1) = p \cdot x^{(\alpha)} \equiv p \cdot \left( \frac{1}{2} - \alpha \right) x + \left( \frac{1}{2} + \alpha \right) x', \quad (2.9) \]

with \( \alpha \) being a parameter [6] to be determined. Decompose the \( p \)-vector such that

\[ p = p_{\parallel} + p_{\perp}, \quad (2.10) \]

where

\[ p_{\parallel} \equiv \frac{p \cdot x^{(\alpha)}}{x^{(\alpha)}}, \]
\[ \left( p_{\perp} \right)_a \equiv \Pi_{ab} \left( x^{(\alpha)} \right) p_b, \quad (2.11) \]

are the parallel and the perpendicular components to the vector \( x^{(\alpha)} \). Then perform the \( p \)-integration to find

\[ \int \frac{d^{D+1}p}{(2\pi)^D} \delta \left( p \cdot x^{(\alpha)} \right) e^{ip \cdot (x-x')} = \frac{1}{|x^{(\alpha)}|} \delta^D \left( x - x' \right)_{\perp}, \quad (2.12) \]

where

\[ (x - x')_{\perp} \equiv \Pi_{ab} \left( x^{(\alpha)} \right) (x - x')^b. \quad (2.13) \]
Therefore the $D$-dimensional $\delta$-function, in the right hand side of (2.12), implies

$$0 = (x - x')_a^a = (x - x')^a - \frac{x^{(a)} \cdot (x - x')}{(x^{(a)})^2} (x^{(a)})^a, \quad (2.14)$$

with the aid of (1.5). The solution is

$$x = x', \quad \text{for } \alpha = 0, \quad (2.15)$$

since the second term of (2.14) vanishes:

$$x^{(\alpha=0)} \cdot (x - x') = \frac{1}{2} \left( x^2 - x'^2 \right) = 0, \quad (2.16)$$

owing to the constraint (2.1). But an additional point emerges if $\alpha \neq 0$

$$x = -x'. \quad (2.17)$$

Thus in $\alpha \neq 0$ the $\delta$-function in (2.12) is double-valued. To avoid the situation we must take $\alpha = 0$, that is, (2.9) turns out to be (2.7).

### 3. A path integral formula in generic cases

In this section we wish to generalize the previous result to $M^D$, given by $f(x) = 0$. Start from (2.3) by putting

$$\phi_2(j) \equiv p(j) \cdot \nabla f(j), \quad (3.1)$$

and study the form of $\nabla f(j)$. The $p(j)$-integral in this case becomes

$$\int \frac{d^{D+1}p(j)}{(2\pi)^D} \delta(p(j) \cdot \nabla f(j)) e^{i\phi(j)\Delta x(j)} = \frac{1}{|\nabla f(j)|} \delta^D(\Delta x_\perp(j)), \quad (3.2)$$

where

$$\Delta x_\perp^a(j) = \Pi_{\perp}(\nabla f(j)) \Delta x_\parallel^a(j) = \Delta x^a(j) - \frac{\Delta x(j) \cdot \nabla f(j)}{\nabla f(j)^2}(\nabla f(j))^a, \quad (3.3)$$

which is again the consequence of the decomposition of $p$'s into the parallel and the perpendicular components with respect to a (still unknown) vector $\nabla f(j)$. 

5
According to the foregoing discussion, (2.14) ∼ (2.17), a sufficient condition for a single-valued δ-function on \( M^D \) is read from (3.3)

\[
\Delta x(j) \cdot \nabla f(j) = 0; \quad \forall x \in M^D. \tag{3.4}
\]

A simple solution therefore is

\[
\Delta x(j) \cdot \nabla f(j) = f(x(j)) - f(x(j - 1)). \tag{3.5}
\]

(This would make sense; since a naive continuum limit, defined by \( x(j), p(j) \to x(t), p(t), \)
\( x(j - 1) \to x(t - dt), \) implies \( \nabla f(j) \to \nabla_x f(x), \) yielding the classical result (1.10).

Write

\[
x(j) = \bar{x}(j) + \frac{\Delta x(j)}{2},
\]

\[
x(j - 1) = \bar{x}(j) - \frac{\Delta x(j)}{2}, \tag{3.6}
\]

and expand the right hand side of (3.5) with respect to \( \Delta x(j) \) to obtain

\[
\nabla f(j) = \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \left( \frac{\Delta x(j) \cdot \nabla \bar{x}}{2} \right)^{2n} \right\} \nabla \bar{x} f(\bar{x}(j)), \tag{3.7}
\]

where \( \nabla \bar{x} \) denotes differentiation with respect to \( \bar{x}(j) \). With this in mind a path integral formula on \( M^D \) is found as

\[
\langle \phi | e^{-iTH} | \psi \rangle \equiv \lim_{N \to \infty} \prod_{j=0}^{N} \int d^{D+1} x(j) \delta(f(x(j)))
\]

\[
\times \prod_{j=1}^{N} \int \frac{d^{D+1} p(j)}{(2\pi)^D} \delta(p(j) \cdot \nabla f(j)) |\nabla_x f(x(j)) \cdot \nabla f(j)|
\]

\[
\times \phi^*(x(N)) \exp \left[ i \sum_{j=1}^{N} \{ p(j) \cdot \Delta x(j) - \Delta t H(p(j), \bar{x}(j)) \} \right] \psi(x(0)). \tag{3.8}
\]

Needless to say, (3.1) with (3.7) matches (2.7), the \( S^D \) case, where symmetry is higher so that the mid-point prescription was valid. But as can be recognized from (3.7) there is no privilege of the mid-point prescription in general cases.

Before closing this section let us argue another aspect of the relation (3.2) with (3.7): on \( M^D, x^a \) can be expressed by some coordinate, say, \( \theta^i (i = 1, 2, \cdots, D): \)

\[
x^a = x^a(\theta), \quad \theta \in M^D. \tag{3.9}
\]
There should be an orthonormal as well as complete set, \( Y_n(\theta) \):

\[
\int d^D\theta \sqrt{g(\theta)}Y_n^*(\theta)Y_n(\theta) = \delta_{n,n'},
\]

(3.10)

\[
\sum_n Y_n(\theta)Y_n^*(\theta) = \frac{1}{\sqrt{g(\theta)}} \delta^D(\theta - \theta'),
\]

(3.11)

where \( n \) represents generic labels and \( g(\theta) \) is the determinant of the induced metric,

\[
g_{ij}(\theta) = \sum_{a=1}^{D+1} \frac{\partial x^a}{\partial \theta^i} \frac{\partial x^a}{\partial \theta^j}.
\]

(3.12)

Specifically, \( Y_n(\theta) \) may be an eigenfunction of the Laplace-Beltrami operator:

\[- \left[ g^{-1/2} \frac{\partial}{\partial \theta^i} \left( g^{ij} g^{1/2} \right) \frac{\partial}{\partial \theta^j} \right] Y_n(\theta) = h(n)Y_n(\theta),
\]

(3.13)

Suppose that Hamiltonian is given by

\[
\hat{H} = -g^{-1/2} \frac{\partial}{\partial \theta} \left( g^{ij} g^{1/2} \right) \frac{\partial}{\partial \theta} + V(\theta),
\]

(4.14)

where the caret denotes operators, then the Feynman kernel,

\[
K(\theta, \theta'; T) \equiv \langle \theta \rangle e^{-iT\hat{H}}|\theta'\rangle = \lim_{N \to \infty} \langle \theta \rangle \left( I - i\Delta t \hat{H} \right)^N |\theta'\rangle,
\]

(3.15)

can be expressed as “path integral”: by inserting the identities, (3.10) and (3.11), which are now read as

\[
\int d^D\theta \sqrt{g(\theta)} |\theta\rangle \langle \theta| = I,
\]

(3.16)

\[
\sum_n |n\rangle \langle n| = I,
\]

(3.17)

with \( I \) being the identity operator,

\[
\langle \theta | \theta' \rangle = \frac{1}{\sqrt{g(\theta)}} \delta^D(\theta - \theta'),
\]

(3.18)

\[
\langle n| n' \rangle = \delta_{nn'},
\]

and \( \langle \theta | n \rangle \equiv Y_n(\theta), \) (3.15) becomes

\[
K(\theta, \theta'; T) = \lim_{N \to \infty} \left( \prod_{j=1}^{N-1} \int d^D\theta(j) \sqrt{g(\theta(j))} \right) \left( \prod_{j=1}^{N} \sum_{n(j)} \right)
\times Y_{n(j)}(\theta(j)) Y_{n(j)}^*(\theta(j-1)) \exp \left[ -i \Delta t \left( h(n(j)) + V(\theta(j)) \right) \right] |\theta(0) = \theta', \theta(1) = \theta \rangle.
\]

(3.19)
However the expression of (3.19) is unsatisfactory as a “path integral” formula if $M^D$ is nontrivial, $g_{ij} \neq \delta_{ij}$; since some of the labels are discrete so that we are left with summation not integration. Moreover $Y_n(\theta)$ is generally far from a plane wave form: in a trivial case, $g_{ij} = \delta_{ij}$, (which is given by an $f(x)$ linear in $x$,) $Y_n(\theta)$ is read as

$$Y_n(\theta) = \frac{1}{(2\pi)^D/2} e^{ip \cdot x}.$$  \hspace{1cm} (3.20)

($n$ and $\theta$ correspond to $P$ and $X$ respectively.) Therefore we obtain a usual path integral formula:

$$K(X, X'; T) = \lim_{N \to \infty} \left[ \prod_{j=1}^{N-1} \int d^D X(j) \prod_{j=1}^{N} \int d^D P(j) \right] \left( \frac{N}{2\pi} \right)^D \exp \left[ i \sum_{j=1}^{N} \left\{ P(j) \cdot \Delta X(j) - \Delta t \left( \hat{h}(P(j)) + V(X(j)) \right) \right\} \right]_{X(0)=X', \text{ } X(N)=X}.$$ \hspace{1cm} (3.21)

(It might be natural, however, to think that the situation is same even in the trivial case if we work with the polar coordinate; since in which there arises the spherical harmonics, being far from the plane wave except the $S^1$ case. But in these cases we can find a desired path integral formula consisting purely of an exponential form as well as integration by means of the canonical transformation \cite{7} from the Cartesian expression (3.21).)

Now it is almost clear that the relation (3.2) with (3.7) cures the above situation for nontrivial cases: according to our discussion, the completeness condition (3.11) can be put into a plane wave type provided solely with integration:

$$\sum_n Y_n(\theta) Y_n^*(\theta') = \frac{1}{\sqrt{g(\theta)}} \delta^D(\theta - \theta') = \left[ \nabla f \right] \left[ \frac{d^{D+1}p}{(2\pi)^D} \delta(p \cdot \nabla f) e^{ip \cdot (x-x')} \right]_M,$$ \hspace{1cm} (3.22)

where from (3.7)

$$\nabla f \equiv \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( \frac{\Delta x \cdot \nabla \varphi}{2} \right)^{2n} \right\} \nabla \varphi f(\varphi),$$ \hspace{1cm} (3.23)

and the subscript $M$ designates that $x$ and $x'$ are on $M^D$. The relation (3.22) thus can be implied as the plane wave representation of the completeness condition on $M^D$. In other words the FS formula is a rigorous consequence from the operator formalism owing to this completeness condition (3.22).
4. Operators from the path integral formula

A similar consideration as in (2.8) leads us to the observation that an expectation value of some operator $\mathcal{O}(\dot{p}, \dot{x})$ can be given, with the aid of the formula (3.8) with (3.7), by

$$
\langle \mathcal{O}(\dot{p}, \dot{x}) \rangle \equiv \langle \varphi | \mathcal{O}(\dot{p}, \dot{x}) | \psi \rangle \equiv \int d^{D+1}x \int d^{D+1}x' \delta(f(x)) \delta(f(x'))
\times |\nabla_x f(x) \cdot \nabla f| \varphi^*(x) \psi(x') \int \frac{d^{D+1}p}{(2\pi)^D} \delta(p \cdot \nabla f) \mathcal{O}(p, \overline{x}) \ e^{ip \Delta x},
$$

(4.1)

where $\nabla f$ is given by (3.23),

$$
\Delta x \equiv x - x',
$$

(4.2)

and

$$
\overline{x} \equiv \frac{x + x'}{2}.
$$

(4.3)

By noting

$$
delta(X)d(Y) = \delta\left(\frac{X+Y}{2}\right) \delta(X - Y),
$$

(4.4)

then using (3.5), (4.1) becomes

$$
\langle \mathcal{O}(\dot{p}, \dot{x}) \rangle = \int d^{D+1}x \int d^{D+1}x' \delta(\overline{f}) \delta(\Delta x \cdot \nabla f) |\nabla_x f(x) \cdot \nabla f| \varphi^*(x) \psi(x')
\times \int \frac{d^{D+1}p}{(2\pi)^D} \delta(p \cdot \nabla f) \mathcal{O}(p, \overline{x}) \ e^{ip \Delta x} = \int d^{D+1}x \int d^{D+1}x'
\times \delta(\overline{f}) \frac{|\nabla_x f(x) \cdot \nabla f|}{(\nabla f)^2} \varphi^*(x) \psi(x') \mathcal{O} \left( -i \frac{\partial}{\partial \Delta x \parallel}, \overline{x} \right) \delta^{D+1}(\Delta x),
$$

(4.5)

where we have introduced the notation,

$$
\overline{f} \equiv \frac{f(x) + f(x')}{2},
$$

(4.6)

and integrated with respect to p’s in a similar manner as before, to find $\delta^D(\Delta x \parallel)$ which is combined with $\delta(\Delta x \cdot \nabla f) \sim \delta\left(\Delta x \parallel \right)$ yielding $\delta^{D+1}(\Delta x)$ finally. Now changing variables $(x, x')$ to $(\overline{x}, \Delta x)$ and performing integration by parts, we find

$$
\langle \mathcal{O}(\dot{p}, \dot{x}) \rangle = \int d^{D+1}\overline{x} \ \mathcal{O} \left( -i \frac{\partial}{\partial \Delta x \parallel}, \overline{x} \right)
\times \left[ \delta(\overline{f}) \frac{|\nabla_x f(x) \cdot \nabla f|}{(\nabla f)^2} \varphi^* \left( \overline{x} + \frac{\Delta x}{2} \right) \psi \left( \overline{x} - \frac{\Delta x}{2} \right) \right]_{\Delta x = 0},
$$

(4.7)
where
\[ \nabla_x f(x) = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\Delta x \cdot \nabla \pi}{2} \right)^n \right\} \nabla \pi f(\pi), \] (4.8)
and
\[ \bar{f} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( \frac{\Delta x \cdot \nabla \pi}{2} \right)^{2n} f(\pi), \] (4.9)
in view of (4.6). (The subscript \( \Delta x = 0 \) designates that \( \Delta x \to 0 \) must be put after all calculations have been done.) Also note that
\[ \frac{\nabla_x f(x) \cdot \nabla f}{(\nabla f)^2} = 1 + \frac{1}{(\nabla f(\pi))^2} \nabla f(\pi) \cdot \nabla f \left( \frac{\Delta x \cdot \nabla \pi}{2} \right) f(\pi) \]
\[ + \frac{1}{3(\nabla f(\pi))^2} \nabla f(\pi) \cdot \nabla f \left( \frac{\Delta x \cdot \nabla \pi}{2} \right)^2 f(\pi) + O(\Delta x^3). \] (4.10)

Let us calculate some examples:

- (i) \( \mathcal{O}(\dot{p}, \dot{x}) \equiv F(\dot{x}) \):
  \[ \langle F(\dot{x}) \rangle = \int d^{D+1}x \delta(f(x)) \varphi^*(x) F(x) \psi(x), \] (4.11)
  where we have written \( x \) for \( \pi \). This shows
  \[ F(\dot{x}) = F(x). \] (4.12)

- (ii) \( \mathcal{O}(\dot{p}, \dot{x}) \equiv \dot{p}_a \):
  \[ \langle \dot{p}_a \rangle = \int d^{D+1}x \Pi_{ab}(\nabla_x f) \left\{ -i \frac{\partial}{\partial \Delta x^b} \right\} \left[ \delta(\bar{f}) \frac{\nabla_x f(x) \cdot \nabla f}{(\nabla f)^2} \right. \]
  \[ \left. \times \varphi^* \left( \frac{\Delta x}{2} \right) \psi \left( \frac{\Delta x}{2} \right) \right] \bigg|_{\Delta x=0} = \int d^{D+1}x \delta(f(x)) \]
  \[ \times \Pi_{ab} \left( \nabla_x f \right) \frac{i}{2} \left\{ \partial_b \varphi^* \left( \frac{\Delta x}{2} \right) \psi \left( \frac{\Delta x}{2} \right) - \varphi^* \left( \frac{\Delta x}{2} \right) \partial_b \psi \left( \frac{\Delta x}{2} \right) + \frac{\partial_b \partial_c f \partial_d f}{(\nabla f)^2} \varphi^* \left( \frac{\Delta x}{2} \right) \partial_b \psi \left( \frac{\Delta x}{2} \right) \right\}, \] (4.13)
where again we have put \( \pi \to x \). The third term in the final expression comes from the differentiation to (4.10). (There remains no effect from differentiating the \( \delta \)-function, in view of (4.9).) Finally integrating by parts with respect to the first term, while paying attention to the property of the projection operator, \( \Pi_{ab} \partial_b \delta(f(x)) = 0 \), we obtain
\[ \langle \dot{p}_a \rangle = \int d^{D+1}x \delta(f(x)) \varphi^*(x) \left[ -i \Pi_{ab}(\nabla_x f) \partial_b \right. \]
\[ - \frac{i}{2} \partial_b \Pi_{ab}(\nabla_x f) - \frac{i}{2} \Pi_{ab}(\nabla_x f) \partial_c \Pi_{bc}(\nabla_x f) \bigg] \psi(x). \] (4.14)
Therefore
\[ \hat{p}_a = -i \Pi_{ab} \nabla_b f \partial_a - \frac{i}{2} \partial_b \Pi_{ab} \nabla_b f + \frac{i}{2} \Pi_{ab} \nabla_b f \partial_a \Pi_{bc} \nabla_c f \]
\[ = -i \Pi_{ab} \nabla_b f \partial_a + \frac{i}{2} \frac{2 \partial_a \partial_b f \partial_c f + \partial_a f \nabla_x^2 f}{(\nabla_x f)^2} - \frac{3i}{2} \frac{\partial_b \partial_c f \partial_a f \partial_c f}{(\nabla_x f)^4}, \]
(4.15)
is the momentum operator. It can be shown by an explicit calculation that \(\hat{p}^2\) satisfies the quantum version of (1.13):
\[
[\hat{\sigma}^a, \hat{\sigma}^b] = 0,
\]
\[
[\hat{\sigma}^a, \hat{p}_b] = i \Pi_{ab} \nabla_b f = i \left( \delta_{ab} - \frac{\partial_a \partial_b f \partial_c f}{(\nabla_x f)^2} \right),
\]
(4.16)
\[
[\hat{p}_a, \hat{p}_b] = \frac{i}{2} \left\{ \left\{ \hat{p}_c, \frac{\partial_a \partial_c f \partial_b f}{(\nabla_x f)^2} \right\} - \left\{ \hat{p}_c, \frac{\partial_b \partial_c f \partial_a f}{(\nabla_x f)^2} \right\} \right\},
\]
where \(\{ \hat{A}, \hat{B} \} = \hat{A} \hat{B} + \hat{B} \hat{A} \).

• (iii) \(\mathcal{O}(\hat{p}, \hat{x}) \equiv \hat{p}^2\): with a similar manner as above, we find
\[
\langle \hat{p}^2 \rangle = \int d^{D+1} x \, \Pi_{ab}(\nabla_x f) \left( -\frac{\partial^2}{\partial \Delta x^a \partial \Delta x^b} \right) \left[ \frac{\delta(f(x))}{(\nabla f)^2} \frac{\nabla_x f(x) \cdot \nabla f}{(\nabla f)^2} \right] \phi^* \phi \bigg|_{\Delta x = 0} 
\]
\[ = \int d^{D+1} x \, \delta(f(x)) \phi^*(x) \left[ -\Pi_{ab}(\nabla_x f) \frac{\partial^2}{\partial \Delta x^a \partial \Delta x^b} \right.
\]
\[ + \left( \Pi_{ab}(\nabla_x f) \frac{\partial_a \partial_c f \partial_b f}{(\nabla_x f)^2} - \partial_a \Pi_{ab}(\nabla_x f) \frac{\partial_c f}{\partial x^c} + \frac{1}{2} \partial_b \Pi_{ab}(\nabla_x f) \frac{\partial_a \partial_c f \partial_b f}{(\nabla_x f)^2} \right)
\]
\[ - \frac{1}{4} \partial_a \partial_b \Pi_{ab}(\nabla_x f) - \frac{1}{6} \Pi_{ab}(\nabla_x f) \frac{\partial_a \partial_b \partial_c f \partial_d f}{(\nabla_x f)^2} \bigg] \psi(x). \]
From this we obtain

\[ \hat{p}^2 = -\Pi_a (\nabla_x f) \frac{\partial^2}{\partial \Delta x^a \partial \Delta x^b} \]

\[ + \left\{ \frac{2\partial_a \partial_b f + \nabla_x^2 f \partial_b f}{(\nabla_x f)^2} - \frac{3\partial_a \partial_c f \partial_b f \partial_b f}{(\nabla_x f)^4} \right\} \frac{\partial}{\partial x^b} \]

\[ + \frac{1}{(\nabla_x f)^2} \left\{ \frac{5}{6} \partial_a \nabla_x^2 f \partial_a f + \frac{1}{4} (\nabla_x^2 f)^2 + \frac{3}{4} \partial_a \partial_b f \partial_a f \partial_b f \right\} \]

\[ - \frac{1}{(\nabla_x f)^4} \left\{ \frac{3}{2} \nabla_x^2 f \partial_a f \partial_b f \partial_c f + \frac{7}{2} \partial_a \partial_b f \partial_a f \partial_c f \partial_c f \right\} \]

\[ + \frac{5}{6} \partial_a \partial_b f \partial_a f \partial_b f \partial_c f \partial_c f \partial_d f. \]  

(4.18)

It should be noted that \( \hat{p}^2 \neq \hat{p}_a \hat{p}_a \) unless \( f(x) \) is linear in \( x \).

5. Discussion

In this paper we have established a form of constraints in the path integral formula given by the time discretization. The main interest is how to incorporate the classical constraint \( p \cdot \nabla_x f = 0 \) into the quantum one: the correct form can be found by requiring that the delta function be single-valued.

The conclusion is unchanged even if we take a nonstandard form of Hamiltonian instead of (1.8) such as

\[ H(p, x) \longrightarrow h(p^2) + V(x), \]  

(5.1)

provided \( h'(p^2) \neq 0 \).

Therefore we have successfully described a ‘local’ form of the path integral formula; where the word ‘local’ must be attached since if manifold is nontrivial and composed of \( G/H \) there emerge induced gauge fields according to recent studies [8][9]. Our formula apparently lacks these informations. There has been a trial [10] but we are still on the way to the final goal.
References


