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AND YANG-BAXTER EQUATION

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LATTICE QUANTUM PHASE SPACE
AND YANG-BAXTER EQUATION¹

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ABSTRACT

In this work, we show that, in the sense of [17], it is possible to construct the quantum group which preserves the *quantum symplectic structure* introduced in the context of the *matrix Hamiltonian formalism* defined in [9]. We also study the braiding existing behind the *lattice quantum phase space*, and present another type of non-trivial solution to the resulting Yang-Baxter equation.

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1. Introduction :

It is well known that Poisson symplectic geometry, [1], provides a complete description of Hamiltonian mechanics.

Furthermore, Since the appearance of quantum mechanics at now several attempts were presented to give a precise interpretation of quantization. At the begining, quantum mechanics was interpreted as a statistical theory over phase space. Founding upon the Weyl quantization procedure, [2], Wigner gave an expression for a phase space distribution function, [3].

Most interesting developments of the Weyl-Wigner approach are due to Moyal, [4], who introduced the *sine-Poisson bracket* (or *Moyal bracket*) for functions on phase space, that firstly corresponds to the commutator of quantum mechanical operators associated to these functions, and secondly, goes into the usual Poisson bracket at the classical limit $\hbar \longrightarrow 0$.

In other respects, it was realized that the Weyl-Wigner-Moyal quantization procedure can be fitted into the *deformation theory of algebraic structures*, [5].

Quantum mechanics has also been described as a *non-commuta-*

tive symplectic geometry, [6].

In a previous paper, [7], we have shown that the quantum phase space (QPS) may be seen as a particular Manin space, [8].

Recently, [9], we have presented a more or less complete description of a matrix (quantum) Hamiltonian formalism defined on a Lattice quantum phase space (LQPS), defining a discrete version of the Weyl-Wigner-Moyal formalism and using the apparatus of the non-commutative geometry of the matrix algebra $M_N(\mathbb{C})$, [10], equipped with the Schwinger basis.

The main purpose of this work is to show the braiding and the Hopf algebra structure existing behind the LQPS, and finally, to give a non-trivial solution to the resulting Yang-Baxter equation.

The paper is organized as follows. In Sections 2 and 3, we present a brief review of the Weyl-Schwinger realization of the Heisenberg group and our matrix (quantum) symplectic formalism respectively. For more details, see [9]. In Section 4, we present a solution of the Yang-Baxter equation issued from some braiding existing behind the LQPS. We end this work with a conclusion.

2. Weyl-Schwinger realization of the Heisenberg group :

The Heisenberg group can be represented as an extension of the abelian double cyclic group $Z_N \oplus Z_N$. Then, finite representations of this group, which is generated by two elements U and V satisfying :

$$V.U = \omega U.V \quad / \quad \omega = e^{i2\pi/N}, \quad (1)$$

are obtained by taking U and V as $N \times N$ matrices such that :

$$U^N = 1_{N \times N} = V^N. \quad (2)$$

For each integer N , there is one realization of the Heisenberg group. Let $\{ |\beta_k\rangle, k \in \mathbb{Z} \}$ be a basis of orthonormalized kets for the space of states. Assume that the $|\beta_k\rangle$ are eigenkets of the operator V :

$$V |\beta_k\rangle = v_k |\beta_k\rangle \quad / \quad v_k = \omega^k \quad (3.1)$$

$$V^m |\beta_k\rangle = \omega^{km} |\beta_k\rangle \longrightarrow V^m = \sum_{k=0}^{N-1} \omega^{km} |\beta_k\rangle \langle \beta_k|. \quad (3.2)$$

Then, U is defined by :

$$U |\beta_k\rangle = |\beta_{k+1}\rangle \quad (4.1)$$

$$U^N |\beta_k\rangle = |\beta_{k+N}\rangle \longrightarrow U^N = \sum |\beta_{k+N}\rangle \langle \beta_k| \quad (4.2)$$

with :

$$|\beta_{k+N}\rangle = |\beta_k\rangle. \quad (4.3)$$

this basis, the matrices V and U are given by :

$$V = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 3 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 3^{N-1} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (5)$$

and verify :

$$\mathbf{v}^n \mathbf{u}^m = \omega^{mn} \mathbf{u}^m \mathbf{v}^n. \quad (6)$$

Let γ_m denote operators that are invariant under this

entry :

$$\gamma_{\vec{m}} = \gamma_{m_1 m_2} = \omega^{m_1 m_2/2} u^{m_1} v^{m_2}, \quad (7)$$

$$\vec{m} = (m_1, m_2) \quad , \quad \vec{m} + \vec{n} = (m_1 + n_1, m_2 + n_2) \quad , \quad k\vec{m} = (km_1, km_2) \quad ,$$

$$\vec{m} \cdot \vec{n} = m_1 n_1 + m_2 n_2, \quad \vec{m} \times \vec{n} = m_1 n_2 - m_2 n_1. \quad (8)$$

The elements γ_m , which satisfy :

$$\gamma_{on} = V^n, \quad \gamma_{mo} = U^m, \quad \gamma_{oo} = \gamma_{oo} = 1, \quad \gamma_{\vec{m}\vec{n}}^{-1} = \gamma_{\vec{n}\vec{m}}^* = \gamma_{-\vec{m}-\vec{n}}, \quad (9.1)$$

form a complete orthonormal basis of the group algebra with the

Associativity : $(\gamma_{\beta_1} \cdot \gamma_{\beta_2}) \cdot \gamma_{\beta_3} = \gamma_{\beta_1} \cdot (\gamma_{\beta_2} \cdot \gamma_{\beta_3})$ (Q.2)

$$\text{Quasi-periodicity: } \gamma_{n_0} = (-1)^n \gamma_{0n}, \quad \gamma_{m_0} = (-1)^m \gamma_{0m} \quad (9.3)$$

Two-foldedness : $\gamma_{NN} = (-1)^N 1$ (9.4)

$$\text{Power : } (\gamma_{\vec{m}})^k = \gamma_{\vec{km}}. \quad (9.5)$$

In the basis $\{|\beta_k\rangle, k \in \mathbb{Z}_N\}$, the operator $\gamma_{\vec{n}}$ take the form :

$$\gamma_m = \sum_{k=0}^{N-1} \omega^{m(2k+m_1)/2} |\beta_{k+m_1}\rangle \langle \beta_k| \quad (10.1)$$

[illegible]

Although the operators U and V have been introduced by Weyl, [2], it was Schwinger, [11], who described quantum mechanics in this formalism. Any operator A belonging to the group algebra will be written in the Schwinger basis $\left\{ \gamma_{\vec{m}} ; m_1, m_2 = 0, 1, \dots, N-1 \right\}$ as follows :

$$A = \frac{1}{N} \sum_{\vec{m}} a_{\vec{m}} \gamma_{\vec{m}} \quad (11)$$

The set of these operators will describe a non-commutative (or quantum) two-torus, [12].

For each value of N , the operators $\gamma_{\vec{m}}$ realize a representation that makes use of a two-cocycle α_2 . Occuring frequently in quantum mechanics, these representations are called ray or projective representations. In fact, if we introduce a Lie algebra structure by means of the standard commutator, we obtain the so-called class of trigonometric sine-Lie algebras or quantum Tori-Lie algebras, [13], with :

$$\gamma_{\vec{n}} \cdot \gamma_{\vec{m}} = \exp[2i\alpha_2(\vec{m}, \vec{n})] \gamma_{\vec{m}} \cdot \gamma_{\vec{n}} = \exp[i\alpha_2(\vec{m}, \vec{n})] \gamma_{\vec{m}+\vec{n}} \quad (12.1)$$

$$[\gamma_{\vec{n}}, \gamma_{\vec{m}}] = 2i \sin[\alpha_2(\vec{m}, \vec{n})] \gamma_{\vec{m}+\vec{n}} \quad (12.2)$$

where the so-called fundamental cocycle α_2 defined as :

$$\alpha_2(\vec{m}, \vec{n}) = -\alpha_2(\vec{n}, \vec{m}) = \frac{\pi}{N} \vec{m} \times \vec{n} \quad (13.1)$$

corresponds in the language of [14] to an element of the first cyclic cohomology group and will serve to define what we will call the simplicial symplectic structure on this class of non-commutative algebras. Using the associativity property (9.2), we get the following consistency condition :

$$\Delta\alpha_2 = \alpha_2(\vec{n}, \vec{p}) - \alpha_2(\vec{m}+\vec{n}, \vec{p}) + \alpha_2(\vec{m}, \vec{n}+\vec{p}) - \alpha_2(\vec{m}, \vec{n}) = 0 \pmod{Z} \quad (13.2)$$

where Δ is a nilpotent derivative (coboundary operator) of some cohomology giving information on the projective representation under consideration, [15].

Quantum mechanics also makes use of a one-cochain α_1 .

Indeed, the action of the operator $\gamma_{\vec{m}}$ on a state $|\beta_k\rangle$ is given by:

$$\gamma_{\vec{m}} |\beta_k\rangle = \exp[i\alpha_1(k; \vec{m})] |\beta_{k+m_1}\rangle \quad (13.3)$$

with,

$$\alpha_1(k; \vec{m}) = \frac{\pi m_2}{N} (2k + m_1) \quad (13.4)$$

If α_1 is exact, i.e. $\alpha_1 = \Delta\alpha_0$ with α_0 some 0-cochain, then α_1 may be eliminated by adding a phase α_0 to the wavefunctions. When α_2 is exact, it may be eliminated by redefining the operators in such a way that they appear state-independent. It turns actually that :

$$\alpha_2 = \Delta\alpha_1 = \alpha_1(k+n_1; \vec{m}) - \alpha_1(k; \vec{m}+\vec{n}) + \alpha_1(k; \vec{n}) \quad (13.5)$$

so that :

$$\Delta\alpha_2 = 0. \quad (13.6)$$

It's clear from (13.1) and (13.4) that α_1 depends effectively on the state-label k while α_2 does not.

Moreover, the fundamental cocycle α_2 will define a simplicial structure on the lattice quantum phase space (LQPS) analog to the classical one on the usual classical phase space (CPS), while the 1-cochain α_1 will play on LQPS the role of the Liouville canonical 1-form θ on CPS. In fact, LQPS is the lattice torus described by the double integer index $\vec{m} = (m_1, m_2)$ that label the operators belonging to the Schwinger's basis $\left\{ \gamma_{\vec{m}} \right\}$, and in which the area of each elementary lattice is equal to $2\pi/N$ tending to zero when N tends to infinity. Therefore, the translations issued from the action of the operator $\gamma_{\vec{m}}$ on LQPS may be interpreted as unit multiple elementary gaps in the two basic direction (U,V).

The choice of the symmetrized Schwinger basis for the Weyl realization of the Heisenberg group is based essentially on its particular interest due to its many remarkable properties. That is, for $N=2$, the $\gamma_{\vec{m}}$'s given by (10.2) reduce to the Pauli matrices :

$$\begin{aligned} \gamma_{00} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = \sigma_0, & \gamma_{10} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \\ \gamma_{11} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2, & \gamma_{01} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \end{aligned} \quad (14)$$

For $N \geq 2$, the Schwinger basis is a preferable basis admitting additive quantum numbers. It provides the finest grading of the

Lie algebra $gl(N; \mathbb{C})$, [16]. Secondly, because of the two-foldedness property (9.4), the $\gamma_{\vec{m}}$'s constitute a double covering $\text{mod}(N)$ of the torus, with (m_1, m_2) playing the role of coordinates $\text{mod}(N)$. U^{m_1} and V^{m_2} may also be seen as global coordinates with values in $\mathbb{Z}_N \otimes \mathbb{Z}_N$ appearing as non-commutative point functions because of the projective character of the representation.

The choice of the Schwinger basis also gives rise to many helpful relations and permits the study of physical situations with various number of quantum degrees of freedom basing upon a prime decomposition of N . Finally, with this choice one may recover the usual situation of position ($\vec{q}$) and momentum ($\vec{p}$) operators by means of a simple correspondence and, the continuous case is obtained by taking to infinity the torus radii while N tends to ∞ . The resulting continuous limit of the operator $\gamma_{\vec{m}}$ is nothing else than the operator $\gamma(\vec{a}, \vec{b})$ of translations on the quantum phase space (QPS) :

$$\vec{q} \longrightarrow \vec{q} + \vec{a}, \quad \vec{p} \longrightarrow \vec{p} + \vec{b} \quad (15.1)$$

$$\gamma(\vec{a}, \vec{b}) = \exp[i(\vec{a} \cdot \vec{p} - \vec{b} \cdot \vec{q})] \quad (15.2)$$

These operators generate the non-commutative Heisenberg

algebra and obey a similar generalized composition as in the discrete case (see equ. (12.2)) :

$$\gamma(\vec{a}, \vec{b}) \cdot \gamma(\vec{a}, \vec{b}) = \exp[i\alpha_2[(\vec{a}, \vec{b}); (\vec{a}, \vec{b})]] \gamma(\vec{a}+\vec{b}, \vec{a}+\vec{b}) \quad (16.1)$$

with :

$$\alpha_2(\vec{a}, \vec{b}) = \frac{\hbar}{2} (\vec{a}, \vec{b}) \times (\vec{a}, \vec{b}) = \frac{\hbar}{2} [\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a}] \quad (16.2)$$

The set $\{ \gamma(\vec{a}, \vec{b}) \}$ may be considered as a Fourier basis and

then become fundamental for the Weyl-Wigner map :

$$Q_F(\vec{q}, \vec{p}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} d\vec{a} d\vec{b} D_F(\vec{a}, \vec{b}) \gamma(\vec{a}, \vec{b}) \quad (17)$$

where $D_F(\vec{a}, \vec{b})$ is the Wigner density associated to the symbol (or quantum observable) $Q_F(\vec{q}, \vec{p})$ of the operator $Q_F(\vec{q}, \vec{p})$. This shows that quantization is deeply tied to Fourier analysis, since any operator is given as Fourier expansion. The relation (17) can be seen as the continuous limit of the relation (11), where the complex coefficients $a_m^{\vec{p}}$ play the role of discrete Wigner densities and one can easily define a discrete Weyl-Wigner-Moyal formalism.

Finally, it is important to point out that the LQPS is a space with matrix structure. This leads us to use the formalism of matrix differential geometry, [10].

3. Matrix symplectic formalism :

The Schwinger basis $\{ \gamma_m^{\vec{p}} \}$ may be considered as a basis of the matrix algebra $M_N(\mathbb{C})$ consisting of hermitian matrices.

Excluding the unit matrix $\gamma_0^{\vec{p}}$, the set $\{ \gamma_m^{\vec{p}} / m \neq 0 \}$ may be viewed as a basis of $\mathfrak{su}(N)$ consisting of N^2-1 unitary traceless $N \times N$ matrices. Then, the relation (11) gives the expansion of any element of $M_N(\mathbb{C})$ in terms of the Schwinger basic matrices $\gamma_m^{\vec{p}}$.

The relation (12.1) represents the multiplication table of these basic elements :

$$\gamma_m^{\vec{p}} \cdot \gamma_n^{\vec{p}} = K_{mn}^{\vec{p}} 1 + [S_{mn}^{\vec{p}} - \frac{1}{2} C_{mn}^{\vec{p}}] \gamma_p^{\vec{p}} \quad (18.1)$$

where $K_{mn}^{\vec{p}}$ and $C_{mn}^{\vec{p}}$ are respectively the Killing metric and structure constants for $\mathfrak{su}(N)$:

$$K_{mn}^{\vec{p}} = \delta_{m+n, 0}^{\vec{p}} \quad (18.2)$$

$$C_{mn}^{\vec{p}} = -2i \sin[\alpha_2(\vec{m}, \vec{n})] \delta_{m+n}^{\vec{p}} \quad (18.3)$$

and the symmetric quantities $S_{mn}^{\vec{p}}$ identically vanish.

The Lie algebra $\text{Der}(M_N(\mathbb{C}))$ of derivations of $M_N(\mathbb{C})$ relatively to the basis $\{ \gamma_m^{\vec{p}} / m \neq 0 \}$ is generated by the basic derivations $e_m^{\vec{p}}$ defined as follows :

$$e_m^{\vec{p}} = \text{ad}(\gamma_m^{\vec{p}}) \quad / \quad e_m^{\vec{p}}(\gamma_n^{\vec{p}}) = [\gamma_m^{\vec{p}}, \gamma_n^{\vec{p}}] \quad (19)$$

We introduce now a basis of 1-forms denoted by $\left\{ \theta^{\vec{m}} / \vec{m} \neq \vec{0} \right\}$

dual to the basis $\left\{ e_{\vec{m}} / \vec{m} \neq \vec{0} \right\}$ i.e. :

$$\theta^{\vec{m}}(e_{\vec{n}}) = \delta_{\vec{n}}^{\vec{m}} \quad (20)$$

In addition, we have the following relations :

$$\theta^{\vec{m}} \wedge \theta^{\vec{n}} = - \theta^{\vec{n}} \wedge \theta^{\vec{m}} \quad (21.1)$$

$$\gamma_{\vec{m}} \theta^{\vec{n}} = \theta^{\vec{n}} \gamma_{\vec{m}} \quad (21.2)$$

$$d\gamma_{\vec{m}} = - \sum_{\vec{n}, \vec{p}} C_{\vec{m}\vec{n}}^{\vec{p}} \gamma_{\vec{p}} \theta^{\vec{n}} \quad (21.3)$$

$$d\theta^{\vec{m}} = - \frac{1}{2} \sum_{\vec{p}, \vec{q}} C_{\vec{p}\vec{q}}^{\vec{m}} \theta^{\vec{p}} \wedge \theta^{\vec{q}} = -i \sum_{\vec{q}} \sin(\alpha_2(\vec{m}, \vec{q})) \theta^{\vec{q}} \wedge \theta^{\vec{m}-\vec{q}} \quad (21.4)$$

Then, the relations (18.1), (21.1)-(21.4) give a presentation of the space $\Omega_{\text{Der}}^1(M_N(\mathbb{C}))$ of matrix forms associated to the Schwinger basis.

The element $\theta \in \Omega_{\text{Der}}^1(M_N(\mathbb{C}))$ defined by :

$$\theta = \sum_{\vec{m}} \gamma_{\vec{m}} \theta^{\vec{m}} \quad (22)$$

will be called the canonical 1-form and will play the role of the Liouville 1-form, when the 2-form :

$$\Omega = - d\theta = \frac{1}{2} \sum_{\vec{m}\vec{n}} \theta^{\vec{m}} \wedge \theta^{\vec{n}} = - \frac{1}{2} \sum_{\vec{m}\vec{n}} C_{\vec{m}\vec{n}}^{\vec{p}} \gamma_{\vec{p}} \theta^{\vec{m}} \wedge \theta^{\vec{n}} \quad (23)$$

will represent the matrix (simplicial) symplectic structure.

It follows that the quantum symplectic matrix $\Omega_{\vec{m}\vec{n}}^{\vec{p}}$ is a $(N^2-1) \times (N^2-1)$ antisymmetric hypermatrix whose entries are the $N \times N$ Schwinger matrices :

$$\Omega_{\vec{m}\vec{n}}^{\vec{p}} = - C_{\vec{m}\vec{n}}^{\vec{p}} \gamma_{\vec{p}} = [\gamma_{\vec{n}}, \gamma_{\vec{m}}] = e_{\vec{n}}(\gamma_{\vec{m}}) = - \Omega_{\vec{n}\vec{m}}^{\vec{p}} \quad (24)$$

For any elements A,B of the operator algebra (see equ. (11)),

we have the following relations :

$$e_{\vec{m}}(A) = - \frac{1}{N} \sum_{\vec{n}} \Omega_{\vec{m}\vec{n}}^{\vec{n}} a^{\vec{n}} \quad (25.1)$$

$$[A, B] = - \frac{1}{N} \sum_{\vec{m}, \vec{n}} a^{\vec{m}} \Omega_{\vec{m}\vec{n}}^{\vec{n}} b^{\vec{n}} \quad (25.2)$$

$$d\gamma_{\vec{m}} = \sum_{\vec{n}} \Omega_{\vec{m}\vec{n}}^{\vec{n}} \theta^{\vec{n}} \quad (25.3)$$

$$dA = \frac{1}{N} \sum_{\vec{m}, \vec{n}} a^{\vec{m}} \Omega_{\vec{m}\vec{n}}^{\vec{n}} \theta^{\vec{n}} \quad (25.4)$$

In other respects, the hypermatrix $\Omega_{\vec{m}\vec{n}}^{\vec{p}}$ has an inverse $\Lambda^{\vec{m}\vec{n}}$

which is another hypermatrix (not antisymmetric) given by :

$$\Omega_{\vec{m}\vec{n}}^{\vec{p}} \Lambda^{\vec{n}\vec{k}} = \Lambda^{\vec{k}\vec{n}} \Omega_{\vec{n}\vec{m}}^{\vec{p}} = \delta_{\vec{m}}^{\vec{k}} \quad (26.1)$$

$$\Lambda^{\vec{m}\vec{n}} = \frac{1}{N^2} \gamma_{\vec{n}} \cdot \gamma_{\vec{m}} = \frac{1}{N^2} K^{\vec{n}\vec{r}} K^{\vec{m}\vec{s}} \gamma_{\vec{r}} \cdot \gamma_{\vec{s}} = \frac{1}{N^2} e^{i\alpha_2(\vec{m}, \vec{n})} \gamma_{\vec{m}+\vec{n}}^{\vec{m}\vec{n}}, \quad (26.2)$$

$$\Lambda^{\vec{n}\vec{m}} = \exp[i\alpha_2(\vec{n}, \vec{m})] \Lambda^{\vec{m}\vec{n}}, \quad (26.3)$$

where $K^{\vec{m}\vec{n}}$ and $\gamma_{\vec{m}}^{\vec{m}}$ are the inverses of the Killing metric $K_{\vec{m}\vec{n}}$ and the Schwinger matrix $\gamma_{\vec{m}}$ respectively :

$$K_{\vec{n}\vec{p}}^{\vec{m}\vec{n}} K_{\vec{p}\vec{n}}^{\vec{n}\vec{m}} = K_{\vec{p}\vec{n}}^{\vec{m}\vec{n}} = \delta_{\vec{p}}^{\vec{m}} \quad , \quad K^{\vec{m}\vec{n}} = \delta^{\vec{m}+\vec{n},\vec{0}} \quad (27.1)$$

$$\gamma_{\vec{n}}^{\vec{m}} = K_{\vec{n}}^{\vec{m}\vec{n}} \gamma_{\vec{n}} \quad , \quad \gamma_{-\vec{m}}^{\vec{m}} = \gamma_{\vec{m}}^{\vec{m}} = \gamma_{\vec{m}}^{-1} \quad (27.2)$$

Notice that, on the one hand, one may define an antisymmetric hypermatrix $\Omega_{\vec{m}\vec{n}}^{\vec{m}\vec{n}}$ such as :

$$\Omega_{\vec{m}\vec{n}}^{\vec{m}\vec{n}} = K_{\vec{m}\vec{n}}^{\vec{m}\vec{n}} K_{\vec{n}\vec{m}}^{\vec{n}\vec{m}} \Omega_{\vec{r}\vec{s}}^{\vec{r}\vec{s}} = [\gamma_{\vec{n}}^{\vec{m}}, \gamma_{\vec{m}}^{\vec{n}}] = N^2 (\Lambda^{\vec{m}\vec{n}} - \Lambda^{\vec{n}\vec{m}}) = -\Omega^{\vec{n}\vec{m}} \quad , \quad (27.3)$$

and on the other hand, invert the relation (25.3) to yield :

$$\theta_{\vec{n}}^{\vec{m}} = \Lambda^{\vec{m}\vec{n}} d\gamma_{\vec{m}}^{\vec{n}} = \frac{1}{N^2} K_{\vec{m}\vec{n}}^{\vec{m}\vec{n}} K_{\vec{r}\vec{s}}^{\vec{n}\vec{m}} \gamma_{\vec{r}}^{\vec{m}} \gamma_{\vec{s}}^{\vec{n}} d\gamma_{\vec{m}}^{\vec{n}} \quad (27.4)$$

If we consider the case $N = 2$, with :

$$\vec{1} = (1,0) \quad , \quad \vec{2} = (1,1) \quad , \quad \vec{3} = (0,1),$$

we obtain :

$$\Omega_{\vec{m}\vec{n}}^{\vec{m}\vec{n}} = 2i \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & \sigma_1 \\ \sigma_2 & -\sigma_1 & 0 \end{pmatrix} \quad , \quad \Lambda^{\vec{m}\vec{n}} = \frac{1}{4} \begin{pmatrix} 1 & -i\sigma_3 & i\sigma_2 \\ i\sigma_3 & 1 & -i\sigma_1 \\ -i\sigma_2 & i\sigma_1 & 1 \end{pmatrix} \quad (28)$$

where σ_i are the Pauli matrices (14).

Furthermore, one may easily verify that, for any vector field $\chi \in \text{Der}(M_N(C))$ given by :

$$\chi = \frac{1}{N} \sum_{\vec{m}} \chi_{\vec{m}}^{\vec{m}} \bullet_{\vec{m}} \quad (29)$$

the quantum canonical 1-form θ defined by (22) is effectively the

basic invariant element of $\Omega_{\text{Der}}^1(M_N(C))$, i.e. :

$$L_{\chi}(\theta) = 0, \quad (30.1)$$

where,

$$L_{\chi} = i_{\chi} d + d i_{\chi} \quad (30.2)$$

is the Lie derivative defined on $\Omega_{\text{Der}}^1(M_N(C))$.

It follows that :

$$L_{\chi}(\Omega) = 0, \quad (30.3)$$

and consequently, all vector fields χ are said, in the classical terminology, to be strictly or globally Hamiltonian, [1].

This means that to each vector field $\chi \in \text{Der}(M_N(C))$ will correspond an element $\alpha = dA$ of $\Omega_{\text{Der}}^1(M_N(C))$, where A is an element of the operator algebra, such that the relation (30.3) holds, and conversely, to each 1-form dA one may associate a Hamiltonian vector field χ_A such that :

$$i_{\chi_A}(\Omega) = dA, \quad (30.4)$$

where the components $\chi_A^{\vec{m}}$ of χ_A in the basis $\{e_{\vec{m}}\}$ coincide with the coefficients $a^{\vec{m}}$ of A in the Schwinger basis $\{\gamma_{\vec{m}}^{\vec{m}}\}$.

In analogy with classical case, the matrix Poisson bracket of two operators A and B is defined by :

$$\langle A, B \rangle_m = \Omega(\chi_A, \chi_B) = \chi_A(B) = -\chi_B(A). \quad (31.1)$$

As expected, this matrix Poisson bracket is just the

commutator :

$$\begin{aligned} \langle A, B \rangle_m &\equiv [A, B] = -\frac{1}{N^2} \sum_{\vec{m}, \vec{n}} a_{\vec{m}}^{\vec{n}} \Omega_{\vec{m}\vec{n}} b_{\vec{n}}^{\vec{m}} \\ &= i \frac{2\pi}{N} C = \frac{2i}{N^2} \sum_{\vec{m}, \vec{r}} a_{\vec{m}}^{\vec{r}} b_{\vec{r}-\vec{m}}^{\vec{r}-\vec{m}} \sin(\alpha_2(\vec{r}, \vec{m})) \gamma_{\vec{r}} \end{aligned} \quad (31.2)$$

and we can read here the discrete version of the Moyal bracket.

In other respects, it is easy to see that the Jacobi identity:

$$[A, [B, C]] + c.p = 0 \quad (32.1)$$

is equivalent to the relation :

$$\begin{aligned} [\chi_A, \chi_B](C) &= \chi_{[A, B]}(C) = -\chi_C([A, B])_m \\ &= dC(\chi_{[A, B]})_m = -dC([A, B])_m(\chi_C) \end{aligned} \quad (32.2)$$

for all operators A, B and C. Finally, in order to complete this

matrix Hamiltonian formalism, we add these three relations :

$$\chi_A = \frac{1}{N} \sum_{\vec{m}} \chi_A^{\vec{m}} e_{\vec{m}}^{\vec{m}}, \quad \chi_A^{\vec{m}} = a_{\vec{n}}^{\vec{m}} = e_{\vec{n}}^{\vec{m}}(A) \Lambda^{\vec{n}\vec{m}}, \quad (33.1)$$

$$\langle A, B \rangle_m = [A, B] = -(\chi_A, \chi_B) = e_{\vec{m}}^{\vec{m}}(A) \Lambda^{\vec{m}\vec{n}} e_{\vec{n}}^{\vec{m}}(B), \quad (33.2)$$

and,

$$\langle A, B.C \rangle_m = \langle A, B \rangle_m.C + B.\langle A, C \rangle_m \xrightarrow[\text{rule}]{\text{Leibnitz}}$$

$$\chi_A(B.C) = \chi_A(B).C + B.\chi_A(C). \quad (33.3)$$

4. Braiding behind LQPS and Yang-Baxter equation :

In the above formalism, we have manipulated $(N^2-1) \times (N^2-1)$ hypermatrices, i.e. matrices whose entries are themselves $N \times N$ matrices. This feature may be considered as a signal of the presence of a Yang-Baxter algebra structure behind the LQPS.

Indeed, the natural place where these hypermatrices may be defined is a direct product space.

So, we will now recall briefly some basic notions of braiding and $\mathcal{R}$ -matrix which we will need in our treatment.

In direct product index notation, the entries of a product of two or three matrices are respectively given by :

$$\langle i|A \otimes B|ab \rangle = (A \otimes B)^{ij}_{ab} = \langle i|A|a \rangle \langle j|B|b \rangle = A^i_a B^j_b \quad (34.1)$$

$$\begin{aligned} \langle ijl|A \otimes B \otimes C|abc \rangle &= (A \otimes B \otimes C)^{ijk}_{abc} \\ &= \langle i|A|a \rangle \langle j|B|b \rangle \langle k|C|c \rangle = A^i_a B^j_b C^k_c. \end{aligned} \quad (34.2)$$

In other respects, the braid group B_n on n strands has $n-1$ generators ρ_i , $i = 1, 2, \dots, n-1$, satisfying the relations :

$$\left\{ \begin{array}{l} \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \\ \rho_i \rho_j = \rho_j \rho_i \text{ if } |i-j| \geq 2. \end{array} \right. \quad (35.1)$$

$$(35.2)$$

Let us consider the case of $n = 3$, and let $\mathcal{B}_{i,i+1}$ denotes the generator ρ_i , i.e. $\rho_1 = \mathcal{B}_{12}$ and $\rho_2 = \mathcal{B}_{23}$.

Then, the braid equation (35.1) reads :

$$\mathcal{B}_{12} \mathcal{B}_{23} \mathcal{B}_{12} = \mathcal{B}_{23} \mathcal{B}_{12} \mathcal{B}_{23} , \quad (35.3)$$

or, in terms of components :

$$\mathcal{B}_{mn}^{ij} \mathcal{B}_{rc}^{hk} \mathcal{B}_{ab}^{mr} = \mathcal{B}_{rm}^{jk} \mathcal{B}_{an}^{ir} \mathcal{B}_{bc}^{nm} . \quad (35.4)$$

This latter equation may also be rewritten in terms of the matrix $\mathcal{R}$ defined by :

$$\mathcal{R} = S \mathcal{B} , \quad (36.1)$$

where S is the permutation map acting on the upper indices, i.e. :

$$\mathcal{R}_{ab}^{ij} = \mathcal{B}_{ab}^{ji} . \quad (36.2)$$

Indeed, with this change we obtain the following Yang-Baxter equation :

$$\mathcal{R}_{mn}^{jl} \mathcal{R}_{rc}^{kn} \mathcal{R}_{ab}^{rm} = \mathcal{R}_{rm}^{kj} \mathcal{R}_{an}^{rl} \mathcal{R}_{bc}^{mn} . \quad (37)$$

If,

$$\mathcal{R} = A \otimes B = \sum_i a_i \otimes b_i , \quad (38.1)$$

then,

$$\mathcal{R}_{12} = A \otimes B \otimes 1 = \sum_i a_i \otimes b_i \otimes 1 , \quad (38.2)$$

$$\mathcal{R}_{13} = A \otimes 1 \otimes B = \sum_i a_i \otimes 1 \otimes b_i , \quad (38.3)$$

$$\mathcal{R}_{23} = 1 \otimes A \otimes B = \sum_i 1 \otimes a_i \otimes b_i . \quad (38.4)$$

Moreover, the matrix elements are given by :

$$\langle ij | \mathcal{R} | ab \rangle = \mathcal{R}_{ab}^{ij} , \quad \langle ijk | \mathcal{R} \otimes 1 | abc \rangle = \delta_c^k \mathcal{R}_{ab}^{ij} . \quad (38.5)$$

Using these notations, the relation (37) reads :

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} . \quad (39)$$

Furthermore, it was shown in [17] that it is possible to define the quantum group which preserves a non-degenerate bilinear form $\mathcal{F}$, in complete analogy with the definition of the ordinary groups. This quantum group is a Hopf algebra defined by a multiplicative matrix (i.e. a matrix quantum group) with an $\mathcal{R}$ -matrix given by :

$$\mathcal{R} = S + \alpha (\mathcal{F}^{-1})^t \otimes \mathcal{F} \quad (40.1)$$

with $\alpha \in \mathbb{C}$, or in terms of components :

$$\mathcal{R}_{ab}^{ij} = \delta_b^i \delta_a^j + \alpha \mathcal{F}_{ab}^{ij} . \quad (40.2)$$

This matrix satisfies the Yang-Baxter equation (37) or (39) if and only if α is a solution of :

$$\alpha + \alpha^{-1} + \mathcal{F}_{ij}^{ij} \mathcal{F}_{ij} = 0 . \quad (40.3)$$

In our matrix symplectic formalism, the hypermatrix $\Omega_{mn}^{\rightarrow}$ will play the role of a non-degenerate bilinear form.

Then, the following $\mathcal{R}$ -matrix :

$$R_{ab}^{\vec{t}\vec{j}} = \delta_{ab}^{\vec{t}} \delta_{ab}^{\vec{j}} + \alpha \Lambda_{ab}^{\vec{j}\vec{t}} \Omega_{ab}^{\vec{t}\vec{j}} \quad (41.1)$$

is a solution of the Yang-Baxter equation (37) provided that α satisfy the following complementary condition :

$$1 + \alpha + \alpha^2 N^2 = 0. \quad (41.2)$$

Here, we have used the following identity :

$$\sum_{\vec{n}, \vec{m}} \Omega_{mn}^{\vec{t}\vec{j}} \Lambda_{mn}^{\vec{t}\vec{j}} = \sum_{\vec{m}, \vec{n}} \Lambda_{mn}^{\vec{t}\vec{j}} \Omega_{mn}^{\vec{t}\vec{j}} = N^2. \quad (41.3)$$

However, we will present hereunder another manner to build a non-trivial solution to the Yang-Baxter equation (37).

It's obvious that the matrices $\gamma^{\vec{m}}$ defined by the relation (27.2) obey a generalized composition law analog to (12.1), i.e. :

$$\gamma^{\vec{m}} \cdot \gamma^{\vec{n}} = \exp[2i\alpha_2(\vec{n}, \vec{m})] \gamma^{\vec{n}} \cdot \gamma^{\vec{m}} = \exp[i\alpha_2(\vec{n}, \vec{m})] \gamma^{\vec{m}+\vec{n}}. \quad (42)$$

It is straightforward to think that this latter relation can be generalized in the following way :

$$\gamma^{\vec{m}} \cdot \gamma^{\vec{n}} = B_{ab}^{\vec{m}+\vec{n}} \gamma^{\vec{a}} \cdot \gamma^{\vec{b}} \quad (43)$$

with commutation coefficients given by :

$$B_{ab}^{\vec{m}+\vec{n}} = \delta_{a+b}^{\vec{m}+\vec{n}} \exp\{i[\alpha_2(\vec{a}, \vec{b}) - \alpha_2(\vec{m}, \vec{n})]\}. \quad (44)$$

Using the associativity property (see (9.2)), i.e. :

$$(\gamma^{\vec{m}} \cdot \gamma^{\vec{n}}) \cdot \gamma^{\vec{k}} = \gamma^{\vec{m}} \cdot (\gamma^{\vec{n}} \cdot \gamma^{\vec{k}}), \quad (45)$$

with,

$$\begin{aligned} (\gamma^{\vec{m}} \cdot \gamma^{\vec{n}}) \cdot \gamma^{\vec{k}} &= B_{ab}^{\vec{m}+\vec{n}} \gamma^{\vec{a}} \cdot (\gamma^{\vec{b}} \cdot \gamma^{\vec{k}}) \\ &= B_{ab}^{\vec{m}+\vec{n}} B_{tj}^{\vec{b}+\vec{k}} (\gamma^{\vec{a}} \cdot \gamma^{\vec{t}}) \cdot \gamma^{\vec{j}} \\ &= B_{ab}^{\vec{m}+\vec{n}} B_{tj}^{\vec{b}+\vec{k}} B_{ot}^{\vec{a}+\vec{t}} \gamma^{\vec{o}} \cdot \gamma^{\vec{t}} \cdot \gamma^{\vec{j}} \end{aligned}$$

and,

$$\begin{aligned} \gamma^{\vec{m}} \cdot (\gamma^{\vec{n}} \cdot \gamma^{\vec{k}}) &= B_{ta}^{\vec{n}+\vec{k}} (\gamma^{\vec{m}} \cdot \gamma^{\vec{t}}) \cdot \gamma^{\vec{a}} \\ &= B_{ta}^{\vec{n}+\vec{k}} B_{ab}^{\vec{m}+\vec{t}} \gamma^{\vec{o}} \cdot (\gamma^{\vec{b}} \cdot \gamma^{\vec{a}}) \\ &= B_{ta}^{\vec{n}+\vec{k}} B_{ab}^{\vec{m}+\vec{t}} B_{tj}^{\vec{b}+\vec{a}} \gamma^{\vec{o}} \cdot \gamma^{\vec{t}} \cdot \gamma^{\vec{j}}, \end{aligned}$$

then, by identification we obtain the braid equation (35.4.3) :

$$B_{ab}^{\vec{m}+\vec{n}} B_{tj}^{\vec{b}+\vec{k}} B_{ot}^{\vec{a}+\vec{t}} = B_{ta}^{\vec{n}+\vec{k}} B_{ab}^{\vec{m}+\vec{t}} B_{tj}^{\vec{b}+\vec{a}} \quad (46.1)$$

$$B_{12} B_{23} B_{12} = B_{23} B_{12} B_{23} \quad (46.2)$$

Diagrammatically, this equation is expressed as follows :

- Fig. 1 -

Before going farther, let us recall that we manipulate double

indices and that the value $\vec{0} = (0,0)$ is excluded.

Examining the matrix elements $\mathcal{B}_{\vec{a}\vec{b}}^{\vec{m}\vec{n}}$ (44), the following cases

arise :

-1- If $\vec{m} = \vec{a}$ and $\vec{n} = \vec{b}$, then :

$$\mathcal{B}_{\vec{a}\vec{b}}^{\vec{a}\vec{b}} = \delta_{\vec{a}}^{\vec{a}} \delta_{\vec{b}}^{\vec{b}}. \quad (47.1)$$

We are here in presence of the identity transformation: $\mathcal{B} = 1$.

-2- If $\vec{m} = \vec{b}$ and $\vec{n} = \vec{a}$, then :

$$\mathcal{B}_{\vec{a}\vec{b}}^{\vec{b}\vec{a}} = \delta_{\vec{a}}^{\vec{b}} \delta_{\vec{b}}^{\vec{a}} \exp[2i\alpha_2(\vec{a}, \vec{b})]. \quad (47.2)$$

This case corresponds to the usual permutation of two strands

since $\mathcal{B}^2 = 1$.

Finally, the other cases correspond effectively to a braiding

and this is essentially tied to the fact that the matrix $\Lambda_{\vec{m}\vec{n}}^{\vec{a}\vec{b}}$ is

not antisymmetric (see equ. (26.3)). Indeed, the relation (43) is

the same as the following one :

$$\Lambda_{\vec{m}\vec{n}}^{\vec{a}\vec{b}} = \sum_{\vec{a}', \vec{b}'} \mathcal{B}_{\vec{a}'\vec{b}'}^{\vec{m}\vec{n}} \Lambda_{\vec{a}\vec{b}}^{\vec{a}'\vec{b}'}. \quad (48)$$

Introduce now, a $\mathcal{R}$ -matrix as defined by (36.1-2) :

$$\mathcal{R}_{\vec{a}\vec{b}}^{\vec{m}\vec{n}} = \mathcal{B}_{\vec{a}\vec{b}}^{\vec{n}\vec{m}} = \delta_{\vec{a}+\vec{b}}^{\vec{n}+\vec{m}} \exp \left\{ i[\alpha_2(\vec{a}, \vec{b}) + \alpha_2(\vec{m}, \vec{n})] \right\}. \quad (49)$$

Replacing into the braid equation (46.1-2), we obtain the

Yang-Baxter equation :

$$\mathcal{R}_{\vec{a}\vec{b}}^{\vec{n}\vec{m}} \mathcal{R}_{\vec{t}\vec{j}}^{\vec{k}\vec{l}} \mathcal{R}_{\vec{s}\vec{i}}^{\vec{l}\vec{a}} = \mathcal{R}_{\vec{t}\vec{a}}^{\vec{k}\vec{n}} \mathcal{R}_{\vec{s}\vec{b}}^{\vec{l}\vec{m}} \mathcal{R}_{\vec{j}\vec{i}}^{\vec{a}\vec{l}}, \quad (50.1)$$

or in compact notation :

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \quad (50.2)$$

Using (1) and (13.1), the relation (49) is rewritten as :

$$\mathcal{R}_{\vec{a}\vec{b}}^{\vec{m}\vec{n}} = \delta_{\vec{a}+\vec{b}}^{\vec{m}+\vec{n}} \omega^{\frac{1}{2}(\vec{a}\vec{b} + \vec{m}\vec{n})}. \quad (51.1)$$

It's clear that the $\mathcal{R}$ -matrices are of $(N^2-1)^2 \times (N^2-1)^2$ type. For

$N = 2$, $\mathcal{R}$ is a 9×9 matrix; For $N=3$, it is a 64×64 -matrix; etc...

Let's consider the case $N=2$, with :

$$\vec{1} = (1,0) \equiv 1, \quad \vec{2} = (1,1) \equiv 2, \quad \vec{3} = (0,1) \equiv 3 \quad \text{and} \quad \omega = e^{i\pi} = -1.$$

The corresponding $\mathcal{R}$ -matrix is then given by :

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & \omega & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \omega^{-1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \omega^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega^{-1} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (51.2);$$

We easily verify that this solution is non-trivial, i.e. $\mathcal{R}^2 \neq 1$.

We will assume that non-commutativity in our above formalism

is absorbed in the coefficients $a_{\vec{m}}$ of the Fourier expansion of an

arbitrary matrix (operator) A :

$$A = \frac{1}{N} \sum_{\vec{m}} a_{\vec{m}} \gamma_{\vec{m}}^{\vec{m}} . \quad (52)$$

One quantum group associated to this $\mathcal{R}$ -matrix is obtained by taking A in the defining non-trivial commutation relations :

$$\mathcal{R} T_1 T_2 = T_2 T_1 \mathcal{R} , \quad (53)$$

where,

$$T_1 = A \otimes 1 , \quad T_2 = 1 \otimes A , \quad (54)$$

to be a 3×3 -matrix of the form :

$$A = \frac{1}{2} \begin{pmatrix} a_3 & a_1 - i a_2 & 0 \\ a_1 + i a_2 & -a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (55)$$

where a_i means $a_{\vec{i}}$ as mentioned above.

Then, using (53) we obtain the following non-commutative Heisenberg algebra relations :

$$a_1 a_2 = - a_2 a_1 , \quad a_1 a_3 = a_3 a_1 , \quad a_2 a_3 = - a_3 a_2 ,$$

$$(a_1)^2 - (a_2)^2 + (a_3)^2 = 1 . \quad (56)$$

We remark the appearance of some unsuspected supersymmetric aspect which can be possibly interpreted in the context of braided algebras and groups, [20].

5. Conclusion :

In this work, we have shown that using the Weyl-Schwinger realization of the Heisenberg group, behind the Lattice quantum phase space there is some braiding and Hopf algebra structure with a non-trivial solution of the Yang-Baxter equation.

However, our $N=2$ $\mathcal{R}$ -matrix is somewhat different from those found in the literature, [18].

To this $\mathcal{R}$ -matrix, one can associate another more general quantum group by taking the matrix A to be the 3×3 -matrix given by (55) with six other indeterminates in the last column and line.

Finally, we plan to study in a future paper other features of the matrix Hamiltonian formalism on LQPS, especially its relation with knot theory, [19], and its extension to braided algebras, [20].

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