Superdiffeomorphisms of the Topological Yang-Mills and Renormalization

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Abstract. We give the superdiffeomorphisms transformations of the four-dimensional topological Yang-Mills theory in curved manifold and we discuss the ultraviolet renormalization of the model. The explicit expression of the most general counterterm is given.

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1 Introduction

Some years ago Witten introduced the topological Yang-Mills model [1] in order to give a quantum field theoretical interpretation of the Donaldson polynomials [2]. These polynomials are global topological invariants, and correspond to the observables of a general covariant field theory on a four manifold, which can be viewed as a twisted version of \( N = 2 \) supersymmetric Yang-Mills field theory.

A further step in this direction was accomplished by the authors of [3], [4] and [5]. Indeed, they reformulated the model in the BRS [6] framework, which, after gauge fixing, yields to the topological Yang-Mills theory. And in fact it was observed that such theory arise after gauge-fixing a topological invariant term. The renormalization of such a theory was considered in [7] – [13].

The physical motivation behind studying the topological quantum field theories (see [14] for a general review), is that they may give us a description of a phase of unbroken diffeomorphisms invariance in quantum gravity.

One motivation of the present paper is to analyse the ultraviolet behaviour of the topological Yang-Mill theory on a curved four dimensional manifold by making use of the superdiffeomorphisms symmetry. This is done in two steps: first we extend the classical analysis of [13] to a four dimensional Riemannian manifold, constructing the superdiffeomorphism transformations, which are nothing else than a generalisation of the vector supersymmetry in a curved space-time [15], [16]. And this will make the subject of section 2.

Second, we discuss the ultraviolet behaviour of the topological Yang-Mills, where we will use the algebraic renormalization techniques, which suppose the existence of a substraction scheme. This is not true for any curved manifold, but at least such substraction scheme exists in manifolds which are topologically equivalent to the four dimensional flat space-time and endowed with asymptotically flat metric. As the topology does not modify the short-distance (the ultraviolet) behaviour of the theory, then in our analysis we expect the same ultraviolet properties as in the flat space-time limit [13]. This will be done in section 3. The paper ends with a conclusion.
2 The Classical analysis in the Landau Gauge

We devote this section to the investigation of the properties of the topological Yang-Mills theory on a curved Riemannian manifold \( \mathcal{M} \) endowed with a metric \( g_{\mu\nu} \).

In the Landau gauge and in the context of such a geometrical background the gauge fixed action is:

\[
\Sigma_{gf} = s \int_{\mathcal{M}} d^4 x \sqrt{g} \left\{ g^{\mu\nu} g^{\rho\sigma} \chi_{\mu\rho}^{\nu} F_{\mu\nu}^{\rho\sigma} - g^{\mu\nu} \left( \partial_\mu \varphi \right) \psi_\mu - g^{\mu\nu} \left( \partial_\mu \tilde{\varphi} \right) A_\nu \right\},
\]

where \( s \), is the nilpotent BRS operator, \( g \) is the determinant of the metric \( g_{\mu\nu} \) and \( g^{-1} \) its inverse. The other fields have their values in the adjoint representation of the gauge group \( G \), which is a Lie group, supposed to be compact. \( f^{abc} \) denotes the structure constants.

We have also:

\[
F_{\mu\nu}^{\rho\sigma} = \frac{1}{2} (F_{\mu\nu} + \tilde{F}_{\mu\nu}),
\]

with

\[
\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} g^{\alpha\beta} g_{\rho\sigma} F_{\alpha\beta},
\]

and

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A_\mu^b A_\nu^c,
\]

being the curvature associated with the gauge connection \( A_\mu^a \). The topological ghost is represented by the field \( \psi_\mu^a \).

The full set of BRS transformations of the fields is:

\[
sA_\mu^a = -(D_\mu \varphi)^a + \psi_\mu^a,
\]

\[
s\psi_\mu^a = f^{abc} c^b \psi_\mu^c + (D_\mu \varphi)^a,
\]

\[
s c^a = \frac{1}{2} f^{abc} c^b c^c + \varphi^a,
\]

\[
s \varphi^a = f^{abc} c^b \varphi^c,
\]

\[
s \tilde{\varphi}^a = b^a, \\
ns \psi_\mu^a = \eta^a, \\
ns \varphi^a = s \eta^a = 0, \\
ns \chi_{\mu\nu}^a = B_{\mu\nu}^a, \\
ns sB_{\mu\nu}^a = 0.
\]

\( c^a \) and \( \varphi^a \) are the two scalar ghost fields corresponding respectively to \( A_\mu^a \) and \( \psi_\mu^a \). We have also the three couples \(( \chi_{\mu\nu}^a, B_{\mu\nu}^a ), (\tilde{\varphi}^a, \eta^a) \) and \(( \tilde{\psi}^a, b^a )\) each consisting of an antighost
field and the corresponding Lagrange multiplier. The Lagrange multiplier $B^a_{\mu\nu}$ enforces the instantonic condition

$$F^a_{\mu\nu} = 0,$$  \hspace{1cm} (6)

whereas $\eta^a$ and $b^a$ enforce the gauge fixings for the two fields $\psi^a_{\mu}$ and $A^a_{\mu}$ respectively. The next step is to introduce the external sources [18] coupled to the BRS transformations which are non linear in the fields. This external sources generate the external part of the action, which we denote by $\Sigma_{ext}$:

$$\Sigma_{ext} = \int_{\mathcal{M}} d^4x \left\{ \Omega^{a\mu} (D_\mu c)^a + \tau^{a\mu} (s\psi^a_{\mu}) + \frac{1}{2} L^a f^{abc} c^b c^c + D^a (s\bar{\phi}^a) \right\},$$  \hspace{1cm} (7)

where $\Omega^{a\mu}$ and $\tau^{a\mu}$ are contravariant vector densities of weight one, whereas $L^a$ and $D^a$ are scalar densities, also of weight one. If we let the sources transform under BRS [13] according to:

$$s\tau^{a\mu} = \Omega^{a\mu}, \hspace{1cm} s\Omega^{a\mu} = 0,$$

$$sD^a = L^a, \hspace{1cm} sL^a = 0,$$  \hspace{1cm} (8)

then $\Sigma_{ext}$ can be written in the simpler form:

$$\Sigma_{ext} = s \int_{\mathcal{M}} d^4x \left( \tau^{a\mu} (D_\mu c)^a + \frac{1}{2} D^a f^{abc} c^b c^c \right),$$  \hspace{1cm} (9)

This leads to the total action:

$$\Sigma = \Sigma_{gf} + \Sigma_{ext}$$  \hspace{1cm} (10)

which is just an $s$-exact expression:

$$\Sigma = s \int_{\mathcal{M}} d^4x \left\{ \sqrt{g} \left( g^{\mu\alpha} g^{\nu\beta} \chi^{a}_{\alpha\beta} F^a_{\mu\nu} - g^{\mu\nu} [\partial_{\mu}\bar{\phi}^a] \bar{\psi}^a_{\nu} - g^{\mu\nu} [\partial_{\nu}\bar{\phi}^a] A^a_{\mu} \right) + \tau^{a\mu} (D_\mu c)^a + \frac{1}{2} D^a f^{abc} c^b c^c \right\};$$  \hspace{1cm} (11)

Note that the operator $s$ is nilpotent.

One can notice that in (11) the metric $g_{\mu\nu}$ is introduced through a trivial BRS variation term, then the physical objects are metric independent. In this case the metric is just a gauge parameter.

The above arguments allow us to extend the BRS transformation [17] by letting $s$ acting on the metric [15] in the following way:

$$sg_{\mu\nu} = \hat{g}_{\mu\nu}, \hspace{1cm} s\hat{g}_{\mu\nu} = 0,$$  \hspace{1cm} (12)

3
Let us now, give in table (1) the dimensions an the ghost numbers of the set of the fields described above:

<table>
<thead>
<tr>
<th></th>
<th>$A^a_\mu$</th>
<th>$\psi^a_\mu$</th>
<th>$e^a$</th>
<th>$\varphi^a$</th>
<th>$\lambda^a_{\mu\nu}$</th>
<th>$B^a_{\mu\nu}$</th>
<th>$\varphi^a$</th>
<th>$\eta^a$</th>
<th>$\bar{\eta}^a$</th>
<th>$h^a$</th>
<th>$\Omega^a_{\mu\nu}$</th>
<th>$\tau^a_{\mu}$</th>
<th>$L^a$</th>
<th>$D^a$</th>
<th>$g_{\mu\nu}$</th>
<th>$\hat{g}_{\mu\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi\Pi$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Dimensions and ghost numbers of the fields

The next step is to generalise the vectors supersymmetry found in [13], to a curved space-time. To this end we propose the following set of transformations with respect to a contravariant vector field $\xi^\mu$ of ghost number +2.

$$
\begin{align*}
\delta^S_{(\xi)} A^a_\mu &= 0, & \delta^S_{(\xi)} \psi^a_\mu &= \mathcal{L}_\xi A^a_\mu, \\
\delta^S_{(\xi)} e^a &= 0, & \delta^S_{(\xi)} \varphi^a &= \mathcal{L}_\xi e^a, \\
\delta^S_{(\xi)} \lambda^a_{\mu\nu} &= 0, & \delta^S_{(\xi)} B^a_{\mu\nu} &= \mathcal{L}_\xi \lambda^a_{\mu\nu}, \\
\delta^S_{(\xi)} \bar{\varphi}^a &= 0, & \delta^S_{(\xi)} \bar{\eta}^a &= \mathcal{L}_\xi \bar{\varphi}^a, \\
\delta^S_{(\xi)} \bar{e}^a &= \mathcal{L}_\xi \bar{\varphi}^a, & \delta^S_{(\xi)} \bar{h}^a &= \mathcal{L}_\xi \bar{e}^a - \mathcal{L}_\xi \eta^a, \\
\delta^S_{(\xi)} \bar{\tau}^a_{\mu} &= 0, & \delta^S_{(\xi)} \Omega^a_{\mu\nu} &= \mathcal{L}_\xi \tau^a_{\mu}, \\
\delta^S_{(\xi)} D^a &= 0, & \delta^S_{(\xi)} L^a &= \mathcal{L}_\xi D^a, \\
\delta^S_{(\xi)} g_{\mu\nu} &= 0, & \delta^S_{(\xi)} \hat{g}_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu}. \\
\end{align*}
$$

(13)

It is straightforward to check that the anticommutator between the BRS operators and the superdiffeomorphisms operator $\delta^S_{(\xi)}$ closes on the Lie derivatives in the direction of $\xi^\mu$:

$$
\{ s, \delta^S_{(\xi)} \} = \mathcal{L}_\xi.
$$

(14)

At the functional level, the superdiffeomorphism transformations, are implemented by means of the following Ward operator $W^S_{(\xi)}$:

$$
W^S_{(\xi)} = \int_M d^4 x \sum_f \delta^S_{(\xi)} \int \delta f, 
$$

(15)

where $f$ describes the set of all the fields transforming under $\delta^S_{(\xi)}$.

Now, let the operator $W^S_{(\xi)}$ acting on the total action (11), and we get:

$$
W^S_{(\xi)} \Sigma = s \int_M d^4 x \left\{ \sqrt{g} g^\mu\nu \mathcal{L}_\xi \left[ \left( \partial_\mu \varphi^a \right) A^a_\nu \right] \right\},
$$

(16)
At this point, one can remark that in the special case where $\xi^\mu$ is a Killing vector ($\mathcal{L}_{\xi} g_{\mu\nu} = 0$) the right hand side of (16) will vanish. But in a general situation such breaking exist, which is quadratic in the quantum fields, this will generate problems at the quantum level, if one would try to quantise this model. Fortunately, one can control such a non-linear breaking by absorbing it in the original action by introducing two auxiliary fields [19], which we will call $L^{\mu\nu}$ and $M^{\mu\nu}$. They are both symmetric contravariant tensors of rank two and weight +1.

By adding to the original action (11) the new term

$$\Sigma_{L,M} = - \int_{\mathcal{M}} d^4 x [L^{\mu\nu} \Xi_{\mu\nu} - M^{\mu\nu} s\Xi_{\mu\nu}].$$

(17)

with:

$$\Xi_{\mu\nu} = (\partial_\mu \bar{\phi}^a) A^a_{\nu}$$

(18)

we will get a vanishing right hand side in (16). So, our action takes the following form:

$$\Sigma = \Sigma_{gf} + \Sigma_{ext} + \Sigma_{L,M}.$$  

(19)

And in order to preserve the BRS invariance of this new action, we impose the following transformations on the two fields $L^{\mu\nu}$ and $M^{\mu\nu}$:

$$sM^{\mu\nu} = L^{\mu\nu}, \quad sL^{\mu\nu} = 0.$$  

(20)

Under the superdiffeomorphisms the auxiliary fields transform as:

$$\delta^S_{(\xi)} M^{\mu\nu} = \mathcal{L}_\xi (\sqrt{g} g^{\mu\nu}),$$

$$\delta^S_{(\xi)} L^{\mu\nu} = \mathcal{L}_\xi [M^{\mu\nu} - s(\sqrt{g} g^{\mu\nu})],$$

(21)

guaranteeing both, the absence of the hard breaking in (16) and the validity of (14).

So, now we have the following identity:

$$\mathcal{W}^S_{(\xi)} \Sigma = 0$$

(22)

where $\mathcal{W}^S_{(\xi)}$ is the operator defined above in (15), and $f$ includes now $L^{\mu\nu}$ and $M^{\mu\nu}$.
In table (2) we give the dimensions and the ghost numbers of the fields $L^{\mu\nu}$ and $M^{\mu\nu}$:

<table>
<thead>
<tr>
<th></th>
<th>$L^{\mu\nu}$</th>
<th>$M^{\mu\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi \Pi$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: dimensions and ghost numbers of the auxiliary fields.

At the functional level, the BRS invariance of the action is characterised by the Slavnov identity:

$$
S(\Sigma) = \int_{\mathcal{M}} d^4 x \left( \psi^a_\mu \frac{\delta \Sigma}{\delta A^a_\mu} - \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta A^a_\mu}{\delta \psi^a_\mu} + \varphi^a \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta c^a}{\delta \check{c}^a} + \frac{\delta \Sigma}{\delta \tau^{a\mu}} \frac{\delta \psi^a_\mu}{\delta \check{c}^a} + \frac{\delta \Sigma}{\delta \varphi^a} \frac{\delta \psi^a}{\delta \tau^{a\mu}} + \frac{\delta \Sigma}{\delta \omega^{a\mu}} \frac{\delta \psi^a_\mu}{\delta \varphi^a} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta \psi^a}{\delta \omega^{a\mu}} + \frac{\delta \Sigma}{\delta \tau^{a\mu}} \frac{\delta \psi^a}{\delta \tau^{a\mu}} + \frac{\delta \Sigma}{\delta \omega^{a\mu}} \right) = 0
$$

from which one can write down the corresponding linearised Slavnov operator:

$$
S_{\Sigma} = \int_{\mathcal{M}} d^4 x \left( \psi^a_\mu \frac{\delta \Sigma}{\delta A^a_\mu} - \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta A^a_\mu}{\delta \psi^a_\mu} + \varphi^a \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta c^a}{\delta \check{c}^a} + \frac{\delta \Sigma}{\delta \tau^{a\mu}} \frac{\delta \psi^a_\mu}{\delta \check{c}^a} + \frac{\delta \Sigma}{\delta \varphi^a} \frac{\delta \psi^a}{\delta \tau^{a\mu}} + \frac{\delta \Sigma}{\delta \omega^{a\mu}} \frac{\delta \psi^a}{\delta \omega^{a\mu}} + \frac{\delta \Sigma}{\delta \tau^{a\mu}} \frac{\delta \psi^a}{\delta \tau^{a\mu}} + \frac{\delta \Sigma}{\delta \omega^{a\mu}} \right).
$$

Beside the BRS and the superdiffeomorphism invariance, the action (19) is also invariant under diffeomorphism transformations, such that:

$$
W_{(\varepsilon)}^{D} \Sigma = \int_{\mathcal{M}} d^4 x \sum_f \mathcal{L} \frac{\delta \Sigma}{\delta f} = 0.
$$

where the Ward operator for the diffeomorphisms is:

$$
W^{D}_{(\varepsilon)} = \int_{\mathcal{M}} d^4 x \sum_f \mathcal{L} \frac{\delta}{\delta f}.
$$

and $f$ stands for the same fields as in (22), $\varepsilon^\mu$ is a contravariant vector, the parameter of the diffeomorphisms transformations.
For reasons, which will be clear later we change the ghost number of $\varepsilon^\mu$ in such a way to render the operator (26) fermionic.

The dimensions and the ghost numbers of the parameters $\varepsilon^\mu$ and $\xi^\mu$ are given in table (3):

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon^\mu$</th>
<th>$\xi^\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\Phi\Pi$</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3: dimensions and ghost numbers of the parameters.

The action (19) is invariant with respect to three symmetries, the BRS, the diffeomorphisms and superdiffeomorphisms, each symmetry generates a Ward operator and the three Ward operators together display the following linear algebra:

\[
\begin{align*}
\{S_\Sigma, S_\Sigma\} &= 0, \\
\{S_\Sigma, W^{P}_{(\varepsilon)}\} &= 0, \\
\{W^{P}_{(\varepsilon)}, W^{P}_{(\varepsilon')}\} &= -W^{P}_{([\varepsilon, \varepsilon'])}, \\
\{S_\Sigma, W^{S}_{(\xi)}\} &= W^{P}_{(\xi)}, \\
\{W^{S}_{(\xi)}, W^{P}_{(\varepsilon)}\} &= W^{S}_{([\xi, \varepsilon])}, \\
\{W^{S}_{(\xi)}, W^{S}_{(\xi')}\} &= 0.
\end{align*}
\]

where we have denoted by $[\ , \ ]$ the graded Lie bracket, such that:

\[
[\varepsilon, \varepsilon']^\mu = \varepsilon^\lambda \partial_\lambda \varepsilon'^\mu + \varepsilon'^\lambda \partial_\lambda \varepsilon^\mu.
\]

and

\[
[\xi, \varepsilon]^\mu = \xi^\lambda \partial_\lambda \varepsilon^\mu - \varepsilon^\lambda \partial_\lambda \xi^\mu.
\]

Furthermore, the total action (19) also satisfies:

(i) two gauge conditions:

\[
\begin{align*}
\frac{\delta \Sigma}{\delta b^a} &= \partial_\mu (\sqrt{\text{det}} g) A^a_{\mu}, \\
\frac{\delta \Sigma}{\delta \eta^a} &= \partial_\mu (\sqrt{\text{det}} g) \psi^a_{\mu} + \partial_\mu (M^\mu A^a_{\nu}).
\end{align*}
\]
(ii) two antighost equations:

\[
\frac{\delta \Sigma}{\delta \bar{\phi}^a} - \partial_\mu (\sqrt{g} g^{\mu \nu} \frac{\delta \Sigma}{\delta \bar{\Omega}^{\mu \nu}}) = -\partial_\mu [s(\sqrt{g} g^{\mu \nu}) A^\nu_a] - \partial_\mu [\sqrt{g} g^{\mu \nu} \psi^a] ,
\]

and

\[
\frac{\delta \Sigma}{\delta \bar{\psi}^a} - \partial_\mu [\sqrt{g} g^{\mu \nu} \frac{\delta \Sigma}{\delta \bar{\psi}^{\mu \nu}}] - \partial_\mu [M^{\mu \nu} \frac{\delta \Sigma}{\delta \bar{\mathcal{A}}^{\mu \nu}}] = \partial_\mu [s(\sqrt{g} g^{\mu \nu}) \psi^a] + \partial_\mu (L^{\mu \nu} A^a_{\mu} - \partial_\mu (M^{\mu \nu} \psi^a) ,
\]

(iii) two ghost equations, present in the Landau type gauge [21]:

\[
\int_\mathcal{M} d^4 x \frac{\delta \Sigma}{\delta \phi^a} = \int_\mathcal{M} d^4 x f^{abc} \frac{\delta \Sigma}{\delta \bar{\phi}^b} = \int_\mathcal{M} d^4 x f^{abc} (\tau^{b \mu} A^a_{\mu} + D^b c^c)
\]

and

\[
\mathcal{G}^a = \Delta^a
\]

where \( \mathcal{G}^a \) writes as:

\[
\mathcal{G}^a = \int_\mathcal{M} d^4 x \left( \frac{\delta}{\delta c^a} + \frac{1}{2} f^{abc} \phi^b \frac{\delta}{\delta \phi^c} + f^{abc} \frac{\delta}{\delta \psi^c} + f^{abc} \frac{\delta}{\delta \psi^c} \right)
\]

and \( \Delta^a \) has the following form:

\[
\Delta^a = \int_\mathcal{M} d^4 x f^{abc} \left( D^b \phi^c - \Omega^b \mu A^a_{\mu} - \tau^{b \mu} \psi^c_{\mu} - L^b c^c \right)
\]

By commuting the ghost equation (34) with the Slavnov identity (23) we get a further constraint [13]:

\[
\mathcal{F}^a = \int_\mathcal{M} d^4 x f^{abc} \left( A^a_{\mu} \Omega^{\mu} + \tau^{c \mu} \psi^c_{\mu} + L^c c^b + D^b \phi^c \right)
\]

where \( \mathcal{F}^a \) denote the following operator:

\[
\mathcal{F}^a = \int_\mathcal{M} d^4 x \left( \frac{\delta}{\delta c^a} - f^{abc} \frac{\delta}{\delta \phi^b} \phi^b f^{abc} \frac{\delta}{\delta \phi^c} - f^{abc} \frac{\delta}{\delta \psi^c} \phi^b f^{abc} \frac{\delta}{\delta \psi^c} f^{abc} \frac{\delta}{\delta \psi^c} \right)
\]

Further, by anticommuting (35) with (23) we get:

\[
\mathcal{R}^a = 0
\]

with:

\[
\mathcal{R}^a = \int_\mathcal{M} d^4 x \sum_\omega f^{abc} \omega^b \frac{\delta}{\delta \omega^c} + \omega^b \frac{\delta}{\delta \omega^c}
\]

Where \( \omega \) stands for all fields possessing a gauge index.

We conclude this section by noting that the above linear algebra (27) generated by the Slavnov operator and the Ward operators for diffeomorphisms and superdiffeomorphisms, has the same structure as in the case of the Chern-Simons theory [15] and the three dimensional BF model [16] on a curved manifold.
3 Quantization

So far we have considered only the classical analysis of the topological Yang-Mills on an arbitrary curved four dimensional space-time. Now, we would like to discuss the possibility to describe the theory at the quantum level.

The first step is the analysis of the stability and the search of possible counterterms. The most general counterterm $\Delta$ is an integrated local field polynomial of vanishing dimension and ghost number, such that the perturbed action takes the form:

$$\Sigma' = \Sigma + \Delta. \quad (42)$$

It is easy to check that the perturbation $\Delta$ obeys the following constraints:

$$\frac{\delta \Delta}{\delta b^a} = 0, \quad (43)$$

$$\frac{\delta \Delta}{\delta \eta^a} = 0, \quad (44)$$

$$\frac{\delta \Delta}{\delta \bar{e}^a} - \partial_\mu (\sqrt{g} g^{\mu\nu} \frac{\delta \Delta}{\delta \Omega^\nu}) = 0, \quad (45)$$

$$\frac{\delta \Delta}{\delta \bar{e}^a} - \partial_\mu (\sqrt{g} g^{\mu\nu} \frac{\delta \Delta}{\delta \bar{\varphi}^\nu}) - \partial_\mu (M^{\mu\nu} \frac{\delta \Delta}{\delta \Omega^\nu}) = 0, \quad (46)$$

$$\int_M d^4x \frac{\delta \Delta}{\delta \bar{e}^a} = 0, \quad (47)$$

$$G^a \Delta = 0, \quad (48)$$

$$F^a \Delta = 0, \quad (49)$$

$$R^a \Delta = 0, \quad (50)$$

$$S_\Sigma \Delta = 0, \quad (51)$$

$$W^{D}_{(\varepsilon)} \Delta = 0, \quad (52)$$

$$W^{S}_{(\varepsilon)} \Delta = 0. \quad (53)$$

The first two equations (43) and (44) imply that $\Delta$ is independent of the fields $b^a$ and $\eta^a$. From the next two equations (45) and (46), it follows that $\Delta$ depends on $\bar{e}^a$, $\Omega^{a\mu}$, $\bar{\varphi}^a$ and $\tau^{a\mu}$ only through the two combinations:

$$\bar{\Omega}^{a\mu} = \Omega^{a\mu} - \sqrt{g} g^{\mu\nu} \partial_\nu \bar{e}^a - M^{\mu\nu} \partial_\nu \bar{\varphi}^a. \quad (54)$$
and,
\[ \tilde{\tau}^{a\mu} = \tau^{a\mu} - \sqrt{g} g^{\mu\nu} \partial_\nu \tilde{\varphi}^a. \] (55)

The last three equations (51), (52) and (53) can be put together [22] in such a way that they generate a single cohomology problem [15] and [16]:
\[ \delta \Delta = 0 \] (56)

where \( \delta \) denotes a nilpotent operator:
\[ \delta^2 = 0 \] (57)
given by:
\[ \delta = \mathcal{S}_\Sigma + \mathcal{W}_{[\varepsilon]}^D + \mathcal{W}_{\xi}^S + \mathcal{U}(\xi) + \mathcal{V}_{[\varepsilon]}. \] (58)

with:
\[ \mathcal{U}(\xi) = \int_M d^4 x \, [\varepsilon, \xi]^\mu \frac{\delta}{\delta \xi^\mu}. \] (59)

and
\[ \mathcal{V}_{[\varepsilon]} = \int_M d^4 x \{ \frac{1}{2} [\varepsilon, \varepsilon]^\mu - \xi^\mu \} \frac{\delta}{\delta \varepsilon^\mu}. \] (60)

In order to solve the cohomology problem (56) it is useful to introduce a filtering operator \(^2\) [23], which will induce a splitting of the operator \( \delta \) as:
\[ \delta = \delta_0 + \delta_1 + \ldots + \delta_N \] (61)

where the \( \delta_n \) increase the homogeneity degree by \( n \) unites. This will reduce the original cohomology problem (56) to a simpler one involving the operator \( \delta_0 \):
\[ \delta_0 \Delta = 0, \] (62)

the nilpotency of the whole operator \( \delta \) (57) implies:
\[ \delta_0 \delta_0 = 0 \] (63)

Now, let us give the explicite expression of the nilpotent operator \( \delta_0 \):
\[ \delta_0 = \int_M d^4 x \left( \psi^a_\mu \frac{\delta}{\delta \psi^a_\mu} + \varphi^a \frac{\delta}{\delta \varphi^a} + \frac{1}{2} B^a_{\mu\nu} \frac{\delta}{\delta B^a_{\mu\nu}} + \left( \eta^a \frac{\delta}{\delta \varphi^a} + b^a \frac{\delta}{\delta \tilde{\varphi}^a} + + \right. \right. \]
\[ + \left. \left. \left. \Omega^a_{\mu\nu} \frac{\delta}{\delta \Omega^a_{\mu\nu}} + L^a \frac{\delta}{\delta L^a} + \frac{1}{2} \tilde{g}_{\mu\nu} \frac{\delta}{\delta \tilde{g}_{\mu\nu}} + \frac{1}{2} L^\mu_\nu \frac{\delta}{\delta M^\mu_\nu} - \xi^\mu \frac{\delta}{\delta \xi^\mu} \right) \right) \] (64)

\(^2\)All fields have homogeneity degree one, except \( c^a, \varphi^a, \tau^{a\mu} \) and \( \Omega^{a\mu} \) to which we attribute homogeneity degree two.
which implies the triviality of the cohomology of $\delta_0$: all the fields transform as a doublet under $\delta_0$. This means that the cohomology of $\delta_0$, which is the space of all solutions of (62) modulo cocycles $\delta_0 \Delta$, is empty. Now, it is important to stress that the cohomology of $\delta$ is isomorphic to a subspace of the cohomology of $\delta_0$ (see for instance [23]). This implies the triviality of the cohomology of $\delta$.

This last result leaves us with the most general counterterm, which is a $\delta$-exact variation solving (56):

$$\Delta = \delta \Delta.$$ (65)

Then the most general counterterm obeying the constraints (43) – (50), takes the following form:

$$\Delta = \delta \int_M d^4x \left( \alpha_1 A_\mu^a [\Omega^a_{\mu \nu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{e}^a - M^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] + \alpha_2 \psi_\mu^a [\tau^{a \mu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] + \alpha_3 \frac{1}{\sqrt{g}} g^{\mu \nu} A^a_\mu A^b_\nu \chi_{\mu \nu} + \alpha_4 \frac{1}{\sqrt{g}} g^{\mu \nu} M^{\mu \nu} A^a_\mu [\sigma^{a \mu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] + \alpha_5 g^{\mu \nu} \tilde{g}^a_\mu A^a_\nu [\sigma^{a \mu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] + \alpha_6 \sqrt{g} g^{\mu \nu} g^{\rho \sigma} A^a_\rho \partial_{\sigma} \chi_{\mu \nu}^a + \alpha_7 [\tau^{a \mu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] \partial_{\mu} e^a + \alpha_8 g^{\mu \nu} g^{\rho \lambda} \tilde{g}^a_\mu \chi_{\rho \lambda}^a \chi_{\mu \nu}^a + \alpha_9 M^{\mu \nu} \tilde{g}^a_\mu \chi_{\mu \nu}^a + \alpha_{10} \sqrt{g} g^{\mu \nu} \tilde{g}^a_\mu \chi_{\mu \nu}^a + \alpha_{11} \sqrt{g} g^{\mu \nu} \tilde{g}^a_\mu \chi_{\mu \nu}^a \right).$$ (66)

where the $\alpha_i$ are constant coefficients. Let us now remark that the parameters of diffeomorphisms and superdiffeomorphisms are present explicitly in the expression of the $\delta$ operator (58). And as the action (19) is independent of $\xi^\mu$ and $\varepsilon^\mu$ the above counterterm has also to be independent of such parameters. This further requirement will reduce (66) to:

$$\Delta = \mathcal{S} \int_M d^4x \left( \alpha_1 A_\mu^a [\Omega^a_{\mu \nu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{e}^a - M^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] + \alpha_2 \psi_\mu^a [\tau^{a \mu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] + \alpha_3 \frac{1}{\sqrt{g}} g^{\mu \nu} A^a_\mu A^b_\nu \chi_{\mu \nu} + \alpha_4 [\tau^{a \mu} - \sqrt{g} g^{\mu \nu} \partial_{\nu} \tilde{\varphi}^a] \partial_{\mu} e^a + \alpha_5 g^{\mu \nu} M^{\mu \nu} \tilde{g}^a_\mu \chi_{\mu \nu}^a + \alpha_6 \sqrt{g} g^{\mu \nu} \tilde{g}^a_\mu \chi_{\mu \nu}^a + \alpha_7 \sqrt{g} g^{\mu \nu} \tilde{g}^a_\mu \chi_{\mu \nu}^a \right).$$ (67)

where we have used the equality (58). Now we can observe that in the flat space limit (67) reduces to the counterterm already found in [13].
The other important point to discuss in the framework of renormalization theory, is the question of the existence of anomalies.

In our case, and due to the triviality of the cohomology of $\delta$, the Slavnov identity as well as the two Ward identities of diffeomorphisms and superdiffeomorphisms are not anomalous. Then they can be promoted to the quantum level.

Concerning the constraints (43) – (46) they can be also promoted to the quantum level by using standard arguments [20].

The two ghost equations possess a linear breacking, and they are also valid at the quantum level [21].

This will conclude our proof of the ultraviolet renormalization of the topological Yang-Mills considered in a curved space-time, topologically equivalent to a flat space-time and endowed with an asymptotically flat metric.

4 Conclusion

We could extend the renormalization of the topological Yang-Mills theory, already present and carried out in the flat space, to a curved manifold by using the strategy of [15], [16]. Where the manifold has to be topologically equivalent to a flat space and possessing an asymptotically flat metric in order to guarantee the existence of a substraction scheme, which has not to be specified in the framework of the algebraic renormalization. The most general counterterm was also constructed, and the results of [13] is in accordance with our analysis in flat space-time limit.

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References


